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The use of notched beams to establish fracture criteria for beryllium

Ronald Allen Mayville
(Ph.D. Thesis)

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Lawrence
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The Use of Notched Beams
to Establish Fracture Criteria for Beryllium

By

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DISSERTATION

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The Use of Notched Beams to Establish
Fracture Criteria for Beryllium

Ph.D.

Mechanical Engineering

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Chairman of Committee

ABSTRACT

Despite the fact that beryllium is brittle, its other properties make it attractive in certain structural applications. This fact has prompted the need for a better understanding of the conditions required to cause fracture in beryllium under arbitrary states of stress. In the present investigation, the fracture of an improved form of Brush-Wellman's S-65 grade pure beryllium was studied under triaxial tensile stresses. This state of stress was produced by testing notched beams, which were thick enough to be in a state of plane strain at the center. A plane strain, elastic-incremental plasticity finite element program was then used to determine the stress and strain distributions at fracture.

A four-point bend fixture was used to load the specimens. It was carefully designed and manufactured to eliminate virtually all of the shear stresses at the reduced section of the notched beams.

The uniaxial properties of the beryllium were obtained by two methods: 1) Conventional tension and compression tests, and 2) A bending test developed in 1910, but curiously unexploited since that time. With this latter technique, the uniaxial stress-strain curves in tension and compression can be obtained from a single bending specimen. The bending experiment was shown to give excellent results for three materials: beryllium, cast iron, and a prestained copper.

The literature was reviewed to determine the precise geometric conditions required to insure a state of plane strain in the notched beams. For the beryllium, however, these conditions were determined by actually measuring the strains at the root of the notch with strain gages for two specimens. Also, it was found that the experimental fracture load cannot be used as input to the plane strain finite element program to give an accurate representation of the stress and strain state in a notched beam which only experiences a limited amount of true plane strain. In these cases, an effective fracture load was estimated from strain gage readings.

A total of 26 beryllium notched beams were tested to fracture. There were nine different geometries, consisting of combinations of four thicknesses and three different root radii. After the specimens were fractured, the location of fracture initiation, which occurred ahead of the notch, was determined by viewing the fracture surfaces in a scanning electron microscope. In this way, the values of the principal stresses and strains at the location of fracture were determined from the finite element analysis.

The maximum tensile stress at fracture in the notched beams was significantly greater than in uniaxial tension. This difference cannot be explained by a statistical variation in strength. Instead, a strain dependent tensile stress criterion of fracture appears to be more suitable.

The information obtained from the notched beam tests was used to closely predict the plane strain fracture toughness of the beryllium. Finally, some useful guidelines for the design against fracture in beryllium are given, and suggestions of some specific experiments are made, which could lead to a more detailed formulation of a fracture criterion.

DEDICATION

I would like to dedicate this work to my father and mother,
whose unending support has made this all possible.

ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Iain Finnie, whose teachings and suggestions, but most of all his friendship, have made this work a satisfying personal experience. I sincerely appreciate the many helpful discussions of Roger Werne and Ronald Streit, and I would like to thank Professor Jack Washburn for reviewing the dissertation.

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The Use of Notched Beams to Establish
Fracture Criteria for Beryllium

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LIST OF SYMBOLS

a	= height of the reduced section in a notched beam.
A1,A2	= notched beam specimen identification (see Fig. 4.1).
b,B	= beam thickness.
B1,B2	= notched beam specimen identification (see Fig. 4.1).
c	= notch depth.
C1,C2	= notched beam specimen identification (see Fig. 4.1).
d	= moment arm.
D1,D2	= notched beam specimen identification (see Fig. 4.1).
e	= strain
$\bar{e}^p$	= effective plastic strain.
E	= Young's modulus.
h	= beam height.
k	= yield stress in pure shear.

- K = applied stress intensity.
- K_{Ic} = plane strain fracture toughness.
- M = bending moment.
- M-S65 = an improved form of S-65 beryllium which satisfies the 1319B specification of Lawrence Livermore Laboratory.
- n = hardening exponent, $\sigma = Ae^n$.
- N = number of elements in the finite element mesh.
- P = applied load.
- S1, S2 = a special notched beam specimen identification (see Fig. 4.1).
- S-65 = a structural grade of beryllium made by Brush-Wellman.
- t = time.
- T = current yield stress.
- T1, T2, T3 = tension specimen identification.

- W = beam height.
- x, y, z = Cartesian coordinates to describe the notch geometry (see Fig. 3.1).
- α = one-half the included notch angle.
- κ = curvature in beam bending.
- μ = coefficient of friction.
- ν = Poisson's ratio.
- ρ = radius of the circular notch in the notched beams,
and radius of curvature in beam bending.
- σ = normal stress.
- σ_{nom} = $6M/Ba^2$.
- σ_0 = initial yield stress in uniaxial tension.
- σ_y = stress in the y direction (see Fig. 3.1).
- τ = shear stress

$$1 \text{ MPa} = 145 \text{ psi}$$

$$25.4 \text{ mm} = 1 \text{ inch}$$

$$4.45 \text{ N} = 1 \text{ pound}$$

CHAPTER 1

INTRODUCTION

Beryllium is the fourth element in the periodic table. When it was first isolated in pure crystalline form in 1898, its unique properties of low density (1.85 gm/cm^3) and high melting temperature (1280 deg C) immediately encouraged research into its possible use as a structural material. Original castings of beryllium were so brittle, however, that it was often impossible to machine them. Efforts made later to render beryllium ductile by eliminating impurities and adding alloying elements met with little success.

It is now generally accepted that beryllium is inherently brittle, and to date the greatest success in achieving ductility has been to produce a very fine-grained material. Commercial grades of pure beryllium are now available with moderate strengths and elongations exceeding 3 percent. This seemingly prohibitive lack of ductility is actually adequate in a number of applications where some of the excellent properties of beryllium (Table 1.1) can be employed. For example, beryllium has been used for disc brakes and nose cones in aircraft because of its high thermal conductivity, as a substrate for mirrors because of its specific stiffness, and in nuclear applications because of its very low cross section for neutron absorption (see Ref. 1).

Although the low ductility is adequate in a number of applications, in structures, beryllium parts must often be designed to perform close to the point of expected failure if the advantages of using it are to be realized. Therefore, a usable fracture criterion for design is required.

As with most brittle materials, a maximum tensile stress criterion has been used to predict fracture for beryllium loaded in tension. This approach appears satisfactory for beryllium subjected to general states of uniform plane stress⁽²⁾.

However, in many beryllium structures a three-dimensional state of stress may lead to fracture. Consequently, there is a need to obtain a greater understanding of the conditions leading to fracture in states other than plane stress. Apparently, the response of homogeneous beryllium to triaxial states of tensile stress, that is three nonzero principal stresses, has not been investigated. The present work was initiated to examine this problem and to develop a fracture criterion for use in design.

Because of beryllium's brittle character, it is instructive to address two problems: 1) the relation between the tensile strength and the stress state to cause fracture under multiaxial stresses, and 2) the question of whether the fracture stress of a part is affected by size. These two problems may be related, since a volume of material in a triaxial stress field is exposed to a higher concentration of tensile stresses. Nevertheless, testing samples with different sizes will aid in the assessment of the size effect in beryllium.

Hydrostatic compression represents one method of inducing triaxial stresses, and has been used on beryllium⁽³⁾. Introducing tensile triaxial stresses in a body, however, is an experimentally difficult problem. Such an investigation requires specimens of relatively simple geometry in order to facilitate the stress analysis and the interpretation of the results.

An investigation by Knott⁽⁴⁾ on cleavage fracture in steels suggests that notched beams in pure bending may provide a means of studying the fracture behavior of beryllium under states of triaxial stress. In his work, Knott sought to verify the critical assumption of the classical theory of notch effects in brittle fracture, which states that the fracture strength of a material is independent of, or at least depends mildly on, temperature.

Knott tested mild steel at low temperatures, for which cleavage fracture is preceded by plastic deformation in notched beams loaded in quasistatic bending. This is the type of behavior exhibited by beryllium at room temperature. The geometry of the notched beam is shown in Fig. 1.1. For the case of a rigid perfectly-plastic material in plane strain, an analytic expression exists for the state of stress across the plane of symmetry of this specimen (see Chapter 3). Realizing that this solution can only be related to the external moment when general yielding occurs, that is, yielding across the entire reduced section, Knott adjusted the geometric parameters of the specimens so that fracture would coincide with general yield for a given temperature. In this way he was able to show that the maximum tensile stress required to initiate cleavage fracture was constant for a wide range of temperatures.

Though Knott's original work was based on a number of approximations, the general trend was clear. In fact, plane strain finite element analyses, made later by Griffiths and Owen (5), demonstrated more convincingly the conclusions reached by Knott. The use of the finite element method (FEM) in this case represents an important tool, for in general it is not possible to have fracture coincide with general yield under conditions of engineering interest. This is true of beryllium. In addition, many metals work harden appreciably after initial yield, making the slip-line field analysis too crude an idealization. The FEM may be used to examine the state of stress and strain in a notched beam specimen for any point along the loading path, including the point of fracture initiation.

For this reason, a finite element analysis was used in the present work to study the response of beryllium to states of tri-axial stress associated with notched beam specimens loaded in four-point bending. Actually, notched beams only provide a limited number of such states, but the information so obtained complements studies made previously with uniaxial and biaxial states of stress and provides the basis for constructing an overall criterion of fracture in beryllium in terms of tensile stresses.

The accuracy of the finite element method depends upon how well the actual material response and boundary conditions in the experiment are modeled in the analysis. By providing a state of pure bending to the gage section of the notched beam, simulation of the

boundary conditions in the finite element analysis becomes particularly simple. Experimentally, this may be accomplished by using a four-point bend apparatus with sufficient degrees of freedom to avoid the introduction of any shear stresses brought about either by twisting or by a nonuniform distribution of moment across the length of the beam. The design and operation of this apparatus is discussed in Chapter 2 and Appendix A.

The multiaxial stress-strain relations to be used in the finite element analysis are derived from uniaxial data. With the assumption of isotropy, the relation between incremental, plastic strain and stress is determined by the relation between effective plastic strain and effective stress as derived from uniaxial data (see Appendix B for details). An accurate determination of the relation between uniaxial stress and strain for tension and compression is thus necessary. As any experimenter realizes, this is far from a trivial task. So, besides using conventional tests to obtain these data, another method was employed, which was proposed in 1910 and curiously unexploited since that time. By measuring the bending moment and the strains at the top and bottom fibers of a rectangular beam loaded in pure bending, it is possible to deduce the uniaxial stress-strain curve in tension and compression from a single specimen. This topic is also treated in Chapter 2.

With the powerful tool of the FEM and satisfactory modeling of the boundary conditions and material behavior it is possible to

determine the state of stress and strain at the point of fracture initiation in the notched beam specimens. The details of the stress analysis are presented in Chapter 3, but there are still a number of important considerations.

In the first place, some parametric measure is required to relate the stress-strain state predicted by the finite element analysis to the state which actually exists in the specimen for any point along the loading path. This parameter must be relatively sensitive and experimentally measurable. Next, it is necessary to determine experimentally the value of this parameter at which fracture initiates in the specimen. This information will provide the state of stress and strain which exists throughout the specimen at fracture initiation so that finally, if the location of fracture initiation is known, the state of stress and strain in its vicinity is determinate.

The experimental approach to these problems is discussed in Chapter 4 together with the results of the beryllium notched beam tests.

Finally, in Chapter 5 the results of the uniaxial, biaxial, and notched beam tests are examined in an effort to determine a workable fracture criterion for design. The microstructural aspects of deformation and fracture in beryllium are also reviewed, with the hope of explaining the observed behavior from a physical standpoint. This latter consideration is used in an attempt to predict the fracture toughness of beryllium.

CHAPTER 2

MATERIAL PROPERTIES

The basic objective of this investigation is to determine the conditions under which beryllium fractures. This is accomplished in part by breaking notched beams in bending and employing finite element analysis to determine the conditions present at fracture.

It is first necessary in this endeavor to obtain a thorough and reasonably accurate knowledge of the uniaxial behavior of the beryllium. Since the simple tension test provides much of the information on which material design considerations are based, it is desirable to seek a relationship between the "fracture" properties obtained from this test, such as tensile strength and elongation, and those properties more directly connected to fracture. It is also necessary to determine the multi-axial stress-strain relations of the beryllium for the stress and strain analysis. With certain assumptions on the plastic behavior of the material, these relations can be derived from uniaxial data.

The uniaxial mechanical properties of the beryllium used in this investigation were obtained in two ways: by conventional tension and compression tests, and by employing rectangular beams in pure bending. This latter technique of obtaining stress-strain relations was first investigated by a German scientist⁽⁷⁾ in 1910 on a suggestion from L. Prandtl.

Except for photographs of the microstructure and a determination of grain size, a discussion of the microstructural aspects of deformation and fracture in beryllium is left until Chapter 5.

Material

A modified form of Brush Wellman's S-65 grade pure beryllium was used in this investigation, hereafter referred to as M-S65 beryllium. It essentially satisfies the Lawrence Livermore Laboratory specification No. 1319B⁽⁸⁾. The chemistry, grain size, and mechanical properties as obtained by Brush Wellman are given in Tables 2.1 - 2.3. Figure 2.1 shows photographs of the microstructure.

The material is produced by special powder metallurgy techniques, which are designed to decrease the beryllium oxide content and provide a very fine-grained structure. For example, comminution is carried out by impacting the larger beryllium particles against a beryllium plate. The final step in the process is the vacuum hot pressing of the powder into a cylindrical block. Table 2.3 reveals that this form of pressing results in slight anisotropy as well as some inhomogeneity, especially near the bottom of the pressing where the mechanical properties deteriorate somewhat. This may result in some scatter in the results of the notched beam experiments.

Unfortunately, the location in the billet from which the specimens were taken is unknown in all cases. In addition, the orientation of the tension and compression specimens is unknown. Most of

the notched beams were cut from the billet so as to have their longitudinal axis coincide with that of the cylindrical block, but some of the 76.2 mm beams are known to have their axis parallel to the transverse direction.

Anisotropy is revealed most significantly by the differences in elongations shown in Table 2.3. In terms of the strength data, however, the beryllium is reasonably homogeneous and isotropic. The large differences exhibited by the bottom transverse tensile data may be due to removing the specimens too close to the end of the billet. X-ray diffraction analysis of three orthogonal planes indicates that there was no texture in the M-S65 beryllium. This information indicates that the assumption of isotropy can be made in the analysis, without affecting the principal conclusions drawn from the results.

Because beryllium is extremely susceptible to machining damage⁽⁹⁾, it is necessary to use special machining practices to insure the integrity of any structural part. These include: machining with successively shallower passes, frequent changing of cutting tools, and the use of an etchant on the final product. This practice and the toxicity of beryllium powder make the processing of parts expensive.

Tension Tests

The geometry of the tension specimens is shown in Figure 2.2. This tapered-end geometry is used conventionally for brittle materials. The gage section also has a slight taper in the center

to insure that fracture will occur in the gage length of the extensometer. Three tension tests were performed.

All of the tests, including the notched beams, were tested at ambient conditions under displacement control on a 50 kN (10,000 lbs) electro-hydraulic, servo-controlled, closed-loop testing machine. The displacement rate for the tension and compression specimens was fixed at 0.27 mm/min (0.005 in/min), which converts in both cases to a strain rate of $0.6 \times 10^{-4} \text{ sec}^{-1}$.

The load was measured by means of a conventional strain gage type load cell and relative displacement was measured with an extensometer. In one test, strain gages were mounted on the specimen in addition to an extensometer. The signals from the load cell and extensometer were amplified and sent to an X-Y-Y Recorder. The X channel was used to measure load and the Y channels were used to measure displacement at two sensitivities. In the case of the strain gaged specimen, a computerized data acquisition system was employed to monitor the strain gages and the load.

Alignment of the specimens was obtained by use of a load train in which one connecting fixture was held rigid and the other was allowed to rotate by means of a spherical seat. The position of the movable connecting fixture was adjusted so that the strains in the outer fibers of the specimen at three locations, separated by 120 degrees, differed by less than five per cent at a stress of 52 MPa (7,500 psi), as determined by an extensometer. In one case, this was accomplished by monitoring the strain gages. All of the procedural requirements outlined in the Publication MAB-205-M⁽¹⁰⁾ on

the testing of beryllium were observed. Furthermore, a line corresponding to the elastic modulus as obtained in the alignment test was drawn on the graph for the actual test to insure that alignment was maintained.

The results of the three tension tests are shown in Figure 2.3. The two tests in which only an extensometer was used to measure strains show behavior different from that obtained for the strain gaged specimen. In addition, the extensometer data for the strain gaged specimen agreed very well with the strain gage data. It is tempting to accept tests T1 and T2 as the best data because of reproducibility, but these tests were performed consecutively and any errors that existed in one probably existed in both. Because of the accuracy of the post-yield strain gages in tension, and for reasons discussed below, the data from specimen T3 are to be preferred.

It is intriguing to note that despite beryllium's brittle character, almost ideal perfect plasticity is exhibited in the first 0.5 percent of strain. After this, hardening follows quite closely a power law relationship, $\sigma = Ae^n$, with $n = 0.23$.

Among all of the tension tests it was not possible to obtain a precise value for the elastic modulus. For the tests in which an extensometer was used, the calculated value never exceeded 262 GPa (38×10^6 psi), while calculations using the average of the three strain gages in the one tension test resulted in a value of 290 GPa (42×10^6 psi). The true value is known to be very close to 303 GPa (44×10^6 psi).⁽⁶⁾

Because it is also of interest to determine Poisson's ratio for the M-S65 beryllium, a simple bending test was performed in which axial and transverse strains at the outer fibers were measured with strain gages. Poisson's ratio for beryllium is quite small, with values reported from 0.03 to 0.1.

A rectangular beam loaded in four-point bending was used for this particular experiment. The beam was loaded ten times at two different stress levels for both tension and compression and the average calculated values of Young's modulus and Poisson's ratio obtained are shown in Table 2.4

It is necessary to load the beam repeatedly, since the strain gages, and perhaps the beryllium itself, apparently suffer some small permanent deformation in initial loading. Also, because of the very small strains in the transverse direction, it is necessary to include the effect of strain gage cross-sensitivity into the calculations with the strain gage output.

Compression Tests

Two compression tests were carried out on the beryllium to determine if the uniaxial behavior is the same in tension and compression and to make comparisons with the bending experiment.

The geometry of the compression specimens is shown in Figure 2.2. As indicated, concentric grooves were machined in the ends to contain lubricant during the test and therefore minimize frictional effects.

The testing conditions for compression were the same as for the tension tests and the criteria for alignment were also the same. In this case alignment was achieved by using a flat plate as the top fixture and at the bottom, a fixture which could be inclined at any small angle to the load-train axis. The bottom fixture consists of a short right cylinder with a plane cut at a very slight angle to the horizontal. The two sections so formed are able to rotate with respect to each other providing a continuous distribution of small inclines.

The resulting stress-strain curves are shown in Figure 2.4 superposed with the results from the tension test. The two curves agree closely up to a strain of approximately one percent, after which the stresses in compression become greater than in tension for the same strain.

Bending Tests

Most readers will recall that given the uniaxial behavior of a material, the moment vs. curvature relationship for a rectangular beam in plane stress, pure bending can easily be determined. What is not so familiar is that given the moment vs. radius of curvature relationship for the same beam geometry, the uniaxial stress-strain curves in tension and compression can be determined. This is possible even if the behavior in tension and compression is different.

The analysis leading to this conclusion is actually quite simple and requires that only a few basic conditions be satisfied.

Analysis: Figure 2.5 illustrates a rectangular beam and the terminology to be used in the following discussion.

First, assume that there is a unique relationship between uniaxial stress and the strain in the same direction represented by,

$$\sigma = f(e)$$

The function f represents compressive and tensile data so that for negative e , σ is also negative.

In pure bending, plane sections must remain plane⁽¹¹⁾. Consequently, the magnitude of the strains at the upper (compressive) and lower (tensile) fibers are given by,

$$e_1 = \frac{y_1}{\rho} \quad , \quad e_2 = \frac{y_2}{\rho} \quad ,$$

where ρ is the radius of curvature. If the strains at the outer fibers are known, then the radius of curvature, or equivalently the curvature $\kappa = \frac{1}{\rho}$, and the location of the neutral axis may be determined as,

$$\kappa = \frac{e_1 + e_2}{h} \quad , \quad y_1 = \frac{e_1 h}{e_1 + e_2} \quad , \quad y_2 = \frac{e_2 h}{e_1 + e_2} \quad ,$$

where the relation $y_1 + y_2 = h$ has been used. Note that if the properties of a material are different in tension and compression

the neutral axis need not coincide with the center of the beam. Equilibrium requires that,

$$\int_{-y_1}^{y_2} \sigma dA = 0 \quad \text{and} \quad \int_{-y_1}^{y_2} \sigma y dA = M \quad ,$$

where dA is an element of area and M is the applied moment. For rectangular cross sections $dA = bdy$ and since $y = e/$ the equilibrium equations become,

$$\int_{-e_1}^{e_2} \sigma de = 0 \quad , \quad \int_{-e_1}^{e_2} \sigma e de = M \frac{\kappa^2}{b} \quad .$$

Consider next a small change in load, and hence curvature. Differentiating the equilibrium equations with respect to curvature by using Leibnitz' Rule, and denoting the magnitudes of the stresses at the outer fibers by σ_1 and σ_2 ,

$$\sigma_2 \frac{de_2}{d\kappa} - \sigma_1 \frac{de_1}{d\kappa} = 0 \quad (a)$$

$$\sigma_2 e_2 \frac{de_2}{d\kappa} + \sigma_1 e_1 \frac{de_1}{d\kappa} = \frac{\kappa}{b} \left[\frac{dM}{d\kappa} + 2M \right] \quad . \quad (b)$$

Multiplying equation (a) by e_1 and adding to equation (b) results in the expression,

$$\sigma_2(e_2 + e_1) \frac{de_2}{d\kappa} = \frac{\kappa}{b} \left[\frac{dM}{d\kappa} + 2M \right] .$$

This is further simplified by recalling that $(e_1 + e_2) = \kappa h$.
Thus,

$$\sigma_2 = \frac{\kappa dM + 2M d\kappa}{hb de_2} \quad (c)$$

The expression for σ_1 is similar.

The stresses and strains can be determined by measuring the moment vs. change in angle between two plane sections in the beam (see Nadai⁽¹²⁾), but it is experimentally convenient and more accurate to measure strains directly and then determine the stresses.

Since $hd\kappa = de_1 + de_2$, expression (c) may be used to represent the stresses entirely by the moment and the strains as,

$$\sigma_\alpha = \frac{dM(e_1 + e_2) + 2M(de_1 + de_2)}{bh^2 de_\alpha} ; \quad \alpha = 1, 2 \quad (d)$$

In 1910, Herbert⁽⁷⁾ demonstrated the validity of this analysis by testing cast iron, which shows substantially different behavior in tension and compression. He used a testing apparatus consisting of hanging weights added in increments to apply load, and

strains were measured with a Marten's extensometer, which employs an optics system to measure displacements.

In the present work this method was originally adopted to obtain the uniaxial properties of the M-S65 beryllium. But the same experiment was conducted on cast iron and prestrained copper, which shows a Bauschinger effect, to examine the general utility of the technique.

It is conceivable that this type of experiment could provide a great deal more information than a conventional tension or compression test. For example, by measuring transverse strains at the top and bottom fibers, Poisson's ratio could be determined. In addition, because of the linear variation in strain across the height, a cross section of the specimen before or after failure could reveal the variation in damage with strain.

Specimen Design: Two beryllium beam specimens were machined from 25 mm x 25 mm x 152 mm (1 in x 1 in x 6 in) beams into the geometry shown in Figure 2.6. The longitudinal axis of these specimens coincided with the axis of the billet. A reduced section was incorporated to insure that failure would occur in the center of the specimen and to require as low loads as possible to avoid roller indentation. In addition, the ratio of the height to width of the reduced section was maintained at 2:1 to insure plane stress. The copper specimen had a similar geometry, but the cast iron bending specimen was a uniform rectangular beam with cross-sectional dimensions of 14 mm x 7 mm (0.56 in x 0.28 in).

The conditions of pure bending and plane stress are easier to insure with large gage sections. In the present case, however, the specimens had to be tailored to fit the four-point bend apparatus originally designed for the testing of the notched beams. This required a rather short gage section.

Four-Point Bending: Simple mechanics of materials calculations show that for a beam loaded in four-point bending, and with equal loads on all points, shear stresses are absent in the section between the inner points of load application. Furthermore, if the thickness of the beam is relatively small and the radius of curvature exceeds about ten times the height, no transverse stresses will exist either. The strains must then vary linearly throughout the height of the beam⁽¹¹⁾.

Eliminating the shear stresses experimentally is actually quite difficult. The careful design of the bend fixture used in the present investigation removes the shear stresses almost completely. The potential sources of shear stress and the design solution for them are given in detail in Appendix A. Figure A.2 shows the four-point bend fixture with one of the beryllium notched beam specimens.

The specimens were tested in a 5 kN testing machine in displacement control at a rate of 6.35×10^{-2} mm/min (2.5×10^{-3} in/min) which is roughly equivalent to a strain rate of 0.5×10^{-6} sec⁻¹.

Strain and Moment Measurements: In order to determine the stress-strain curves from a bending test it is necessary to measure the

bending moment and the strains at the outer fibers. It is also necessary to know the rate of change of the strains for a change in moment at all times in the loading of the beam (see equation (d)).

In the present case, strain gages were mounted on the top and bottom of the beams for the measurement of strain. Post-yield Micro Measurement EP strain gages were used in the axial direction, and the more standard EA strain gages were used in the transverse direction with the hope of also measuring Poisson's ratio. Single active-arm current-excited Wheatstone bridges were employed for the measurement of output voltages, which were recorded by a computerized data acquisition system. The load voltage was also recorded in this way. The data acquisition system was programmed to scan the output from the current excitation source, the strain gages, and load cell at certain regular time intervals and to record the results on magnetic tape.

Conversion of Data: The data acquisition system employed in this experiment is capable of reading a large number of data points. A computer program was thus written to convert the data-voltages to load and strains, and finally to stresses. This is also useful for the computation of the differential quantities in equation (d).

The computer program was structured in the following way. The output voltages are read as a function of time. The bending moment and strains are then computed from this data and, therefore, are also functions of time. The calculation of the differentials was performed using the numerical four-point differentiation formula,

$$df(n) = \left[f(n-2) + 8f(n-1) - 8f(n+1) - f(n+2) \right] / 12 ,$$

where $f(n)$ refers to the n th data point.

In the present case, this is essentially differentiation with respect to time considering the way in which the data were taken. However, it is not necessary for time to appear explicitly, because of the form of equation (d); it is only necessary that $e(n)$ and $M(n)$, the n th data points of strain and moment, be recorded at the same time.

In some tests, slow loading rates resulted in small variations in the load and strain at any particular time. When data acquisition was relatively fast in these cases, the resulting differential calculations actually showed large variation. To circumvent this, an option was built into the program whereby every n th data point could be used in the calculations. Leaving this parameter as an input variable essentially provided a device to smooth the resulting stress-strain curve. Note that it would be incorrect to fit a curve to the load or strain vs. time curves, since, in general, they consist of linear and nonlinear portions.

The program was also written to compute Poisson's ratio.

Results and Discussion of the Bending Tests: Figures 2.7 and 2.9 - 2.11 show the results of the bending tests for the three materials in comparison to data obtained from the conventional tension and compression tests.

Data from the first beryllium bend specimen show excellent agreement with the data obtained from the conventional tests. The tension data agree almost identically, and the major differences are a significantly higher strain to fracture - four percent as opposed to 2.1 percent - and a somewhat higher tensile strength - 402 MPa (59×10^3 psi) compared to 375 MPa (54×10^3 psi). The compression stress-strain curve obtained in bending gives a greater value of stress for a given strain, but never by more than seven percent. This is somewhat surprising, since frictional stresses in the conventional compression test usually tend to give a higher stress than the true value. It is also curious that the tension data agree so well, since the calculation of the tension and compression stresses are related.

These differences are probably due to the testing techniques used for the conventional tests, and the inability of strain gages to measure compressive strains accurately. An examination of Figures 2.7 and 2.9 - 2.11 show that the differences between the data obtained from the bending and conventional tests do not follow a consistent pattern. This indicates an inability to maintain alignment in the conventional tests, because with misalignment an extensometer will read either a lower or higher displacement than the average, depending on where it is located around the circumference. On the other hand, deformation of thin foil strain gages in compression is not likely to be the same as in tension, especially as the strains become large.

Control over these sources of error could have been maintained by employing three strain gages on all of the conventional tension and compression specimens.

Scanning electron micrographs of the fracture surfaces for the first bending specimen and one of the tension specimens are shown in Figure 2.8. The plasticity in the bending specimen is evident in this figure.

The results of the second beryllium bend specimen, shown in Figure 2.9, are at greater variance with the data obtained from the conventional tests. The tension data are very similar, but the compression data differ from the beginning and become erratic after about two percent strain. Because of the relation between the calculated tension and compression stresses obtained from the bending test, the erratic compression data affects the tensile data after two percent strain as well.

Examination of the transverse strains obtained from these two bending tests failed to reveal a constant value of Poisson's ratio for any portion of the stress-strain curve prior to yielding. This may be due to microplasticity and could be overcome with cyclic loading. However, such practice is not usually recommended for post-yield strain gages.

The results of the cast iron and prestrained copper experiments are shown in Figures 2.10 and 2.11. One bending specimen and two each of the tension and compression specimens were tested in each case, and an extensometer was used in the conventional tests to measure relative displacement.

The versatility of using a bending test to obtain uniaxial stress-strain curves is well demonstrated by the cast iron bending results, which were obtained from a simple uniform rectangular beam. Figure 2.10 shows that the strain to tensile fracture for cast iron in bending is also greater than in tension, and, in fact, it was not possible to break the cast iron bending specimen because of the displacement constraint of the four-point bend fixture.

The Bauschinger effect corresponds to the lower yield stress often observed with the reversed loading of a material from the plastic range. A Bauschinger effect was induced into the copper by compressing a 50 mm x 50 mm x 165 mm (2 in x 2 in x 6.5 in) block in the longitudinal direction to four percent strain. The specimens were then cut so that their longitudinal axes coincided with the direction of straining. Figure 2.11 shows that the tension data obtained from the conventional and bending tests agree quite closely up to strains as large as seven percent. The compression data agree well up to about one percent strain, after which the results differ markedly.

Only in the case of copper was it possible to obtain a constant value of Poisson's ratio from the bending test. The value given in compression was $\nu = 0.38$; a physics table gave a value of $\nu = 0.35$.

Biaxial Test Results

Test results obtained by Lindholm, et al,⁽²⁾ with thin-walled tubes made of Brush-Wellman's S-65 grade beryllium, are shown in

Figure 2.12. The basic form of the uniaxial stress-strain curve for this particular material is the same as shown in Figure 2.3, but the yield and tensile strengths are different as can be read from Figure 2.12. The data points in this figure correspond to experiments in which the tubes were subjected to either combined axial loading with internal pressure, or combined axial tension and torsion. From these data, Lindholm, et al, concluded that S-65 beryllium yields according to a Mises criterion and that fracture in the tension-tension quadrant is governed by a maximum stress criterion.

Summary of M-S65 Beryllium Uniaxial Behavior

The results of three tension tests show some variation. Because of the difficulty in obtaining alignment and accurate displacement measurements, the average stress vs. average strain determined from the strain gaged specimen is believed to represent the uniaxial tension behavior most closely. This is further supported by the results of the bending tests, which agree almost identically with these data.

The uniaxial compression results agree with the tension results up to about one percent strain. Because the notched beams never experience a compressive strain of this magnitude, the assumption of identical behavior in tension and compression is satisfactory.

As mentioned, the tension results of the bending tests are in excellent agreement with the results obtained from the conventional tests. The significant differences given by the bending tests include a closer prediction of the true elastic modulus and a much

greater strain to fracture. Finite element calculations, presented later, indicate that some of the beams experience tensile strains in excess of the elongations observed in the conventional tension tests. For this reason, the stress-strain data obtained from the first beryllium bending test were used as input to the finite element code. The piecewise-linear curve used in the numerical analysis is shown in comparison to the actual data in Figure 2.13.

CHAPTER 3

STRESS AND STRAIN ANALYSIS OF THE NOTCHED BEAM IN BENDING

A difficult task in studying notched beams with finite root radii is the choice of the method of stress analysis. Virtually every investigation of this type in the literature employs the rigid, perfectly-plastic analysis of Hill⁽¹³⁾. This plane strain analysis is based on the assumption that slip-lines emanate from the circular boundary of the notch root as illustrated in Fig. 3.1. The theory of slip-lines then predicts that the stress across the reduced section ahead of the notch is given by

$$\sigma_y = 2k \left[1 + \ln \left(1 + \frac{x}{\rho} \right) \right], \quad (a)$$

where k is the yield strength in pure shear, ρ is the radius of curvature of the notch, and x is the distance ahead of the notch. This equation is valid until the plastic zone grows to the point where the slip-lines impinge on the straight boundaries of the notch. Beyond this point, the stress remains constant up to the plastic zone boundary at the maximum level given by

$$\sigma_{\max} = 2k \left(1 + \frac{\pi}{2} - \alpha \right) \quad (b)$$

The distance ahead of the notch at which this stress first occurs is,

$$x_0 = \rho \left[\exp\left(\frac{\pi}{2} - \alpha\right) - 1 \right] . \quad (c)$$

Figure 3.1 shows the predicted shape of the plastic zone and the resulting stress distribution.

The shortcomings of this analysis in predicting the stress distribution in an arbitrary notched beam are the requirement of a rigid, perfectly plastic material behavior and the inability to relate the stresses to the externally applied bending moment. The latter can, in fact, only be done after general yielding has occurred; that is, yielding across the entire reduced section.

Attempts have been made to overcome these difficulties by including corrections for hardening⁽¹⁴⁾, and combining known elastic solutions for bodies of similar geometry⁽¹⁵⁾.

The application of the finite element method to this problem, however, represents the greatest improvement in accurately analyzing the state of stress and strain. Such an analysis was carried out first by Griffiths and Owen⁽⁵⁾.

Their work entails a thorough investigation of the stress and strain distributions in a notched beam of linear hardening material subjected to a variety of loads prior to and exceeding general yield. Furthermore, comparisons with available experimental results showed good agreement for the prediction of the plastic zone sizes, the general yield load, and the strains at the root of the notch.

The stress distributions obtained by Griffiths and Owen, when compared to the prediction of Hills's analysis, revealed a surprising difference. In contrast to the predictions of the analytic treatment, the numerical analysis demonstrated that a true maximum occurs, not at the elastic-plastic interface, but at some distance behind it (see, for example, Fig. 3.4).

In view of the success of the Griffiths and Owen analysis, plane strain finite element analysis was adopted in the present investigation as a means of examining the stress and strain distributions in the beryllium notched beams. It is still important, however, to examine the accuracy of the computer program. This depends most critically on how well the multiaxial stress-strain relations are represented and to what extent a state of plane strain exists in the beryllium notched beams.

A discussion of the implications and validity of the plane strain assumption is presented next. The details of the finite element program are then described, followed by some specific analyses of beryllium notched beams. Some of these analyses were used originally to investigate the effects of different geometric parameters on the stress state for the design of the specimens. Other examples illustrate the typical stress and strain distributions which occurred in the notched beams tested, and will be used to interpret some of the test results.

The Assumption of Plane Strain

The consequences of having a state of plane strain in the notched beams are very important in this investigation. Besides allowing a two-dimensional analysis to be made, the stress state at fracture will consist of three nonzero tensile principal stresses; a situation which apparently has not been investigated.

The magnitude of the transverse stress in the case of plane strain, where $\epsilon_z = 0$, is dependent upon the mode and extent of deformation, but lies in the range,

$$\nu(\sigma_x + \sigma_y) \leq \sigma_z \leq \frac{1}{2}(\sigma_x + \sigma_y) \quad .$$

The stress σ_z is equal to the expression on the left during elastic deformation and approaches the expression on the right as plastic deformation accumulates. The low value of Poisson's ratio for beryllium, Table 2.4, shows that the transverse stress will experience a wide transition with increasing deformation.

It is important to realize that, in general, plane strain in the notched beam specimens exists only in the vicinity of the notch; the rest of the beam is in a state of plane stress. If the majority of the beam remains elastic, the bending moment required to provide a given nominal stress is greater in plane strain than in plane stress by a factor of $1/(1-\nu^2)$. Consequently, if the experimental situation is modeled by a plane strain analysis with the same load, the analysis will predict a lower displacement and stress than what

actually occurs in the experiment. In view of the exceptionally low value of Poisson's ratio for beryllium, no corrections for this effect were usually necessary. Only in the case of the smallest beams was plastic deformation extensive enough to require the consideration of this factor.

Plane strain can be expected to exist in some region of the specimen, because of the constraint imposed by the material adjacent to the highly strained region of the notch root. But a transverse stress of the magnitudes given above cannot exist through the entire thickness in the notched beams, since the lateral boundaries are stress free. The question is: at what distance away from the boundaries does the transverse stress approach the value given by a plane strain analysis? This information is necessary for the design of the specimens and the interpretation of the results.

The precise geometric requirements for a body to experience a state of plane strain, with only limited plastic deformation, have only been obtained in a few cases. For example, Sternberg and Sadowsky⁽¹⁶⁾ have carried out an analytic, three-dimensional analysis of an elastic, infinite plate containing a circular cylindrical hole, loaded in tension. Their results show that conditions of plane strain are attained at a distance from the plate surface equal to 1.5 hole diameters for $\nu = 0.1$, and 3.0 for $\nu = 0.5$. In other words, a thickness-to-radius ratio from six to twelve is required to guarantee the existence of a region of plane strain.

This range of conditions on geometry for the circular hole problem is consistent with experimental observations on notched beams. Weiss⁽¹⁷⁾ has pointed out that an analogy between flat

plates in bending and notched bars in tension may be made for the determination of the conditions required for plane strain. The analogy is made by comparing the width-to-height ratio of a beam in pure bending to the thickness-to-root radius ratio for the notched bar. In the former case, experiments show that plane strain at the center of the beam occurs when the width exceeds eight times the height.

Pranti⁽¹⁸⁾ has recently carried out experiments on notched beams made of lead and loaded in four-point bending. The lateral and transverse strains at the root of the notch were measured by observing the displacement of lines which were scribed into the material. Results were obtained for only one root radius but with a variety of notch depths and thicknesses, and indicate that the thickness-to-root radius ratio must exceed ten for plane strain to exist at the center of the notched beam.

It may also be possible to make an analogy with a phenomenon which occurs in fracture toughness testing. In the testing of aluminum alloys, for example, the size of the shear lips are essentially independent of thickness. These shear lips are indicative of the distance from the sides required to attain a state of plane strain. Knott⁽⁴⁾ has shown that this distance can be explained by assuming that transverse stresses are relaxed over a distance equal to the radius of the plane stress plastic zone.

The finite element analyses, presented below, indicate that the radius of the plane strain plastic zone at fracture in the beryllium notched beams is roughly two to three times the root radius. With a

factor of two to account for the larger size plane stress plastic zone, the above argument implies that the thickness-to-root radius must exceed twelve for plane strain conditions.

Based on these results, the beryllium notched beams were designed so that the thickness-to-root radius was always greater than ten. However, the limited information on the necessary conditions for plane strain to exist in notched beams, certainly warrants further investigation into this problem. This could be done in a direct but expensive way, by employing a three-dimensional finite element program. Such a program was not available during this work, but instead an experiment on a beryllium notched beam was performed in which strains at the root of the notch were measured with strain gages. This experiment is described in Chapter 4 and was originally designed to test the accuracy of the finite element code in predicting the longitudinal strain at the root of the notch. It also afforded the opportunity to investigate the distribution of transverse strain along the thickness of the notched beam.

Two M-S65 beryllium notched beams were tested, with a root radius of 2.54 mm (0.1 in.) and a thickness of 38.1 mm (1.5 in.), which gives a thickness-to-root radius ratio of 15. Four transverse strain gages were placed along the root of the notch at successively greater distances from the center of the beam. The results from the two experiments agreed very well and are shown in Fig. 3.2. The data indicate clearly that plane strain conditions are attained at a distance of about four root radii from the beam surface. Thus, a

thickness-to-root radius of eight appears to be sufficient to insure plane strain at the center of the beryllium notched beams at fracture.

For information on the conditions required for plane strain to exist at general yield in a notched bar, see McClintock⁽¹⁹⁾.

The Finite Element Analysis

A plane strain, elastic-incremental plasticity finite element program, developed at the Lawrence Livermore Laboratory, was used to determine the state of stress and strain in the notched beams at fracture. The details of the computational method are given in Appendix B, but the important assumptions, besides plane strain, are that yielding occurs according to Mises criterion and that isotropic hardening prevails. These conditions allow the uniaxial stress-strain behavior to be used in the formation of the multiaxial stress-strain relations.

As discussed in Chapter 2, initial yielding in beryllium may be characterized by the Mises criterion. On the other hand, beryllium has been shown to exhibit a Bauschinger effect⁽²⁾. The fact that a high degree of symmetry exists in the notched beams and that loading is monotonic, however, indicate that the errors introduced by making the isotropic hardening assumption will have little bearing on the interpretation of the results.

Input to the finite element program consists of the zoning of the notched beam into elements, and the specification of the

boundary conditions and the effective stress-effective plastic strain relation which, in this case, was derived from uniaxial data.

The zoning of specimen B2-1 (see Fig. 4.1 for the dimensions) is shown in Fig. 3.3. Because of symmetry, only half of each notched beam was analyzed, and this is done by fixing the y displacements at the centerline. The geometry of the mesh at the root of the notch and, generally, the size of the elements outside of this region were maintained the same in the analysis of all of the beryllium notched beams. This provided some means of control, although refinement of the zoning had little effect on the stress and strain distributions obtained. Because of this, however, the number of elements required for any given analysis varied from 400 for the smallest specimen to 2000 for the largest.

The boundary conditions used to simulate four-point bending consisted of fixing the x displacement of the node corresponding to the outer roller and applying a point load at the node corresponding to the inner roller. Load per unit thickness was specified as a piecewise linear ramp in time with approximately equal increments for each step. In most cases, the final load corresponded to the load at which fracture in the actual specimen occurred.

The uniaxial effective stress-effective plastic strain relation was derived from the data given by the first bend specimen and was specified as a piecewise linear curve. A comparison of the input curve, after adding elastic strains, to the curve of the first bend specimen was presented in Fig. 2.13.

The accuracy of the finite element code in predicting the state of stress and strain in the notched beams was examined by actually measuring the strains at the root of the notch in two of the notched beams. The procedure and results of these experiments are described in Chapter 4, in light of the general mechanical behavior of the beams. It can be said here that the results of the experiment and the analysis compared very well.

The Effects of Notch Geometry on the Stress and Strain Distributions In The Beryllium Notched Beams

Earlier in this chapter, a thickness-to-root radius ratio of eight was shown to guarantee a state of triaxial stress in the beryllium notched beams. The exact form of the stress state, however, depends on the notch geometry, which includes the root radius, the notch depth, and the included notch angle. By varying these parameters, one would expect specimens of otherwise similar geometry to fracture at different loads. An examination of the differences in stress and strain state would then provide information for the establishment of an overall fracture criterion. In the opposite sense, a study of the effects of notch geometry before testing allows one to induce different states of stress and strain through the careful design of specimens.

In the present work, a limited investigation of this kind was conducted for beryllium notched beams by using a finite element program. Input to the program consisted of material properties obtained from other batches of M-S65 beryllium, and loads which provided maximum stresses close to the tensile strength of the material.

Although the original conclusions for specimen design were drawn from these calculations, the important aspects of the study are illustrated by data obtained from analyses made on the actual test specimens.

In the following discussion, the stress and strain distributions given always correspond to the center of the beam; that is $y = 0$.

First, consider the form of the stress and strain distributions in a beryllium notched beam loaded in four-point bending. Figure 3.4 illustrates the σ_y distribution in the vicinity of the notch for a series of loads. The loads are represented in the non-dimensional form,

$$\frac{\sigma_{\text{nom}}}{\sigma_0} = \frac{1}{\sigma_0} \frac{6M}{Ba^2} ,$$

where σ_{nom} is the nominal outer fiber stress of an elastic uniform rectangular beam of the same reduced cross-sectional dimensions, and σ_0 is the uniaxial yield stress, equal to 260 MPa (37.7×10^3 psi). This figure shows that with increasing plastic deformation the location of maximum stress moves away from the notch surface but always remains roughly in the center of the plastic zone. A comparison with equation (a) of Hill's analysis, also given in Fig. 3.4, shows seemingly good agreement for the prediction of maximum stress. But if the plastic zone size was used with equation (a) to obtain the maximum stress (see for example Ref. 15), there would be a large overestimation in this case. Figure 3.5 shows the shape of

the plane strain plastic zone for the same beryllium notched beam, as determined by a contour of effective stress equal to the yield stress.

A number of important observations can be made from Fig. 3.6, which shows the distributions of principal stresses and strains across the center of the beam in the vicinity of the notch. For example, the most severe location in terms of tensile stresses lies between one and two root radii ahead of the notch, where the maximums in all of the principal stresses occurs. The ratio of the principal stresses in this region is roughly 2:1:1. This behavior in stress is exhibited by all of the beryllium notched beam specimens, including the beams with a root radius of 2.54 mm (0.1 in.). The variation in stress space represented by the fracture data obtained from the notched beams is, therefore, limited.

Figure 3.6 also illustrates that the maximum strain, by any measure, occurs at the root of the notch, but decays very rapidly ahead of the notch.

Consider next the effect of notch geometry on the stress and strain distributions in the beryllium notched beams. For materials exhibiting rigid, perfectly plastic behavior, the analytic treatment of Hill indicates that the maximum attainable stress in a notched beam is independent of root radius (see equation (b)), but instead depends upon the included notch angle. The smallest angle, $\alpha = 0$, gives a maximum stress of $\sigma_{\max} = 2.57 (2k)$. The analysis of Hill does predict that the influence of a smaller root radius is to increase the stress gradient and to move the location of maximum stress closer to the notch root.

Figure 3.7 shows the stress and strain distributions in the y direction, as obtained by the FEM, for two beryllium notched beams which differ only in root radius. Actually, the load on the beam with the smaller root radius is less than on the second beam, but this serves to illustrate that the effect of a smaller root radius is to provide a greater maximum stress at the same load. In addition, the stress gradient is higher and the maximum stress occurs closer to the notch for the beam with the smaller root radius. The location of maximum stress in both cases, however, occurs in the range of one to two root radii ahead of the notch. The maximum strain is significantly greater for the smaller root radius, but decays more quickly.

The included notch angle was found to have no effect on the form or the magnitudes of the stress and strain distributions in the beryllium notched beams. It must be emphasized, however, that this conclusion was reached by examining included notch angles that ranged from 15 to 90 degrees, and at maximum stress levels close to the tensile strength of the beryllium. Obviously, at very large angles, the stress distribution will change significantly.

This conclusion is contrary to Hill's analysis, but it is easy to see why. In the first place, the uniaxial tensile strength was attained in the beams studied before the plastic zones impinged on the straight boundaries of the notch. But even at higher loads, one would not necessarily expect the conclusions based on Hill's analysis to be valid, since beryllium work-hardens and the location of maximum stress is not at the elastic-plastic interface.

Plasticity in a notched beam in four-point bending begins at the root of the notch and progresses in a roughly radial direction. With continued deformation, plasticity begins at the back, compressive surface and eventually joins the region extending from the root of the notch. The load at which these zones connect is called the general yield load. It can be predicted accurately for mild steels by the theory of slip-lines, but it is necessary that plastic deformation does not extend to the upper lateral surface before this load is reached. The geometric requirements for this condition vary with the included notch angle (see Knott⁽⁴⁾), but for an included angle of 45 degrees, the notch depth must be at least 33 percent of the beam height. This restriction on notch depth was not placed on beryllium because the preliminary analyses showed that fracture would occur before general yield.

Varying the notch depth around a value of 33 percent of the beam height showed virtually no effect on the form of the stress and strain distributions in the beryllium notched beams. Instead, a rather surprising result was obtained. By decreasing the notch depth, the same stress distribution could be obtained at a lower load. This is curious, since one would expect the larger reduced section to support a higher load. This phenomenon has been demonstrated for elastic behavior⁽¹⁵⁾ and is limited to a certain range of notch depth-to-beam height ratios. It can be explained by realizing that shallower notches provide greater nominal strains.

Summary

The notched beam specimen can provide a limited range of tri-axial stress states. The geometric conditions necessary for this situation are not well defined, but analytic and experimental treatments of this problem in the literature indicate a thickness-to-root radius ratio of ten to be sufficient. A thickness-to-root radius ratio of eight was shown experimentally to be sufficient for the beryllium notched beams.

A limited investigation of the effects of notch geometry on the stress and strain distributions in the beryllium notched beams, as obtained by finite element analysis, reveal that the root radius has the most significant effect. In fact, for the same load, smaller root radii increase the stress and strain gradients, give higher strains at the root of the notch and greater maximum stresses, and move the location of maximum stress closer to the notch. In addition, the included notch angle has little effect on the stress and strain distributions, but the effect of decreasing the notch depth from a value of $1/3$ the beam height is to require less load to attain the same state of stress and strain.

CHAPTER 4

THE NOTCHED BEAM EXPERIMENTS

The previous chapter contained the information necessary for the design of the beryllium notched beams. These specimens must provide as great a variation as possible in both size and the states of stress and strain. In addition, very careful specimen preparation is required, which includes special machining, etching, and a close inspection of the final dimensions.

The analysis presented in Chapter 3 provides some indication of the behavior to be expected in the notched beam experiments. However, the plane strain finite element results cannot describe the effect of a mixed plane strain-plane stress state on the overall deformation, or the nature of fracture in the beryllium notched beams. This information can only be obtained from actual tests.

Since only part of each notched beam is in a state of plane strain, it is necessary to find an experimental parameter which is representative of the behavior in this region. It is also desirable to know if fracture initiation leads directly to instability, for otherwise the finite element analysis cannot be used to describe the conditions leading to unstable fracture. Finally, but of extreme importance, is the need to determine, experimentally, the precise location of fracture initiation in the nonuniform stress and strain fields of the notched beam.

Specimen Design

A total of 26 beryllium notched beams were tested in this investigation. The geometries of 24 of the beams are listed in Figure 4.1; the other two beams were designed specifically to test the accuracy of the finite element program and the assumption of plane strain, but also provided some valuable fracture data.

The 24 specimens consist of three each of eight different geometries: four sizes, each with two root radii. The four different sizes are illustrated in Figure 4.2. They range in thickness from 6.35 mm (0.25 in.) to 38.1 mm (1.5 in.) and in length from 38.1 mm (1.5 in.) to 229 mm (9 in.). The two root radii are 0.254 mm (0.01 in.) and 0.508 mm (0.02 in.), and provide geometric similarity between adjacent sets of beam sizes. It would be interesting to test beams with smaller root radii, but these cannot be machined accurately with present techniques.

These different geometries result in thickness-to-root radius ratios ranging from 12.5 to 150. The notch depths were machined at 25 percent of the beam height in order to minimize the fracture loads, and the included notch angle was maintained at 45 degrees.

The other two beryllium notched beams were machined from 38.1 mm X 38.1 mm X 229 mm (1.5 in. X 1.5 in X 9 in.) beams, so that the root radius was 2.54 mm (0.1 in.), the notch depth was 33 percent of the beam height, and the included notch angle was 45 degrees. The large root radius facilitated the installation of strain gages.

Specimen Preparation

The specimens were machined and etched according to the guidelines set by the Materials Advisory Board ⁽¹⁰⁾, and the dimensions of each beam were then recorded. The value of the root radius was first checked by viewing the contour of the notch in an optical comparator. Since this method is insensitive to variations along the thickness, silicone rubber molds were made and inspected for each notch. After fracture, the point of initiation along the thickness was noted and a thin slice from the mold corresponding to this location was then viewed in the optical comparator. This second method of determining the value of the root radius revealed in some cases significant differences from the first method. Nevertheless, slices taken along the thickness for a few of the molds showed no detectable variation. The root radii obtained by examining the molds were used in the finite element calculations.

As mentioned earlier, it is necessary to experimentally measure some parameter, such as load, deflection, or strain, which can be used to predict the stress-strain state at fracture using a plane strain finite element analysis. To investigate this, at least one strain gage was mounted on the back, compressive surface of one each of the four sizes of notched beams so that the axis of the gage coincided with the y direction of the beam. A transverse gage was also mounted on the back surface of the largest beam.

Experimental Apparatus

The beryllium notched beams were tested in a 50 kN (10000 lb) electro-hydraulic, servo controlled testing machine. The specimens were loaded in four-point bending to fracture with the fixture described in Appendix A. The moment arms used for each specimen are listed in Figure 4.1.

Each specimen was tested in displacement control with rates ranging from 0.64×10^{-2} mm/min (0.25×10^{-3} in/min) for the smallest beams to 5.1×10^{-2} mm/min (2×10^{-3} in/min) for the largest beams. The respective displacements at fracture in the tests ranged from 20×10^{-2} mm (8×10^{-3} in.) to 70×10^{-2} mm (28×10^{-3} in.). The applied load and ram displacement were monitored on an X-Y recorder. For the strain gaged notched beams, a computerized data acquisition system was used to record the load and the strain gage data on magnetic tape.

Results

The Four Strain Gaged Specimens: The four strain gaged beryllium notched beams were tested first and then analyzed thoroughly. The root radius for each of these beams was 0.508 mm (0.02 in.).

Figure 4.3 illustrates the plane strain plastic zones at fracture for these four specimens, as determined by the finite element program using a contour of effective stress equal to the initial yield stress, 260 MPa. One can see that as the specimen becomes

smaller, there is much more plasticity. There are two reasons for this. In the first place the size of the plastic zone around the notch in each specimen, which is governed mainly by the root radius, is a larger percentage of the reduced cross section for the smaller specimens. Secondly, less of the region along the notch in the smaller specimens is in a state of pure plane strain. The thickness-to-root radius ratios of the four strain gaged specimens, A2-1, B2-1, C2-1, and D2-1, are 12.5, 25, 50, and 75 respectively. Thus, the percentage of the thickness at the notch root for each specimen that is in a state of pure plane strain, is 36, 68, 84, and 89 percent respectively. The relatively small amount of material in a state of plane strain in the smaller specimens results in larger deflections than would be predicted from a plane strain analysis.

This point is well demonstrated by the differences in the back surface strains as measured experimentally and as obtained by the plane strain finite element program. These are shown in Table 4.1. The strain predictions from the finite element analysis were obtained by using the experimental fracture load as input. It was also necessary to make a correction to the strains predicted by the plane strain analysis, since the back of each beam is in a state of plane stress. A correction in the opposite sense -- to the experimental strains -- is easier to visualize. It involves the determination of the transverse strain, e_z , in the experiment, which may be simply estimated with a knowledge of the total axial strain and the uniaxial compression stress-strain curve as,

$$e_z = \left(\frac{1}{2} - \nu \right) \frac{\sigma}{E} - \frac{1}{2} e_y^T$$

where σ is the compressive stress corresponding to the total axial strain, e_y^T at the back surface. Data from a transverse strain gage mounted on the back surface of the largest beam showed good agreement with the above equation. The next step in the correction is to calculate the effect on axial strain of applying a transverse strain of, $-e_z$.

The results presented in Table 4.1 are to be viewed as estimates, but it is clear that the behavior of specimen A2-1 is not well represented by the plane strain finite element analysis when the experimental fracture load is used as the input parameter. Since displacements are what in fact give rise to stresses, the finite element analysis will underestimate the distribution of stresses in the actual beam. Thus, the true stress distribution in the region of plane strain could be better estimated by using the applied displacement at fracture as input. However, the experimental determination of the relative displacement of points on the specimen is difficult. It is easy to measure the ram displacement, but this includes the deformation of the testing system which includes the rollers and the blocks in the test fixture.

Instead, an equivalent displacement for specimen A2-1 was obtained by increasing the fracture load specified in the finite element analysis until the back surface strain predicted by the analysis was roughly eight percent less than the value observed in the experiment. This is consistent with the behavior exhibited by the other three strain gaged specimens. The load required to do

this was roughly 3.5 percent greater than the experimental load, and the increases in maximum strain and maximum stress were 20 percent and 10 percent respectively. This procedure is supported by the fact that the resulting stress and strain distributions predicted for specimen A2-1 at fracture agreed almost identically with those predicted for the other three strain gaged specimens.

To be consistent, this procedure should also be applied to the other specimens, but to a lesser degree as the thickness-to-root radius ratio and beam height increase. For example, in specimen B2-1 the percentage of the thickness at the root of the notch in a state of plane strain is twice that for specimen A2-1. In addition, the plastic zone size in specimen B2-1 represents a smaller percentage of the reduced section than for specimen A2-1. Therefore, an increase in the input load of one percent over the experimental load would appear to be a sufficient correction for specimen B2-1. However, even with an increase in load of 3.5 percent, the changes in the predicted strains and stresses for specimen B2-1 were insignificant. In other words, the experimental fracture load can be used in the finite element program to give accurate results for all of the beryllium notched beams except specimens A2. For these specimens, the input load to the finite element program should be greater than the experimental fracture load by roughly 3.5 percent.

Fracture in each of the four specimens was quite sudden. This indicates that instability is either associated with crack nucleation or crack growth on a microscale. Therefore, it appears that the finite element results can be used to describe the conditions which lead to unstable fracture.

The location of fracture initiation in the four strain gaged specimens was easily discernible. The fact that fracture initiated ahead of the notch was obvious by visual inspection, but precise measurements of the distance ahead of the notch at which fracture initiated were made by viewing each specimen in a scanning electron microscope. Figure 4.4 shows scanning electron micrographs of the location of fracture initiation for specimen B2-1. The figure shows the magnification, which is within 10 percent of being exact.

The information obtained from the four strain gaged specimens may be summarized as follows. Fracture is very sudden and for each of the four notched beams, except beam A2-1, the experimental fracture load can be used as input to the finite element program to obtain the distribution of stress and strain at fracture. For specimen A2-1 the load specified in the finite element analysis was 3.5 percent higher than the experimental load. The distance ahead of the notch at which fracture initiated was easily determined by viewing each specimen in a scanning electron microscope.

Tests on the Two Special Notched Beams: The testing of the two special beryllium notched beams, which had a very large root radius, provided additional information for the general analysis of the notched beams. These specimens were originally designed to examine the accuracy of the finite element analysis by measuring the strains at the root of the notch with strain gages and comparing the results to the predictions of the computer program. But in view of the low thickness-to-root radius ratio for these specimens, equal to

15, the displacement effect spoken of earlier will certainly make the finite element analysis predictions an underestimate.

Figure 4.5 (a) illustrates the configuration of the strain gages in the root of the notch for one of the specimens. Two fixed axial- y direction - and four transverse - z direction - Micro Measurement EA strain gages were used, each with a grid length equal to 0.79 mm (0.031 in.). These beams were tested in a 100 kN (20000 lb) testing machine in four-point bending and in displacement control; the rate was set at 5.1×10^{-2} mm/min (2×10^{-3} in/min). The load and strain gage data were recorded on magnetic tape with a computerized data acquisition system. The results from the transverse strain gages were presented in Chapter 3, Figure 3.2.

The results from the axial strain gages are shown in Figure 4.5 (b), compared to the predictions of the plane strain finite element program in which the experimental loads were used as input. As expected, the computer program predictions underestimate the values given by the strain gages with a maximum difference of about 15 percent at fracture. If the strain gage data are considered accurate, and this is reasonable in view of the success of the bending experiments discussed in Chapter 2, the best estimate of the stress distribution at fracture should be obtained by sufficiently increasing the fracture load specified in the computer program. In this case, the load specified is increased until the predicted axial strain at the root of the notch is the same as that given by the strain gages. An increase in load of approximately six percent was required to do this, which resulted in an increase in maximum stress

of less than two percent. This increase in load for the special beams is greater than for specimens A2, because the notch depth is larger for the special beams: 33 percent as opposed to 25 percent.

In view of the correction required for the mixed state of plane strain and plane stress, it is difficult to make a precise statement about the accuracy of the finite element program. Nevertheless, three facts indicate that the plane strain computer analysis is very accurate: the predicted strains are underestimated, as expected; the correction required to make the experimental and predicted strains equal is supported by other experiments; and the changes in stresses with the correction are small for all of the specimens but those with a very small thickness-to-root radius ratio.

Beryllium Notched Beam Fracture Data: Once the precise geometry and the fracture load of each beryllium notched beam was determined, a finite element analysis was performed to obtain the stress and strain distributions at fracture. This information was used with the scanning electron microscope observations to determine the conditions which led to fracture for each specimen. The stress and strain distributions at fracture for specimen B2-1 and the location of fracture initiation were shown in Figures 3.6 and 4.4 respectively. It is impractical to include the fracture information in this form for each specimen. Instead, the data which best represents the distributions of stress and strain at fracture are listed in Tables 4.2 and 4.3.

Table 4.2 lists the experimentally obtained data for each beryllium notched beam. This includes the actual root radius and notch depth, the total applied load at fracture, and the location of fracture initiation. The location of fracture is defined by the distance ahead of the notch in root radii, x/ρ , and the relative distance from the center of the beam, $2|z|/B$.

A variety of root radii, ranging from 0.127 mm (0.005 in.) to 0.533 mm (0.021 in.), resulted from the machining and etching process in the regular specimens. The high stress and strain gradients in the specimen with the smallest root radius, D1-3, made it difficult to determine the precise location of fracture initiation. However, in the other specimens fracture always initiated ahead of the notch and the location was easy to determine. The three exceptions to this were traced to machining flaws. A simple calculation also shows that initiation always occurred in the region of plane strain. For example, plane strain exists in specimens A2, B2, C2, and D2 when $2|z|/B$ is less than 0.36, 0.68, 0.84, and 0.89 respectively.

Scanning electron microscopy of the fracture surfaces in the beryllium notched beams showed that fracture occurred by transgranular cleavage. Initiation frequently originated from a grain with a diameter of roughly $20\text{ }\mu\text{m}$, but in some cases originated from a grain with a diameter closer to $50\text{ }\mu\text{m}$. In the former case other cleaved grains of the same size and orientation were usually in the immediate vicinity of the initiation site. This indicates the possibility that unstable fracture occurs when a large grain cracks or

when a number of favorably oriented grains crack and coalesce. Examples of these two cases are shown in Figure 4.6 and 4.7.

The exceptionally high fracture loads for specimens D were not expected. In fact, it was not possible to break specimen D2-3 in the 50 kN testing machine.

Table 4.3 contains the data obtained from the finite element analyses. These include the maximum strain, which always occurs at the root of the notch, the maximum stress and its location, and the stress, σ_y , strain, e_y , and stress ratio, $\sigma_y:\sigma_x:\sigma_z$, at the location of fracture initiation. Since shear stresses are assumed to be absent along the centerplane, $y = 0$, the data in Table 4.3 correspond to principal stresses and strains. Therefore, parameters such as the effective stress, the mean stress, and the maximum shear stresses at the location of fracture initiation can be calculated. In addition, the accumulated effective plastic strain can be obtained by using the effective stress with the effective stress-effective plastic strain curve of Figure 2.13.

Again, the experimental fracture load was used as the input to the finite analysis for each beam except specimens A2; in these cases the input load was 3.5 percent greater than the experimental load. Table 4.3 shows that, in general, the maximum stresses and the strains at the location of fracture initiation are greater in the specimens with a 0.254 mm (0.01 in.) root radius than in the specimens with a 0.508 mm (0.02 in.) root radius. This effect of root radius is convincingly demonstrated by the data from the special notched beams, S-1 and S-2.

The spatial distribution of maximum stress, σ_y , at fracture may be characterized by the location of maximum stress. As this location moves away from the notch, the stress gradient becomes lower and, in general, the distance ahead of the notch covered by a certain stress level becomes greater. For example, the distances ahead of the notch covered by a tensile stress of 400 MPa (58000 psi) at fracture for specimens B1-1, B2-1, and S-1, are 1.04 mm, 1.35 mm, and 5.7 mm respectively. For notched beams with the same root radius, but different beam heights, the distributions of stress and strain at fracture are almost identical.

Summary

The results of testing 26 M-S65 beryllium notched beams in four-point bending to fracture are listed in Tables 4.2 and 4.3. For each beam the critical event leading to fracture was well defined and the location of initiation was easy to determine. In addition, the detailed experimental investigation of some specially prepared specimens indicates that the results of the plane strain finite element calculations can be viewed as accurately representing the actual states of stress and strain in the experiments.

CHAPTER 5

FRACTURE CRITERIA FOR BERYLLIUM

The fracture behavior of beryllium has now been studied for a number of different situations. In the present investigation, M-S65 beryllium was tested in uniaxial tension, plane stress pure bending, and triaxial tension as produced by notched beams in four-point bending. In addition, the literature contains data obtained from experiments on thin-walled tubes made of the similar S-65 grade beryllium. Together, these tests provide a wide variation in stressed volume and states of stress at fracture.

By examining the results of these tests, and by drawing on the present understanding of the physical processes of deformation and fracture in polycrystalline beryllium, it is possible to establish a fracture criterion for multiaxial states of tensile stresses. Such a criterion allows the conditions leading to fracture under arbitrary states of stress to be predicted. As an example, the currently available stress analyses for a sharp crack permit the fracture toughness of beryllium to be estimated.

Uniaxial Tension

Uniaxial tension experiments on M-S65 beryllium were conducted by the manufacturer and the author. The results from conventional tension and bending tests indicate that the maximum stress and strain at fracture are approximately 400 MPa (58,000 psi) and four

percent, respectively. Necking was never observed in these tests, so the stress and strain distributions at fracture were uniform.

There was very little variation in the ultimate strength in the tension tests for any particular location in the billet. This indicates that the statistical variation in strength for specimens of the same size is quite small. No tests on uniaxial tension specimens of different size were conducted, so nothing can be said about a size effect at this point.

In the tension and bending tests performed in this work, fracture appeared to initiate by cleavage. However, in the bending tests, scanning electron microscopy of the fracture surface revealed evidence of plasticity. Micrographs of the fracture surfaces in the vicinity of initiation for one of the tensile and bending tests were shown in Figure 2.8.

Biaxial Stress

Although biaxial stress experiments on M-S65 beryllium have not been conducted, the results of Lindholm, et al⁽²⁾ on S-65 beryllium thin-walled tubes are useful in establishing a fracture criterion for M-S65 beryllium. The results were presented in Figure 2.12.

These data cannot be used quantitatively, since the uniaxial strengths of the S-65 differ considerably from the strengths of the M-S65. However, the basic material is the same, and the form of the uniaxial stress-strain curve is very similar; in both cases, near perfect plasticity is exhibited up to strains of roughly 0.5

percent, after which hardening occurs until fracture. Therefore, it is reasonable to generalize the biaxial tensile stress fracture behavior of S-65 beryllium to M-S65 beryllium.

In the thin-walled tube tests, different ratios of axial tension-to-transverse tension were obtained by varying the axial load and the internal pressure. In every case, the maximum tensile stress at fracture was constant. Because of this, the maximum tensile strain ranged from 2.6 percent for pure axial tension to 1.5 percent for an axial tension of two times the circumferential tensile stress.

Therefore, one of the main differences between uniaxial tension and biaxial tension is the magnitude of the strains. Nevertheless, for the S-65 beryllium the strains in biaxial tension still represent a large percentage of the maximum strain in uniaxial tension.

Again, no necking was observed and so the stress and strain distributions were uniform.

As in the present investigation, little variation in the uniaxial ultimate strength was observed in the S-65 thin-walled tube tests. All of the tubes were of identical size, so it is not possible to make a direct statement about a size effect. However, besides a size effect, a statistical theory of strength predicts that the biaxial stress failure envelope is rounded at the corner $\sigma_1 = \sigma_2$. This is because with equal biaxial tension, a greater number of the randomly oriented flaws are exposed to the full tensile stress than in uniaxial tension. The fact that this effect is not observed in the thin-walled tube results, gives further

evidence against a significant statistical variation in strength for S-65 and M-S65 beryllium.

There is another important result of the thin-walled tube tests. In pure shear, brought about either by pure torsion or equal tension and compression, fracture occurred at a tensile stress of 20 percent less than the uniaxial tensile strength. Thus, fracture in pure shear does not follow the maximum tensile stress criterion exhibited by the biaxial tension tests. This may be due to the fact that the shear strains at fracture in pure shear exceeded those in uniaxial tension by almost 100 percent. As will be discussed in more detail later, larger plastic strains could result in greater damage on a microscale and, therefore, a lower tensile stress would be required to cause unstable fracture.

Scanning electron micrographs of the thin-walled tube fracture surfaces show that, in every case, fracture was initiated by cleavage, although some plasticity is evident in the torsion specimens.

Triaxial Tension

The results of the M-S65 beryllium notched beam tests were presented in Table 4.3. Because of the nonuniform distributions of stress and strain ahead of the notch, it is much more difficult to interpret these data.

The maximum stress at fracture in each of the notched beams occurred at a distance between 1.0 and 1.5 root radii ahead of the notch. The ratio of the three principal tensile stresses in this

region, $\sigma_y:\sigma_x:\sigma_z$, was roughly 1:0.5:0.5. For every beryllium notched beam the location of fracture initiation was in the vicinity of the maximum tensile stress. Furthermore, the tensile stress, σ_y , at this location exceeded the uniaxial tensile strength by at least 20 percent, and in some cases by 40 percent.

The maximum strains in the notched beams occur at the root of the notch, but decrease very rapidly away from the notch for the specimens with small root radii. At the location of fracture initiation, the strain, e_y , ranged from 0.25 to 0.7 percent. The plastic strains at this location are even smaller, because of the high triaxial stresses. The distributions of principal stresses and strains in the beryllium notched beams at fracture were illustrated in Figure 3.6.

For specimens with different root radii, the maximum tensile stress at fracture was observed to decrease with increasing root radius. For specimens with root radii close to 0.254 mm (0.01 in.), 0.508 mm (0.02 in.), and 2.54 mm (0.1 in.), the average of the maximum tensile stresses at the location of fracture initiation exceeded the uniaxial tensile strength by roughly 35, 25, and 20 percent, respectively. This effect of root radius has also been observed in mild steel at low temperatures⁽¹⁵⁾.

Figure 3.7 illustrates the differences in the stress and strain distributions at fracture for specimens with different root radii. The main differences in stress and strain are the higher gradients for specimens with smaller root radii. In terms of stress, this generally means that less material is exposed to a given stress level. In terms of strain, a higher value occurs at the root, but

the strain ahead of the notch decreases more rapidly. Therefore, except for the region very close to the notch, the strains at any distance ahead of the notch are smaller for specimens with smaller root radii.

A size effect is an attractive argument to explain the greater maximum tensile stress carried by the notched beams, and the effect of the root radius on the maximum tensile stress at fracture. However, it is doubtful that a statistical effect alone can account for these significant differences.

The size effect is a consequence of a statistical variation in strength. Such a variation has not been exhibited by any of the tests on M-S65 beryllium. The results from three specimens for each of the notched beam geometries were quite reproducible. Generally, when significant variation in fracture loads did exist, the cause could be traced to differences in geometry. Further evidence against a statistical variation in strength is provided by the fact that notched beams with the same root radius but with different thicknesses, fractured at virtually the same maximum stress.

It appears that a physical argument is required to explain the differences in maximum tensile stress at fracture.

A Fracture Criterion for Beryllium

For many nominally brittle polycrystalline metals, the process of fracture initiation begins with microcrack nucleation caused by

plastic deformation. This was demonstrated convincingly by Low⁽²⁰⁾, who showed that for a range of temperatures, fracture in mild steel at low temperatures occurred only after the yield stress was attained. In this case, plastic deformation was inhibited because of the lack of thermal activation at the low temperatures. Another way to inhibit plastic deformation is to increase the triaxiality of the applied stresses. Because plastic deformation is governed by the differences in the principal stresses, that is, the shear stresses, a high degree of triaxiality allows the tensile stresses to attain relatively large values before yielding occurs.

Unless the tensile stresses are extremely high, initial plastic deformation is unlikely to lead directly to crack nucleation and unstable fracture. Instead, crack nucleation, or more generally, damage, probably increases in frequency and severity with continued plastic deformation, until enough damage has accumulated to lead to unstable fracture at the current value of the tensile stress. The importance of tensile stress in the fracture of beryllium must be emphasized. Fracture in everyone of the notched beams initiated very close to the location of maximum stress. The type and possible sources of damage in polycrystalline beryllium are discussed in the next section.

The argument just presented explains a number of the experimental observations. In particular, the triaxial state of stress in the notched beams inhibits plastic deformation, and consequently, allows higher tensile stresses to be attained before the critical level of plastic deformation, or damage occurs. In contrast, the plastic deformation in uniaxial and biaxial tension is

so large compared to that which occurs in triaxial tension, that a much lower tensile stress is required for unstable fracture. Furthermore, the lower tensile stress at fracture observed for pure shear may be explained by the considerably higher plastic deformation which occurs in this test.

The use of a strain dependent fracture criterion to explain the effect of root radius on the maximum tensile stress at fracture, is not as clear. For a given distance ahead of the notch, it is true that the strains are lower for specimens with smaller root radii. But since the maximum tensile stress occurs at different distances ahead of the notch for specimens with different root radii, it is possible for the strains at the locations of maximum stress to be the same. This is demonstrated for specimens B1-1 and B2-1 in Figure 3.7. Nevertheless, the greater overall magnitudes of the strains in the specimens with larger root radii, would appear to increase the probability of fracture initiation, and therefore, account for the effect of root radius on the maximum tensile stresses at fracture.

This probabilistic argument is not the same as the one used to explain the statistical variation in strength for a material. Fracture apparently initiates when a certain combination of damage, or plastic strain, and tensile stress occurs. At some value of the externally applied load this becomes possible over a limited volume of the specimen. Apparently, this volume is greater for specimens with larger root radii, because of the overall higher strains. This argument is supported by the data of Table 4.3, which shows that the strain at the location of fracture becomes nearly constant for the thicker specimens.

Without knowing the precise relationship between strain, damage, and the tensile stress required to initiate fracture, it is very difficult to establish a quantitative criterion of fracture. However, a strain dependent criterion of fracture has a useful implication. The plane strain finite element analyses of the notched beams indicate that the ratio of the tensile stresses at fracture is independent of root radius. Therefore, the plastic deformation in a notched beam in plain strain will always be considerably less than in uniaxial tension. Consequently, a limit to the decrease in maximum tensile stress at fracture should occur, and this value would be the most suitable for general design.

Microstructural Features of Deformation and Fracture in Beryllium

Beryllium has an hexagonal close-packed crystal structure with the smallest c/a ratio among all of the hexagonal close-packed metals. Slip at room temperature occurs predominantly on the basal planes and the prism planes of the first kind, both with directions in the basal plane (See Partridge⁽²¹⁾ for a review of the crystallography of hexagonal close-packed metals.)

Beryllium's inability to slip with a Burger's vector out of the basal plane is believed to be the cause of its room temperature brittleness. In order for a crystal to undergo an arbitrary homogeneous shape change, it is necessary that it have five independent slip systems. Beryllium normally only has four at room temperature.

Fracture in beryllium occurs mainly by cleavage on the basal planes. The process of fracture initiation is believed to require dislocation interaction; that is, plastic deformation. The most popular mechanism at present is called the "bend plane splitting mechanism" (see Conrad, et al⁽²²⁾). This mechanism is appealing, because it accounts for the fact that beryllium slips and cleaves most easily on the same plane. There is substantial evidence that this mechanism is operative in beryllium single crystals, but there is virtually no evidence that it operates in polycrystalline beryllium.

Without specifying an explicit mechanism of microcrack initiation, it is still possible to examine the microstructural origins of unstable fracture in polycrystalline beryllium. Generally, the nucleation process results in a crack whose length is equal to one grain diameter. For a very large-grained material, this could be sufficient to cause complete fracture. But for a fine-grained material, such as M-S65 beryllium, the grain boundaries and the differing orientations of the neighboring grains probably provide the obstacle required to prevent further propagation. Cooper⁽²³⁾ has commented that isolated, cracked grains have often been observed in cross sections of fractured polycrystalline beryllium, but two adjacent cracked grains have never been observed. He, therefore, postulated that the critical event was the cracking of two neighboring grains.

In the present investigation, the fracture surfaces of each of the M-S65 beryllium notched beams, and one each of the tension and bending specimens, were examined with a scanning electron

microscope. In almost every notched beam, fracture initiated from an area which appeared relatively smooth with respect to the surroundings. This size of this area was generally 50 μ (0.002 in.) in diameter and consisted of either two or more cleaved grains, (Figure 4.6) or of one relatively large grain (Figure 4.7). The first case provides evidence for Cooper's prediction.

In the tension test, fracture initiated at the surface, and in the bending tests, the fracture surface showed evidence of plasticity (see Figure 2.8). Consequently, it was not possible to draw conclusions about the degree of damage required to initiate fracture in these tests.

The Prediction of Plane Strain Fracture Toughness for M-S65 Beryllium

The determination of the conditions leading to fracture for a structure that contains a sharp crack is a technically important problem in engineering. For a wide class of materials and load and crack configurations, linear elastic fracture mechanics can be used quite successfully to predict these conditions. The requirements for such an analysis are a knowledge of the stress intensity factor, which depends on load and geometry, and the plane strain fracture toughness, which is a material property. The stress intensity factor for virtually any situation can be closely estimated by using known, tabulated solutions. The experimental procedure for determining the plane strain fracture toughness of a material is well established, and ideally, requires only one test. However, this is not the case for beryllium.

Prefatiguing of beryllium generally requires compressive as well as tensile loads, and the tensile loads must usually exceed the limits set in the standards. Special standards are now being written⁽²⁴⁾ to obtain consistent and representative values of the plane strain fracture toughness for beryllium, but it would be helpful if another independent method could be used to obtain this property.

Recently, it has been proposed⁽²⁵⁾ that the plane strain fracture toughness of steels which fracture by cleavage can be estimated from strength values obtained from notched bars. This approach can now be applied to the M-S65 beryllium. For this prediction it is first necessary to have an accurate knowledge of the stress and strain distribution ahead of a sharp crack. Both analytic and finite element, plane strain, elastic-plastic solutions exist for this problem. Two of these solutions are illustrated in Figure 5.1 for different values of the hardening parameter n ; where $T = A(\bar{\epsilon}^P)^n$ represents the effective stress-effective plastic strain relation for the material. For the M-S65 beryllium, n is approximately equal to 0.2.

Since the material around a sharp crack experiences a high degree of tensile triaxiality, it is also necessary to have a fracture criterion for the material under these conditions. Such a criterion was established for M-S65 beryllium in the present investigation. The maximum tensile stress at fracture in the beryllium notched beam with the smallest root radius was 575 MPa (83,400 psi). Considering the effect of root radius on the maximum tensile stresses at fracture in the notched beams, 575 MPa is

probably an underestimate of the stress required to cause fracture for a sharp crack. In addition, observations of the notched beam fracture surfaces indicate that a flaw with a diameter of 50 μm (0.002 in.) is required to initiate fracture in a state of triaxial stresses. A flaw of this size could practically be considered as a point for the stress gradients experienced by the notched beam specimens. On the other hand, the stress gradient ahead of a sharp crack is much higher. Therefore, one may assume that fracture at a sharp crack occurs when a stress of 575 MPa occurs over a distance of 50 μm .

In Figure 5.1 the stress distribution is represented on the ordinate by the ratio of the stress, σ_y , to the initial yield stress σ_0 , and on the abscissa by the distance ahead of the crack normalized by the ratio $(K/\sigma_0)^2$, where K is the applied stress intensity. At fracture, K is equal to the plane strain fracture toughness. Therefore, since $\sigma_y/\sigma_0 = 2.2$ and $X/(K_{IC}/\sigma_0)^2 = 0.035$,

$$K_{IC} = 260(50 \times 10^{-6}/0.035)^{1/2} = 9.8 \text{ MPa } \sqrt{\text{m}} \text{ (8.9 ksi } \sqrt{\text{in}}).$$

The plane strain fracture toughness of a different batch of M-S65 beryllium has been obtained by testing the standard three-point bend fracture toughness specimen⁽²⁶⁾. Very reproducible results were obtained from three tests, giving a value close to 11 MPa $\sqrt{\text{m}}$ (10 ksi $\sqrt{\text{in}}$).

The agreement between the predicted and experimental results is exceptional. In view of the difficulty in performing fracture

toughness tests on beryllium, and the relative ease of performing a notched beam test, the use of notched beam test to estimate the fracture toughness should be considered as an attractive alternative.

Summary

The tensile stress required to cause fracture in the triaxial tension stress field of M-S65 beryllium notched beams is significantly higher than in uniaxial tension. The difference increases from 20 to 40 percent as the notch becomes sharper. Therefore, a maximum tensile stress criterion of fracture is unsatisfactory as an accurate design tool.

This conclusion is contrary to biaxial tension tests on the similar S-65 grade beryllium. However, the differences can be explained by assuming that the tensile stress required to cause fracture is dependent upon the amount of damage in the material which is a function of plastic strain. In contrast to the uniaxial and biaxial states of stress, the high triaxiality associated with a notch leads to considerably less plastic deformation. Furthermore, sharper notches can withstand higher maximum tensile stresses, because the overall magnitudes of the strains adjacent to the notch are smaller.

The damage is a result of the accumulated plastic strains and occurs in the form of microcracks. The size of the critical microcrack appears to be reasonably constant in the notched beams, and occurs when either a large grain cracks or when one or more adjacent cracked grains coalesce.

Finally, the size of the critical microcrack can be used with the maximum tensile stresses observed in the notched beams at fracture to successfully predict the plane strain fracture toughness of beryllium.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

The use of notched beams in four-point bending proved to be a highly successful method of investigating the fracture behavior of beryllium under tensile, triaxial states of stress.

Four-point bending is not a trivial method of load application, but with a close examination of the possible errors in this technique, a fixture was designed so that virtually all shear stresses were eliminated at the reduced section of the notched beams. This same fixture was used to show that the uniaxial tension and compression stress-strain curves of a material can be obtained from a single bend specimen. The accuracy and versatility of this method were demonstrated by testing beryllium, cast iron, and prestrained copper, and comparing the results to conventional tension and compression tests. As a consequence the beryllium stress-strain relations for the finite element program were derived from a bending test.

The plane strain finite element program, NIKE2D, provides an accurate tool for investigating the state of stress and strain in the beryllium notched beams. This was shown by actually measuring the strains at the root of the notch with strain gages for some specimens. These same experiments revealed that the geometric conditions required for a state of plane strain to exist in the

center of the beryllium notched beams, is a thickness-to-root radius ratio greater than or equal to eight. When the thickness-to-root radius is only slightly greater than this value, a limited region of the specimen is in plane strain and the experimental load cannot be used in the plane strain finite element program to accurately describe the actual state of stress and strain. Instead, relative displacement, or an effective load must be used.

The root radius has the greatest effect in determining the strength of a notched beam. Sharper notches elevate the peak stresses and cause fracture to occur at lower loads. A reduction of the notch depth from a value of one-third the beam height also reduces the loads required for fracture, but the effect of the included notch angle is minimal, except for very large values.

The lack of variability in the strength of the uniaxial test specimens and the notched beams for any given geometry, indicates that the statistical size effect in M-S65 beryllium is insignificant.

The maximum tensile stress at fracture in the triaxial tension stress field of the M-S65 beryllium notched beams attained significantly higher values than in uniaxial tension. In addition, the maximum tensile stress at fracture increases as the root radius decreases. These two phenomena show that a maximum tensile stress criterion of fracture cannot be used for the accurate design of an M-S65 beryllium structure.

The differences in the maximum tensile stresses at fracture in uniaxial and triaxial tension, and the effect of root radius, can be explained by using a strain dependent tensile stress criterion of fracture. With higher plastic strains, more damage in the form of

microcracks occurs and unstable fracture can initiate at lower tensile stresses. A lack of understanding of the microstructural processes leading to microcracking did not allow a quantitative relationship between the plastic strain, damage, and tensile stress required to initiate fracture to be made. Nevertheless, it was possible to use information about damage obtained from the close examination of the notched beam fracture surfaces together with strength data obtained from the notched beam tests, to closely predict the plane strain fracture toughness of the M-S65 beryllium.

Considerations in the Design Against Fracture of Beryllium

Although a quantitative fracture criterion was not established, it is still possible to make a number of statements about the conditions required to cause fracture in arbitrary states of stress, and their relationship to data obtained from a uniaxial test.

Since the plastic strain is governed by the effective stress,

$$\bar{\sigma} = \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2} / \sqrt{2},$$

the most general statement possible is;

1. As the effective stress beyond yield corresponding to a given stress state decreases, the maximum tensile stress at fracture should increase, and vice versa.

The data of Lindholm, et al⁽²⁾, indicate that the difference in plastic strains for states of plane tensile stress is insufficient to observe a difference in maximum tensile stresses at fracture. Therefore;

2. The uniaxial tensile strength should be used to predict fracture in tensile states of plane stress.

In states of triaxial tension, as produced by a stress concentration, the effective stress can be considerably less than in uniaxial tension. If the part is thin, plane stress prevails. For thicker parts the state of stress approaches plane strain, and while the degree of triaxiality increases, the effective stress decreases. Therefore;

3. The uniaxial tensile strength can be used as a conservative fracture criterion for a part which is not in plane stress and contains a stress concentration.

For a more accurate estimate of the tensile stress required to cause fracture, a notched beam with the same thickness and notch radius should be tested and analyzed.

For states of stress consisting of tensile and compressive components of stress, the effective stress is greater than in uniaxial tension. If all other forms of instability can be ruled out, such as buckling, then;

4. The fracture strain in uniaxial tension can be used as a conservative estimate to predict fracture when the stress state consists of compressive as well as tensile components of stress.

The size of an M-S65 beryllium part need not be considered in calculating the conditions which will lead to fracture, provided the size of the part is relatively small with respect to the size of the billet.

Recommendations for Future Research

The apparent dependence of the critical tensile stress upon the plastic strain should be investigated further.

It would first be useful to determine the relation between plastic strain and microstructural damage. This could be accomplished, for example, by testing a beam in pure bending, which would provide a linear variation in strain over the beam height. After the beam is bent to fracture, a region adjacent to the fracture could be polished to reveal the damage.

There are fracture data available on thin notched bars of S-65 beryllium tested in tension⁽²⁹⁾, which could be analyzed to test the uniaxial tensile stress criterion for nonuniform states of plane stress. With a plane stress finite element code, the maximum tensile stresses at fracture could be determined. According to the

strain dependent tensile stress criterion of fracture, the maximum tensile stresses at fracture should be close to the uniaxial tensile strength.

With a three-dimensional finite element program, tests could be conducted on beryllium notched beams which do not satisfy the plane strain criterion. The maximum tensile stresses at fracture in these specimens should lie between the uniaxial tensile strength and the maximum stress at fracture observed in a sharply notched beam. This information could provide the basis for establishing a quantitative criterion of fracture for beryllium.

A three-dimensional finite element calculation would also be useful to verify the plane strain calculations made in the present work. Actually, the data obtained from the tests on the special beams provide an experimental test for a three-dimensional finite element code.

Finally, it would be very instructive to conduct tests in which both tensile and compressive stresses exist. The easiest such test would be torsion of solid cylinders.

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TABLE 1.1

PHYSICAL PROPERTIES OF BERYLLIUM(6)

Atomic Number	4
Atomic Weight	9.015
Crystal Structure	h.c.p.
Density (gm/cm ³)	1.848
Melting Point (C Degrees)	1283
Boiling Point (C Degrees)	2770
Specific Heat (cal/gm C Degrees)	0.45
Thermal Conductivity (cal/cm sec C Degrees)	0.38
Coefficient of Linear Expansion (10 ⁻⁶ /C Degrees)	12.4
Neutron Absorption Cross Section (barns/atom)	0.009

TABLE 2.1
CHEMICAL ANALYSIS OF THE M-S65 BERYLLIUM

Element	Weight, Percent
Be	99.1
BeO	1.07
Fe	0.072
C	0.045
Al	0.018
Mg	0.002
Si	0.043
Cr	0.005
Mn	0.002
Ni	0.005
Cu	0.005
U	0.006

TABLE 2.2
GRAIN SIZE OF THE M-S65 BERYLLIUM (μm)

	Average	Maximum
Top	8.4	43
Bottom	12.9	40

TABLE 2.3
UNIAXIAL MECHANICAL PROPERTIES OF THE M-S65 BERYLLIUM
AS OBTAINED BY BRUSH-WELLMAN

Location From Billet	Direction	Temp. C Degrees	Yield Strength MPa (ksi)	Tensile Strength MPa (ksi)	Elongation %
Top	Long	20	297 (43.1)	417 (60.5)	3.2
Top	Long	20	297 (43.1)	418 (60.6)	3.7
Top	Trans.	20	372 (39.5)	420 (60.9)	5.6
Top	Trans.	20	279 (40.4)	422 (61.2)	4.4
Top	Trans.	370	195 (28.3)	279 (40.5)	38.0
Top	Trans.	650	94 (13.6)	128 (18.6)	19.0
Bottom	Long	20	274 (39.7)	387 (56.1)	3.2
Bottom	Long	20	284 (41.2)	398 (57.7)	3.5
Bottom	Trans.	20	232 (33.7)	368 (53.4)	5.7
Bottom	Trans.	20	231 (33.5)	363 (52.7)	5.9
Bottom	Trans.	370	181 (26.3)	246 (35.7)	40.0
Bottom	Trans.	650	83 (12.1)	128 (18.6)	23.0

TABLE 2.4

THE ELASTIC MODULUS AND POISSON'S RATIO FOR
M-S65 BERYLLIUM AS OBTAINED FROM A BENDING TEST

	E(GPa)	ν
Tension	303	0.054
Compression	313	0.052

TABLE 4.1
A COMPARISON OF THE BACK SURFACE STRAINS
OBTAINED EXPERIMENTALLY AND PREDICTED BY THE FEM

Specimen	$\frac{B}{\rho}$	Strain Gage Result (10^{-6})	Corrected Strain (10^{-6})	FEM Result (10^{-6})	% Error
A2-1	12.5	-4710	-3750	-2860	24
B2-1	25.0	-2390	-2000	-1840	8
C2-1	50.0	-1440	-1280	-1210	5
D2-1	75.0	-750	-740	-690	7

TABLE 4.2
ROOT RADIUS, NOTCH DEPTH, LOAD AT FRACTURE, AND
LOCATION OF FRACTURE INITIATION IN THE BERYLLIUM
NOTCHED BEAMS

Specimen	Root radius-mm	Notch depth-mm	Fracture load- kN	Location of initiation (x/ρ , $2 z /B$)
A1-1	0.229	1.64	2.13	(1.30, 0.04)
A1-2	0.229	1.63	2.06	(0.67, 0.53)
A1-3	0.203	1.52	2.16	(1.20, 0.34)
A2-1	0.508	1.60	2.31	(1.50, 0)
A2-2	0.406	1.69	2.02	(0.57, 0.34)
A2-3	0.508	1.63	2.26	(0.62, 0)
B1-1	0.254	3.21	7.76	(1.15, 0.37)
B1-2	0.254	3.15	7.56	(0.74, 0.61)
B1-3	0.254	3.21	7.74	(0.94, 0.29)
B2-1	0.533	3.21	8.24	(1.40, 0.26)
B2-2	0.508	3.19	8.26	(0.65, 0.21)
B2-3	0.508	3.21	8.35	(1.25, 0.45)
C1-1	0.203	6.28	18.9	At a flaw in the notch
C1-2	0.254	6.51	24.9	(2.50, 0.07)
C1-3	0.229	6.36	20.6	At a flaw in the notch
C2-1	0.508	6.39	27.3	(1.40, 0.02)
C2-2	0.508	6.41	23.4	(1.50, 0.43)
C2-3	0.508	6.35	25.9	(1.80, 0.21)
D1-1	0.191	9.61	39.2	At a flaw in the notch
D1-2	0.254	9.59	42.1	(1.85, 0.45)
D1-3	0.127	9.58	34.0	? (0.94, 0.16)
D2-1	0.483	9.66	39.7	(1.15, 0.47)
D2-2	0.508	9.67	44.3	(1.30, 0.32)
D2-3	0.457	9.69	>46.3	No fracture at this load
S-1	2.51	15.5	45.7	(1.10, 0.37)
S-2	2.54	15.5	47.0	(0.93, 0.21)

TABLE 4.3a
DATA ON THE STRESS-STRAIN DISTRIBUTIONS AT FRACTURE
FOR THE BERYLLIUM NOTCHED BEAMS

Specimen	Root radius-mm	Maximum stress-MPa	max $e_y \cdot 10^{-2}$	Location of max stress - x/ρ	At the location of fracture initiation			
					x/ρ	σ_y -MPa	stress ratio $\sigma_y:\sigma_x:\sigma_z$	$e_y \cdot 10^{-2}$
A1-1	0.229	550	1.95	1.30	1.30	550	1:0.50:0.47	0.35
A1-2	0.229	545	1.75	1.20	0.67	510	1:0.37:0.64	0.61
A1-3	0.203	550	2.00	1.25	1.20	550	1:0.46:0.58	0.37
A2-1	0.508	524	1.75	1.20	1.50	490	1:0.49:0.43	0.23
A2-2	0.406	510	1.35	1.10	0.57	465	1:0.34:0.64	0.53
A2-3	0.508	524	1.75	1.20	0.62	485	1:0.36:0.65	0.59
B1-1	0.254	572	2.15	1.50	1.15	555	1:0.45:0.66	0.43
B1-2	0.254	572	2.15	1.50	0.74	535	1:0.40:0.66	0.68
B1-3	0.254	572	2.15	1.50	0.94	545	1:0.43:0.67	0.52
B2-1	0.533	531	1.50	1.30	1.40	530	1:0.47:0.54	0.27
B2-2	0.508	531	1.50	1.20	0.65	485	1:0.36:0.65	0.54
B2-3	0.508	540	1.50	1.30	1.25	540	1:0.46:0.57	0.30

TABLE 4.3b
DATA ON THE STRESS-STRAIN DISTRIBUTIONS AT FRACTURE
FOR THE BERYLLIUM NOTCHED BEAMS

Specimen	Root radius-mm	Maximum stress-MPa	max $e_y \cdot 10^{-2}$	Location of max stress - x/ρ	At the location of fracture initiation			
					x/ρ	σ_y -MPa	stress ratio $\sigma_y:\sigma_x:\sigma_z$	$e_y \cdot 10^{-2}$
C1-2	0.254	580	2.40	1.65	2.50	535	1:0.58:0.47	0.22
C2-1	0.508	545	1.65	1.40	1.40	545	1:0.46:0.58	0.30
C2-2	0.508	530	1.20	1.15	1.50	510	1:0.50:0.47	0.21
C2-3	0.508	545	1.65	1.40	1.80	510	1:0.54:0.44	0.22
D1-1	0.191	580	2.20	1.65	0	495	1:0:0.5	2.2
D1-2	0.254	560	1.95	1.55	1.85	545	1:0.52:0.52	0.25
D1-3	0.127	575	2.40	0.7-1.75 (constant)	? 0.94	575	1:0.43:0.66	0.49
D2-1	0.483	510	1.10	1.05	1.15	505	1:0.44:0.55	0.30
D2-2	0.508	525	1.30	1.15	1.30	515	1:0.48:0.50	0.28
D2-3	0.457	540	1.45	1.30	-	-	-	-
S-1	2.51	490	1.00	1.00	1.10	485	1:0.43:0.48	0.25
S-2	2.54	495	1.10	1.05	0.93	490	1:0.41:0.55	0.30

Figure 1.1 The geometry of a notched beam loaded in four-point bending.

(a)

200x

(b)

500x

Figure 2.1 Photomicrographs of the M-S65 beryllium microstructure: (a) grain structure, (b) inclusions (probably BeO).

Figure 2.2 The geometry of the uniaxial tension and compression specimens.

Figure 2.3 Results of the three uniaxial tension tests.

Figure 2.4 Comparison between the uniaxial tension and compression tests.

Figure 2.5 Terminology used in the analysis for the bending experiment to generate uniaxial stress-strain curves.

(a)

(b)

Figure 2.6 Geometry of the beryllium beams used in the bending experiments to generate uniaxial stress-strain curves: (a) beam dimensions, (b) photograph of the strain gages.

Figure 2.7 Comparison of the tension and compression uniaxial stress-strain curves for M-S65 beryllium as obtained by conventional tests and the first bending test.

(a)

(b)

Figure 2.8 Scanning electron micrographs of the probable location of fracture initiation in: (a) a conventional tension specimen, and (b) in a bending specimen. Magnification and a marker are supplied at the bottom of each photo.

Figure 2.9 Comparison of the tension and compression uniaxial stress-strain curves for M-S65 beryllium as obtained by conventional tests and the second bending test.

Figure 2.10 Comparison of the tension and compression uniaxial stress-strain curves for Meehanite cast iron as obtained by conventional tests and the bending test.

Figure 2.11 Comparison of the uniaxial data for prestrained copper as obtained from conventional and bending tests: (a) 0-1 percent strain, (b) 0-10 percent strain.

S-65 beryllium thin-walled tubes

Figure 2.12 Biaxial plane stress yield and fracture data on S-65 beryllium(2): first quadrant - axial tension and internal pressure; second quadrant - axial compression and internal pressure; fourth quadrant - torsion and axial tension .

Figure 2.13 The M-S65 mechanical properties used as input to the finite element program NIKE2D.

(a)

(b)

Figure 3.1 A representation of: (a) the slip-line field for a rigid-perfectly plastic notched beam with a circular root, and (b) the resulting stress distribution.

Figure 3.2 The experimentally measured transverse strain, e_z , at the root of the notch at various distances from the center of the beam and for different loads. Data from two specimens were identical. Fracture occurred at $\sigma_{nom} / \sigma_0 = 1.7$.

Figure 3.3 The basic element configuration and boundary conditions used to analyze the notched beams in the finite element program NIKE2D.

Figure 3.4 The distribution of stress, σ_y , in the vicinity of the notch vs. the distance ahead of the notch normalized by the root radius, i.e., x/ρ , for a number of different loads. The figure also shows a comparison to the slip-line theory prediction.

Figure 3.5 The plane strain plastic zone in a beryllium notched beam as determined by the finite element program.

Figure 3.6 The distribution of principal stresses and strains in the vicinity of the notch vs. the distance ahead of the notch normalized by the root radius, i.e., x/ρ .

Figure 3.7 The distribution of stress, σ_y , and strain, ϵ_y , in the vicinity of the notch vs. distance ahead of the notch, for two beryllium notched beams with different root radii.

<u>Specimen</u>	<u>ρ</u>	<u>L</u>	<u>B = W</u>	<u>c</u>	<u>a</u>	<u>d</u>
A1	0.254	38.1	6.35	1.59	4.76	10.6
A2	0.508					
B1	0.254	76.2	12.7	3.18	9.52	21.2
B2	0.508					
C1	0.254	152.4	25.4	6.35	19.0	42.3
C2	0.508					
D1	0.254	228.6	38.1	9.52	28.6	63.5
D2	0.508					
S	2.54	228.6	38.1	15.2	22.9	63.5

Figure 4.1 The basic dimensions of the different notched beam geometries tested in this investigation. There were three specimens of each geometry, except S.

Figure 4.2 An illustration of the four different notched beam sizes. The dimension shown refers to the length. See Figure 4.1 for the other dimensions.

Figure 4.3 The relative size of the plane strain plastic zones at fracture in four different sizes of notched beams with the same root radius (0.508mm). The plastic zones were obtained from the finite element program.

(a)

(b)

Figure 4.4 Scanning electron micrographs of the fracture surface of specimen B2-1: (a) shows the location of fracture initiation ahead of the notch, which is the vertical line on the left, (b) shows a close-up of the location of fracture initiation.

Figure 4.5 Figure shows: (a) the configuration of the strain gages at the root of the notch in the special beams, and; (b) the results of the experimental and predicted strains.

Figure 4.6 Scanning electron micrographs of the fracture initiation site in specimen D2-1. The figure illustrates the probable coalescence of favorably oriented grains.

Figure 4.7 Scanning electron micrographs of the fracture initiation site in specimen C1-2. The figure illustrates fracture from a large grain.

Figure 5.1 The distribution of stress, σ_y , ahead of a sharp crack for different hardening exponents.

APPENDIX A

THE DESIGN AND OPERATION OF THE FOUR-POINT BEND FIXTURE

In the study of the mechanical behavior of brittle materials it is necessary to take great care in testing. Besides their limited ability to deform, many brittle materials are also very stiff. Consequently, small errors in testing can account for a significant portion of the possible deformation and strength of a specimen.

A distinct advantage of bending over conventional tension and compression tests is the potential ease of uniform load application. Four-point bending is attractive because of the absence of shear stresses between the inner points of load application; shear stresses are always present in three-point or cantilever bending.

There are, however, a number of possible problems associated with four-point bending. (See, for example the paper by Hoagland, Marschall, and Duckworth ⁽¹⁾.) One of the most serious of these occurs if the points of load application are held rigid. With deflection of the beam, the fibers in tension extend and the fibers in compression contract. Unless the points of load contact are free to move in these directions, shear stresses arise which oppose the applied moment as shown in Figure A.1. The applied moment is then reduced to

$$M = P(d - \mu h)$$

Even if lubrication is used, a significant error can result which is difficult to assess.

In order for the shear stresses in the central portion of a beam loaded in four-point bending to be absent, it is at least necessary that the loads on each of the outer sets of contact points be the same. If the moment arms on each end of the beam are also the same, then the loads on all four of the contact points must be equal.

The problems in attaining this condition arise in two ways. First, if the fixture for transferring load from the testing machine to the upper or lower points of load contact is capable of transmitting moment, the loads on these two points will be different unless there is perfect alignment. Secondly, if the loads on the side of load transfer are equal but the moment arms are not, then the loads on the opposite side will be different. This effect is minimized by having large moment arms. Both of these effects introduce shear stresses and make the determination of the exact applied moment difficult.

Another important problem occurs if the specimen is not machined properly or if the lines of load contact are not perfectly parallel. In both cases, torsional stresses will be induced into the specimen. The maximum shear stress due to the torsional loading of an elastic beam of rectangular cross section occurs where the minor axis of the cross section meets the boundary, and is given by⁽²⁾

$$\tau_{\max} = 2kG\theta a,$$

where k is a numerical factor which varies from 0.675 for square cross sections to unity for very slender rectangular sections, G is the shear modulus, θ is the angle of twist per unit length, and a is one half the dimension of the short side. As an example of the magnitude of stresses which can arise, consider a beam made of beryllium, 152.4 mm (6 inches) long with cross-sectional dimensions 25.4 mm x 12.7 mm (1 inch x 0.5 inch) and rotated by 1 degree. The maximum shear stress is then approximately 390 MPa (57,000 psi).

A final consideration is the effect of the stress distribution under the points of load application. Unless the inner points of contact are very close together and transmit exceedingly high loads, there should be little effect on the state of stress in the center section of the beam. However, at one point in this investigation the author was concerned about the possibility of premature failure adjacent to the rollers due to tensile stresses. As it turns out, the Hertzian theory of contact stresses predicts that no tensile stresses exist on the surface of the body next to the roller⁽³⁾. Nevertheless, in the design of bending specimens, care should be taken to avoid high loads which could lead to load point indentation into the specimen or the fixture. This would certainly increase the frictional effects discussed earlier.

The apparatus used in this investigation has been designed to eliminate as much as possible the problems mentioned above. A diagram of the fixture and a photograph of it in use are shown in Figures A.2 and A.3.

Friction at the points of load contact can be overcome by using rollers which are free to roll in the appropriate direction: outward for the bottom and inward for the top. Rubber bands are used to hold the rollers in place, but soft springs could be used as well.

Torsional stresses are compensated for by allowing three of the four rollers to rotate independently in a plane perpendicular to the axis of the beam specimen. This is brought about for one of the bottom rollers by employing a roller bearing on one end of the support block as shown. One of the bottom rollers is always positioned at this point while the other bottom roller can be located in four different positions. The two top rollers are each connected to a separate support block which may rotate about a dowel running along the common center of the two blocks as shown. These blocks are not connected, but are situated in intimate contact prior to loading.

In order to obtain equal loads on the top two rollers, the dowel which runs along the center of the separate blocks also acts as a means of load transfer from another block. This block can rotate around an axis perpendicular to the specimen axis via a second dowel, and insures that no moment is transferred to the load points.

Two different upper block configurations allow four different support arrangements to be used such that the moment arms and the distance between the inner rollers are equal, but many other positions are possible.

Other features of the fixture include hardened seats for the rollers and alignment bars.

APPENDIX A

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Figure A.1 A representation of the frictional stresses which arise in four-point bending when the loading points are held rigid.

Figure A.2 A schematic diagram of the four-point bend fixture used in this investigation.

Figure A.3 A photograph of the four-point bend fixture in use with a beryllium notched beam.

APPENDIX B

THE COMPUTATIONAL METHOD USED IN NIKE2D

The two-dimensional finite element program NIKE2D was developed at Lawrence Livermore Laboratory⁽¹⁾, and is capable of treating a variety of problems encountered in solid mechanics. These include a number of material models, static, dynamic, and large deformation problems, and geometries where materials are allowed to slide against each other. The present description consists of only those aspects of the computational method which are employed to solve the particular problem of beryllium notched beams in quasistatic bending.

The reader is assumed to be familiar with the finite element method, but Bathe and Wilson⁽²⁾, and Zienkiewicz⁽³⁾ represent good sources of information.

Displacement Calculations

NIKE2D is a displacement formulated finite element program. The basic method of solution consists of solving for the displacements at all of the nodal points and deriving from these the stresses and strains throughout the body. The positions of the nodal points define the partitioning of the body into elements, and the displacements within an individual element are determined by the displacements at the nodes. Because of this, it is possible to write the strains in an element in terms of the element, nodal displacements in the matrix form,

$$\underline{e} = \underline{B} \underline{u} \quad . \quad (a)$$

In this expression, $\underline{u}$ is a vector consisting of the nodal displacements, and $\underline{B}$ is a matrix which depends on the kinematics of the element type, the geometry of the element, and the precise location within the element at which the components of the strain vector, e , are to be evaluated. Four-node isoparametric elements were used in the present work which provide a linear variation in displacement -- and hence a constant strain -- and insure compatibility between adjacent elements.

Since the strains throughout the entire body are determined by the displacements at the nodes, it is convenient to generate the equations of equilibrium with an application of the principle of virtual work. This may be written mathematically as

$$\int_V \underline{\hat{e}}^T \sigma dV = \underline{\hat{u}}^T \underline{P} \quad , \quad (b)$$

where the integration is over the volume of the body, $\underline{\hat{u}}$ corresponds to the virtual displacements from which the strains $\underline{\hat{e}}$ are derived, $\underline{P}$ corresponds to the externally applied tractions, and the vector quantities at this time refer to components, not discrete values ($\underline{\hat{e}}^T$ is the transpose of $\underline{e}$).

For an individual element, the tractions are point loads applied at the nodes. The displacements at the nodes can then be used on the right-hand side of equation (b) and, with equation (a),

in the integral expression to represent the strains. With a suitable choice of virtual displacements and summing expression (b) for all of the elements that make up the body, equilibrium is insured by satisfying the matrix expression,

$$\sum \int_V \underline{B}^T \underline{\sigma} dV = \sum \underline{P} \quad , \quad (c)$$

where N represents the number of elements.

Furthermore, if it is possible to relate the stresses and strains in matrix form as,

$$\underline{\sigma} = \underline{C} \underline{e} = \underline{C} \underline{B} \underline{u} \quad ,$$

then expression (c) may be written as,

$$\sum \left\{ \left[\int_V \underline{B}^T \underline{C} \underline{B} dV \right] \underline{u} \right\} = \sum \underline{P} \quad (d)$$

In problems of small deformation elasticity, the set of equations (d) may be solved directly to obtain the nodal displacements from which strains and stresses are simply calculated. This is not possible when finite or plastic deformations are allowed, since, in general, changes occur in the

geometry dependent terms, $\underline{B}$ and V , and in the relation between stress and strain.

One approach to the plasticity problem incorporates the deformation theory of plasticity in which stresses are considered to be unique but nonlinear functions of strain.

In NIKE2D an iterative approach is used to determine the displacements and stresses for a given load increment. Suppose the body in question is in equilibrium at time $t = n$ as determined by equilibrium equations of the form (d). At $t = n + 1$, the external loads change. Writing the stresses at $t = n + 1$ as $\underline{\sigma}^{n+1} = \underline{\sigma}^n + \Delta \underline{\sigma}$, the first estimate for the displacements at $t = n + 1$ is obtained by solving the linearized equations,

$$\sum^N \left\{ \left[\int_V \underline{B}^T \underline{C} \underline{B} dV \right] \Delta \underline{u} \right\} = \sum^N \left[\underline{p}^{n+1} - \int_V \underline{B}^T \underline{\sigma}^n dV \right], \quad (e)$$

where the quantities $\underline{B}$, V , and $\underline{C}$ correspond to the geometry and material properties at $t = n$. In the present case, $\underline{C}$ may be considered as describing the relation between increments of stress and strain; a more detailed description follows.

For example, in the very first time step $\underline{\sigma}^n = 0$ and $\underline{C}$ corresponds to the elastic matrix so that the elastic solution is obtained.

In general, the increments of displacement obtained from the solution of equations (e) consist of elastic and plastic

deformations. Because of this, iterations within any time step are required. The Δu obtained are first used to calculate the increments in strain, which are then used to compute the constitutive matrix, C , and the corresponding increments in stress as described below. With this information, all of the quantities on the right-hand side of equations (e) are updated. This process is continued until the right-hand side becomes zero, which is equivalent to the satisfaction of the equilibrium equations. Actually, the process is continued until a norm of the increments Δu becomes smaller than some specified fraction of the norm of the total displacements.

Within any time step, the constitutive matrix, C , is, in general, not updated. This leads to convergence provided there is no unloading and the stress-strain curve has a decreasing slope. Otherwise, the displacements predicted will be too large, and in some cases the iterative process will diverge. In these cases, the stiffness matrix is reformed within the time step by reevaluating the constitutive matrix with the last estimate of strains.

NIKE2D does include the effects of stress rotation and geometry changes, but in the case of beryllium notched beams in bending, such effects are negligible.

Stress-Strain Relations

Under most conditions, the increment in plastic strain may be written according to the so-called associated flow rule,

$$de_{ij}^P = d\lambda \frac{df}{d\sigma_{ij}} ,$$

where $d\lambda$ is a scalar which varies with strain history, and f is the yield function which must always be less than or equal to zero; it is zero while plastic deformation occurs.

In NIKE2D the yield function is assumed to be represented by the Mises condition,

$$f = \frac{1}{2} t_{ij} t_{ij} - \frac{1}{3} T^2 ,$$

where the deviatoric stresses are

$$t_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}$$

and $\sqrt{\frac{2}{3}} T$ is the radius of the yield surface which varies with effective plastic strain. T is equal to the yield stress in uniaxial tension.

In this case the stress-strain relations become,

$$de_{ij} = de_{ij}^E + de_{ij}^P \quad (f)$$

$$de_{ij}^E = \frac{1}{2G} dt_{ij} \text{ (deviatoric elastic strains)} \quad (g)$$

$$de_{ij} = de_{ij} - \frac{1}{3} \delta_{ij} de_{kk} \quad (h)$$

$$e_{kk}^E = \frac{1}{3K} \sigma_{kk} \quad (i)$$

$$de_{ij}^P = d\lambda t_{ij} \quad (j)$$

$$de_{kk}^P = 0$$

where G and K are the shear and bulk moduli.

Hardening is assumed to be isotropic (NIKE2D also handles kinematic hardening) and governed by an equivalent plastic strain-or-work hypothesis. The equivalent plastic strain is determined from,

$$\bar{e}^P = \int d\bar{e}^P$$

where

$$d\bar{e}^P = \left[\frac{2}{3} de_{ij}^P de_{ij}^P \right]^{1/2}$$

It reduces to the plastic principal strain in uniaxial tension. The relation between T and $\bar{e}^P$, which is assumed to be unique, can then be obtained from a simple tension test. This is illustrated in Figure B.1.

Calculation of Strains and Stresses

The first step in the calculation of strains and stresses is to test for yield. The strains derived from the increments in displacement obtained from equation (e) are used to compute elastic trial stresses. The current yield function is then evaluated to determine if yielding has occurred. If not, the time step is complete. Otherwise, it is necessary to compute the stresses and plastic strains corresponding to the latest increment in displacements. This procedure itself consists of a number of subiterations. First an expression is required to determine the plastic strain increments from the total strain increments.

During plastic deformation,

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial T} dT = 0$$

or

$$t_{ij} d\sigma_{ij} - \frac{2}{3} T dT = 0 \quad (k)$$

The stress increments can be described as,

$$d\sigma_{ij} = C_{ijkl} de_{kl}^E = C_{ijkl} (de_{kl} - de_{kl}^P),$$

and defining the tangent to the effective stress - effective plastic strain curve as,

$$H \equiv \frac{dT}{d\bar{e}^P}, \quad (\text{see Figure B.1})$$

equation (k) becomes,

$$t_{ij} C_{ijk1} (de_{k1} - de_{k1}^P) = \frac{2}{3} T H d\bar{e}^P$$

Now,

$$de_{k1}^P = d\lambda t_{k1}$$

and

$$d\bar{e}^P = \frac{2}{3} d\lambda T$$

So

$$t_{ij} C_{ijk1} (de_{k1} - d\lambda t_{k1}) = \frac{4}{9} T^2 H d\lambda$$

By solving for $d\lambda$ the plastic strains may be written as,

$$de_{ij}^P = \left[\frac{t_{ij} C_{ijk1} de_{k1}}{t_{ij} C_{ijk1} t_{k1} + \frac{4}{9} T^2 H} \right] t_{ij} \quad (1)$$

In the subiteration process, the following steps are repeated M times, where M is larger for greater values of the yield function when evaluated with the trial stresses.

1. Equation (1) is evaluated with de_{k1}/M and the updated values of t_{ij} , T , and H obtained from the last subiteration.

2. The increments in stress are computed from equations (f) - (i) using de_{k1}/M and the plastic strain increments obtained from 1.
3. The increments in plastic strain, effective plastic strain, and stress are added to the old totals, and the values of T and H are recalculated (see Fig. B.1.).

Again, these subiterations are carried out until all of the total strain corresponding to the latest displacement increment is accounted for. The new stresses are then used to update the right-hand side of equation (e) and the next increment in displacements for the current time step is made.

APPENDIX B

REFERENCES

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Figure B.1 An illustration of how the parameters T , and H are determined from the uniaxial stress - plastic strain curve, given the accumulated effective plastic strain, $\bar{\epsilon}^p$.