Nonlinear Saturation of the Internal Kink Mode

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A numerical study shows that in a cylindrical tokamak the internal kink mode ($m = 1$) develops nonlinearly into a helical equilibrium state that possesses a singular current sheet. In the large aspect ratio limit, the neighboring equilibria obtained agree well with the asymptotic analytic theory of Rosenbluth et al.

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The nonlinear behavior of modes having regimes of linear instability and growth rates which interfere with the operation of a tokamak is of great interest. The nonlinear behavior of low beta tearing modes is now fairly well understood.^{1,2} The tearing mode studies, most of which utilize the reduced tokamak equations,^{3,4} describe the plasma behavior when the ideal internal mode growth is not important. In this letter, we present a study of the other limit; i.e., that in which the linear resistive modes are negligible compared to the ideal internal kink mode.^{5,6} The usual reduced tokamak equations^{3,4} are not of high enough order in the inverse aspect ratio to resolve the internal kink in a circular section tokamak, while the full magnetohydrodynamic equations are difficult to use numerically due to the fast compressional time scale. We, therefore, use the full set of ideal incompressible magnetohydrodynamic equations for the numerical study. The incompressibility condition is strictly valid for a tokamak plasma in the large aspect ratio limit.⁶ Here in fact the tokamak is treated as a cylinder and, therefore, the problem can be further simplified by treating fixed helicity. The internal kink is found to develop nonlinearly into a neighboring equilibrium state which possesses a singular current sheet. This singularity would seriously affect the tearing process in a real experimental situation, and the resulting modification of the sawtooth oscillations⁸ will be the subject of a further study.

To resolve the singularity numerically, an 'artificial field' method is introduced. In the large aspect ratio limit, the saturated equilibria obtained agree well with the asymptotic (large aspect ratio) analytic theory of Rosenbluth, Dagazian, and Rutherford.⁶

Assuming incompressibility and fixed helicity the velocity and magnetic field can be described by a stream function U and the helical flux A ,

$$\vec{V} = g \vec{\nabla} U \times \hat{e} + v_e \hat{e}, \quad (1)$$

$$\vec{B} = g \vec{\nabla} A \times \hat{e} + B_e \hat{e}. \quad (2)$$

Here $g \equiv [1 + \epsilon^2 (r/a)^2]^{-1/2}$, ϕ is a helical coordinate, and $\hat{e} = \hat{r} \times \hat{\phi}$ is the unit vector in the symmetry direction. All the scalar components on the unit basis vectors $(\hat{r}, \hat{\phi}, \hat{e})$ are functions only of r and ϕ . For a cylindrical torus with minor radius a , poloidal angle θ , and toroidal periodicity length $z = 2\pi R$, we can identify $\phi = \theta + \epsilon(z/a)$. Setting the helicity equal to the inverse aspect ratio $\epsilon \equiv a/R$ corresponds to the symmetry of the $m = 1$, $n = 1$ linear internal kink for this system. The following exact equations of incompressible magnetohydrodynamics with fixed helicity can then be derived (in rationalized emu units with minor radius $a = 1$ and homogeneous density $\rho_0 = 1$);

$$dA/dt = 0 , \quad (3)$$

$$(d/dt)(gB_e) = \vec{B} \cdot \nabla(gV_e) + 2\epsilon g^4 \vec{V} \cdot \nabla A , \quad (4)$$

$$\begin{aligned} (d/dt)(\Delta^* U) = & -\vec{B} \cdot \nabla(gJ_e) + \epsilon^2 g^2 [\partial(V_e^2 - B_e^2)/\partial\phi] \\ & - 2\epsilon \vec{V} \cdot \nabla(g^3 V_e) , \end{aligned} \quad (5)$$

$$(d/dt)(V_e/g) = \vec{B} \cdot \nabla(B_e/g) . \quad (6)$$

Here $\nabla f = (\partial f/\partial r)\hat{r} + (1/rg)(\partial f/\partial\phi)\hat{\phi}$, $\Delta^* f \equiv \nabla \cdot (g^2 \nabla f) = (1/r)[(\partial/\partial r)(rg^2 \partial f/\partial r)] + (1/r^2)(\partial^2 f/\partial\phi^2)$, $J_e = -(\Delta^* A/g + 2\epsilon g^2 B_e)$, $\vec{V} \cdot \nabla f = (\nabla f \times g \nabla U) \cdot \hat{e}$, $\vec{B} \cdot \nabla f = (\nabla f \times g \nabla A) \cdot \hat{e}$ and d/dt is the convective derivative. Note that the usual reduced tokamak equations³ can be readily obtained from these equations by applying consistent ordering $a \nabla B_e/B_e \sim \epsilon$ and $V_e/\nabla U \sim \epsilon$, and then taking only the lowest order terms [Eqs. (3) and (4) become decoupled from the rest]. For the present study, we numerically integrate the Eqs. (3) - (6) using Fourier decomposition in ϕ and finite difference methods in r . The numerical methods used are similar to those employed for the 3-D code HIB which was used to study high β ballooning stability.⁷

We have chosen a set of initial azimuthally symmetric equilibrium states which are defined by $\Delta^* A = -a_1 r^2 + a_2$ ($a_1 = 2$, $a_2 = -0.49$) and $B_e = \epsilon^{-1} g$. The fixed boundary wall is set at $r = 1$. The singular surface ($q = 1$ or equivalently $\partial A/\partial r = 0$)

occurs at $r = 0.7 = s$. A q profile is shown in Fig. 1(a) for $\epsilon = 0.245$ ($\epsilon^2 = 0.06$). The linear modes are studied using a linearized version of the code. The fastest growing linear eigenmode for $\epsilon = 0.245$ is shown in Fig. 1(b), in which the contours of the stream function U are plotted on a (r, ϕ) plane. The linear growth rates γ for various ϵ values are plotted in Fig. 2. In the limit of $\epsilon \rightarrow 0$, the growth rate approaches $0.55\epsilon^2$. This value agrees well with the asymptotic analytic theory of Rosenbluth et al.,⁶ which gives

$$\lim_{\epsilon \rightarrow 0} \gamma = \frac{\pi}{(\partial B_\phi / \partial r)_s^2} \int_0^s -G dr = 0.523\epsilon^2,$$

where $G = 2\epsilon^2 r^2 (\partial p / \partial r) + 4\epsilon^3 r^2 B_\phi B_e - \epsilon^2 r B_\phi^2$.

For the nonlinear study, the same initial equilibrium states are perturbed with the fastest growing linear eigenmode. In the subsequent nonlinear evolution of the plasma, a singular current sheet develops on the $q = 1$ surface. (The singularity in the current will be shown later.) To resolve this singularity numerically, we introduce an 'artificial field' $\vec{B}_a$ into the Eqs. (3) - (6). $\vec{B}_a$ is analogous to but independent of the real field $\vec{B}$. The force on the plasma is the linear superposition of forces due to $\vec{B}$ and $\vec{B}_a$. A nonvanishing ϕ component of the artificial field on the singular layer resolves the singularity. Let A_a denote the corresponding artificial helical flux; $\vec{B}_a = g \nabla A_a \times \hat{e}$. For A_a we take

$$dAa/dt = 0, \quad (7')$$

and the vorticity Eq. (5) is modified through

$$\begin{aligned} (d/dt)(\Delta^* U) = & -\vec{B} \cdot \nabla (gJ_a) + \varepsilon^2 g^2 [\partial (v_e^2 - B_e^2) / \partial \phi] \\ & - 2\varepsilon \vec{V} \cdot \nabla (g^3 v_e) + \vec{B} a \cdot \nabla (\nabla^* Aa). \end{aligned} \quad (5')$$

In this modified system of equations [Eqs. (3), (4), (5'), (6), (7')], the total energy

$$E = (1/2) \int (g^2 |\nabla U|^2 + v_e^2 + g^2 |\nabla A|^2 + B_e^2 + g^2 |\nabla Aa|^2) d\tau$$

is conserved. Thus, an equilibrium state can be obtained by dissipating the kinetic energy in the nonlinear evolution. One such equilibrium state ($\varepsilon = 0.245$) with a small artificial field in the vicinity of the singular layer is shown in Fig. 3. Figure 3(a) shows the helical flux A contours on a (r, ϕ) plane and Fig. 3(b) the current density J_a on the symmetry line $[\phi = 0, (\pi/2)]$. The flux contours inside the $q = 1$ surface remain almost concentric circles despite the overall shift. A developing current sheet is also apparent. The actual shift ξ of the magnetic axis in the saturated equilibrium state without artificial field is obtained by extrapolating a series of equilibrium values with decreasing strength of artificial field, as shown in Fig. 4(a). The converged value of this graph, in

turn, determines one point ($\epsilon = 0.245$) in Fig. 4(b) where ξ/ϵ^2 is plotted versus the inverse aspect ratio ϵ . In the limit of $\epsilon \rightarrow 0$, ξ/ϵ^2 approaches 3.0. This value also agrees well with the asymptotic analytic theory of Rosenbluth et al.,⁶ which gives

$$\lim_{\epsilon \rightarrow 0} \frac{\xi}{\epsilon^2} \approx \frac{13\gamma_8}{\pi^{1/2} (\partial B_\phi / \partial r)} = 3.09 .$$

This shows that in the limit of $\epsilon \rightarrow 0$, the plasma evolves into the neighboring equilibrium state found by Rosenbluth et al. From the way this state was numerically found, we can also conclude that this equilibrium state is stable to the ideal modes.

When the artificial field was reduced by half, the equilibrium state shown in Fig. 3 evolved into another equilibrium shown in Fig. 5(a) where the current density J_e profile is plotted. The numerical calculation uses a variable grid which can be altered as the singularity develops. To obtain the resolution of the current sheet in Fig. 5(a), 350 grid points were used across the minor radius with about 40 in the current sheet region. Thirty ϕ harmonics are used at this stage. The current sheet appears to narrow without limit as the artificial field decreases for fixed ϵ . To determine whether the current sheet becomes truly singular when the artificial field vanishes, we perform a convergence study analogous to the one in Fig. 4(a). In Fig. 5(b) is plotted the square root of the inverse of the current sheet height versus strength of the

artificial field B_a . As $B_a \rightarrow 0$, the height of the current sheet indeed goes to ∞ even for finite ϵ . This singularity in the saturated kinked equilibrium state for finite ϵ value is consistent with the asymptotic analytic result of Rosenbluth et al.⁶ who found that neighboring equilibrium state with singular current sheet (thickness order ϵ^3 or smaller) exists in the limit of $\epsilon \rightarrow 0$. We find the total integrated current in the sheet to approach a finite value as $B_a \rightarrow 0$.

The formation of a current singularity should critically enhance the nonlinear tearing process in high temperature tokamak experiments, thereby affecting the time scale of the sawtooth oscillation.^{8,1,9} In particular, finite nonlinear tearing may occur even in the limit of vanishing resistivity. To answer these questions, a very accurate resistive internal kink study is necessary. We have begun such an investigation using a resistive version of the incompressible magnetohydrodynamic Eqs. (3) - (6).

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FIGURE CAPTIONS

Fig. 1. (a) q profile for $\epsilon = 0.245$. (b) Contours of the stream function U for the fastest growing $m = 1$ linear eigenmode.

Fig. 2. The computed growth rate γ (open circles) versus the inverse aspect ratio ϵ . At $\epsilon = 0$, the analytic result (solid circle) of Rosenbluth et al. is also shown. Solid line represents best fit to the numerical results.

Fig. 3. An equilibrium state with a small artificial field; (a) Contours of the helical flux A . (b) Profile of the current density in the symmetry direction, J_e .

Fig. 4. (a) ξ/ϵ^2 (ξ : the shift of the magnetic axis in an equilibrium) versus the artificial field B_a for a fixed ϵ value. The converged value of ξ/ϵ^2 as $B_a \rightarrow 0$ determines one point in (b). (b) The converged ($B_a \rightarrow 0$) values of ξ/ϵ^2 versus various values of ϵ . At $\epsilon = 0$, the analytic result of Rosenbluth et al. is also shown.

Fig. 5. (a) The equilibrium J_e profile obtained when the artificial field is reduced by half from the state shown in Fig. 3. (b) $\sqrt{1/J}$ (J : The height of the current sheet) versus the artificial field B_a for a fixed ϵ

value. This graph shows that the current sheet becomes singular when $Ba = 0$.

Fig. 1. 802011

Fig. 2. 802016

Fig. 3. 802012

Fig. 4. 802015

Fig. 5. 802014