TITLE: RENORMALIZING THE STRONG-COUPLING EXPANSION FOR QUANTUM
FIELD THEORY: PRESENT STATUS

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RENORMALIZING THE STRONG-COUPLED EXPANSION
FOR QUANTUM FIELD THEORY: PRESENT STATUS
(Presented by Fred Cooper)

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1. INTRODUCTION

In this talk we review the progress we have made in determining the re-normalized strong coupling expansion in quantum field theory. We restrict ourselves here to $\lambda\phi^4$ field theory in d dimensions. Our starting point is the lattice version of the path integral representation for the Green's functions.

We first determine the unrenormalized strong coupling expansion on the lattice. This is a series in the dimensionless parameter $x = \frac{M_0^2 a^2}{\sqrt{\lambda a^{4-d}}}$, where a is the lattice spacing and M_0 is the bare mass. We derive a simple set of graphical rules for determining the Green's functions of the theory in a power series in x . For finite field theories ($d < 2$) we present a scheme for extrapolating to zero lattice spacing ($a \rightarrow 0$). This scheme determines a sequence of approximations to the unrenormalized strong coupling series in the form of a series in $\frac{M_0^2}{\sqrt{2/(4-d)}}$.

We next consider the problem of renormalization. For the lattice field theory this consists of eliminating x in favor of $y = \frac{M_R^2 a^2}{\sqrt{\lambda a^{4-d}}}$, where M_R is the renormalized mass. The region of small y is also the critical region, since λ large is equivalent to M_R small. For $d < 2$, $M_R^2 = 0$ occurs when $M_0^2 \rightarrow \infty$, and thus the path integral is dominated by instantons. Thus, although the unrenormalized strong coupling expansion is valid for the single well, $M_0^2 \ll \lambda^{2/(4-d)}$, the renormalized strong coupling expansion is valid for the double well where $M_R^2 \ll \lambda^{2/(4-d)}$. We obtain the renormalized strong coupling expansion via two methods. One is to extrapolate the renormalized lattice series into the critical regime and then take the continuum limit. The second is to exactly determine the renormalized continuum strong coupling expansion using

instantons for $d < 2$. We find our extrapolation procedure for obtaining the continuum limit of the renormalized lattice field theory works well in zero, one and two dimensions when compared with the exact instanton results.

11. DERIVATION OF THE LATTICE STRONG COUPLING EXPANSION.

The vacuum persistence functional for d dimensional $\lambda\phi^4$ field theory in the presence of an external source $J(x)$ can be expressed in Euclidean space as a functional integral

$$Z = \langle 0_+ | 0_- \rangle^J = N \int [D\phi] \times \exp \left[- \int_{-\infty}^{\infty} d^d x \left[1/2 (\dot{\phi})^2 + \frac{M_0^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 - J\phi \right] \right] \quad (2.1)$$

The definition of the Path integral is as follows: on the lattice

$$\begin{aligned} \dot{\phi} &= \frac{\phi_n - \phi_{n-1}}{a} \\ \int d^d x &= a^d \sum_n \quad , \quad \phi(x) = \phi(n) \\ \phi_1(x) &= \phi\left(n_1 + \frac{1}{a}\right) = \phi(\dot{n}) \\ \int [D\phi] &= \lim_{\substack{a \rightarrow 0 \\ n \rightarrow \infty, na = x}} \frac{1}{n} \int d\dot{n} K(\dot{n}) \end{aligned} \quad (2.2)$$

Thus on the lattice

$$\begin{aligned} Z[J] &= \frac{1}{N} \int dK(1) \exp \left[a^{2d} \sum_{n,m} \phi(n) G_0^{-1}(n,m) \phi(m) \right] \\ &\times \exp \left[-a^d \left(\frac{\lambda}{4} \phi(1)^4 + \frac{M_0^2}{2} \phi^2(1) - J(1)\phi(1) \right) \right] \end{aligned}$$

where

$$G_0^{-1}(n,m) = \sum_{\mu=1}^d \frac{(\delta_{n,m+\hat{\mu}} + \delta_{m,n+\hat{\mu}}) - 2\delta_{nm}}{a^{d+2}} \quad (2.3)$$

is the second difference operator. The lattice strong coupling expansion is derived by first treating the kinetic energy as a perturbation and noticing that what remains is a product of ordinary integrals due to the local nature of the interaction.¹⁻⁷ That is

$$Z[J] = \exp \left[\sum_{n,m} J(n) G_0^{-1}(nm) \frac{\partial}{\partial J(m)} \right] \tau_1 F[J(1)] \quad (2.4)$$

where

$$\begin{aligned} F(x) &= \int_{-\infty}^{\infty} \exp \left[-a^d \left(\frac{\lambda}{4} x^4 + \frac{M_0^2}{2} x^2 - vx \right) \right] dx \\ &= \sum \Lambda_{2n} \frac{x^{2n}}{2n!} \end{aligned}$$

and

$$\Lambda_{2n} = \int_{-\infty}^{\infty} \exp \left[-a^d \left(\frac{\lambda}{4} x^4 + \frac{M_0^2}{2} x^2 \right) \right] x^{2n} (a^d)^{2n} \quad (2.5)$$

To obtain the lattice strong coupling expansion⁵ we expand eq.2.5 assuming $M_0^2/a^2 \ll \sqrt{\lambda a^{4-d}}$ and obtain⁸

$$\Lambda_{2n} = \left(\frac{\lambda}{a^d} \right)^{n/2} \sum \left(\frac{-M_0^2/a^2}{\sqrt{\lambda a^{4-d}}} \right)^k \frac{\Gamma(n/2 + \frac{1}{4} + k/2)}{k!} \quad (2.6)$$

The theory at zero internal and external moment give the vertices of the theory. Calling

$$Z_0[J] = \pi \int \mathcal{F}[J(i)] = \exp \sum_i \epsilon_n \mathcal{F}[J(i)]$$

$$\epsilon_n Z_0[J] = \sum_i \epsilon_n \left(\sum_n \frac{\Lambda_{2n}}{\Lambda_0} \frac{J^{2n}(i)}{2n!} \right) = \sum_i \sum_n \frac{L_{2n}}{2n!} J^{2n}(i) \quad (2.7)$$

The L_{2n} are the vertices of the theory: at each point there is the possibility of exciting $2n$ bosons with strength $\left(\frac{1}{\Lambda_d^{4-d}} \right)^{n/2}$. The L_{2n} have the following structure

$$L_{2n} = \left(\frac{1}{\Lambda_d^{4-d}} \right)^{n/2} \sum_{k=0}^n h_{2n}^{(k)} \left(\frac{K_0^2 a^2}{\sqrt{\Lambda_d^{4-d}}} \right)^k \quad (2.8)$$

The diagrammatic expansion of any Green's function $G(x_1, x_2, \dots, x_n)$ in $\left(\frac{1}{\Lambda_d^{4-d}} \right)^{n/2}$ is obtained by applying

$$\exp \left(\sum_i \frac{\partial}{\partial J(i)} G_{0nm}^{-1} \frac{\partial}{\partial J(m)} \right) \frac{\partial}{\partial J(1)} \cdots \frac{\partial}{\partial J(n)} \quad (2.9)$$

to $Z_0[J]$.

Since G_{0nm}^{-1} connects nearest neighbors on the lattice the strong coupling expansion is also an expansion in how many lattice sites are interacting.

Since the L_{2n} are power series in $\epsilon = \frac{1}{\sqrt{\Lambda_d^{4-d}}}$ starting at ϵ^n one sees that in order ϵ^n there are only a finite number of terms contributing to $\epsilon_n Z[J]$ in that order. In particular the contribution to the $2k$ point function $G(x_1, \dots, x_{2k})$ at order ϵ^n will contain exactly $n-k$ internal lines

corresponding to factors of G_0^{-1} connecting vertices L_{2m} with $m \leq n$.

For example the first few terms contributing to

$$G_2(1j) = \int \mathcal{D}[\phi] \phi(1) \phi(j) e^{-S[\phi]} \quad , \quad (2.10)$$

are shown diagrammatically in figure one.

In figure one the solid lines represent G_0^{-1} . The series for all the G_{2k} can be organized in terms of diagrams which are one particle irreducible with respect to G_0^{-1} . Call Λ_{2n} the n point function which is one particle irreducible with respect to G_0^{-1} . The basic graph-lets up to 4 G_0^{-1} entering in Λ_{2n} are shown in figure two as well as their evaluation at zero external momenta. One can write

$$\begin{aligned} G_2 &= \Lambda_2 + \Lambda_2 G_0^{-1} \Lambda_2 + \Lambda_2 G_0^{-1} \Lambda_2 + \dots \\ &= (-G_0^{-1} + \Lambda_2^{-1})^{-1} \quad , \end{aligned} \quad (2.11)$$

Fourier transforming G_2

$$\begin{aligned} W_2(P) &= a^d \sum \exp[-i\vec{P} \cdot a(\vec{\omega} - \vec{k})] \cdot G_2(\vec{m}, \vec{k}) \\ W_2^{-1}(P) &= \epsilon_E^2 + 1/\Lambda(P_E^2) \end{aligned} \quad (2.12)$$

where $\Lambda_2^{(?) = L_2 - L_4 A/2 + L_4 L_2 B + L_6 A^2 + L_4 L_4 C + \dots$

and the A, B, C, etc. are the graphlets of fig.(2).

Since the particle spectra are the zeros of the real part of the Inverse Green's functions in Minkowski space and $-G_0^{-1} = p_E^2 = -p_M^2$, the particle spectra is the solution of

$$-p_M^2 + \frac{1}{\Lambda(-p_M^2)} = 0 \quad (2.13)$$

For the first zero in G_2^{-1} , which is the renormalized mass we get a series of the form

$$M_R^2 a^2 \epsilon = 1/2R + \sum_{k,l}^{k+l=n} a_{kl}(d) (M_0^2 a^2 \epsilon)^k (\epsilon)^l \quad (2.14)$$

where $R = \frac{\Gamma(3/4)}{\Gamma(1/4)} \approx .338$ and $a_{kl}(d)$ depend only on the dimension of space time.

Similarly by looking at graphs one can show that

$$G_4 = \Lambda_4 (1 + G_0^{-1} G_2)^4 = \Lambda_4 (\Lambda_2^{-1})^4 G_2^4 \quad (2.15)$$

thus the 1PI in G_2 4 point function

$$\Gamma_4 = G_4 (G_2^{-1})^4 = \Lambda_4 (\Lambda_2^{-1})^4, \text{ where } \Lambda_4 \text{ has the structure} \quad (2.16)$$

$$\Lambda_4(0) = L_4 + L_6 \Lambda + L_8 \Lambda^2 + L_4'' B + L_6 L_2 B + \dots$$

The renormalized coupling constant, $-\lambda_R = Z_3^2 \Gamma^4(p=0)$ where

$$Z_3 = \left. \frac{dG_2^{-1}(p^2)}{dp^2} \right|_{p^2=0} \quad (2.17)$$

We find for the dimensionless renormalized coupling constant the following strong coupling series on the lattice

$$\frac{\lambda_R}{M_R^{4-d}} \epsilon^{\frac{d}{2}} = \frac{12R^2-1}{6R^4} + \sum_{k, \ell} b_{k\ell}(d) (M_0^2 a^2 \epsilon)^k \epsilon^\ell \quad (2.18)$$

Similarly one can calculate in terms of the graphlets of figure 2 all the coefficients V_n in the expansion of the potential about $\phi=0$.

$$V[\phi] = \sum \frac{V_{2n}}{2n!} \phi^{2n}$$

The V_n which are the scattering amplitudes at zero external momentum are easily calculated in terms of $V_{2n}(0)$ which are constructed from the graphlets of figure 2.

We notice that in the series eq. 2.14 and 2.18 $\epsilon \rightarrow \infty$ as $a \rightarrow 0$ and thus every term in these series diverges when the lattice spacing a goes to zero.

III. EXTRAPOLATING TO ZERO LATTICE SPACING

Even in a finite theory such as $d=1$ dimensional field theory (Quantum Mechanics) the lattice strong coupling series for finite quantities such as M_R or λ_R become infinite term by term as $a \rightarrow 0$ (or $c \rightarrow \infty$). Thus we must be able to extrapolate such truncated series in order to go to the continuum after having calculated only a finite number of orders. To see the nature of the series for say the renormalized mass we replace a^{-2} by $(\lambda \epsilon^2)^{2/(4-d)}$ to obtain from eq. 2.14.

$$\frac{M_R^2}{\lambda^{2/(4-d)}} = f_1(c) + f_2(c) \frac{M_0^2}{\lambda^{2/(4-d)}} + f_3(c) \left(\frac{M_0^2}{\lambda^{2/(4-d)}} \right)^2 + \dots \quad (3.1)$$

The $f_n(c)$ have the structure in M 'th order perturbation theory

$$f_n^M(c) = c^a \sum_{k=0}^M b_k^{(n)} c^k \quad (3.2)$$

For the anharmonic oscillator a canonical transformation yields

$$H(M_0^2, \lambda) = \lambda^{1/3} \bar{H}(M_0^2/\lambda^{2/3}, 1) \quad (3.3)$$

thus one expects for $d=1$

$$\lim_{c \rightarrow 0} \lim_{M \rightarrow \infty} f_n^M(c) = f_n(\infty) \quad , \quad (3.4)$$

where $f_n(\infty)$ is finite. Our problem is to take $c \rightarrow \infty$ when we have only a finite number of terms in the series. In general we want to study the series

$$Q(c) = c^a \sum_{n=0}^{\infty} a_n c^n \quad a \neq 0 \quad (3.5)$$

derived assuming ϵ is small. We assume $Q(\epsilon)$ has a finite limit as $\epsilon \rightarrow \infty$. If we truncate the series after the ϵ^N term, for small ϵ we can raise the expression

$$Q_N^{(\epsilon)} = \epsilon^\alpha \sum_{n=0}^N a_n \epsilon^n \quad , \quad (3.6)$$

to the N/α power and write the result in the form of a ratio

$$[Q_N^{(\epsilon)}]^{N/\alpha} = \frac{\epsilon^N}{\sum_{n=0}^N b_n \epsilon^n} = G_N(\epsilon) \quad , \quad (3.7)$$

where the b_n are determined by the a_n .

Now we can take the limit $\epsilon \rightarrow \infty$ and we define

$$Q_N^{(\infty)} = \left(\frac{1}{b_N} \right)^{\alpha/N} \quad , \quad (3.8)$$

as the N th approximant.

The $Q_N^{(\infty)}$ in many cases lead to a convergent sequence of approximants, however, in other cases the sequence is only asymptotic to the exact continuum answer. More work needs to be done to understand this question⁹.

For the anharmonic oscillator the series 3.1 at $\epsilon = \infty$ is known numerically to be¹⁰

$$-\frac{M_R}{\lambda^{1/3}} = 1.9308 + .3399 \frac{M_0^2}{\lambda^{2/3}} + \dots \quad (3.9)$$

The first three extrapolants Q_n to 1.080 are

$$Q_1 = 1.1194, Q_2 = 1.1021, Q_3 = 1.0973, \quad (3.10)$$

which appears monotonically to converge.

For the anharmonic oscillator we have calculated the most terms (12) in the series for the derivative of the ground state energy for $H_0^2 = 0$

$$4\lambda \frac{dE}{d\lambda} = .569473 \lambda^{1/3}. \quad (3.11)$$

Evaluating $Q_n(\epsilon)$ at $\epsilon = N^{3/2}$ via $Q_n(N^{3/2}) = [G_n(N^{3/2})]^{3/N}$ we get 12 apparently converging approximants.

$$\begin{aligned} &.477786, .535123, .548817, .554843, .558673, .561548, \\ &.563809, .565563, .566845, .567683, .568114, .568195. \end{aligned} \quad (3.12)$$

We have tested this method of taking the lattice spacing to zero most thoroughly in solving differential equation boundary layer problems¹¹. We first discretize the equation, perturb in the equivalent to the kinetic energy, and then take the lattice spacing to zero. The boundary layer problem relevant to $\lambda\phi^4$ field theory is the kink which is a solution of the one dimensional Euclidean classical field equation. Introducing the dimensionless time $u = \tilde{t}$ and rescaling the fields,

$$\phi(\tau) = \sqrt{\frac{M}{\lambda}} \tilde{\phi}(u\tau)$$

the classical field equation with b.c. $\tilde{\phi} = \begin{cases} +1 & \text{at } +\infty \\ 0 & \text{at } 0 \end{cases}$,

$$\frac{d^2 \tilde{\phi}}{d\tilde{\tau}^2} + \tilde{\phi} - \tilde{\phi}^3 = 0 \quad (3.13)$$

has the continuum solution

$$\tilde{\phi} = \tanh^{-1} \left(\frac{\tilde{\tau}}{\sqrt{2}} \right) \quad (3.14)$$

The lattice problem is $\phi(\tilde{\tau}) \rightarrow \phi(n)$

$$\epsilon [\phi(n+1) + \phi(n-1) - 2\phi(n)] + \phi(n) - \phi(n)^3 = 0 \quad (3.15)$$

where $\epsilon = 1/\Delta^2$.

Assuming $\phi(n) = \sum_{n=0}^{\infty} \phi_n(n)$

with $\phi_0(n) = \begin{cases} 1 & n \geq 1 \\ 0 & n = 0 \\ -1 & n \leq -1 \end{cases}$. Then

$$\phi_m(0) = \frac{\phi(1) - \phi(0)}{\epsilon} = \frac{1}{2} \sum_{n=1}^m b_n \epsilon^n \quad (3.16)$$

The exact value of the first derivative of eq. 3.14 at $\tau=0$ is $1/\sqrt{2} \approx .7071$. We have been able to calculate over 40 terms in the series for the derivative at the origin (3.16). In this case the extrapolants whether evaluated at $\epsilon = \infty$ or $\epsilon = R$ seem to be an asymptotic sequence. The first 10 are 1, .7409, .7819, .75724, .74076, .73121, .72393, .7190, .71515, .71211. Then the extrapolants overshoot, reach a minimum at $Q_{24} = .70198$ and then continue rising and at $R = 41$, one has $Q_{41} = .7080$.

We have tried to understand the divergence structure of the unrenormalized theory in the strong coupling regime for $d \geq 2$ by using this same extrapolation procedure. If we consider the derivative of the ground state energy density for $M_0^2 = 0$, then

$$4\lambda \frac{dE}{d\lambda} = \delta(0) - \int d^d y W_2(x, y) G_0^{-1}(y, x) = \lambda f(\epsilon) \quad .$$

For $d=2$, the sequence of Q_N involves taking fractional roots of negative numbers after $N=3$, and returns imaginary answers. Thus the extrapolants are clever enough to know that $f(\epsilon)$ does not have a finite limit as $\epsilon \rightarrow \infty$. In fact they seem to be able to determine the actual manner of divergence (here logarithmic) of $f(\infty)$.

If one takes the series for $4 \frac{dE}{d\lambda}$ and divide by the divergent series for $\ln(1 + \epsilon)$ one gets a seemingly convergent sequence of extrapolants when evaluated at $\epsilon_N = N$.¹² We find for $4(dE/d\lambda)/\ln(1 + \epsilon)$ that the $Q_N(\epsilon_N)$ are .3171, .3639, .3766, .3858, .3932, .3987, .4031, .4069, .4102, .4132, .4161, .4189.

Thus it seems that the extrapolants Q_N provide in many cases quantitatively good approximations to the correct lattice limit. In some cases the Q_N converge very slowly (solution of the diffusion equation) or are asymptotic (solution to the kink equation) and it is clear that more work must be done in understanding these extrapolations. What is comforting is that the Q_N are smart enough to know when one assumes the wrong physics such as an infinite quantity being finite.

IV. OBTAINING THE RENORMALIZED STRONG COUPLING EXPANSION [$d < 4$]

In finite theories the unrenormalized strong coupling expansion is a series in $\frac{g_0^2}{\Lambda^{4-d}}$ on the lattice and a series in $\frac{g_0^2}{\Lambda^{(4-d)}}$ in the continuum. Mass renormalization consists of eliminating the unphysical parameter m_0 , the bare mass parameter, in favor of m_K the position of the lowest pole in the 2 point Minkowski Green's function (the physical energy of the first excited state minus the ground state energy). For $d = 2$, mass renormalization is a necessity since physical quantities such as m_K are not finite functions of $\frac{g_0^2}{\Lambda^2}$ as the lattice spacing a goes to zero. What we are interested in is how physical dimensionless quantities behave for small values of $\frac{g_0^2}{\Lambda^{4-d}}$ on the lattice and for small values of $\frac{g_0^2}{\Lambda^{(4-d)}}$ in the continuum. It is clear that the regime of large Λ fixed m_K is equivalent to the regime fixed $\frac{g_0^2}{\Lambda^{4-d}} \rightarrow 0$, and it is thus the regime where on the lattice the correlation length $\xi = 1/m_K a$ is going to infinity. Thus one expects the possibility that certain quantities become singular as $\xi \rightarrow \infty$. Thus obtaining the renormalized lattice strong coupling expansion means we are interested in the lattice field theory near the critical regime ($\frac{g_0^2}{\Lambda^{4-d}} \rightarrow 1$). However, if we look at equation 2.14 we notice that where the unrenormalized strong coupling series is valid, $\frac{g_0^2}{\Lambda^{4-d}} \rightarrow 1$, $\frac{g_0^2}{\Lambda^{4-d}} \rightarrow 1$ is of order 1, and even in the continuum strong coupling series for m_K in the anharmonic oscillator (Eq. 1.9) we notice that $\frac{m_K}{m_0} \rightarrow 1$ for $\frac{g_0^2}{\Lambda^{4-d}} \rightarrow 1$. Thus the place where our unrenormalized strong coupling expansion is valid is far from the regime where the renormalized dimensionless parameters $\frac{g_0^2}{\Lambda^{4-d}}$ or $\frac{g_0^2}{\Lambda^{(4-d)}}$ are small. Thus we have two choices, one is to try to perform mass renormalization by inverting Eq. (2.14) to obtain a series of the form $(1 - 1/\Lambda^{4-d})$

$$M_0^2 a^2 = \sum_n l_n(c,d) (M_R^2 a^2 = \frac{1}{2R_0} + \dots)^n \quad (4.1)$$

and try to extrapolate the series in $(M_R^2 a^2 = \frac{1}{2R_0} + \dots)$ to the region $M_R^2 \rightarrow 0$ taking into account the possibility of singular behavior.

The second method is to try to obtain a different unrenormalized strong coupling expansion which is valid near the critical region. For example in the anharmonic oscillator we know that the regime where $\frac{E_0}{\hbar\omega} \rightarrow 0$ (λ fixed) is the regime where the first excited state and the ground state becomes degenerate. This clearly occurs in a double well with $M_0^2 \rightarrow -\infty$. The almost infinite double well potential has its physics determined by instantons, and for the lattice field theory one is tempted to make the change of variables $M_0^2 \rightarrow -\mu^2 \left(\frac{1}{\sqrt{2}} a^{4-d} \right)$ so that $M_0^2 \rightarrow -\infty$ with μ . In the new variable μ^2 a steepest descent calculation about the point $\phi(n) = \frac{1}{\sqrt{2}} a^{4-d}$ leads to a new lattice strong coupling expansion in the variable μ

$$\left(\frac{1}{\sqrt{2}} a^{4-d} \right)^{-1/d} = [\mu a]^{2-d} \quad (4.2)$$

For $d=2$ this new expansion has the property that $M_R \rightarrow 0$ as $\mu \rightarrow 0$ so that one is already near the critical regime when μ is large (μ fixed using the strong coupling expansion valid for the double well).

First let us discuss the method of extrapolation of the mass renormalization formula valid for $\frac{M_0^2 a^2}{\sqrt{2}} \rightarrow 1$. First look at the trivial case of field theory in zero dimensions (no kinetic energy) where one does not have to deal with taking the lattice spacing to zero limit. Defining

$$\frac{1}{M_R^2} = l_2(d=0) \quad , \quad \frac{\lambda_R}{M_R^4} = -l_4(d=0) \times M_R^4 \quad (4.3)$$

we obtain for the unrenormalized series for M_R^2 and λ_R

$$\frac{M_R^2}{\lambda} = \frac{1}{2R} + \frac{(4R^2-1)}{8R^2} \frac{M_0^2}{\lambda} - \frac{(8R^2-1)}{32R^2} \left(\frac{M_0^2}{\lambda} \right)^2 + \dots \quad (4.4)$$

$$R = 1.438 \quad \frac{1}{2R} = 1.4793$$

$$\frac{\lambda_R}{M_R} = .811 + .278 \frac{M_0^2}{\lambda} + .0349 \left(\frac{M_0^2}{\lambda} \right)^2 + \dots \quad (4.5)$$

where we have calculated 15 terms in the convergent power series in M_0^2/λ .

We notice $M_R^2/\lambda \approx 1.48$ when $M_0^2/\lambda = 0$. Here mass renormalization consists of inverting equation (4.4) to obtain

$$\frac{M_0^2}{\lambda} = -1.68 \left(\frac{M_R^2}{\lambda} - \frac{1}{2R} \right) + .112 \left(\frac{M_R^2}{\lambda} - \frac{1}{2R} \right)^2 + .177 \left(\frac{M_R^2}{\lambda} - \frac{1}{2R} \right)^3 + \dots \quad (4.6)$$

Inserting eq.(4.6) into eq.(4.5) we obtain the renormalized series

$$\frac{\lambda_R}{M_R} = .811 + .000 \left(\frac{M_R^2}{\lambda} - \frac{1}{2R} \right) + .192 \left(\frac{M_R^2}{\lambda} - \frac{1}{2R} \right)^2 + \dots \quad (4.7)$$

which is clearly valid for $M_R^2/\lambda < 1/(2R)$. If we evaluate the series for λ_R/M_R truncated at 8 terms at $M_R^2/\lambda = 0$, we get the following series or approximations to λ_R^* the dimensionless renormalized coupling constant at

$$\lambda = 0 : .811, 1.505, 1.925, 2.148, 2.260, 2.321, 2.352, 2.362, 2.360, \\ 2.351, 2.345, 2.331, 2.298, 2.2526, 2.217, 2.178 \quad (4.8)$$

These numbers reach a maximum at 2.362 and then steadily decrease. The maximum is within 1% of the exact answer for $\lambda_R^* = 2$ which we will derive by the double well method below. Thus direct evaluation of (4.7) at $M_R^2=0$

leads to a series quite similar to that for the kink, (i.e., asymptotic)

We can also try to estimate the way in which $M_0^2 \rightarrow -\infty$ as $M_R^2 \rightarrow 0$ ($\xi \rightarrow \infty$). To do this we assume a power law singularity¹³

$$\lim_{M_R^2 \rightarrow 0} M_0^2 \sim -\Lambda [M_R^2]^{-\gamma} \quad (4.9)$$

so that

$$\lim_{M_R^2 \rightarrow 0} M_R^2 \frac{d^2 \ln M_0^2}{dM_R^2} = -\gamma \quad (4.10)$$

[γ is a critical index]. Thus we consider the series ($Z = M_0^2 / (\lambda - 1/2 R)$)

$$\{Z + \frac{1}{2R}\} \frac{d^2 \ln M_0^2}{dZ} = 1.579 + .70Z + .657Z^2 + .692Z^3 + \dots \quad (4.11)$$

Direct evaluation of the series truncated at N terms evaluated at $Z = -1/(2R)$ yield for γ the following values

$$\begin{aligned} &1, .291, .337, .382, .466, .517, .574, .636 \\ &.714, .790, .8786, 1.073 \end{aligned} \quad (4.12)$$

which are all in the ball park of the exact answer of 1 which we will obtain later by the double well expansion. We have not tried to improve on these results by "Padé'ing" first the series in Z before extrapolating.

Thus in this simple case we see that we can get qualitatively good results by naively extrapolating our renormalized series

valid near $M_R^2/\sqrt{\lambda} \sim 1$ directly to the point $M_R^2/\sqrt{\lambda} = 0$.

For $d > 0$ the series for λ_R/M_R^{4-d} has the form after mass renormalization (see Eq. 4.1)

$$\frac{\lambda_R}{M_R^{4-d}} = \sum_n c_n(\epsilon, d) [M_R^2 a^2 - \frac{1}{2R\epsilon} + \dots]^n \quad (4.13)$$

Our strategy for calculating λ_K^* here is the first take $M_R^2 a^2 \rightarrow 0$ obtaining a series of the form

$$\frac{\lambda_R}{M_R^{4-d}} = \epsilon^{a(d)} \sum_n a_n(d) \epsilon^n \quad (4.14)$$

where $\epsilon = 1/\sqrt{\lambda_d^{4-d}}$ and ϵ and a_n just depend on the dimensionality of space time. This is exactly a series of the form Eq. (4.5) and we take the lattice spacing to zero by looking at the sequence of extrapolants $\lambda_K^{*R}(\epsilon)$ where R is the order of perturbation theory. For $d = 1$ and $d = 2$ this strategy produces a seemingly convergent sequence of extrapolants for λ_K^* .

For $d = 1$ we get the sequence

$$2.424, \quad 2.678, \quad 4.644, \quad 55.551, \quad 9.624, \quad 51.885 \quad (4.15)$$

which appears to converge to **6**.

whereas for $d = 2$ we obtain the sequence

$$4.743, \quad 8.210, \quad 11.758, \quad 14.173, \quad 15.982, \quad 17.150 \quad (4.16)$$

14

G. Baker and Kincaid have calculated these quantities using 12 terms in a high temperature expansion and have obtained 6 in $d=1$, 14.5 in $d=2$ which qualitatively agrees with these results.

For $d = 3$, however one does not obtain a meaningful sequence for λ_R^* by first taking $M_R^2 = 0$ and then taking $a \rightarrow 0$. George Baker suggests that this is because in λ_R^* is sensitive to the path one takes in trying to take both M_R and a to zero. In his high temperature expansion approach it was necessary to have $M_R \rightarrow 0$ gently with a to get sensible answers (he claims $\lambda_R^* = 25$). We have not yet tried this type of extrapolation.

It thus appears that more work needs to be done in trying to understand how to do double extrapolations ($a \rightarrow 0$, $M_R \rightarrow 0$) starting from the unrenormalized strong coupling expansion. For $d = 2$ at least one can avoid the problem of the second extrapolation ($M_R^2 a^2 \epsilon^2 \rightarrow 1$ to $M_R^2 a^2 \epsilon^2 \rightarrow 0$) by considering a second strong coupling expansion which we will now describe.

For $d = 2$ the place where $M_R^2 a^2 \epsilon^2 \rightarrow 0$ is at $M_0^2 \rightarrow -\infty$. Thus we can try to evaluate Eq. 2.5

$$\Lambda_{2n} = 2 \int_0^\infty \exp \left[-a^d \left(\frac{\lambda}{4} x^4 + \frac{M_0^2}{2} x^2 \right) \right] x^{2n} (a^d)^{2n} \quad (4.17)$$

in that vicinity by letting $M_0^2 = -\mu^2 \left(\frac{1}{4-d} \right)$ (this change of variables was suggested by work of Imamura¹⁵ on the anharmonic oscillator).

Thus we can write

$$A_{2n} = \left[\frac{2}{a^{d-2}} \frac{\mu^{3-d}}{\lambda^{d-2}} \right]^n \left(\frac{\partial}{\partial \mu} \right)^n z_0(\mu^2) \quad (4.18)$$

$$\text{where } z_0(\mu^2) = \int dx \exp \left[- a^d \lambda \left(\frac{x^4}{4} - \frac{x^2 \mu^{d-2}}{2} \right) \right] \quad (4.19)$$

Since $\lambda \rightarrow \infty$ we can evaluate $z_0(\mu^2)$ by Laplace's method, the minima being at $x = \pm \mu^{(d-2)/2}$. We find

$$z_0(\mu^2) \sim 4\sqrt{\pi} \exp \left[- \frac{(\mu a)^d}{4} + \frac{\lambda}{\mu^{4-d}} \right] \frac{\mu}{(\mu a)^{d/2} \sqrt{\lambda}} \quad (4.20)$$

$$\left[\left(1 + \frac{3}{4} \left(\frac{\mu^{4-d}}{\lambda} \right) + \frac{1}{(\mu a)^d} + \frac{105}{12} \left(\frac{\mu^{4-d}}{\lambda (\mu a)^d} \right)^2 + \frac{3465}{128} \left(\frac{\mu^{4-d}}{\lambda (\mu a)^d} \right)^3 + \dots \right] \right.$$

Thus the A_{2n} have the structure

$$A_{2n} = [\mu^{d-2}]^n \sum_k a_k^{(n)} \left(\frac{\mu^{4-d}}{\lambda (\mu a)^{d-2}} \right)^n \quad (4.21)$$

Thus dimensionless quantities will have unrenormalized series of the form

$$A \left(\frac{\mu^{4-d}}{\lambda}, \bar{c} \right) = \sum f_n(\bar{c}) \left(\frac{\mu^{4-d}}{\lambda} \right)^n, \quad (4.22)$$

where $\bar{c}$ is $(\mu^4/\lambda)^{d-1}$ and $f_n(\bar{c})$ has the usual form of Eq. (3.2).

In zero dimensions one can now directly calculate the renormalized strong coupling series. Using Eq. (4.3) we obtain

$$\begin{aligned} M_R^2 = \mu^2 \cdot \left[1 + \frac{\mu^4}{\lambda} + 4 \left(\frac{\mu^4}{\lambda} \right)^2 + 31 \left(\frac{\mu^4}{\lambda} \right)^3 + 364 \left(\frac{\mu^4}{\lambda} \right)^4 + 5746 \left(\frac{\mu^4}{\lambda} \right)^5 \right. \\ \left. + 113944 \left(\frac{\mu^4}{\lambda} \right)^6 \right]. \end{aligned} \quad (4.23)$$

Since $\mu^2 = \frac{-\lambda}{M_U^2}$, we see that as $M_R^2 \rightarrow 0$ λ fixed

$$\frac{M_U^2}{\sqrt{\lambda}} = - \frac{\sqrt{\lambda}}{M_R^2}. \quad (4.24)$$

Thus $M_U^2 \rightarrow -\infty$ as $-1/M_R^2$ in the critical regime, directly giving -1 as the critical index. For $\frac{\lambda_R}{M_R^4}$ we have

$$\begin{aligned} \frac{\lambda_R}{M_R^4} = 2 - 2 \frac{\mu^4}{\lambda} - 6 \left(\frac{\mu^4}{\lambda} \right)^2 - 40 \left(\frac{\mu^4}{\lambda} \right)^3 - 434 \left(\frac{\mu^4}{\lambda} \right)^4 \\ - 6552 \left(\frac{\mu^4}{\lambda} \right)^5 - 126412 \left(\frac{\mu^4}{\lambda} \right)^6. \end{aligned} \quad (4.25)$$

Since $\mu^4/\lambda \rightarrow 0$ when $M_R^4/\lambda \rightarrow 0$, this series immediately gives

$$\lambda_R^* = 2.$$

Inverting Eq. 4.23 gives the mass renormalization series

$$\begin{aligned} \mu^2 = M_R^2 \left(1 - \frac{M_R^4}{\lambda} - \left(\frac{M_R^4}{\lambda} \right)^2 - 11 \left(\frac{M_R^4}{\lambda} \right)^3 - 139 \left(\frac{M_R^4}{\lambda} \right)^4 \right. \\ \left. - 2373 \left(\frac{M_R^4}{\lambda} \right)^5 - 50181 \left(\frac{M_R^4}{\lambda} \right)^6 + \dots \right) \end{aligned} \quad (4.26)$$

For $\frac{\lambda_R}{M_R^4}$ we have obtained the renormalized strong coupling series by inverting Eq. 4.26 into 4.25 to obtain

$$\begin{aligned} \frac{\lambda_R}{M_R^4} = 2 - 2 \frac{M_R^4}{\lambda} - 2 \left(\frac{M_R^4}{\lambda} \right)^2 - 14 \left(\frac{M_R^4}{\lambda} \right)^3 - 166 \left(\frac{M_R^4}{\lambda} \right)^4 \\ - 2714 \left(\frac{M_R^4}{\lambda} \right)^5 - 55866 \left(\frac{M_R^4}{\lambda} \right)^6 \dots \end{aligned} \quad (4.27)$$

For $d=1$, the anharmonic oscillator we again expect that $\mu^2 \rightarrow 0$ as $M_R^2 \rightarrow 0$ so that the lattice series in μ^2 is already valid in the critical regime. Thus to obtain λ^* one need only set $\mu^2 = 0$ and extrapolate to $a \rightarrow 0$. This calculation is in progress.

For the anharmonic oscillation we expect that the continuum strong coupling series for λ_R can be obtained directly via an instanton calculation. The reason for this is as follows. Isaacson has shown using the canonical transformation

$$x = \frac{\mu x'}{(g-1)^{1/4}} \quad p = \frac{(g-1)^{1/4}}{\mu} p' \quad , \quad g \approx \frac{\lambda}{\mu^2}$$

that

$$H = \mu(g-1)^{1/2} \left[(p')^2 + v \left[(x')^2 - \frac{1}{2v} \right]^2 + \frac{1}{4v} \right] \quad (4.28)$$

where

$$v = \frac{g}{(g-1)^{3/2}} .$$

Thus as $\lambda \rightarrow \infty$ μ fixed, the problem becomes 2 separate harmonic wells at $x = \pm \frac{1}{\sqrt{2}v}$ with almost degenerate energy levels

$$E_n \sim \mu(g-1)^{1/2} (2n+1) . \quad (4.29)$$

Also as $\lambda \rightarrow \infty$ the renormalized mass goes to zero exponentially fast with λ ¹⁵

$$M_R \leq \mu(g-1)^{1/2} e^{-Bg^{1/2}} , \quad (4.30)$$

B an unknown constant.

However the regime μ fixed $g = \frac{\lambda}{M_0^2} \rightarrow \infty$ can be reexpressed in terms of the original bare mass parameter $M_0^2 = -\lambda/\mu$. The dimensionless coupling g is also

$$g = - \left(\frac{M_0^2}{\lambda} \right)^2 . \quad (4.31)$$

Thus large g is equivalent to weak coupling in the double well situation, and Eq. 4.30 can be written

$$\frac{M_R}{\lambda^{1/3}} \leq \frac{|M_0|}{\lambda^{1/3}} e^{-B \frac{|M_0^3|}{\lambda}} \quad (4.32)$$

which is to be compared with the instanton evaluation of the renormalized mass due to Polyakov¹⁶ and Gildiner and Patrascioiu¹⁷ of

$$\frac{M_R}{\lambda^{1/3}} = \sqrt{2} \frac{|M_0|}{\lambda^{1/3}} \frac{16\sqrt{2}}{\pi} \left(\frac{|M_0|^3}{\lambda} \right)^{1/2} \exp \left[-\frac{2\sqrt{2}}{3} \frac{|M_0|^3}{\lambda} \right] \quad (4.33)$$

derived assuming $\lambda/|M_0|^3 \ll 1$ (dilute gas of instantons).

We see that the regime where the instanton calculation is valid $\lambda/|M_0|^3 \ll 1$ is the place where $M_R/\lambda^{1/3}$, the expansion parameter of the renormalized strong coupling series is small. It is also the place where $g = \lambda/\mu^3$ is large.

It is therefore expected that expanding about the classical minimum

$$\phi(\tau) = \left(\frac{\bar{M}_0^2}{\lambda} \right)^{1/2} \tanh^{-1} \left(\frac{\bar{M}_0(\tau-a)}{\sqrt{2}} \right) \quad (4.34)$$

where $M_0^2 = -\bar{M}_0^2$, one can directly evaluate W_4 in the pseudoparticle approximation and determine the exact strong coupling expansion for λ_R

To calculate λ^* it is not necessary to do a complete instanton calculation. We have

$$\langle 0 | \phi(\tau) \phi(0) | 0 \rangle = \sum_n \langle 0 | \phi(\tau) | n \rangle \langle n | \phi(0) | 0 \rangle = \sum_n |a_n|^2 e^{-(E_n - E_0)\tau}$$

In the regime of strong coupling we have

$$E_n - E_0 \sim \mu^2/g, \quad n \neq 0 \quad (4.35)$$

$$a \sim e^{-\Lambda/g}, \quad n = 1$$

Thus as $g \rightarrow \infty$ only the $n = 1$ term contributes. Similarly if we break up the time orderings in the four point function and insert states we have terms of the form

$$\sum_{n,m,l} \langle 0 | \phi | n \rangle \langle n | \phi | m \rangle \langle m | \phi | l \rangle \langle l | \phi | 0 \rangle \quad (4.36)$$

When $g \rightarrow \infty$, n, l can be only 1 and m can be only zero. Thus the 4 point Wightman functions factor into products of two point functions.

When we put in the theta functions in τ space and integrate over τ to obtain the four point function at zero momentum we obtain

$$\lambda^* = 0 \quad (4.37)$$

It thus seems that the exact answer for $d \geq 2$ is just $\lambda^* = \Gamma(d+3)$. In table one we compare the first six extrapolants in dimension 0, 1/2, 1, 3/2, 2 with $\Gamma(d+3)$. We see that the extrapolants are an asymptotic sequence: they eventually overshoot the exact answer by say 15% at the maxima and then diverge. This is clear in zero and 1/2 dimension and is expected in the higher dimensions.

Summary.

We have shown that in $d \leq 2$ we can extrapolate the lattice strong coupling series to zero lattice spacing and into the regime where $\frac{M_R^2}{\lambda^{2/(4-d)}}$ is small. The extrapolants give 15% accuracy to the exact answer for λ^* , the critical coupling constant which we determine by an instanton calculation. We hope to find a way to improve the 15% accuracy by finding a way to handle the asymptotic nature of the sequence. What is interesting is that for $d < 2$ the renormalized strong coupling series is directly calculable in terms of instanton physics. This happens because the critical region ($M_R^2 \rightarrow 0$) occurs in the infinite double well regime ($M_0^2 \rightarrow \infty$). For $d > 2$ one expects for the lattice field theory near $M_R^2 = 0$ the following behavior

$$M_R^2 \sim (M_0^2 - M_0^{*2})^\alpha, \quad (4.3b)$$

with M_0^{*2} finite for fixed lattice spacing a . We, however, only can evaluate L_{2n} analytically for $M_0^2 \rightarrow 0$ and $M_0^2 \rightarrow \infty$. By doing a two point Padé we hope to extrapolate to the critical region and zero lattice spacing to obtain the continuum strong coupling series. This calculation is in progress. Obviously, our next project is to understand the continuum limit of Lattice Gauge field theory, hoping to set the scale of the theory in pure Yang-Mills using the mass scale of Glueballs.

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TABLE 1.

DIMENSION	$\Gamma(N+1)$	1	2	3	4	5	6
0	2	.911	1.505	1.924	2.168	2.323	2.362
$\frac{1}{2}$	1.72135	1.879	2.024	2.137	2.219	2.273	2.315
1	6	2.425	3.078	4.046	5.251	6.626	8.155
$1\frac{1}{2}$	11.63173	3.275	5.179	7.079	8.118	9.878	12.336
2	24	6.743	9.519	11.753	14.173	16.882	19.830

FIGURE CAPTIONS

Figure 1: First three orders in the strong coupling expansion of the two point function $G_2(l,l)$.

Figure 2: Graphlets with less than five internal lines G_0^{-1} used in constructing $\Lambda_{2n}(p_1=0)$.

	Order	Number of G_0^{-1}	Symmetry Number
ϵ	0	1
ϵ^2	1	$\frac{1}{2}$
ϵ^2	1	1
ϵ^3	2	$\frac{1}{3}$
ϵ^3	2	$\frac{1}{2}$
ϵ^3	2	$\frac{1}{2}$
ϵ^3	2	1
ϵ^3	2	$\frac{1}{2}$

Fig. 1

Fig. 2