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Determination of the Radius of a Self-Pinched Beam from its Energy Integral

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ABSTRACT

The total transverse energy (kinetic plus potential) of a self-pinched beam may be used to predict the final equilibrium radius when the beam is mismatched at injection. The dependence of potential energy on the current profile shape is characterized by a dimensionless parameter $C(z)$, variations of which are correlated with the change of emittance.

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MOTIVATION

In experiments with charged particle beams and also their simulation, it often happens that there is a large mismatch with respect to the equilibrium conditions at injection. It has been the (unfounded) practice to predict the final equilibrium radius by applying the Bennett pinch condition assuming emittance is conserved. The result can easily be in error by an order of magnitude. On the other hand it is desirable to be able to predict the radius within a factor of two, for example to design a differentially pumped transition section. It has been found that the results of simulation are in fair agreement with predictions based on the conservation of transverse energy rather than conservation of emittance. There is an analogy with an ideal gas released suddenly into a vacuum vessel. The resultant equilibrium pressure of the gas, may be determined by equating the initial and final energies. The entropy of the gas which has a resemblence to emittance, increases in the process. It is stressed that energy conservation is only approximate for the beam because eddy currents are induced in the background plasma; these represent an external drain. This problem is partly removed by allowing the eddy current to be described by a neutralization factor. But the factor must be constant to obtain a conserved energy. Thus the results will only be approximate, but they appear to be much more reliable than those obtained by assuming conservation of emittance.

PHYSICAL MODEL

We consider a thin disk of N particles moving at constant velocity (βc) in the z direction. The equations of motion for the transverse coordinates and velocity of the ith particle are

$$\frac{d\underset{\sim}{r}^{(i)}}{dz} = \underset{\sim}{v}^{(i)} \quad , \tag{1}$$

$$\frac{d\underset{\sim}{v}^{(i)}}{dz} = - k_\beta^2 \left(|\underset{\sim}{r}^{(i)}|, z \right) \underset{\sim}{r}^{(i)} \quad . \tag{2}$$

The betatron frequency is written in the convenient form

$$k_\beta^2(r, z) = \frac{2 I_{br}}{r^2 I_A} (1 - f_m) , \tag{3}$$

where $I_A = \beta\gamma mc^3/q$ is the Alfvén current and

$$I_{br} = \int_0^r dr' \; 2\pi r' \; J_b(r', z) \tag{4}$$

is the beam current contained inside radius r. The effect of a counterflowing eddy current is represented by the constant neutralization factor $(1 - f_m)$. The quantity I_{br} is a function of both z and r, but $I_{br}(r \to \infty) = I_b$, the constant net beam current.

Various mean quantities are defined for the N particles of the disk, e.g.,

$$\overline{r^2} = R^2 = \frac{1}{N} \sum_1^N |\underset{\sim}{r}^{(i)}|^2 \quad = \quad \text{mean squared radius,} \tag{5}$$

$$\overline{v^2} = V^2 = \frac{1}{N} \sum_1^N |\underset{\sim}{v}^{(i)}|^2 \quad = \quad \text{mean squared velocity.} \tag{6}$$

If an averaged quantity is a function of $\underset{\sim}{r}^{(i)}$ and z only, then the mean can be written as an integral over r, weighted by the beam current density. We have for example,

$$R^2(z) = \int_0^\infty dr \left(\frac{2\pi r \; J_b}{I_b} \right) r^2 \; . \tag{7}$$

The quantity $\overline{k_\beta^2 r^2}$ is also of this form and can actually be explicitly evaluated independent of the form of J_b and the value of R:

$$\overline{k_\beta{}^2 r^2} = \int_0^\infty dr \left(\frac{2\pi r\ J_b}{I_b}\right)\left(\frac{2\ I_{br}}{r^2 I_A}\right)\left(1 - f_m\right) r^2 \tag{8}$$

$$= \frac{I_b}{I_A}\left(1 - f_m\right)\int_0^\infty dr \left(\frac{\partial}{\partial r}\frac{I_{br}}{I_b}\right)\left(\frac{2\ I_{br}}{I_b}\right)$$

$$= \frac{I_b}{I_A}\left(1 - f_m\right) = \text{constant}\ . \tag{9}$$

If a function depends on $\underset{\sim}{v}^{(i)}$ then its mean cannot generally be cast into an integral over r.

The Energy Integral

If f_m is constant then the N particles of the disk behave as a closed system and there is a conserved total energy integral (W). The form of W is most easily derived starting from the velocity moment of Eq. (2):

$$\underset{\sim}{v}^{(i)} \cdot \frac{d\underset{\sim}{v}^{(i)}}{dz} = - k_\beta{}^2(|\underset{\sim}{r}^{(i)}|\ ,\ z)\left(\underset{\sim}{v}^{(i)} \cdot \underset{\sim}{r}^{(i)}\right). \tag{10}$$

The disk mean of Eq. (10) is

$$\frac{d}{dz}\frac{v^2}{2} = - \overline{k_\beta{}^2\ \underset{\sim}{v} \cdot \underset{\sim}{r}}\quad . \tag{11}$$

The rhs of Eq. (11) depends only linearly on $\underset{\sim}{v}^{(i)} \cdot \hat{e}_r$; this velocity dependence may be removed as follows. At each point r we define the mean radial velocity

$$\langle v_r \rangle = \frac{\sum' \underset{\sim}{v}^{(i)} \cdot \hat{e}_r}{\sum' 1^{(i)}}\quad . \tag{12}$$

The primed sums over (i) include only particles close to a particular value of r. Thus $\langle v_r \rangle$ is a function of r and z, and is the local mean flow velocity if the disk is considered to be a fluid. Equation (11) takes the form

$$\frac{d}{dz}\ \frac{\overline{v^2}}{2} = -\ \overline{k_\beta{}^2 \langle v_r \rangle\, r}$$

$$= -\int_0^\infty dr \left(\frac{2\pi r\ J_b}{I_b}\right) k_\beta{}^2 \langle v_r \rangle\, r\ . \tag{13}$$

The local mean flow velocity $\langle v_r \rangle$ is related to the beam profile through the continuity equation

$$\frac{\partial J_b}{\partial z} = -\ \frac{1}{r}\frac{\partial}{\partial r}\, r\ J_b \langle v_r \rangle\ . \tag{14}$$

We integrate once in r to obtain

$$r\ J_b \langle v_r \rangle\ = -\int_0^r dr'\ r'\ \frac{\partial J_b(r',\ z)}{\partial z}\ . \tag{15}$$

The rhs of Eq. (15) is conveniently written using I_{br} (see Eq. 4):

$$2\pi r\ J_b \langle v_r \rangle\ = -\ \frac{\partial I_{br}}{\partial z}\ , \tag{16}$$

which states the obvious fact that the flux of particles through a surface at radius r equals the negative rate of change of the number of particles inside radius r.

Eliminating $\langle v_r \rangle$ from Eq. (13) with Eq. (16) we have

$$\frac{d}{dz}\frac{\overline{v^2}}{2} = \int_0^\infty dr\ \frac{k_\beta{}^2 r}{I_b}\ \frac{\partial I_{br}}{\partial z}\ . \tag{17}$$

Substituting from Eq. (3) for $k_\beta{}^2$, this becomes

$$\frac{d}{dz}\frac{\overline{v^2}}{2} = \int_0^\infty dr\left(\frac{2\ I_{br}}{r^2\ I_A}\right)\left(1 - f_m\right)\frac{r}{I_b}\ \frac{\partial I_{br}}{\partial z}$$

$$= \frac{I_b}{I_A}\left(1 - f_m\right)\int_0^\infty \frac{dr}{r}\left(\frac{2\ I_{br}}{I_b}\right)\left(\frac{\partial}{\partial z}\ \frac{I_{br}}{I_b}\right)\ . \tag{18}$$

A cutoff at large radius r = b is inserted to avoid having to deal with an infinite constant--this does not affect any physical result. Then the rhs of Eq. (128) is a complete derivitive in z:

$$\frac{d}{dz}\frac{V^2}{2} = \frac{I_b}{I_A}\left(1 - f_m\right)\frac{d}{dz}\int_0^b \frac{dr}{r}\left(\frac{I_{br}}{I_b}\right)^2 \quad . \tag{19}$$

The conserved total energy is

$$W = \frac{V^2}{2} - \frac{I_b}{I_A}\left(1 - f_m\right)\int_0^b \frac{dr}{r}\left(\frac{I_{br}}{I_b}\right)^2 = \text{constant} \; . \tag{20}$$

Relation of W to Envelope Equation and Emittance Change

The beam disk is also described by an envelope equation for R; for the case at hand this is[1]

$$\frac{\partial^2 R}{\partial z^2} = \frac{E^2}{R^3} - \frac{\overline{k_\beta^{\,2} r^2}}{R} \quad , \tag{21}$$

where the emittance (E) is defined as

$$E^2 = R^2\left[V^2 - \left(\frac{dR}{dz}\right)^2\right] . \tag{22}$$

It is known that if the beam profile does not change shape then E^2 is constant. When the profile does change we may use the energy integral to evaluate the change of emittance.

The energy can be written in the following form:

$$W = \frac{V^2}{2} - \overline{k_\beta^{\,2} r^2}\left[\log\left(\frac{b}{R}\right) - \frac{C}{2}\right] \quad . \tag{23}$$

Here we have substituted $\overline{k_\beta^{\,2} r^2} = (1 - f_m)\, I_b/I_A$ and explicitly removed the dominant log(R) dependence from the integral over r. The dimensionless parameter C(z) characterizes the profile shape and may be calculated for any particular $J_b(r, z)$ from its defining formula:

$$C \equiv -2 \int_0^b \frac{dr}{r}\left(\frac{I_{br}}{I_b}\right)^2 + \log\left(\frac{b^2}{R^2}\right) . \tag{24}$$

Example: (flat profile)

$$J_b = \begin{cases} \dfrac{I_b}{\pi a^2} & 0 < r < a \ , \\ 0 & r > a \ , \end{cases} \tag{25a}$$

$$\frac{I_{br}}{I_b} = \begin{cases} \dfrac{r^2}{a^2} & 0 < r < a \ , \\ 1 & r > a \ , \end{cases} \tag{25b}$$

$$R^2 = \frac{a^2}{2} \ . \tag{25c}$$

Then Eq. (24) gives

$$C_{flat} = -2 \int_0^a \frac{dr}{r}\left(\frac{r^2}{a^2}\right)^2 - 2 \int_a^b \frac{dr}{r}\,(1)^2 + \log\left(\frac{b^2}{a^2/2}\right)$$

$$= -\frac{1}{2} - \log\left(\frac{b^2}{a^2}\right) + \log\left(\frac{2b^2}{a^2}\right)$$

$$= -\frac{1}{2} + \log(2) = +\ .193147 \ . \tag{25d}$$

Other profiles give values of C in the range (see Table 1)

$$0 \le C \le C_{flat} \ , \tag{26}$$

with non-pathological cases in the range

$$C = (.15 \pm .04) \ . \qquad (27)$$

From Eqs. (22) and (23) we have

$$2W = \frac{E^2}{R^2} + (R')^2 - 2\,\overline{k_\beta^2 r^2}\left[\log\left(\frac{b}{R}\right) - \frac{C}{2}\right] . \qquad (28)$$

Here the prime notation denotes d/dz. Taking the derivitive of Eq. (28) and substituting from the envelope equation for R" we have

$$0 = 2W' = \frac{E^{2\prime}}{R^2} - \frac{2E^2}{R^3} R' + 2R'\left[\frac{E^2}{R^3} - \frac{\overline{k_\beta^2 r^2}}{R}\right] - 2\,\overline{k_\beta^2 r^2}\left[-\frac{R'}{R} - \frac{C'}{2}\right]$$

$$= \frac{E^{2\prime}}{R^2} + \overline{k_\beta^2 r^2}\, C' \ . \qquad (29)$$

Thus we have the desired relation

$$E^{2\prime} = -\,\overline{k_\beta^2 r^2}\, R^2\, C' \ . \qquad (30)$$

As shape changes, emittance also changes. The correlation of emittance change with R must be such that damping of oscillations occurs in the small amplitude domain. This has been modeled with the phenomenological relation(1)

$$E^{2\prime} = -\frac{2\alpha\, \overline{k_\beta^2 r^2}\, R^2 R''}{\left(\frac{E}{R} + \overline{k_\beta^2 r^2}\,\frac{R}{E}\right)} \ . \qquad (31)$$

Comparison with Eq. (30) yields

$$C' = +\frac{2\alpha R''}{\left(\frac{E}{R} + \overline{k_\beta^2 r^2}\,\frac{R}{E}\right)} \ . \qquad (32)$$

The appropriate value of α has been estimated to be [1]

$$\alpha^2 = \left(\frac{\overline{k_\beta^4 r^2}\, R^2}{\left(\overline{k_\beta^2 r^2}\right)^2} - 1 \right) . \tag{33}$$

Table 1 includes values of α along with values of C for several beam profiles.

TABLE I

Profile Name	J_b/I_b	I_{br}/I_b*	R	C	α
Flat	$\frac{H(a-r)}{\pi a^2}$	$\frac{r^2}{a^2}$	$\frac{a}{\sqrt{2}}$	.193147	0
Parabolic	$\frac{2}{\pi a^2}\left(1-\frac{r^2}{a^2}\right)H(a-r)$	$2\left[\left(\frac{r^2}{a^2}\right)-\frac{1}{2}\left(\frac{r^4}{a^4}\right)\right]$	$\frac{a}{\sqrt{3}}$	.181946	.149071
Bowl	$\frac{2}{\pi a^2}\left(\frac{r^2}{a^2}\right)H(a-r)$	$\frac{r^4}{a^4}$	$\sqrt{\frac{2}{3}}\,a$	.155465	.258199
Gaussian	$\frac{e^{-r^2/R^2}}{\pi R^2}$	$\left(1-e^{-r^2/R^2}\right)$	R	.115931	.388237
$(\text{Bowl})^4$	$\frac{5}{\pi a^2}\left(\frac{r^2}{a^2}\right)^4 H(a-r)$	$\left(\frac{r^2}{a^2}\right)^5$	$\sqrt{\frac{5}{6}}\,a$	.0823215	.436436
Annulus	$\frac{\delta(r-a)}{2\pi a}$	$H(r-a)$	a	0	indeterminate

* $\frac{I_{br}}{I_b} = 1$ when $r > a$ = beam edge

Determination of Beam Radius Resulting from a Mismatch at Injection

We consider a beam which is not matched to its equilibrium state at z = 0, i.e. the conditions

$$0 = R'(o) \quad , \tag{34a}$$

$$0 = R''(o) = \frac{{E_o}^2}{{R_o}^3} - \frac{\overline{{k_\beta}^2 r^2}}{R_o} \quad . \tag{34b}$$

are not satisfied (for given initial emittance E_o). The beam then oscillates in radius (with profile changing in some complicated manner we do not compute). Eventually the beam settles into an equilibrium state of its own chosing, satisfying Eqs. (34a, 34b) but with new values of the variables (R_f and E_f). W is constant (we assume eddy currents have not ruined the constancy of W). Thus we have four equations for the final state:

$$R_f' = 0 \ , \tag{35a}$$

$$V_f^2 = \frac{{E_f}^2}{{R_r}^2} + \left(R_f'\right)^2 \ , \tag{35b}$$

$$\frac{{E_f}^2}{{R_f}^3} - \frac{\overline{{k_\beta}^2 r^2}}{R_f} = 0 \ , \tag{35c}$$

$$\frac{{V_f}^2}{2} - \overline{{k_\beta}^2 r^2} \left[\log\left(\frac{b}{R_f}\right) - \frac{C_f}{2}\right] = W_f = W_o$$

$$= \frac{{V_o}^2}{2} - \overline{{k_\beta}^2 r^2} \left[\log\left(\frac{b}{R_o}\right) - \frac{C_o}{2}\right] . \tag{35d}$$

These equations are sufficient to determine the four quantities R_f', V_f, E_f, and R_f if $(C_f - C_o)$ is known. Generally $(C_f - C_o)$ is not known but it may be small enough to neglect--this is especially the case for a large amplitude mismatch because the possible range of C is small (see table 1).

Example: Beam injected cold at a neck. We have:

$$E_o = 0 \quad , \tag{36a}$$

$$R_o' = 0 \quad , \tag{36b}$$

$$R_o \neq 0 \quad , \tag{36c}$$

$$V_o^2 = \frac{E_o^2}{R_o^2} + (R_o')^2 = 0 \quad . \tag{36d}$$

Equations 35 (a-d) yield

$$R_f' = 0 \quad . \tag{37a}$$

$$V_f^2 = \frac{E_f^2}{R_f^2} = \overline{k_\beta^2 r^2} \quad , \tag{37b}$$

$$\frac{\overline{k_\beta^2 r^2}}{2} - \overline{k_\beta^2 r^2}\left[\log\left(\frac{b}{R_f}\right) - \frac{C_f}{2}\right] = -\overline{k_\beta^2 r^2}\left[\log\left(\frac{b}{R_o}\right) - \frac{C_o}{2}\right] \tag{37d}$$

Solving for R_f we have

$$\log\left(\frac{R_o^2}{R_f^2}\right) = 1 + (C_f - C_o) \quad . \tag{38}$$

Assuming $(C_o - C_f) << 1$,

$$\frac{R_o}{R_f} \approx \sqrt{e} = 1.64872 \; , \tag{39}$$

$$E_f^{\,2} = \overline{k_\beta^{\,2} r^2}\, R_f^{\,2} = \frac{\left(\overline{k_\beta^{\,2} r^2}\, R_o^{\,2}\right)}{e} \; . \tag{40}$$

By contrast, if we had assumed E^2 = constant = 0 instead of constant energy, we would have found $R_f = 0$.

References

1. E. P. Lee and S. S. Yu "Model of Emittance Growth in a Self-Pinched Beam ," Lawrence Livermore Laboratory Report UCID 18330, December 3, 1979.