UCRL--89811

DE84 000185

UCRL-89811
PREPRINT

CONF-8207100--1

PROBLEMS IN ASTROPHYSICAL
RADIATION HYDRODYNAMICS

John I. Castor

To appear in the proceedings of the NATO
workshop "Astrophysical Radiation Hydro-
dynamics" held in Munich, Germany July
1982; K.-H. Winkler and M. Norman, eds.

September 14, 1983

This is a preprint of a paper intended for publication in a journal or proceedings. Since changes may be made before publication, this preprint is made available with the understanding that it will not be cited or reproduced without the permission of the author.

The logo of the Lawrence Livermore National Laboratory, featuring a stylized 'L' symbol and the text 'Lawrence Livermore National Laboratory' arranged in a chevron shape.

Lawrence
Livermore
National
Laboratory

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

PROBLEMS IN ASTROPHYSICAL RADIATION HYDRODYNAMICS

John I. Castor

Lawrence Livermore National Laboratory

1. ABSTRACT

The basic equations of radiation hydrodynamics are discussed in the regime that the radiation is dynamically as well as thermally important. Particular attention is paid to the question of what constitutes an acceptable approximate non-relativistic system of dynamical equations for matter and radiation in this regime. Further discussion is devoted to two classes of application of these ideas. The first class consists of problems dominated by line radiation, which is sensitive to the velocity field through the Doppler effect. The second class is of problems in which the advection of radiation by moving matter dominates radiation diffusion.

2. INTRODUCTION

The objective of my lectures in this workshop is to try to illuminate some of the phenomena that arise in the coupling of radiation with matter in the presence of a material velocity field. My discussion will be entirely analytic, and often very approximate, so perhaps a few words are in order about the relationship of this kind of discussion to the intensely computational subject addressed by the majority of the speakers.

The first reason I would advance why analytic theory is helpful in a computational discipline is that it can be very difficult to calculate phenomena correctly if one does not have a fairly good idea of what to expect ahead of time. That is,

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

240

the choice of zoning, time steps, frequency bin structure, discrete angles, and also of the dominant physical processes, depends on the nature of the solution of the problem, especially if the result is to be obtained with reasonable economy. Thus the availability of an analytic caricature of the problem that displays the main phenomena may be a great aid in setting up the numerical calculation. The second reason is that in many cases the rough analytic answer may be good enough. An answer that is correct in the order of magnitude may be the most refined theory that is warranted when the data are at least that inaccurate; when the data are indeed very good, a rough calculation may suffice to show that a certain theory is hopeless. A somewhat extreme statement of this point of view is: Computational methods exist to provide accurate answers, when these are needed, to problems whose approximate solutions are already known from other means. While this is certainly arguable, it is a viewpoint that represents a counterweight to the computing-nature-as-it-is philosophy, which can result in the waste of alarming amounts of computer time on ill-conceived problems. Of course, an intelligent combination of analytic estimation and economical computation is best.

My first lecture will be devoted to a general discussion of the equations of radiation hydrodynamics, and in particular of the photon transport equation for a moving medium. These remarks are intended to be complementary to the thorough discussion by D. Mihalas, also in this workshop. Even at this late date there appears to remain some confusion about the choices the theoretician can make of the best form of transport equation for his own problem, and even some confusion about the fact that choices do exist, and must be made, either explicitly or inadvertently. So the first lecture will begin with a discussion of the most general form of the transport equation, then proceed to its $O(v/c)$ reduction, and a further simplification of that to a practical system for use in cases of modest velocity. The limiting forms of the radiation hydrodynamics equations in optical thick and optically thin regimes will be discussed, with further approximations that can be made in these cases.

The second lecture will emphasize transport of spectral line radiation, for which the effects of a velocity field can easily be very large. The methods of approximation breaks into categories for subsonic, transonic and hypersonic problems. The second category is naturally the most difficult, and I will call attention to the lack of a generally useful computational method for such problems. The hypersonic class is the one for which Sobolev's approximation is designed, and I will discuss some interesting phenomena that arise when radiation treated in the Sobolev approximation is coupled to matter. As pointed out by

Karp, Lasher, Chan and Salpeter, there are many astrophysical flow problems in which the medium is optically thick, but the velocity gradient is large enough to smear the lines until they cover a substantial part of the spectrum and thereby affect the radiation diffusion rate substantially. The method they devised to deal with this situation is a presently underutilized tool.

The third lecture is devoted to a discussion of two examples of "dynamic diffusion" of radiation. This is the nomenclature introduced by Mihalas to distinguish two types of diffusion. In the diffusion (optically thick) limit, the dynamical equation for the radiation energy density as a function of inertial coordinates has the form of an advection-diffusion equation with a source term and a sink term. The relative importance of advection versus diffusion gives two cases: "static diffusion" (diffusion dominates) and "dynamic diffusion" (advection dominates). The dynamic diffusion regime with decoupled matter and radiation is relatively unfamiliar to astrophysicists, so a discussion of these two examples may be a useful illustration of the phenomena that can occur. The examples, drawn from the winds-and-accretion literature of high-energy astrophysics, are of the influence of cyclotron line radiation on neutron star accretion, specifically the work of Langer and Rappaport, and the wind associated with supercritical disc accretion, studied by Meier.

3. FIRST LECTURE -- EQUATIONS OF RADIATION HYDRODYNAMICS

There are two kinds of radiation hydrodynamics: the ordinary kind and the subtle kind. The ordinary kind is represented by problems for which the Boltzmann number $B_0 = F_m/F_R$ is of order unity, in which F_m is a material heat flux $\rho C_p T v$, and F_R is the radiative flux σT^4 . The typical ratio of radiation energy density to matter energy density is $(F_R/c)/(\rho C_p T) = (v/c) B_0^{-1}$, so in this kind of problem radiation energy and radiation pressure are $O(v/c)$ and play only minor roles. This has the consequence that every term in the transport equation involving the velocity or a time derivative can be dropped with no great effect. This leads to the simple static transfer equation with which we are all familiar. The subtle kind of radiation hydrodynamics is represented by problems for which the Boltzmann number is small, of order v/c , so the radiation energy density and matter energy density are comparable, while the radiative energy flux overwhelms the other fluxes. In this case most of the terms in the transport equation that could be dropped before must now be retained, since otherwise a completely wrong answer would be obtained. In ordinary radiation hydrodynamics the thermal effect of radiation is considerable, but its mechanical effect

is not. In subtle radiation hydrodynamics the mechanical effect of radiation is also important. The subtle kind of radiation hydrodynamics is both more trouble for the theorist and more interesting, and it is that I want to discuss.

3.1. Relativistic photon transport

I would like to begin the discussion of the equations of radiation hydrodynamics with the photon Boltzmann equation in its relativistic form. Please note: the use of some of the methods of general relativity (GR) in the development does not mean that this work is tied to typical GR problems. In fact, I am not going to say anything about gravity, field equations, etc., etc. The point is that the covariant language of relativity is a handy way of getting answers to otherwise awful problems and avoiding some conceptual pitfalls along the way. This language also incorporates special relativity, which of course is essential in a proper formulation.

Definitions: The intensity function is the product of the occupation number per photon state and a phase-space factor involving the frequency that changes from one observer's frame to another. The occupation number itself is a Lorentz invariant, so this is taken as the invariant intensity,

$$\mathcal{I} = \frac{I}{\nu^3}. \quad (1)$$

Likewise, frequency-dependent factors appear in the quantum expressions for the absorption coefficient k (length⁻¹) and the emission coefficient j_ν (energy/(volume time frequency·solid angle)). Dividing out these factors puts them into invariant form:

$$\sigma = \nu k_\nu \quad (2)$$

and

$$\eta = j_\nu / \nu^2. \quad (3)$$

Now, what does it mean to say that the variables $\mathcal{I}$, σ , and η are invariant? It does not mean that they are independent of any of the variables space, time, frequency or direction, although they may be. What it does mean is that different observers in relative motion will agree on their values, after they have properly worked out the correspondences of frequency and direction.

The properly covariant transport equation is the following:

$$\frac{d\mathcal{I}}{d\lambda} = -\sigma\mathcal{I} + \eta, \quad (4)$$

in which λ is a scalar parameter measuring displacement along the photon's world line, that is,

$$\frac{dx^\alpha}{d\lambda} = M^\alpha, \quad (5)$$

in which $(x^\alpha) = (x^1, x^2, x^3, t)$ is the position four-vector and $(M^\alpha) = \sqrt{n^1, n^2, n^3, 1}$ is the photon momentum four-vector. I will use the conventions that α runs from 1 to 4, time being the fourth coordinate, and that the Minkowski metric is $ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + dt^2$. The 3-vector $\underline{n} = (n_1, n_2, n_3)$ is the ordinary unit direction vector of the photon. Now although M^α is a 4-vector, it is a null vector and so it has only 3 functionally independent components, for example M^1, M^2, M^3 . Not to prejudice the formalism, suppose that the four components of M^α are functions to be specified later of three variables y^1, y^2 and y^3 . Thus the invariant intensity, as well as the invariant opacity and emission coefficient, depend on the seven variables $x^1, x^2, x^3, t, y^1, y^2$, and y^3 . Doing out the derivative in equation (4) using the chain rule, and making use of equation (5), leads to

$$\frac{d\mathcal{I}}{d\lambda} = M^\alpha \frac{\partial \mathcal{I}}{\partial x^\alpha} + \frac{dy^a}{d\lambda} \frac{\partial \mathcal{I}}{\partial y^a}, \quad ()$$

where the roman index a runs from 1 to 3.

The hard part of obtaining the right transport equation lies in finding the derivatives $dy^a/d\lambda$. The physical basis for this task is the assumption that the photon momentum is unchanged, in some sense, as the photon moves along its path. Now, it is not literally unchanged unless space-time is flat and the metric is constant. More general cases invoke the concept of parallel transport. What does this mean? It is a rule for carrying a 4-vector such as M^α along any curve in space-time. In order to do the carrying from the point on the curve labelled by λ to the point labelled $\lambda + d\lambda$, a transformation is made locally to a coordinate system for which the metric is constant up to terms of (displacement)¹. With respect to this special coordinate system, M^α should have the same components at λ and at $\lambda + d\lambda$. When this condition is expressed in terms of the components in the general coordinate system, it is

$$\frac{\delta M^\alpha}{\delta \lambda} = \frac{dM^\alpha}{d\lambda} + \Gamma_{\beta\gamma}^\alpha M^\beta \frac{dx^\gamma}{d\lambda} = 0, \quad (7)$$

in which the Γ symbols are the Christoffel symbols derived from the metric tensor.

Now we can say how y^a should change with λ . Let the 4-momentum $M^\alpha(x^\alpha, y^a)$ at λ be parallel-transported to $\lambda + d\lambda$. Scalar products and lengths of 4-vectors are unchanged by being parallel-transported, so if M^α was a null vector at λ , its transported self is a null vector at $\lambda + d\lambda$. As such, it must agree with $M^\alpha(x^\alpha + M^\alpha d\lambda, y^a + dy^a)$ for some choice of the perturbations dy^a to the null-cone coordinates y^a . This leads to the equation

$$M^\beta \frac{\partial M^\alpha}{\partial x^\beta} + \Gamma_{\beta\gamma}^\alpha M^\beta M^\gamma + \frac{dy^a}{d\lambda} \frac{\partial M^\alpha}{\partial y^a} = 0, \quad (8)$$

or, in more compact form,

$$\frac{dy^a}{d\lambda} = y_\beta^a M^\beta, \quad (9)$$

where

$$M_{;\beta}^\alpha + \frac{\partial M^\alpha}{\partial y^a} y_\beta^a = 0, \quad (10)$$

and $M_{;\beta}^\alpha$ is the usual covariant derivative of the contravariant vector M^α with the parameters y^a held fixed. For each of the four values of β equation (10) represents a set of four linear equations for the three unknowns y_β^a . That these have a solution is ensured by the conditions that M^α be a null vector at all x^α , whatever the choice of y^a . Finally, we find the transport equation

$$M^\alpha \frac{\partial \mathcal{I}}{\partial x^\alpha} + M^\beta y_\beta^a \frac{\partial \mathcal{I}}{\partial y^a} = -\alpha \mathcal{I} + \eta. \quad (11)$$

A summary of where we are: We use the invariant intensity, and absorption and emission coefficients, to free us from reference to any particular frame. We can choose any coordinate system x^α that we find to be convenient; it certainly does not have to be the rest frame of the fluid (not a practical choice in multidimensional problems anyway). Then we can choose a functional representation of the photon 4-momentum in terms of 3 felicitously chosen variables y^a . The choice of these

variables is in no way tied to the choice of our coordinate frame. In fact, for many problems the most convenient choice of frame is the Eulerian one of a fixed inertial frame, and the most convenient choice of y^a would be the 3-momentum of the photon in the co-moving frame, expressed in polar coordinates. The functional form of M^α in term of space-time and null-cone coordinates, plus some algebra, leads to the desired transport equation. If it is desired, the factors of v can be reinstated in the transport equation, as known functions of y^a , to give an equation for the intensity as seen by an observer in some frame. If y^a is based on the comoving-frame 3-momentum, it would be natural to use the intensity in the comoving frame as well.

In numerical computations there is a further choice that must be made. This is of the computational mesh. One choice is the Eulerian one of a mesh that is fixed at discrete values of the chosen coordinates x^α . However, the mesh may also move in time, and if the nodes of the mesh are required to move with the local fluid velocity the Lagrangian system of equations results. It must be emphasized that one is using a Lagrangian mesh in this case, not Lagrangian coordinates. Lagrangian coordinates, in the sense of an orthogonal coordinate system with respect to which the fluid is everywhere and at all times at rest, do not exist for fluid flows unless the vorticity vanishes. This is true even if orthogonality only between space and time is desired.

The Lagrangian approach really consists of a formulation of dynamical equations with respect to inertial coordinates, followed by the appropriate transformation of the time derivative to give the evolution equations with respect to the moving mesh. Thus a perfectly reasonable way of doing radiation hydrodynamics is to solve on a Lagrangian mesh the equations with respect to inertial coordinates for the radiation field in the comoving frame! And of course other combinations are equally possible.

We can proceed further in reducing the transport equation if we assume a linear relation between y^a and the x^α -frame components of M :

$$y^a = A^a_\alpha M^\alpha, \quad (12)$$

which is to say that we have selected 3 covariant vectors A^a_α , $a = 1$ to 3, that must be space-like, and define y^a as the projection of the 4-momentum onto these. If we add a fourth time-like vector A^4_α , then the four covariant vectors A^a_α ,

with $\alpha' = 1$ to 4, form a covariant basis set at the point x . This basis set replaces the natural covariant basis $\partial/\partial x^\alpha$, and presumably we can choose the new basis in such a way as to make the transport equation nicer. If the components of the four covariant vectors are arranged as the rows of a 4×4 matrix, this matrix gives the transformation from the natural basis to the new one, for contravariant vectors. The matrix inverse of $(A^\alpha_{\alpha'})$ is another matrix $(A^{\alpha'}_{\alpha})$ of which the columns comprise four contravariant vectors that form a contravariant basis at x^α . The inverse relation to equation (12) is then

$$M^\alpha = A^\alpha_{\alpha'} y^{\alpha'}, \quad (13)$$

in which a term in y^4 also enters, but y^4 is determined by the requirement that M^α be null:

$$g_{\alpha\beta} A^\alpha_{\alpha'} A^\beta_{\beta'} y^{\alpha'} y^{\beta'} = 0, \quad (14)$$

a quadratic equation for y^4 , given values of y^a , with one positive root.

Equation (13) is the representation of M^α in terms of y^a that is needed for the transport equation. Carrying out the algebraic steps needed to find y^a_α gives the result

$$y^a_\alpha = -A^a_\beta A^\beta_{\alpha';\alpha} y^{\alpha'} = A^a_{\beta;\alpha} A^\beta_{\alpha'} y^{\alpha'} = M^\beta A^a_{\beta;\alpha}, \quad (15)$$

where the second form follows from the first since the product of inverse matrices is the constant identity, of which the derivative vanishes. Substitution into equation (11) gives this form for the transport equation:

$$M^\alpha \frac{\partial \mathcal{J}}{\partial x^\alpha} + M^\alpha M^\beta A^a_{\beta;\alpha} \frac{\partial \mathcal{J}}{\partial y^a} = -\sigma \mathcal{J} + \eta. \quad (16)$$

This gives a physical interpretation of the coefficient $dy^a/d\lambda$ of $\partial \mathcal{J}/\partial y^a$: it is the strain in the M direction associated with a field of displacements along the a^{th} basis vector.

For the further specialization of the transport equation we assume (1) that spacetime is flat and that the x^α frame is Minkowskian; (2) that transformation (13) is a Lorentz transformation, so the y basis is orthonormal, i.e., the y^a are tetrad components; (3) the Lorentz transformation is a simple boost by the fluid velocity, of which the 3-space components are v^i . The tetrad components then consist of a time-like one, which is just the projection on the fluid 4-velocity (that was

the object of the transformation), and space-like ones that are the projections on the original space-like unit vectors boosted by v^i . The transformation of the time-like component is the Doppler effect, and the transformation of the space-like ones is the aberration effect. Working out the partial derivatives of the usual Lorentz transformation formulae leads to the following result

$$\frac{d\mathcal{G}}{d\lambda} + \frac{dy^a}{d\lambda} \frac{\partial \mathcal{G}}{\partial y^a} = -\sigma \mathcal{G} + \eta, \quad (17)$$

where

$$\begin{aligned} \frac{dy^a}{d\lambda} = & \frac{\gamma^2}{\gamma + 1} y^i \left(v^a \frac{dv^i}{d\lambda} - v^i \frac{dv^a}{d\lambda} \right) \\ & - y^4 \left(\gamma \frac{dv^a}{d\lambda} + \frac{\gamma^3}{\gamma + 1} v^i \frac{dv^i}{d\lambda} v^a \right) \end{aligned} \quad (18)$$

(with $\gamma = (1 - v^2)^{-1/2}$), and the operator $d/d\lambda$ can have any of the following forms

$$\frac{df}{d\lambda} \equiv y^{\alpha'} f_{,\alpha'} = A_{\alpha'}^{\alpha} f_{,\alpha} y^{\alpha'} = M^{\alpha} f_{,\alpha}, \quad (19)$$

in which $A_{\alpha'}^{\alpha}$ is the Lorentz transformation matrix. $f_{,\alpha'}$ represents derivatives in the tetrad directions, while $f_{,\alpha} = \partial f / \partial x^{\alpha}$. When the A's are substituted equation (17) becomes quite long, so I will not write it here.

The terms in equation (18) arise from the Lorentz transformation from the tetrad defined by v^i to the one defined by $v^i + (dv^i/d\lambda)d\lambda$. This transformation consists of a boost by roughly dv^i plus a rotation by a small angle around the axis $\underline{y} \times d\underline{y}$. The latter effect is essentially the Thomas precession. Specifically, the y^i term in equation (18) is the Thomas precession term, and the y^4 term represents the Doppler and aberration effects associated with the boost by $\approx dv^i$. The boost is not precisely by dv^i , as we see. In fact, the coefficient of y^4 in equation (19) consists of the space-like tetrad components of the 4-vector $du^{\alpha}/d\lambda$, where u^{α} is the fluid 4-velocity. If the velocity referred to a local almost comoving frame is $\underline{v}'$, these components do form the vector $d\underline{v}'/d\lambda$.

3.2. $O(v/c)$ transport equation

When the terms in the fluid velocity higher than the first are discarded, the non-relativistic (NR) limit, the equations simplify greatly. The Thomas precession vanishes, and the infinitesimal boost is precisely by dv^i . Furthermore, the transformation to the tetrad components is linear in v . The transport equation becomes

$$(y^4 + y^i v^i)_{,4} \mathcal{G} + (y^i + y^4 v^i)_{,i} \mathcal{G} - y^4 (y^4 v^i_{,4} + y^j v^i_{,j}) \frac{\partial \mathcal{G}}{\partial y^i} = -\sigma \mathcal{G} + \eta. \quad (20)$$

This equation is equivalent to the one given by Buchler (1982). Restoring the factors of c required for dimensional consistency and using a compact notation gives Buchler's form:

$$\begin{aligned} & \left(1 + \frac{\underline{n} \cdot \underline{v}}{c}\right) \frac{1}{c} \frac{\partial I_v}{\partial t} + \left(\underline{n} + \frac{\underline{v}}{c}\right) \cdot \underline{\nabla} I_v \\ & - \left(\frac{\underline{a} \cdot \underline{n}}{c} + \underline{n} \cdot \underline{\nabla} \underline{v} \cdot \underline{n}\right) \frac{v}{c} \frac{\partial I_v}{\partial v} \\ & - \frac{\partial}{\partial \underline{n}} \cdot \left[P \left(\frac{\underline{a}}{c^2} + \frac{\underline{n} \cdot \underline{\nabla} \underline{v}}{c} \right) I_v \right] + \left(\frac{\underline{a} \cdot \underline{n}}{c^2} + \frac{\underline{\nabla} \cdot \underline{v}}{c} \right) I_v \\ & = -k_v I_v + j_v, \quad (21) \end{aligned}$$

where the new symbols are the operator P that projects a vector perpendicular to $\underline{n}$, the differential operator $\partial/\partial \underline{n}^*$, which is the divergence operator on the unit sphere, and the acceleration vector $\underline{a} = d\underline{v}/dt$. The notations $\underline{n}$ and v are now used for the direction and frequency in the comoving frame. This is equivalent to the following conservation-law form:

$$\begin{aligned} & \frac{1}{c} \frac{\partial}{\partial t} \left[\left(1 + \frac{\underline{n} \cdot \underline{v}}{c}\right) \frac{I_v}{v} \right] + \underline{\nabla} \cdot \left[\left(\underline{n} + \frac{\underline{v}}{c}\right) \frac{I_v}{v} \right] \\ & + \frac{\partial}{\partial v} \left[- \left(\frac{\underline{a} \cdot \underline{n}}{c^2} + \frac{\underline{n} \cdot \underline{\nabla} \underline{v} \cdot \underline{n}}{c} \right) v \frac{I_v}{v} \right] \\ & + \frac{\partial}{\partial \underline{n}} \cdot \left[- P \left(\frac{\underline{a}}{c^2} + \frac{\underline{n} \cdot \underline{\nabla} \underline{v}}{c} \right) \frac{I_v}{v} \right] = -k_v \frac{I_v}{v} + j_v. \quad (22) \end{aligned}$$

The quantity that is being conserved is $(1 + \underline{n} \cdot \underline{v}/c)$ times the photon number density as measured in the comoving frame. It is not hard to show that the Doppler factor in this expression accounts for the difference in number density when the volume element is fixed in the inertial frame rather than fixed in the comoving frame. Thus the conserved quantity is indeed the inertial frame number density of photons in a unit comoving frame momentum volume. Equation (22) is a multidimensional continuity equation, with the advection "velocities"

$$\frac{d\underline{r}}{dt} = \frac{\underline{nc} + \underline{v}}{1 + \frac{\underline{n} \cdot \underline{v}}{c}}, \quad (23)$$

$$\frac{d\underline{v}}{dt} = - \left(\frac{\underline{a} \cdot \underline{n}}{c} + \underline{n} \cdot \underline{\nabla} \underline{v} \cdot \underline{n} \right) \underline{v}, \quad (24)$$

and

$$\frac{d\underline{n}}{dt} = - \underline{p} \left(\frac{\underline{a}}{c} + \underline{n} \cdot \underline{\nabla} \underline{v} \right), \quad (25)$$

to first order in $\underline{v}$.

An analysis of equation (21) by Buchler, and of the similar one dimensional equation by Mihalas and Weaver (1982), shows that certain of the terms are never important in any low-velocity problem, regardless of the time scales and optical thickness. These are all the terms with c^2 in the denominator in equation (22): the acceleration terms and the Doppler factor in the time derivative. Dropping them gives this equation

$$\begin{aligned} \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{I_v}{v} \right) + \frac{1}{c} \underline{\nabla} \cdot \left(\underline{v} \frac{I_v}{v} \right) + \underline{\nabla} \cdot \left(\underline{n} \frac{I_v}{v} \right) \\ + \frac{\partial}{\partial v} \left(- \frac{\underline{n} \cdot \underline{\nabla} \underline{v} \cdot \underline{n}}{c} I_v \right) + \frac{\partial}{\partial \underline{n}} \cdot \left(- \underline{p} \frac{\underline{n} \cdot \underline{\nabla} \underline{v}}{c} \frac{I_v}{v} \right) \\ = -k_v \frac{I_v}{v} + \frac{j_v}{v}. \end{aligned} \quad (26)$$

There are four terms in equation (26) with c in the denominator. These are responsible for the dynamical effects of radiation. They are the time derivative term, which we may call the retardation effect; the divergence term with the fluid velocity, which we may call the advection effect (for the tendency of the radiation to be carried with the fluid, owing to

this term); the frequency derivative, which we may call the Doppler effect; and the angular divergence, which we may call the aberration effect.

3.3. Radiation moments and coupling to matter

The first two moments of equation (26) with respect to angle are very important, since they form the basis of many computational methods, as well as leading to the momentum and energy conservation laws. In the first, unweighted, moment, the source-sink terms and the spatial transport term are subject to severe cancellation, and for this reason the $1/c$ terms can play a major role; this is not true of the μ -weighted moment, for which only the time derivative term among the $1/c$ terms can be important. This leads to this reduced set of moment equations:

$$\frac{\partial E_v}{\partial t} + \nabla \cdot (\nabla E_v) + \nabla \cdot F_v - v \frac{\partial P_v}{\partial v} : \nabla v = -k_v c E_v + 4\pi j_v \quad (27)$$

$$\frac{1}{c} \frac{\partial F_v}{\partial t} + c \nabla \cdot P_v = -k_v E_v, \quad (28)$$

where P_v is the monochromatic pressure tensor. Their frequency-integrated form is

$$\begin{aligned} \frac{\partial E}{\partial t} + \nabla \cdot (\nabla E) + \nabla \cdot F \\ + P : \nabla v = \int_0^\infty (-k_v c E_v + 4\pi j_v) dv \end{aligned} \quad (29)$$

$$\frac{1}{c} \frac{\partial F}{\partial t} + c \nabla \cdot P = - \int_0^\infty k_v E_v dv. \quad (30)$$

The equations of radiation hydrodynamics are completed by adding to the foregoing dynamical equations for the radiation field the fluid equations for the material. In the form compatible with our v/c approximation of the radiation equations these are

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 v) = 0 \quad (31)$$

$$\rho_0 \left(\frac{\partial E_m}{\partial t} + \underline{v} \cdot \underline{\nabla} E_m \right) + p_m \underline{\nabla} \cdot \underline{v}$$

$$= - \int_0^\infty (4\pi j_v - k_v c E_v) dv \quad (32)$$

and

$$\left(\rho_0 + \frac{\rho_0 E_m}{c^2} + \frac{p_m}{c^2} \right) \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right) + \underline{\nabla} p_m$$

$$+ \frac{\underline{v}}{c^2} \frac{\partial p_m}{\partial t} = \frac{1}{c} \int_0^\infty k_v \underline{F}_v dv, \quad (33)$$

where the new variables are ρ_0 , the density of proper mass, not including excitation energy, in the fluid frame, E_m , the material internal energy per unit proper mass, and p_m , the material pressure. To allow for the possibility that the material may be thermally relativistic (the sound speed approaches c even though $v \ll c$) the enthalpy contribution to the inertial mass is retained, as is the correct form for the fluid-frame pressure gradient. The reason for this is seen when $(1 + E_m/c^2)\underline{v}$ times equation (31) and $\underline{v}/c^2$ times equation (32) are added to equation (33). The result, neglecting a v^2/c^2 term, is

$$\frac{\partial}{\partial t} \left[\left(\rho_0 + \frac{\rho_0 E_m}{c^2} + \frac{p_m}{c^2} \right) \underline{v} \right]$$

$$+ \underline{\nabla} \cdot \left[\left(\rho_0 + \rho_0 \frac{E_m}{c^2} + \frac{p_m}{c^2} \right) \underline{v} \underline{v} \right] + \underline{\nabla} p_m$$

$$= \frac{1}{c} \int_0^\infty k_v \underline{F}_v dv - \frac{\underline{v}}{c^2} \int_0^\infty (4\pi j_v - k_v c E_v) dv. \quad (34)$$

We see that the momentum coupling term between radiation and matter has different forms in equations (33) and (34). In both cases the force that appears on the right-hand side consists of the space part of the 4-force coupling radiation and matter, but in the acceleration equation (33) the fluid-frame space part appears, while in the momentum conservation equation (34) the

inertial frame space part appears. The additional term in the force in equation (34) can be thought of as the reaction force due to isotropic absorption and emission in the fluid frame, which is anisotropic in the inertial frame owing to the aberration effect. This correction can dominate the one due to the fluid-frame flux if the medium is transparent and emits strongly.

The form of radiation momentum conservation comparable with equation (34) is

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} (\tilde{F} + \tilde{v}E + \tilde{p} \cdot \tilde{v}) \\ + \tilde{v} \cdot \left(\tilde{p} + \frac{\tilde{v}F + F\tilde{v}}{c^2} \right) = - \frac{1}{c} \int_0^\infty k_v \tilde{F}_v dv \\ + \frac{\tilde{v}}{c^2} \int_0^\infty (4\pi j_v - k_v cE_v) dv, \quad (35) \end{aligned}$$

and the corresponding radiation energy equation in conservation-law form is

$$\begin{aligned} \frac{\partial}{\partial t} \left(E + \frac{2v}{c^2} \cdot F \right) + \tilde{v} \cdot (\tilde{F} + \tilde{v}E + \tilde{p} \cdot \tilde{v}) \\ = \int_0^\infty (4\pi j_v - k_v cE_v) dv - \frac{\tilde{v}}{c} \cdot \int_0^\infty k_v \tilde{F}_v dv. \quad (36) \end{aligned}$$

Some problems arise with these equations, however. First of all, in order to put the radiation moments into conservative form, all the c^{-1} and c^{-2} terms must be kept in the transport equation. The c^{-2} terms are needed to get the correct velocity terms under the time derivatives, and all the c^{-1} terms are needed to get the velocity terms under the space derivatives. For this reason equations (35) and (36) cannot be obtained from equations (29) and (30). Also, while the right-hand sides of equations (35) and (34) cancel, thus ensuring total momentum conservation, the material total energy equation, the right-hand side of which cancels the right-hand side of equation (36), can only be obtained relativistically. I have not been able to find a conservative system of dynamically non-relativistic but thermally relativistic hydro equations. The problem is that the relativistic variation of mass with velocity contributes a term to the total energy just equal to the change in the kinetic energy density due to the

variation of mass density with the internal energy. The correct total energy equation cannot be obtained unless both effects are included.

It is not hard to construct an example for which these difficulties arise. Consider a steady flow with a uniform high velocity, which is everywhere transparent but which emits uniformly and isotropically (in the fluid frame) within a fixed slab perpendicular to the flow. The velocity is large enough and the slab is transparent enough that v/c is large compared with the optical thickness of the slab. In this case, the reaction or recoil force, proportional to the velocity times the emissivity, dominates the usual force proportional to the opacity times the flux. In this situation the force is dynamically significant only if the matter is very hot: the equilibrium radiation energy density should be larger than the rest-energy density. If the simplified equations (27) and (28) are used to solve this problem, the correct leading terms are found for the expansions of E and F in v/c . However the subsequent terms are in error, and as a result the leading terms in ∇E and $\nabla \cdot \underline{P}$ are wrong. For the correct solution, $c \nabla \cdot \underline{P}$ does not balance $-\int k_\nu F_\nu dv$, it balances $-\underline{v} \cdot \nabla F/c$. The body force in the comoving frame, the quantity on the right-hand side of equation (33), does come out right, since the leading term in F is right, but this agrees with the left side of equation (30) only if the incorrect $\underline{P}$ is used there. If the incorrect $\underline{P}$ is used, the left side of equation (35) will not agree with the right side. Thus consistent use of the simplified equations (27) and (28) for the radiation, together with the material mass, acceleration and internal energy equations, (31), (32), and (33), will give essentially the right radiation field, and essentially the right motion and temperature of the material, at the cost of conserving momentum and energy exactly. This example is one in which the material energy and momentum densities are small compared with those of the radiation field, so the total conservation equations are not appropriate for finding the small material parts.

With cold -- thermally non-relativistic -- material, finding a conservative system is easier. The c^{-2} terms in all the material equations can be dropped. It is then immaterial whether the radiation force in the comoving frame or the fluid frame is used. The simple forms (29) and (30) of the radiation conservation laws can be used, which allows momentum to be conserved exactly, and energy to be conserved except for a negligible error $(\underline{v} \cdot \partial \underline{E} / \partial t) / c^2$. If the time derivative term is dropped in equation (31), which then becomes the static flux equation, both momentum and energy can be conserved. An inspection of the equations to see what has been lost shows that the radiation momentum density has disappeared. Indeed, radiation momentum density can never be important in a thermally non-relativistic problem.

3.4. The right approximation for the problem

A primary question to be answered when the equations of radiation hydrodynamics are being formulated is: what complexities should be retained in the theory when it is applied in different situations? The different situations we have in mind are specified by different strengths of coupling between matter and radiation, or otherwise put, different optical thicknesses. Radiation hydrodynamics is a theory of a two-component system. One component consists of all the material particles, which are lumped together and called matter for our purpose, and the other component consists of the photons. The three coupling regimes are: strong coupling, which produces radiation-matter equilibration, weak coupling, for which the radiation field can be specified a priori, and the in-between case of moderate coupling.

I will call the strong coupling regime the stellar interior limit. The two conditions that must be met in order for this regime to apply are (1) that the distance radiation diffuses before exchanging appreciable energy with the matter is small compared with the characteristic length of the system, and (2) that the time required for this to occur is small compared with the characteristic time. Diffusion theory can then be applied in the local comoving frame. This leads to the result that the radiation field is the equilibrium blackbody field plus small corrections of order the mean free path times the Planck function gradient, the mean free time times the time derivative of the Planck function, and the mean free path and mean free time times respectively the velocity gradient divided by c and the acceleration divided by c , both multiplied by the Planck function. The velocity and acceleration corrections represent the effects of radiative viscosity. Since the radiation stress-energy tensor is known explicitly in terms of the state of the matter, it can be added to the matter stress-energy tensor, so it is no longer necessary to carry the radiation-matter coupling terms. In the fluid frame the radiation corrections consist of adding $aT^4/3$ to the pressure, adding aT^4/ρ to the internal energy, accounting for the diffusion energy flux, and including the viscosity-like velocity gradient and acceleration terms in the internal energy and pressure tensor. In stating this, which seems to be obvious and represents the way stellar interiors theory has been done since the time of Eddington, I am making the point that this approach is fully consistent with even a relativistic treatment of radiation transport. The only question would be whether additional v^2/c^2 terms should enter if the flow is dynamically relativistic. But since the velocity enters the transport equation in the comoving frame only through gradient terms and time derivatives, it must enter the diffusion theory in combination with the mean free path, thus v^2/c^2 terms

come in only when the diffusion theory is taken to second order in the mean free path. This assumes that relativistic kinematics is used to evaluate the velocity gradient and acceleration in the comoving frame in terms of the inertial frame velocity.

The weak and moderated coupling regimes I will call the atmospheres limit. Weak coupling results when the system is transparent to essentially all radiation. Then there are two cases for how the radiation is produced. One case is that the radiation is externally imposed from presumably known sources. Then the radiation transport equation can be discarded, since the radiation is known, and the radiation hydrodynamics problem becomes a pure hydrodynamics problem, but with source terms of momentum and energy given by the radiation-matter coupling. Optical thinness ensures that the coupling terms have a negligible effect on the radiation field. The coupling terms can still be significant if the Boltzmann number for the problem is small enough, but since the coupling is weakened by a factor τ , the optical thickness of the system, the Boltzmann number must be less than τ for radiation to be energetically important, and less than $\tau v/c$ for radiation to be dynamically important. These conditions are easily realized in a tenuous gas next to a bright radiation source, such as a stellar atmosphere.

The second case of weak coupling is the one in which the radiation is internally generated in the medium. The magnitude of this radiation contains a factor τ , so the upper limits on the Boltzmann number for radiative heating and momentum coupling to be significant are further reduced by a factor τ . In fact, radiative heating is moot, since radiative cooling is sure to overwhelm it in this case. In the rare case that the material is so hot that momentum coupling to the internally-generated radiation is significant, the problem must be treated in the same way as for moderate coupling.

In the strong coupling and weak coupling regimes the two-component problem reduces to a one-component problem, either by equilibration of the two components (strong coupling) or by having one component -- radiation -- fixed by boundary conditions (weak coupling). The moderate coupling regime is the one of true radiation hydrodynamics, since the full system of material and radiation equations must be solved simultaneously. Of course any problem that spans a large range of density will likely have regions that are in the strong coupling limit, and other regions that are in the weak coupling limit, with interfaces between them that fall in the moderate coupling regime. It may be that the entire problem must be treated with the general methods in this case. More interesting in the sense of creating new effects are problems that contain extended regions of moderate coupling. This can arise in a variety of ways, but I will focus on two: an

atmosphere in which the spectral lines are opaque, but the continuum between the lines is transparent; and a continuum scattering problem in which the system is opaque to scattering, but rare enough that the radiation is not thermalized.

In the thick lines/thin continuum problem, the large majority of radiation energy and momentum resides in the continuum, and does not couple to the matter. For this reason the total radiation conservation laws are not useful, and a simpler procedure is to take the continuum radiation as given and solve only for the radiation field at line frequencies. The transport equation can then be simplified by omitting all c^{-2} terms and all c^{-1} terms except for the Doppler term, which is enhanced in importance owing to the rapid frequency variation of the radiation field in the lines.

In the continuum scattering problem the full radiation equations as discussed above (viz., equations [27] and [28]) must be used, but owing to the optical thickness of the problem, the diffusion approximation is successful. The frequency spectrum of the radiation can be expected to depart significantly from Planckian, so a frequency-dependent treatment is needed.

A discussion of these two problems will occupy the balance of these lectures.

4. SECOND LECTURE -- METHODS FOR LINE-DOMINATED PROBLEMS

At frequencies that lie within a spectral line the absorption coefficient may be orders of magnitude greater than at continuum frequencies, with the result that regions of even modest optical depth in the continuum become so opaque in the lines that the flux at line frequencies, given by the diffusion approximation, is quite negligible, and the contribution made by these frequencies to the momentum and energy coupling terms (viz., equations [29] and [30]) is to the continuum contribution as the fraction of the spectrum they occupy. If the lines are sparse in the spectrum, their effect in this limit is negligible. However, as the region becomes partially transparent in the lines, the flux is no longer inversely related to the absorption coefficient, but tends to be comparable in and out of the lines, and the lines have an importance in the coupling terms far greater than their relative bandwidth. This is the regime that needs to be investigated more carefully.

The effect of lines is also greatly enhanced by a velocity gradient. Such a gradient acts like a source of line broadening, increasing the line width and reducing the peak opacity. The fraction of the spectrum influenced by lines is increased, as is

their effect on the coupling terms since their saturation is reduced with the opacity. At typical temperatures and densities lines normally have widths $\Delta\nu/\nu$ that are 10^{-4} or less, so the effect of a velocity of only 30 km s^{-1} will be major for transport in the lines. The sensitivity to advection, aberration and retardation is no greater at line frequencies than at any others, so the Doppler term overwhelms the other c^{-1} effects in the co-moving frame (CMF) transport equation.

The transport equation (26), simplified by the omission of retardation, advection and aberration, becomes

$$\underline{n} \cdot \underline{\nabla} I_\nu - \frac{1}{c} \underline{n} \cdot \underline{\nabla} \underline{v} \cdot \underline{n} \nu \frac{\partial I_\nu}{\partial \nu} = j_\nu - k_\nu I_\nu. \quad (37)$$

The comparison of the transport terms on the left side -- spatial transport and frequency transport -- leads us to define three velocity regimes since different calculational methods are used in each regime. These are (1) static regime, $v \ll v_{th}$; (2) transition or, roughly, transonic regime, $v \approx v_{th}$; and (3) Sobolev or hypersonic regime, $v \gg v_{th}$. Here v_{th} stands for the intrinsic line width in velocity units. In the most common case, the principal mechanism of line broadening is the thermal Doppler effect, and v_{th} is $(2/3)^{1/2}$ times the rms speed of a line-emitting atom or ion. This is a few times less than the speed of sound, depending on the atomic weight of the species. The calculational methods differ in that in the static case the Doppler term is negligible, and in the Sobolev case the spatial transport term is negligible. These are relatively easy to treat, since equation (37) becomes an ODE. In the transition case the two transport terms are comparable and the PDE character of equation (37) is essential.

4.1. Line transport in the static regime

In the static regime the transport equation is the familiar equation of transfer, and methods for solving it in spectral lines are discussed by, for example, Mihalas (1978). Analytic approximations for the momentum coupling term -- the force -- due to a line are given by Castor (1974). The behavior of the force due to a line in a plane-parallel stellar atmosphere can be understood from the following rough formula:

$$f_{line} \approx - \Delta\nu \frac{4\pi}{3c} \frac{dJ_{\nu_0}}{dz}, \quad (38)$$

in which $\Delta\nu$ is the equivalent width of the line (essentially the width of the frequency interval in which line opacity exceeds the continuous opacity) and J_{ν_0} is the angle-averaged intensity at

the center of the line. We see that it is the depth dependence of the mean intensity that determines the line force. Deep in the atmosphere the mean intensity will be closely equal to the Planck function at the local temperature, both at line center and at other frequencies. Then we see that the line force is very small in comparison with the force due to the continuum opacity, since the equivalent width of the line is a small part of the whole spectrum. However, near the edge of the atmosphere, where the radiation escapes freely, the line-center mean intensity can be substantially less than the mean intensity in the continuum or the Planck function. Just where the line-center intensity begins to depart from the continuum intensity, and how much less it is at the edge of the atmosphere, depends on the details of the formation mechanism of the line. One general rule is that the stronger the line is, i.e., the greater its opacity is, the closer to the edge of the atmosphere the drop in line-center intensity is. Thus strong lines and weak lines all have about the same pressure jump across the whole atmosphere, but for a strong line the pressure gradient occurs in low density material and therefore produces a larger force per unit mass than for weaker lines. The line force has its maximum at the edge where all the radiation can escape, and the maximum value can approach the multiple of the star's gravity given by

$$\frac{f_{\text{line}}}{\rho g} \approx \frac{\kappa_{\text{line}}}{\kappa_{\text{es}}} \frac{v_{\text{th}}}{c} \frac{L}{L_{\text{ED}}} \quad (39)$$

in terms of the ratio of line opacity to the electron scattering opacity and the ratio of the stellar luminosity L to the Eddington limit luminosity $L_{\text{ED}} = 4\pi G M c / \kappa_{\text{es}}$. This limit for the force is approached if the emergent flux at line center is not too much less than the continuum flux. The point to be noted about this formula is that the opacity ratio can be 10^7 for the resonance line of an abundant ion, and v_{th}/c is about 10^{-4} , so the force due just to a single line could be 10^3 times greater than gravity for a star not far from the Eddington limit! This idea was the departure point for thinking about radiatively-driven stellar winds (Castor, Abbott and Klein 1975; referred to below as CAK). What this estimate shows is that for some stars a static state is impossible. The wind itself has a velocity much larger than v_{th} , and the Sobolev method or the transitional methods are needed to treat it.

4.2. Transonic line transport

In the transitional case the space and frequency derivative terms are comparable, and the transport equation must be solved as a PDE in those variables. In 1-D slab symmetry the intensity is a function of z , the comoving-frame frequency ν , and μ , the z component of the direction vector. It obeys this equation

$$\mu \frac{\partial I}{\partial z} - \frac{\mu^2}{c} \frac{dv}{dz} \nu \frac{\partial I}{\partial \nu} = k_L \phi(\nu) (S - I_\nu) + k_c (S_c - I_\nu), \quad (40)$$

in which k_L is the integrated line absorption coefficient, k_c is the continuous absorption coefficient, $\phi(\nu)$ is the normalized line profile, S_c is the known continuum source function and S is the line source function, which in the usual case of complete frequency redistribution has the form

$$S = (1 - \epsilon) \int_0^\infty d\nu \phi(\nu) \frac{1}{2} \int_{-1}^1 I_\nu(\mu) d\mu + \epsilon B_\nu \quad (41)$$

in terms of the intensity, the Planck function and a thermalization parameter ϵ .

Methods to solve equations (40) and (41) have been presented by Noerdlinger and Rybicki (1974) and Mihalas, Kunasz and Hummer (1975, 1976). The latter authors considered the case of spherical symmetry. All used the Feautrier transformation, that is, they separated the intensity into the sum of an even function of μ and an odd function of μ , which obey coupled equations. The equations are spatially differenced in a staggered fashion: the even components are centered in zones, and the odd components at zonal interfaces. This differencing of $\partial/\partial z$ is second-order accurate. Noerdlinger and Rybicki used centered implicit frequency differencing, while Mihalas, Kunasz and Hummer used a donor-cell type frequency differencing. The donor-cell scheme is only first-order accurate, but is more robust for optically thick lines, when equation (40) becomes stiff. There is some question about the accuracy of the centered frequency differencing in this case.

Equation (40) is to be solved between certain limits in z , the boundaries of the problem in physical space, and between certain limits in ν outside of which the opacity contribution of the line in question is negligible. In other words, the equation is to be solved within a rectangle in the z, ν plane. The boundary conditions that may be applied to make this problem properly posed are to specify the intensity at one point on each characteristic of the equation, the point where that characteristic, traced in the direction of radiation flow, first crosses the perimeter of the rectangle. Therein lie the difficulties that afflict these methods for complex problems. While it is easy to see that the boundary data can always be given at the inflow boundary in z , i.e., at $z_{\min}$ for $\mu > 0$ and

at $z_{\max}$ for $\mu < 0$, the inflow boundary in v is defined only when dv/dz has constant sign. If dv/dz is positive everywhere, then $v_{\max}$ is the inflow boundary; if dv/dz is negative everywhere, then $v_{\min}$ is the inflow boundary. If $v(z)$ is not monotonic, i.e., if dv/dz changes sign, then data may be given at $v_{\max}$ for some range of z , and at $v_{\min}$ for another range of z , but for still other values of z the characteristics that enter the rectangle at $v_{\max}$ or $v_{\min}$ are the same ones that left the rectangle at some other z , so the incident intensity at the first z is equal to the emergent intensity at the second z . This kind of mapping function for the boundary data would be very cumbersome to handle. In practice, only for the monotonic velocity case have results been given in the literature.

All the previous authors have used direct solution methods for the large linear system that results from the differencing of equations (40) and (41). Noerdlinger and Rybicki (1974) and Mihalas, Kunasz and Hummer (1976) solved the system in block tri-diagonal form for which the blocks are indexed by z and the internal structure of each block refers to the frequencies and directions. Mihalas, Kunasz and Hummer (1975) solved the system by formally expressing the intensity in terms of the line source function at each frequency and direction, then at the end solving a full system for the source function vector. The first method of solution is called the Feautrier elimination scheme, from the paper by Feautrier (1964) in which was proposed, and the second method is called the Rybicki elimination scheme, from Rybicki (1971).

The problem of the proper frequency boundary conditions can be circumvented by making the frequency range computed, from $v_{\min}$ to $v_{\max}$, large enough -- more than twice the full range of the flow velocity. Then the frequency boundary data are irrelevant, and can be set to zero at the inflow frequency for each z . To avoid an excessive total number of frequencies the size of the frequency increment can be graded from a fraction of a Doppler width at the center of the line to a fraction of the flow velocity Doppler shift at $v_{\max}$ and $v_{\min}$. This would only roughly double the number of frequency points from the number required for the line proper. There still remains the problem that the stable donor-cell frequency differencing requires the correct choice of donor frequency, and in the Mihalas, Kunasz and Hummer (1975) method this must be the same at all z , since their method consists of a forward and reverse sweep in z within a unidirectional sweep in v .

If we return to the first-order form of the transport equation, that is, abandon the Feautrier even-odd parts of the intensity in favor of the intensity itself at positive and negative angles, the calculation can be organized as a

unidirectional sweep in v within a unidirectional sweep in z , for each angle. This allows the correct direction of the v sweep to be chosen for each z . The Rybicki elimination scheme can still be used, with the same operation count, and the only price that must be paid is either some loss of accuracy of the spatial differencing or the necessity of using exponentials, for example, in the difference formulae.

Another approach to the problem is to make it an iterative one. The PDE solution as just described can be used as a method for evaluating the intensity given the source function. This can be used within an iteration at each step of which the difference between an accurately calculated intensity and an intensity calculated using a crude intensity-source function relation is used, with the crude relation, to improve the current source function estimate and make another accurate intensity calculation. Iterative methods of this kind have been presented by Cannon (1973a,b) and Scharmer (1981). Convergence is obtained, in favorable cases, in a few iterations, and the total time required is largely spent in the accurate intensity calculations. This can result in a severalfold improvement in speed compared with even the Rybicki elimination scheme, which is usually the fastest of the direct solution methods.

In the preceding paragraphs I have discussed the limitations of the presently available methods for solving line transport problems in the transitional -- transonic -- regime, and have made some suggestions of directions for further work. This remains an area that is ripe for development.

4.3. Line transport in the Sobolev regime

The Sobolev or hypersonic regime is the one in which the Doppler term in the transport equation overwhelms the spatial transport term. The Sobolev approximation (Sobolev 1947; Castor 1970, 1974) consists of simply neglecting spatial transport. The picture one has is that spatial transport occurs only in the continuum, and can be treated trivially if the continuum is optically thin, as we assume here. The Doppler shift due to the fluid flow then feeds radiation from the continuum into the line. The simplified form of equation (37) for the Sobolev approximation is

$$-\frac{n \cdot \nabla v \cdot n}{c} v \frac{\partial I_v}{\partial v} = k_L \phi(v) (S - I_v) . \quad (42)$$

The line width is assumed negligible compared with v , so the factor v is treated as a constant. The dependence on the line profile function is absorbed by defining a new frequency variable y :

$$y = \begin{cases} \int_v^\infty \phi(v') dv' & \underline{n} \cdot \underline{\nabla} \underline{v} \cdot \underline{n} > 0 \\ \int_0^v \phi(v') dv' & \underline{n} \cdot \underline{\nabla} \underline{v} \cdot \underline{n} < 0 \end{cases} \quad (43)$$

which runs from 0 at the "inflow" side of the line to 1 at the "outflow" side. The intensity thus satisfies the following ODE:

$$\frac{\partial I_v}{\partial y} = \tau_S(\underline{n}) (S - I_v) \quad (44)$$

in which $\tau_S(\underline{n})$ is the Sobolev optical depth defined by

$$\tau_S(\underline{n}) = \frac{k_L c}{v |\underline{n} \cdot \underline{\nabla} \underline{v} \cdot \underline{n}|} \quad (45)$$

The solution of equation (44) is

$$I_v = I_c e^{-\tau_S y} + S (1 - e^{-\tau_S y}) \quad (46)$$

where I_c is the intensity in the continuum on the inflow side of the line. The dependence of I_c , I_v and τ_S on $\underline{n}$ has not been shown.

The results for the frequency-averaged angle moments of the intensity are expressed in terms of escape probabilities. The angle dependent escape probability $\beta(\underline{n})$ is defined as the probability that a photon emitted in the direction $\underline{n}$ with an initial frequency v in the comoving frame sampled from the distribution $\phi(v)$ will escape to infinity, or at least to a region where the velocity is different by many line widths, without being reabsorbed. The expression for $\beta(\underline{n})$ is

$$\beta(\underline{n}) = \frac{1 - e^{-\tau_S(\underline{n})}}{\tau_S(\underline{n})} \quad (47)$$

The average of $\beta(\underline{n})$ over direction is the angle-mean escape probability, or simply the escape probability, $\bar{\beta}$ given by

$$\bar{\beta} = \int \beta(\underline{n}) \frac{d\Omega}{4\pi} = \int \frac{d\Omega}{4\pi} \frac{1 - e^{-\tau_S(\underline{n})}}{\tau_S(\underline{n})} . \quad (48)$$

The key results of the Sobolev theory are expressions for $\bar{J}$, the angle and frequency-averaged intensity which enters the non-LTE kinetic equations for atomic level populations, and the line force. These are

$$\bar{J} = S(1 - \bar{\beta}) + \int \frac{d\Omega}{4\pi} \beta(\underline{n}) I_c(\underline{n}) \quad (49)$$

and

$$f_{\text{line}} = \frac{k_L}{c} \int d\Omega \beta(\underline{n}) \underline{n} I_c(\underline{n}) . \quad (50)$$

When equation (49) for $\bar{J}$ is combined with equation (41) for S in terms of $\bar{J}$ the result is an explicit formula for S in terms of the temperature, thermalization parameter ϵ , and the locally-defined optical depth. There is no coupling to other depths, and there are no differential equations to solve. This simplicity makes the Sobolev method very attractive. Expression (50) for the line force is even simpler, since the force does not even depend on the local source function (owing to the near isotropy of the locally-emitted radiation) and can be computed directly from the continuous spectrum, given the local optical depth.

The order of magnitude of the force per unit volume due to a single line is that given in equation (39), if the Sobolev optical depth τ_S is less than unity, and smaller by a factor τ_S when τ_S is larger than unity. In the latter case the force contributed by the line is independent of the strength of the line, but proportional to the velocity gradient. When the force is summed for all the lines in the spectrum, a total is obtained which is essentially the force due to one strong line multiplied by the number of lines for which $\tau_S > 1$. It is a non-linear function of the velocity gradient. In the case that the distribution function of line strengths is a power law, which is often a reasonable approximation, the total force is also a power law in the velocity gradient, with an exponent between 0 and 1 (cf., CAK). The force due to the lines can be non-dimensionalized by expressing it as a multiple of the force due to electron scattering of the continuum radiation:

$$f_{\text{line}} = \left(\frac{\kappa_{\text{es}}^{\text{DF}}}{c} \right) \frac{1}{F} \int d\Omega M \left(\frac{\kappa_{\text{es}}^{\text{DF}} v_{\text{th}}}{|\underline{n} \cdot \underline{v}|} \right) \underline{n} I_c(\underline{n}) \quad (51)$$

where $M(t)$ is a dimensionless function of t , which is not the time but a measure of the continuum optical thickness of the layer across which the velocity changes by the intrinsic line width. The function M depends implicitly on the ionization and excitation state of the material, since these determine the opacities of the various lines. The most complete and accurate calculations of $M(t)$ have been made by Abbott (1977, 1982), and the results are presented graphically in that paper; they can indeed be fitted to a power law: $M \propto t^{-0.6}$ approximately. There is an additional weak dependence on p that arises from the variation of the ionization state.

4.4. Dynamical effect of line radiation

In order to illustrate the dynamical effects that can occur when the force due to an ensemble of lines is substantial, let's consider the following problem. Let there be a flow in the radial direction in a possibly stratified atmosphere with a collimated beam of radiation parallel to the flow, for example a stellar wind. The energy coupling between matter and radiation we suppose to be strong enough to hold the material at a fixed temperature. We want to consider what kinds of waves are produced when this flow is perturbed. The following discussion is similar to that of Abbott (1980). The equations of motion are written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (52)$$

and

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{a^2}{\rho} \nabla \rho = \frac{f_{\text{line}}}{\rho} \mathbf{e}_r \quad (53)$$

where a is the isothermal sound speed and $\mathbf{e}_r$ is the unit radial vector. We suppose that f_{line}/ρ is a function of $(dv_r/dr)/\rho$, as suggested by equation (51). Equations (52) and (53) can be linearized around the hypothesized flow, and the dispersion relation obtained for short wavelength perturbations (i.e., wavelength small compared with the scale of the stratification) is found to be

$$\frac{\omega}{k} = v_{\text{or}} \cos \theta + \Delta V \quad (54)$$

where θ is the angle between the propagation vector k and the radius, and the velocity shift ΔV satisfies the cubic equation

$$(\Delta V)^3 + A \cos \theta (\Delta V)^2 - a^2 \Delta V - a^2 A \cos \theta \sin^2 \theta = 0, \quad (55)$$

in which $A = \{d(f_{\text{line}}/\rho)/d(\rho^{-1}dv_r/dr)\}/\rho_0$.

If there were no line force, the wave modes found from equations (54) and (55) would be just the usual acoustic wave modes, and a root at $\Delta V = 0$ representing the degenerate internal waves which cannot be supported by the assumed isothermal variations. If instead f_{line}/ρ is comparable to the acceleration vdv/dr and v is hypersonic, then the factor A is of order $v \gg a$, and the roots of the cubic give the following for the wave modes

(radiation wave mode)

$$\Delta V \approx -A \cos \theta \quad (56)$$

$$\frac{\delta \rho}{\rho_0} \approx \frac{\delta v_r}{\Delta V} \quad (57)$$

$$\underline{u} \approx \underline{v}_0 - A \underline{e}_r \quad (58)$$

and

(acoustic modes)

$$\Delta V \approx \pm a \sin \theta \quad (59)$$

$$\delta \underline{v} \approx \frac{\delta \rho}{\rho_0} a \hat{k}_1 \quad (60)$$

$$\underline{u} \approx \underline{v}_0 \pm a \hat{k}_1 \quad (61)$$

In these formulae the quantities prefixed by δ are the wave amplitudes, the ones subscripted by 0 are the unperturbed values, $\hat{k}_1$ is the unit vector in the direction of the projection of $\underline{k}$ perpendicular to the radius. $\underline{u}$ is the group velocity of the wave in each case. What we see is that the acoustic waves can propagate only transverse to the radius, while the radiation wave propagates radially, and with a velocity that is inward with respect to the flow for the case we are considering.

4.5. Unstable radiation waves

The wave modes that have been described above are the ones that are obtained when the Sobolev approximation is valid, so that the force is a function of the local velocity gradient. This approximation is valid only so long as the spatial wavelength of the fluctuations is large compared with the distance over which the velocity varies by the line width, v_{th} . Let's call this distance ℓ . When the wavelength is smaller than ℓ the fluctuation in the force depends not on the velocity gradient,

but on the velocity fluctuation itself. Both cases are obtained by letting the fluctuating force be proportional to the fluctuation of the velocity gradient averaged over a region of size about equal to ℓ located just interior to the radius of interest. The dispersion relation for the radiation mode for the case $\alpha = \theta = 0$ is then

$$\omega_{\text{real}} \approx k \left[v_0 - \frac{A}{1 + (k\ell)^2} \right] \quad (62)$$

$$\omega_{\text{imag}} \approx \frac{A}{\ell} \frac{(k\ell)^2}{1 + (k\ell)^2} \quad (63)$$

Indeed, equation (63) shows that when the wavelength is not very large compared with ℓ the wave has exponential growth in time. Even for small k , large wavelengths, there is exponential growth at a rate proportional to k^2 , which is characteristic of negative viscosity. The corresponding negative kinematic viscosity coefficient is $A\ell$. Since A is typically the same order as the flow velocity, the negative Reynolds number for the problem is of order L/ℓ , where L is the relevant length scale. This ratio varies from about one, near the sonic point of the wind, to about 100 when the wind is approaching full velocity. We can conclude that the negative viscosity should be important, at least near the sonic point, and that instabilities will grow and presumably produce shock waves. These are essentially the conclusions that have been reached by MacGregor, Hartmann and Raymond (1979), Lucy and White (1980) and Owocki and Rybicki (1983).

4.6. Radiatively driven steady winds

A steady stellar wind is a flow that satisfies equation (53), or its equivalent, without the time derivative term. This becomes an ordinary differential equation for the radial component of velocity, and, as discussed by Parker (1958) and many others, a smooth flow can be obtained only if a regularity condition is obeyed at every point where the velocity of one of the linearized wave modes vanishes. For the case considered by Parker, this means the sonic point in the flow, and a complete analogy exists with the deLaval nozzle for which the velocity vanishes at the throat of the nozzle. Furthermore, in Parker's theory the only physically acceptable wind solution is the one which passes through such a sonic point. In general the flow also contains a shock outside the sonic point, which provides the correct pressure match of the wind to the interstellar medium.

When the equation of motion includes the force due to the momentum transfer in line scattering, as in equation (53), the situation becomes more complex. That is because the line force depends globally on the velocity field, so equation (53) is no longer a simple ODE for the velocity. Certain idealizations of this dependence can be made, treating the line force in different ways, and each of these leads to a Parker-type ODE with a singular point like Parker's sonic point, but very different ones for the different idealizations. If f_{line} is treated as a known function of either the radius or the local velocity, then it has no effect on the wave speeds (only on their growth), so the critical point is just the sonic point, and it turns out that the regularity condition says in effect that f_{line} balances the gravity at that point. If f_{line} is treated as a known function of the local velocity gradient, as in the Sobolev approximation, then the appropriate wave speed becomes that of the radiation wave described by equations (56)-(58), and the regularity condition is more complicated. In each case, the singular point is the outermost one that can send signals both inward and outward.

The paradoxical situation that arises is that if indeed there is a smooth steady flow that obeys equation (53) with an accurately calculated f_{line} , it should be described well by all the idealizations. It should obey the regularity condition at the sonic point expected if $f_{\text{line}}(r)$ is fixed, and also the CAK regularity condition at the Sobolev approximation singular point, where v is nearly equal to A , defined above. Indeed, this seems to be true for the models that have been made in the two ways. For the Sobolev CAK model f_{line} does in fact nearly balance the gravity at the sonic point (guaranteed by the ODE). Weber's (1981) wind models, which use the CMF method in place of the Sobolev approximation and which also differ from CAK in other respects (discrete line groups, accurate angle integration), have the same characteristics as matched Sobolev models. If we accept that the same flow can be modelled in quite different ways, what about the critical point? We may ask, what really is the outermost point that can send signals upstream? The answer is: what kind of signals? If the radiation field is frozen, then indeed, an ordinary sound wave is the signal that can be sent, and the sonic point is the singular point. If the radiation field is affected, then it becomes the medium that can carry a signal upstream much more quickly than sound can, and it is not surprising that the wave speed and singular point are changed. We are led once again to consider the wavelength of the wave to decide what kind of role the radiation plays, and come again to the dispersion relations of equations (62) and (63). Equation (62) shows that the wave speed is A , directed inward relative to the fluid, for wavelengths large compared with λ , and small, i.e., of order a , for wavelengths small compared with λ . So the conclusion is that an observer situated outside the

sonic point and inside the Sobolev critical point of a wind can send medium and long pulses inward, but not short pulses.

The rather complex picture that emerges from a consideration of the stellar wind driven by line radiation, as indicated above, is that the relative stability of the Sobolev wind model to long-wavelength perturbations may mean that this model gives a valid picture of the wind structure in the large, meaning smoothed over moderate increments in space and time, but that the instability to short wavelengths will necessarily produce small-scale fluctuations around that model. It is virtually impossible to quantify this statement, owing to the lack of a time-dependent hydrodynamic calculation of the wind that can treat the nonlinear development of the short-wavelength instabilities. This is an area that calls for the application of better numerical techniques for radiation hydrodynamics in line-dominated problems, such as an improved comoving-frame method, as discussed above.

4.7. Velocity-modified diffusion

The final subject in the area of line-dominated problems that I want to discuss is what I will call velocity-modified diffusion (VMD). This is a version of the Sobolev theory that is adapted to solve the following kind of problem: a region filled with material in motion such that the region is optically thick in the continuum, that is, the continuum mean free path is small compared with the size L of the region, but with a large enough velocity gradient that the continuum optical thickness associated with the thickness ℓ of the local constant-velocity surface is small, that is, the continuum mean free path is larger than ℓ . The Sobolev approximation discussed earlier assumed free-streaming radiation in the continuum; velocity-modified diffusion extends that to the case of opaque continua. If the continuum mean free path diminishes to the point that it no longer exceeds ℓ , then the effect of the velocity gradient on the diffusion process becomes negligible and the ordinary diffusion limit is recovered. The intermediate case exists only if the velocity span V across the region is hypersonic, i.e., large compared with the intrinsic line width. This is actually the case in many astrophysical situations, and in fact conditions in moving stellar envelopes often fall into the VMD regime, as pointed out by Karp, Lasher, Chan and Salpeter (1977), to whom this theory is due.

The heart of the VMD approximation is the calculation of the mean free path of a photon emitted at a particular fluid-frame frequency ν_0 . For this calculation the approximation is taken of an infinite homogeneous medium with a uniform velocity gradient. Then there is a constant ratio between displacement in space and displacement in the logarithm of the fluid-frame frequency for the photon, so the mean free path can be expressed as

$$\lambda^{eff}(\nu_0) = \frac{dx}{d \ln \nu} \left\langle \ln \frac{\nu_{final}}{\nu_0} \right\rangle. \quad (64)$$

in which x is the distance along the photon path. The mean can be calculated assuming an exponential distribution for the running optical depth $\tau(\nu)$ along the path:

$$\lambda^{eff}(\nu_0) = \int_0^\infty \frac{1}{\kappa_{\nu_0}} e^{-\tau(\nu)} d\tau(\nu) \quad (65)$$

and

$$\tau(\nu) = s \int_\nu^{\nu_0} \frac{\kappa_{\nu'}}{\kappa_c} \frac{d\nu'}{\nu'}, \quad (66)$$

in which the constant s , the same as c/v_{th} times the variable t of equation (51), is

$$s = \frac{\kappa_c \rho c}{\bar{n} \cdot \bar{v} \cdot \bar{n}}. \quad (67)$$

It should be noted that these relations hold for either sign of s . If the velocity gradient is one of expansion for the direction in question, then s is positive, and the frequency ν steadily decreases from the initial frequency ν_0 . If the velocity gradient is compressive, s is negative and the frequency increases from ν_0 . The range in $|s|$ for which the VMD approximation is useful is from the lower limit c/V , below which the continuum radiation is free-streaming, to the upper limit c/v_{th} , above which the ordinary diffusion theory is applicable.

Karp, Lasher, Chan and Salpeter evaluate the integral in equation (65) by assuming a negligible intrinsic line width, so that $\tau(\nu)$ has the form of a linear ramp with a jump discontinuity at each line frequency. Their result is

$$\lambda^{eff}(\nu_0) = \lambda_c \left(1 - \epsilon_{\nu_0} \right), \quad (68)$$

in which λ_c is the continuum mean free path and the correction factor ϵ_{ν_0} is given by

$$\epsilon_{\nu_0} = \sum_{j=J}^{\infty} \left(\frac{\nu_j}{\nu_0} \right)^s \exp \left(- \sum_{i=J}^{j-1} \tau_i \right) (1 - e^{-\tau_j}). \quad (69)$$

The indices i and j in equation (69) run over the list of absorbing lines, and the ordering is such that for positive (negative) values of s , the lines are ordered in decreasing (increasing) order of frequency, and the term J in the summations stands for the line with frequency closest to but less (greater) than ν_0 . The variable τ_i is the Sobolev optical depth of the i th line, given by equation (45).

Karp, Lasher, Chan and Salpeter supply a table of values of the Rosseland mean of ϵ_{ν_0} , which demonstrates that the lines can reduce the mean free path by a factor 2 for s of order 1000, and a factor 4 or more for s of order 100. (The effect is inversely related to s .)

The conclusion to be drawn from this work is that a person who is contemplating doing a diffusion calculation of a moving medium should have a look at the value of s for the problem. If this is significantly smaller than the limit c/v_{th} , then the VMD correction factor $1 - \epsilon_{\nu_0}$ should be applied. This can be in the form of a Rosseland average for the whole spectrum (a result given by Karp et al.) or separate Rosseland averages for individual coarse frequency groups. Of course, if one chooses to use so many frequency groups that all the lines can be resolved, the CMF approach is applicable. The use of diffusion coefficients based on the VMD approximation requires one further comment. Since all the quantities entering equation (69) are angle-dependent, the effective mean free path is also, which means that the diffusion is anisotropic and is described by a diffusion coefficient tensor. The easiest way of obtaining this tensor is by a low-order angle quadrature of the (probably tabular) angle-dependent mean free path. The quadrature procedure will require special care in the event that the three principal values of the strain tensor include both positive and negative values.

5. THIRD LECTURE -- DYNAMIC DIFFUSION PROBLEMS

A second class of radiation hydrodynamics problem that requires a full two-fluid treatment is of problems for which the continuum is optically thick but effectively thin, and such that advection of radiation competes with radiation diffusion. The distinction between optical and effective thickness arises if the continuous opacity is dominated by scattering. Then being optically thick means that the size of the region is many scattering mean free paths, while being effectively thin means

that the size of the region is less than the distance a photon would diffuse by scattering before suffering an absorption or other thermalizing collision. In this kind of problem the lines are often saturated, as discussed above, to the degree that their role is minor in energy and momentum coupling to the matter. It is the effective thinness of these problems which prevents assuming equilibration of radiation and matter, as in the stellar interior regime. However, the optical thickness of the problem allows using the Eddington approximation, so equations (27) and (28), closed by neglecting $\partial \underline{E}_V / \partial t$ and replacing $\underline{E}_V$ by $E_V/3$, give an adequate description.

In these scattering-dominated problems the coupling of matter and radiation is relatively weak, and is provided by absorption, such as inverse bremsstrahlung, and by the Compton effect. For the latter, the energy exchanges between photons and electrons can be described using a Fokker-Planck approximation, which leads to the Kompaneets (1957) equation. If these are taken for the coupling terms in equation (27), we find the following dynamical equation for the radiation field:

$$\begin{aligned} \frac{\partial E_V}{\partial t} + \left[\underline{\nabla} \cdot (\underline{v} E_V) - \frac{1}{3} \underline{\nabla} \cdot \underline{v} \nu \frac{\partial E_V}{\partial \nu} \right] \\ - \underline{\nabla} \cdot \left(\frac{\lambda_V c}{3} \underline{\nabla} E_V \right) = \kappa_{abs}(\nu) \rho (4\pi B_V - c E_V) \\ + \kappa_{es} \rho c \frac{kT}{mc^2} \nu \frac{\partial}{\partial \nu} \left[\nu \frac{\partial E_V}{\partial \nu} + \left(\frac{h\nu}{kT} - 3 \right) E_V \right. \\ \left. + \frac{c^3}{8\pi \nu^2 kT} E_V^2 \right] \quad (70) \end{aligned}$$

The terms in equation (70) have been divided so that the ones on the left side of the equation represent transport in the most general sense, while the ones on the right describe coupling to the matter. With the exception of the Doppler term (with the frequency derivative), the operators on the left are independent of frequency in the large part of the spectrum where Compton scattering is the dominant opacity. We may think of this left-side operator as governing the dependence on space and time of the field of photons, created through the action of a source term on the right side, that have suffered a variable number N of scatterings. This part of the problem is geometry dependent, and the important processes are advection and diffusion of the photons. But this description does not require any knowledge of the frequency spectrum of the photons. Conversely, the right

side operator governs the evolution of the spectrum of the photons as a function of the number of scatterings that have occurred. This description does not depend on the geometry except insofar as the temperature, etc., are inhomogeneous. (The Doppler term, though part of the transport operator, must be included with the right-side terms in this dichotomy.)

Whether the problem in question is one of temporal evolution, spatial diffusion or spatial advection is established by the competition of the left-side terms. For each of these cases there will be the appropriate distribution function of the number of scatterings for the photons seen by an observer at the selected observation point. The terms on the right side that compete are the absorption and Comptonization terms, and the comparison of these introduces a transitional frequency where the two effects are comparable. For a particular number of scatterings, the photon spectrum will be equilibrated to a Bose or Planck distribution to a degree that depends on the frequency as well as the number of scatterings. Folding the distribution of the number of scatterings, determined by the left-side operator, with the spectrum vs. number of scatterings determined by the right-side operator, gives the observed spectrum.

5.1. Comptonization

The Comptonization problem is the problem of the right side of equation (70). It has been discussed thoroughly by Chapline and Stevens (1973) and Rybicki and Lightman (1979) and in several other references cited by Rybicki and Lightman. A recent refined analytic treatment is the one by Caflisch and Levermore (1983). Since this subject can become rather involved, I will describe only the main features here.

The strength of the Comptonization operator is measured by the so-called Compton parameter y defined by

$$y = \frac{4kT}{mc^2} N. \quad (71)$$

When $y \ll 1$, Comptonization is unimportant with a bremsstrahlung source spectrum, and the spectrum evolved through N scatterings is simply described by saying that it equals the equilibrium Planck spectrum at frequencies below ν_{cr} defined by

$$\frac{\kappa_{es}}{\kappa_{abs}(\nu_{cr})} = N, \quad (72)$$

and equals the accumulated source of photons at frequencies above ν_{cr} . When $y = 1$, ν_{cr} coincides with the frequency ν_{coh} (cf., Rybicki and Lightman) defined by

$$\frac{\kappa_{es}}{\kappa_{abs}(\nu_{coh})} = \frac{mc^2}{4kT} \quad (73)$$

ν_{coh} is the dividing frequency for the importance of absorption vs. Comptonization. At frequencies below ν_{coh} absorption dominates, and at frequencies above ν_{coh} Comptonization dominates. So when $y \gg 1$, the photon spectrum is equilibrated at frequencies below ν_{coh} . Above ν_{coh} the spectrum is more complicated. If N is not sufficiently large for the entire spectrum to be equilibrated, i.e., $N < \kappa_{es}/\kappa_{abs}(4kT/h)$, then the spectrum above ν_{coh} consists of a rather flat energy distribution on which is superimposed a Wien peak. The amplitude of the Wien peak is reduced from the blackbody peak by about the factor $N\kappa_{abs}(4kT/h)/\kappa_{es}$. For still larger values of N the entire spectrum is blackbody.

If the advection term on the left side is dominant, then the effect of the Doppler term on the evolution of the spectrum should also be considered. For large y the Doppler term cannot compete with the Comptonization term, and so has little effect. For small y the effect of the Doppler term is that for frequencies above ν_{cr} the spectrum is the accumulated photon source Doppler shifted by the amount corresponding to the bulk strain experienced by the matter between the points of photon creation and observation.

5.2. Static vs. dynamic diffusion

The three terms on the left side of equation (70) apart from the Doppler term comprise an advection-diffusion operator. The question that immediately arises is whether advection or diffusion dominates. Assuming that Compton scattering is the dominant opacity, that question is answered as follows:

$$\text{diffusion (static diffusion)} \quad \tau_{es} \frac{v}{c} \ll 1 \quad (74)$$

$$\text{advection (dynamic diffusion)} \quad \tau_{es} \frac{v}{c} \gg 1, \quad (75)$$

where τ_{es} is the typical Compton optical thickness of the system.

The radiation field that evolves according to equation (70) has an initial transient that lasts a time

$$t_{diff} = \frac{\tau_{es}^2}{\kappa_{es} \rho c} \quad (76)$$

in the static diffusion regime, and a time

$$t_{\text{flow}} = \frac{\tau_{\text{es}}}{\kappa_{\text{es}} \rho v} \quad (77)$$

in the dynamic diffusion regime, and in general the smaller of the two times. Apart from the transient behavior, the temporal evolution proceeds on the time scale t_{evol} of the source and boundary data. The comparison of the three time scales leads to the three regimes

$$\begin{array}{ll} \text{temporal evolution (infinite medium)} & t_{\text{evol}} \ll t_{\text{flow}}, t_{\text{diff}} \\ \text{diffusion (static)} & t_{\text{diff}} \ll t_{\text{evol}}, t_{\text{flow}} \\ \text{advection (static)} & t_{\text{flow}} \ll t_{\text{evol}}, t_{\text{diff}} \end{array}$$

The representative value of N for each of these cases is just the number of scatterings that occur in the appropriate length of time:

$$\begin{array}{ll} \text{temporal evolution} & N = \kappa_{\text{es}} \rho c t_{\text{evol}} \\ \text{diffusion} & N = \kappa_{\text{es}} \rho c t_{\text{diff}} = \tau_{\text{es}}^2 \\ \text{advection} & N = \kappa_{\text{es}} \rho c t_{\text{flow}} = \tau_{\text{es}} \frac{c}{v} \end{array} \quad (78)$$

These are the values of N to use in equations (71) and (72) to estimate the observed spectrum. The spectrum in fact depends not only on the typical N but also on the distribution of N . For example, in the diffusion case the probability distribution is of the form $p(N) \sim N^{-1/2}$ up to the typical N listed above. This power-law behavior leads to the modified blackbody spectrum $F_{\nu} \sim (\kappa_{\text{abs}}(\nu))^{1/2} B_{\nu}$ discussed by Rybicki and Lightman. This power-law part of the distribution function of N is absent for the temporal and dynamic cases.

The production of the observed spectrum can be summarized by saying that the spectrum can be calculated in a geometry- and kinematics-independent way in terms of N (except for the effect of the Doppler term in a limited domain), as described in the references. The geometry and kinematics enter in finding the appropriate distribution of N , which is not a difficult problem in general.

For some problems the mechanical effect of radiation dominates that of the matter, i.e., $B_0 \ll v/c$, and the dynamical part of the radiation hydrodynamics problem can be divorced from

the question of the formation of the spectrum. This is true to a greater or lesser degree of the two examples that I will discuss next.

5.3. First example -- supercritical wind

Circumstances exist in which a more or less spherical star has a luminosity that exceeds the Eddington limit $L_{ED} = 4\pi G M c / \kappa_{es}$. These include, for example, a compact object that is accreting matter from a disk at a rate such that the release of gravitational energy exceeds L_{ED} . Another example is an X-ray burst source, a similar compact object that undergoes a thermonuclear outburst after a critical amount of nuclear fuel has accumulated on its surface. It is impossible for a steady structure to exist for such stars in which this super-Eddington luminosity is removed by radiation alone, since the super-Eddington flux would produce a force on the matter in excess of gravity. If there is a steady structure with a total energy loss $L_T > L_{ED}$ then it must incorporate a wind so that the sum of the wind energy loss and the sub-Eddington radiative energy loss equals L_T . (I am putting aside the possibility of energy transport by other mechanisms such as convection and waves.) For objects that exist only a short time in the high-power state, such as the X-ray burst sources alluded to above, a wind may or may not be produced, depending on the comparison of the lifetime of the high-power state with the energy storage time scale for the portion of the star through which the large luminosity must flow. But for the present purpose I will assume that the lifetime of the high-power state is indeed very long, and that a wind results.

The discussion of supercritical winds by Meier (1982a) provides a nice illustration of the ideas of dynamic vs. static diffusion and of decoupling between matter and radiation, so I will use this paper for my example.

The model to be considered is a compact star into a certain outer layer of which energy is injected at a rate L_T by either release of gravitational energy of an accreting wind (with subsequent spreading over the whole surface) or by thermonuclear burning; it is not critical which energy source applies. The hypothesis is that in and near the region of the energy injection the radiative flux in the comoving frame rises to a value close to L_{ED} . The excess of L_T above L_{ED} is carried by the wind, so

$$L_T - L_{ED} \approx \dot{M} \left[\frac{v^2}{2} - \frac{GM}{r} + E_T + \frac{P_T}{\rho} \right], \quad (79)$$

where E_T and P_T are the total energy per unit mass and the total pressure. Another assumption is that the radiative energy and pressure are overwhelming, so E_T can be replaced by E/ρ and P_T by P , where E and $P \approx E/3$ are the radiation energy density and pressure. A consequence of the dominance of radiation is that the net energy exchange between matter and radiation must be small, since the matter is unable to retain an appreciable amount of energy. Thus the radiation energy density obeys the sourceless equation

$$\underline{v} \cdot \underline{\nabla} E + \frac{4}{3} (\underline{v} \cdot \underline{v}) E - \underline{v} \cdot \left(\frac{c}{3\kappa_{es}\rho} \underline{\nabla} E \right) = 0, \quad (80)$$

and the only other equations needed to describe the problem are mass conservation

$$4\pi\rho v r^2 = \dot{M} \quad (81)$$

and the material momentum equation

$$\underline{v} \cdot \underline{\nabla} v + \frac{1}{3\rho} \underline{\nabla} E = \underline{g}. \quad (82)$$

The fact that energy exchange with the matter is negligible in the radiation (not matter) energy balance allows us to put aside the question of the radiation spectrum in considering the dynamics.

The first two terms in equation (80) are the same order, namely, the order of the advection operator, while the third term is the diffusion operator. The relative magnitudes of these terms gives the comparisons of equations (74) and (75), with τ_{es} estimated as $\kappa_{es} \rho r$. We notice that since $\rho v r^2$ is a constant, from equation (81), the quantity $\tau_{es} v / c$ decreases with r . Thus the dynamic diffusion regime exists at small r , while the static diffusion regime exists at large r . The crossover radius is called by Meier the trapping radius, r_t . (Trapping is another name for the situation in which the radiation is advected with the flow faster than it diffuses.) Interior to the trapping radius the first two terms of equation (80) dominate the third, therefore they must cancel each other, and the radiation obeys the $\gamma = 4/3$ adiabatic law

$$E \propto \rho^{4/3}. \quad (83)$$

Exterior to the trapping radius the third term should dominate. This can occur only if the flux is nearly divergence-free,

$$\underline{v} \cdot \underline{F} = - \underline{v} \cdot \left(\frac{c}{3\kappa_{es}\rho} \underline{\nabla} E \right) \approx 0, \quad (84)$$

which is the flux-freezing condition. But Meier notes that the momentum equation itself ensures the vanishing of the flux divergence if the velocity is very much subsonic, in which case the first two terms of equation (80) must balance and equation (83) is still valid. Thus the adiabatic radius, defined as the outermost radius where equation (83) applies, cannot be inside the trapping radius, but it can be outside the trapping radius if the latter occurs in the subsonic region.

The density of the wind determines the ordering of the sonic and trapping radii, and the actual photosphere or radius of last scattering, r_{sc} defined by $\kappa_{es}\rho r_{sc} = 1$. The various orderings give the cases called A, B, C1, C2, and D by Meier. The densest wind is for the largest ratio of L_T to L_{ED} , and this is Meier's case D. This is the case for which the trapping radius is in the supersonic region, and therefore the adiabatic radius coincides with the trapping radius.

For the whole structure of the wind out to and beyond the sonic radius in Meier's case D, the adiabatic relation of E to ρ may be used, which turns equations (81) and (82) into Parker's standard adiabatic wind model, with $\gamma = 4/3$. All the details of the wind are determined once two parameters are given, which can be taken to be the total wind energy flux, fixed by equation (79), and the constant radiation entropy per unit mass. The further condition that the wind must be in pressure balance with the compact star's envelope across the injection region fixes the entropy.

The structure of the wind derived from Parker's model is the following for the subsonic region:

$$\begin{aligned} v &\propto r \\ r < r_s & \quad E \propto r^{-4} \\ L &= L_{ED} \end{aligned} \quad (85)$$

(L is the comoving-frame radiative luminosity), and for the supersonic region:

$$\begin{aligned} v &= \text{constant} \\ r_s < r < r_{ad} & \quad E \propto r^{-8/3} \\ L &\propto r^{1/3} \end{aligned} \quad (86)$$

For the structure of wind outside the adiabatic radius, we see that the velocity remains constant, while E changes from the

variation given by equation (83) to that given by equation (84). Thus L remains constant at the value it has at the adiabatic radius, and $E(r)$ adjusts itself accordingly. The structure is

$$\begin{aligned} v &= \text{constant} \\ r_{\text{ad}} < r < r_{\text{sc}} & \quad E \propto r^{-3} \quad (87) \\ L &= \text{constant} \end{aligned}$$

between the adiabatic radius and the photosphere, and

$$\begin{aligned} v &= \text{constant} \\ r > r_{\text{sc}} & \quad E \propto r^{-2} \quad (88) \\ L &= \text{constant} \end{aligned}$$

exterior to the photosphere.

One interesting feature of the energetics of this model is that the dominant energy loss mechanism changes from the advection of radiation at the base of the wind to kinetic energy far out in the wind, and at no point is the comoving-frame radiative luminosity dominant. Another feature is that the emergent radiative luminosity is in fact larger than L_{ED} . The factor by which it is larger, from equation (86), is the cube root of $r_{\text{ad}}/r_{\text{S}}$. The ratio of the final kinetic energy loss to L_{ED} is just $r_{\text{ad}}/r_{\text{S}}$, so a super-Eddington luminosity can be obtained from a supercritical wind provided the mechanical power is even larger, as the cube (Rees 1977).

Obtaining the emergent spectrum from the wind is a good bit more complicated than finding the dynamical structure (Meier 1982b). In rough outline the spectrum is that described by Rybicki and Lightman and others, with regions of Planck shape, modified Planck shape, and a Wien peak normalized in this case to the known value of the emergent flux. What is difficult is that the relevant temperature for the different parts of the spectrum should be the material temperature at some appropriate depth. (Perhaps the depth where N , the number of scatterings to escape, is $\kappa_{\text{es}}/\kappa_{\text{abs}}(v)$ for the Planck and modified Planck regions, and $mc^2/4kT$ for the Wien peak.) The run of material temperature can only be found by solving for it self-consistently with the run of the radiation spectrum; a discussion of this is too involved to give here, and the reader is referred to Meier (1982b).

5.4. Second example -- cyclotron line emission in X-ray pulsars

The mechanism by which X-ray pulsars produce their radiation has been investigated intensively, and several competing models

If there is no loss of line radiation whatever, then the structure of the column follows equations (82) and (83) discussed in connection with the supercritical wind. Since the flow is subsonic, the structure approximates that of an $n = 3$ polytropic static envelope model. Of course, this would fail to properly match the conditions at the surface of the neutron star. The height of the shock above the neutron star surface must be sufficient for the line radiation to have time to escape, which produces a much more rapid increase of density with decreasing radius as required to match the neutron star. Thus the steady-state height of the shock is determined by the time necessary before a sufficient number of non-resonant scatterings has occurred to lead to the escape of the photon.

The gravitational energy of the accreting material is finally lost to the system as radiation from the side of the column (or possibly through its hollow core) near its base, as the photons that have been non-resonantly scattered escape from the line. The spectral distribution that would be expected is roughly that of an emission line with perhaps a central dip at the resonant frequency. At this time it is not clear whether this spectrum is consistent with that observed for X-ray sources such as Her X-1 in which line features have been seen. Another question that must be considered is whether this model gives a fair representation of the pulse profile.

6. ACKNOWLEDGMENT

This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48.

7. REFERENCES

- Abbott, D.C. 1977, unpublished Ph. D. thesis, University of Colorado.
- Abbott, D.C. 1980, *Astrophys. J.*, 242, p. 1183.
- Abbott, D.C. 1982, *Astrophys. J.*, 259, p. 282.
- Buchler, J.-R. 1982, Los Alamos National Laboratory report, No. LA-UR-82-1336.
- Caflisch, R.E. and Levermore, C.D. 1983, in preparation.
- Cannon, C.J. 1973a, *J. Quant. Spectr. Rad. Transf.*, 13, p. 627.
- Cannon, C.J. 1973b, *Astrophys. J.*, 185, p. 621.
- Castor, J.I. 1970, *Mon. Not. Roy. Astr. Soc.*, 149, p. 111.
- Castor, J.I. 1974, *Mon. Not. Roy. Astr. Soc.*, 169, p. 279.
- Castor, J.I., Abbott, D.C. and Klein, R.I. 1975, *Astrophys. J.*, 195, p. 157 (CAK).
- Chapline, F.G. and Stevens, J. 1973, *Astrophys. J.*, 184, p. 1041.

- Feautrier, P. 1964, C.R. Acad. Sci. Paris, 258, p. 3189.
- Karp, A.H., Lasher, G., Chan, K.L. and Salpeter, E.E. 1977, *Astrophys. J.*, 214, p. 161.
- Klein, R.I., Arons, J. and Lea, S.M. 1983, presented at the Santa Cruz workshop on High Energy Transients.
- Kompaneets, A.S. 1957, *Sov. Phys. JETP*, 4, p. 730.
- Langer, S.H. and Rappaport, S. 1982, *Astrophys. J.*, 257, p. 733.
- Lucy, L.B. and White, R.L. 1980, *Astrophys. J.*, 241, p. 300.
- MacGregor, K.B., Hartman, L. and Raymond, J.C. 1979, *Astrophys. J.*, 231, p. 514.
- Meier, D.L. 1982a, *Astrophys. J.*, 256, p. 681.
- Meier, D.L. 1982b, *Astrophys. J.*, 256, p. 693.
- Mihalas, D. 1978, "Stellar Atmospheres", second edition (San Francisco: Freeman).
- Mihalas, D., Kunasz, P.B. and Hummer, D.G. 1975, *Astrophys. J.*, 202, p. 465.
- Mihalas, D., Kunasz, P.B. and Hummer, D.G. 1976, *Astrophys. J.*, 210, p. 419.
- Mihalas, D. and Weaver, R. 1982, Los Alamos National Laboratory report No. LA-UR-82-606.
- Noerdlinger, P. and Rybicki, G. 1974, *Astrophys. J.*, 193, p. 651.
- Owocki, S.P. and Rybicki, G. 1982, *Bull. Amer. Astr. Soc.*, 14, p. 920.
- Parker, E.N. 1958, *Astrophys. J.*, 128, p. 664.
- Rees, M.J. 1977, presented at the NATO Advanced Study Institute "Active Galactic Nuclei", Cambridge, England.
- Rybicki, G.B. 1971, *J. Quant. Spectr. Rad. Transf.*, 11, p. 589.
- Rybicki, G.B. and Lightman, A.P. 1979, "Radiative Processes in Astrophysics" (New York: Wiley-Interscience).
- Scharmer, G.B. 1981, *Astrophys. J.*, 249, p. 720.
- Sobolev, V.V. 1947, English edition 1960, "Moving Envelopes of Stars" (Cambridge, Mass.: Harvard University Press).
- Ventura, J. 1979, *Phys. Rev.*, D19, p. 1684.
- Weber, S.V. 1981, *Astrophys. J.*, 243, p. 954.

DISCLAIMER

This document was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor the University of California nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial products, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government thereof, and shall not be used for advertising or product endorsement purposes.