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INFLUENCE OF GYRORADIUS AND DISSIPATION
ON THE ALFVEN-WAVE CONTINUUM

BY

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Influence of Gyroradius and Dissipation
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ABSTRACT

It is well known that in ideal magnetohydrodynamics there is a continuous spectrum of real frequencies associated with a singularity of the shear Alfvén waves on the surface $k_{\parallel} v_A = \omega$. It is also known that the introduction of first-order gyroradius effects eliminates the continuum. In the present work we examine the influence of the full gyroradius response and of dissipation on the continuum. In the absence of dissipation we first confirm that if only first-order gyroradius effects are incorporated, the continuum disappears. However, when the full gyroradius response is included, this discrete spectrum vanishes, and a new continuum (associated with singularities at $k_{\perp} v_A = 0$) appears. The introduction of collisional dissipation removes the original MHD continuum leaving discrete modes whose frequency tends to zero with the collision rate as $\nu^{1/3}$. Collisions also remove the new continuum of the full gyroradius model leaving discrete modes whose frequency tends to zero as $(\log \nu)^{-1}$. Collisionless Landau damping has a similar effect.

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I. INTRODUCTION

It is well known that in ideal magnetohydrodynamic (MHD) theory, the eigenmodes of linear oscillation about a plane-slab in a sheared magnetic field possess a continuous frequency spectrum. For perturbations $\phi(x) \exp(ik_y y - i\omega t)$ the linear oscillations are described by the equation

$$\frac{d}{dx} (\omega^2 - \omega_A^2(x)) \frac{d\phi}{dx} - k_y^2 [\omega^2 - \omega_A^2(x)] \phi = 0, \quad (1)$$

where $\omega_A(x) = k_y x v_A / L_S$, v_A is the Alfvén speed, and L_S is the shear length with $B_y = x B / L_S$.

The continuous spectrum is associated with the singularities at $\omega_A(x) = \omega$. Despite the conservative nature of the basic MHD equations, the presence of the continuum gives rise to a form of dissipation leading to so-called Alfvén-wave heating.¹ If first order gyroradius effects are added to the basic MHD description, the equation for linear oscillations of a plane slab takes the form²

$$\omega^2 \rho^2 \frac{d^4 \phi}{dx^4} + \frac{d}{dx} [\omega^2 - \omega_A^2(x)] \frac{d\phi}{dx} - k_y^2 [\omega^2 - \omega_A^2(x)] \phi = 0, \quad (2)$$

where

$$\rho^2 = \left(1 + \frac{3}{4} \frac{T_i}{T_e}\right) \frac{m_i T_e}{e^2 B},$$

is related to the ion gyroradius. The singularities are removed by this gyroradius term and the continuous spectrum is replaced by a discrete spectrum of real frequencies. These discrete modes may become unstable due to resonant excitation by high energy ions.² The sensitivity of the plane-slab spectrum

to small effects, such as finite gyroradius, and the change from a continuum (associated with Alfvén wave damping) to discrete modes (associated with possible instability) makes the spectrum of normal oscillations a matter of considerable interest - despite the highly idealized nature of the plane-slab configuration.

In the present paper we examine the effect of gyroradius and dissipation on the continuous MHD spectrum of the plane slab. We find that when there is no dissipation and gyroradius effects are introduced only to first order, as in Ref. 2, we recover the result that the singular modes and the continuum are removed and replaced by a discrete spectrum of real frequencies. However, when the full ion gyroradius response is incorporated in a nondissipative plasma model, this discrete spectrum disappears and a continuum reappears! This new continuum is associated with singularities at $\omega_A(x) = 0$ and does not seem to have been considered hitherto.

We then examine the effect of dissipation on the spectrum. When collisional damping is introduced into the MHD model, we find that the Alfvén continuum disappears leaving only discrete resistive modes whose frequency tends to zero with the collision rate (in fact as $\nu^{1/3}$). When collisional dissipation is incorporated together with full gyroradius effects, its continuum also disappears leaving discrete modes whose frequency again tends to zero with the dissipation but at a slower (logarithmic) rate. Finally, we show that collisionless (Landau) damping produces a similar effect.

II. BASIC EQUATIONS

We consider here only the conventional plane-slab configuration in which the equilibrium magnetic field lies in the y-z plane with $\vec{B} = (0, x/L_s, 1)B$. The magnetic shear represents the only source of inhomogeneity (i.e., we

ignore diamagnetic effects). We consider finite but small β (i.e., $m_e/m_i < \beta < 1$) so that the perturbed electromagnetic field can be described by the electrostatic potential ϕ and $A_{\parallel}$ (the component of the vector potential along B).

To provide a unified description, suitable for the investigation of first order and full gyroradius effects and of dissipation, the ion response is calculated directly from the Vlasov equation and the electron response from a drift kinetic equation.

(a) Ion Response

For the perturbed ion distribution $f(\tilde{r}, \tilde{v}, t)$ we have

$$\frac{\partial f}{\partial t} + \tilde{v} \cdot \nabla f + \frac{e}{m} \left(\delta \mathbf{E} + \frac{1}{c} \tilde{\mathbf{v}} \times \delta \mathbf{B} \right) \cdot \frac{d\mathbf{F}}{d\tilde{\mathbf{v}}} - \omega_c \frac{\partial f}{\partial \phi} = 0 \quad (3)$$

where F is the Maxwellian function

$$F_m = n \left(\frac{m}{2\pi T} \right)^{3/2} \exp \left(-\frac{m\tilde{v}^2}{2T} \right) ,$$

ω_c is the gyrofrequency, and ϕ is the gyrophase angle. The velocity is expressed in components along and transverse to the equilibrium field as

$$\tilde{\mathbf{v}} = v_{\parallel} \hat{\mathbf{n}} + v_{\perp} (\hat{\mathbf{e}}_{\perp} \cos \phi + \hat{\mathbf{n}} \times \hat{\mathbf{e}}_{\perp} \sin \phi) ,$$

and the perturbation itself as

$$f = f(\tilde{v}, x) \exp(ik_y y - i\omega t) ,$$

with $-\infty < x < \infty$. Since we are specifically interested in finite gyroradius

effects, we must consider transverse variations of f on the scale of the ion gyroradius, whereas the variations along the field are on the macroscopic equilibrium scale, i.e.,

$$k_{\parallel} = -i\eta \cdot \nabla = k_y \frac{x}{L_s} \sim 0 \left(\frac{1}{L_s} \right) .$$

A systematic expansion of Eq. (3) with the ordering $k_{\parallel}/k_{\perp} \sim \omega/\omega_c \ll 1$ yields in leading order

$$\frac{\partial f}{\partial \phi} = v_{\perp} \cdot \nabla f + \frac{e}{T} (v_{\perp} \cdot \nabla \phi) F_m . \quad (4)$$

By introducing the Fourier transform in the x -direction,

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dk_x \hat{f}(k_x) \exp(ik_x x) , \quad (5)$$

we obtain

$$\hat{f} = \frac{-e\phi}{T} F_m + \hat{h}(v_{\perp}, v_{\parallel}, k_x) \exp(ip) , \quad (6)$$

with

$$p = \frac{v_{\perp}}{\omega_c} (k_x \sin \phi - k_y \cos \phi) .$$

The next order yields an equation for $\hat{h}$ as an integrability condition on $\hat{f}^1$,

$$(i\omega - k_y \frac{v_{\parallel}}{L_s} \frac{d}{dk_x}) \hat{h} = \frac{i\omega e}{T} F_m J_0 \left(\frac{k_{\perp} v_{\perp}}{\omega_c} \right) \left(\hat{\phi} - \frac{v_{\parallel}}{c} \hat{A}_{\parallel} \right) , \quad (7)$$

where $k_{\perp}^2 = k_x^2 + k_y^2$. Hence, in the usual limit, $k_{\parallel} v_{ti} \ll \omega$, we obtain the perturbed ion density,

$$\hat{n}_i(k_x) = \frac{-e\hat{\phi} n_0}{T_i} [1 - \Gamma_0(b)] \quad , \quad (8)$$

where $b = k_{\perp}^2 \rho_i^2 / 2$ with $\rho_i^2 = 2m_i T_i / e^2 B^2$ and $\Gamma_0(b) = I_0(b) \exp(-b)$. The perturbed ion current is negligible compared to the perturbed electron current which is derived below.

(b) Electron Response

For the electrons a convenient description is the drift-kinetic equation

$$i(\omega - k_{\parallel} v_{te}) f = \frac{-|e|}{m} E_{\parallel} \frac{\partial F_m}{\partial v_{\parallel}} - C(f) \quad , \quad (9)$$

where C is the collision operator. In the collisional, fluid-like limit a systematic expansion with $\omega \ll v$, $\omega v \sim k_{\parallel}^2 v_{te}^2$ (ignoring temperature fluctuations) leads to an electron response (Ohm's law)

$$\hat{j}_{\parallel}(k_x) = -\frac{k_y}{L_s} \frac{\partial \hat{\phi}}{\partial k_x} + i\omega \hat{A}_{\parallel} + \frac{T_e}{n_0 |e|} \frac{k_y}{L_s} \frac{\partial \hat{c}}{\partial k_x} \quad , \quad (10)$$

(where c is the Spitzer conductivity, $\sigma = ne^2/mv$, and v is the appropriate collision frequency). We also have the continuity equation

$$\hat{n}_e(k_x) = \frac{ik_y}{\omega |e| L_s} \frac{\partial \hat{j}_{\parallel}}{\partial k_x} \quad . \quad (11)$$

Using the perturbed response functions (8), (10), and (11) in the charge neutrality condition, $n_i = n_e$, and Ampere's law, $\nabla^2 A_{\parallel} = -j_{\parallel}$, we obtain a single eigenvalue equation for the spectrum of the plane slab:

$$\frac{d}{dk} \frac{(1+k^2)}{\{1+(i\lambda/\Omega)(1+k^2)\}} \frac{d\hat{\phi}}{dk} + \frac{\Omega^2}{b_0} \left(\frac{1-\Gamma_0}{1+\tau(1-\Gamma_0)} \right) \hat{\phi} \approx 0, \quad (12)$$

with

$$\begin{aligned} \Gamma_0 &= I_0(b) \exp(-b), & b &= b_0'(1+k^2), & b_0' &= \frac{k^2 y^2}{2} \\ \tau &= \frac{T_e}{T_i}, & \Omega &= \frac{\omega L}{v_A}, & \lambda &= \frac{\omega L}{v_{te}} \frac{b_0' \tau}{2} \\ k &= \frac{k_x}{k_y}, & \hat{\phi} &= \{ \tau(\Gamma_0 - 1) - 1 \} \hat{\psi}. \end{aligned}$$

Equation (12) forms the basis for the investigation of nondissipative and collisional modes in Secs. III and IV. The conventional MHD models are obtained by taking the limit $b_0 \rightarrow 0$. The first order ion gyroradius effects are incorporated by retaining one further term in the expansion of Γ_0 (i.e., $\Gamma_0 = 1 - b + 3b^2/4$) while full ion gyroradius effects are incorporated when Γ_0 is retained in its entirety. Naturally the nondissipative versions are obtained by setting $\lambda = 0$.

It should be noted that in order to deal conveniently with the gyroradius effects, we consider the eigenvalue problem in k -space where modes which are singular in x are recognized by their behavior as $|k| \rightarrow \infty$. Note that the basic Eq. (12) has no singularities for finite real k .

For the collisionless kinetic theory, incorporating electron Landau damping, the electron response is obtained directly from Eq. (9) as

$$j_{\parallel} = \frac{\omega_0^2 e^2}{2 T_e} [1 + Z(\zeta)] \left(\frac{\omega_A}{c} - k_{\parallel} \phi \right), \quad (13)$$

and

$$n = -k_{\parallel} j_{\parallel} / \omega |e| \quad , \quad (14)$$

where $\zeta = \omega / k_{\parallel} v_{te}$ and $Z(\zeta)$ is the plasma dispersion function. These equations replace Eqs. (10) and (11) and together with charge neutrality and Ampere's law again lead to an eigenvalue equation for the oscillations of the plane slab. Because of the appearance of the plasma dispersion function in the electron response it is more convenient in this case to express the eigenvalue problem in x-space namely,

$$x^2 \frac{d^2 \phi}{dx^2} + \left(1 + \frac{1(\epsilon_0 - 1)}{W(\epsilon)} \right) \phi + \frac{\gamma^2}{b_0} (1 - \epsilon_0) \phi = 0 \quad , \quad (15)$$

where

$$W(\zeta) = -[1 + \zeta Z(\zeta)] \quad ,$$

and ϵ_0 is to be interpreted as an operator, defined through its Fourier transform, i.e.,

$$\epsilon_0 \phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \phi(x') \int_{-\infty}^{\infty} dk \epsilon_0(b) \exp[ik(x - x')] \quad , \quad (16)$$

III. SPECTRUM OF NONDISSIPATIVE MODELS

In this section we consider the spectrum of normal modes of the plane slab when there is no dissipation. The nondissipative modes are governed by the equation

$$\frac{d}{dk} (1 + k^2) \frac{d\hat{\phi}}{dk} + \frac{\Omega^2}{b_0} \frac{(1 - \Gamma_0)}{[1 + \Gamma(1 - \Gamma_0)]} \hat{\phi} = 0 \quad (17)$$

The general features of the spectrum and the striking changes produced by first order and full gyroradius effects can be illustrated by expressing Eq. (17) as a Schrodinger equation for a particle in a potential. Then it becomes

$$\frac{d^2 \psi}{dk^2} + [E - V(k)] \psi = 0 \quad (18)$$

where $\psi = (1 + k^2)^{1/2} \phi$. The effective potential is

$$V(k) = \frac{1}{(1+k^2)^2} - \frac{\Omega^2}{(1+k^2)} \frac{1}{b_0} \frac{(1 - \Gamma_0)}{[1 + \Gamma(1 - \Gamma_0)]} \quad (19)$$

and the required solution of Eq. (18) corresponds to $E = 0$. We shall refer to this as the equivalent Schrodinger problem.

The MHD model, corresponding exactly to Eq. (1), is obtained from Eq. (17) by taking the limit $b_0 \rightarrow 0$. Then the effective potential in the equivalent Schrodinger problem is

$$V^{MHD}(k) = \frac{1}{(1+k^2)^2} - \Omega^2 \quad (20)$$

which tends to a constant value, $-\Omega^2$, as $|k| \rightarrow \infty$. In this case it is clear from Fig. 1 that $E = 0$ is never a bound state of the equivalent Schrodinger problem but is always part of the continuum, and Ω^2 also lies in a continuum.

The inclusion of first-order gyroradius effects, in the form corresponding to Eq. (2), can be achieved by expanding the potential in powers of the gyroradius parameter b_0 and retaining terms up to and including those linear in b_0 . Then the effective potential is

$$V_{(k)}^{GR} = \frac{1}{(1+k^2)^2} - \Omega^2 \left[1 - \left(\frac{3}{4} + \tau \right) b_0 (1+k^2) \right] . \quad (21)$$

As shown in Fig. 1, this potential increases indefinitely as $|k| \rightarrow \infty$ so that for this case $E = 0$ lies in the discrete spectrum of bound states of the equivalent Schrodinger problem.

Finally, when the full gyroradius response is retained, the potential (Fig. 1) tends to zero as $|k| \rightarrow 0$, and $E = 0$ is on the lower edge of the continuous spectrum of the equivalent Schrodinger problem.

More formally, the nature of the spectrum can be inferred from the asymptotic form of the two independent solutions of Eq. (17) as $|k| \rightarrow \infty$. We consider each model in turn.

(a) MHD

For the MHD model the basic eigenvalue equation reduces to

$$\frac{d}{dk} (1+k^2) \frac{d\hat{\phi}}{dk} + \Omega^2 (1+k^2) \hat{\phi} = 0 , \quad (22)$$

and the two asymptotic forms as $|k| \rightarrow \infty$ are

$$\hat{\phi}^{\pm} \sim \frac{1}{k} \exp(\pm i\Omega x) . \quad (23)$$

Thus for all real Ω , both independent solutions tend to zero as $|k| \rightarrow \infty$. The solutions have no singularities for real k and their Fourier transforms decay as $|x| \rightarrow \infty$. Consequently, eigenmodes (in fact an infinity of such modes) can be constructed for all real values of Ω which therefore form a continuous spectrum. It should be noted that the Fourier-transform of $\hat{\phi}(k)$ will diverge at the points $x = \pm \Omega$ so that the eigenmodes of the continuum are singular in x -space at these points. They correspond exactly to the singular solutions of Eq. (1).

(b) First Order Gyroradius

When first order gyroradius effects are incorporated into the MHD model, the eigenvalue equation becomes

$$\frac{d}{dk} (1 + k^2) \frac{d\hat{\phi}}{dk} + \Omega^2 (1 + k^2) \left[1 - \left(\frac{3}{4} + \tau \right) b_0 (1 + k^2) \right] \hat{\phi} = 0 \quad , \quad (24)$$

and the two asymptotic forms for $\hat{\phi}(k)$ as $|k| \rightarrow \infty$ are

$$\hat{\phi}^{\pm}(k) \sim \frac{1}{k^{3/2}} \exp\left[\pm \left[\left(\frac{3}{4} + \tau \right) b_0 \right]^{1/2} \frac{\Omega^2 k^2}{2} \right] \quad . \quad (25)$$

In this case, for a general value of Ω , a solution of Eq. (24) will contain both exponentially growing and decaying parts as $|k| \rightarrow \infty$. The requirement that the solution contain only the decaying exponential at both $k \rightarrow +\infty$ and $k \rightarrow -\infty$ provides a discrete eigenvalue condition. The Fourier-transforms of these discrete eigenfunctions converge for all x and are nonsingular. Their discrete, real, frequencies can be estimated by a WKB approximation to be

$$\Omega = 4 \left[\left(\frac{3}{4} + \tau \right) b_0 \right]^{1/2} \left(n + \frac{1}{2} \right) \quad . \quad (26)$$

The spacing of the modes is therefore proportional to the gyroradius, and Eq. (26) is equivalent to the result of Rosenbluth and Rutherford² based on Eq. (2).

(c) Full Gyroradius Response

When the full gyroradius response is retained, the two asymptotic forms for $\hat{\phi}(k)$ as $|k| \rightarrow \infty$ are

$$\hat{\phi}^{\pm}(k) \sim \frac{1}{k^{1/2}} k^p, \quad p = \pm \left(\frac{1}{4} - \Omega^2\right)^{1/2}. \quad (27)$$

For all real Ω both solutions again tend to zero for $|k| \rightarrow \infty$ (indeed for $\Omega^2 > 1/4$ they do so at identical rates). Consequently, it is again possible to construct eigenmodes for all real Ω . In this case, however, the Fourier-transform diverges only at $x = 0$, so that these eigenfunctions differ from those of MHD in that they are regular at $x = \pm \Omega$ but singular at $x = 0$!

IV. SPECTRUM OF MODELS WITH COLLISIONAL DISSIPATION

It is clear from Eq. (12) that even a small collision rate will have a significant effect on the behavior of $\hat{\phi}(k)$ at large $|k|$. Consequently, a small dissipation will modify the continuous spectra of the "simple MHD" and "MHD plus full gyroradius" models, (whose eigenfunctions decay slowly as $|k| \rightarrow \infty$). We consider this modification in detail below. However, the effect of dissipation on the discrete spectrum of the "MHD plus first order gyroradius" model will be negligible and is not considered further.

(a) Spectrum of Resistive MHD

When collisions are retained, we obtain from Eq. (12) the "resistive MHD" plasma model whose spectrum is determined by

$$\frac{d}{dk} \frac{(1+k^2)}{[1+(i\lambda/\Omega)(1+k^2)]} \frac{d\hat{\phi}}{dk} + \Omega^2(1+k^2)\hat{\phi} = 0. \quad (28)$$

In the limit of weak collisions we write $y^2 = i\lambda k^2/\Omega$ and $p^2 = i\Omega^3/\lambda$ with $\lambda \rightarrow 0$ to obtain

$$\frac{d}{dy} \frac{y^2}{(1+y^2)} \frac{d\hat{\phi}}{dy} - p^2 y^2 \hat{\phi} = 0. \quad (29)$$

The substitution

$$\hat{\phi} = \frac{1}{y^2} \frac{dP}{dy} , \quad (30)$$

reduces Eq. (29) to the simpler eigenvalue equation

$$y^2 \frac{d}{dy} \frac{1}{y^2} \frac{dP}{dy} - p^2 (1 + y^2) P = 0 , \quad (31)$$

which has power-series solutions

$$P = \exp\left(\frac{py^2}{2}\right) \sum_{m=0}^{\infty} a_m y^{m+3} , \quad (32)$$

where the coefficients a_m satisfy the recurrence relation

$$\frac{a_{m+2}}{a_m} = \frac{(p-2m-5)p}{(m+2)(m+5)} . \quad (33)$$

At large m , this ratio is the same as for the coefficients of $\exp(-py^2)$. Consequently, unless the series terminates, the solution will contain both growing and decaying exponentials. Thus the eigenvalue condition is

$$p = (2m + 5) , \quad (34)$$

and the bounded eigenfunctions correspond to discrete stable frequencies

$$\Omega = \lambda^{1/3} (2m + 5)^{2/3} (3^{1/2} - i)/2 . \quad (35)$$

Thus, in the presence of a small amount of dissipation the continuous MHD spectrum is replaced by a discrete spectrum of low frequency resistive modes

whose frequency tends to zero with the collisional damping rate as $\nu^{1/3}$. Numerical solutions of the basic Eq. (28) obtained from a shooting code for a wide range of parameter values indicate excellent agreement with the analytic formula given by Eq. (35).³

(b) Spectrum with collisions and full ion gyroradius

With the full ion gyroradius response and collisions retained, the spectrum is determined by the eigenvalue Eq. (12), i.e.,

$$\frac{d}{dk} \frac{(1+k^2)}{\left[1 + \frac{i\lambda}{\Omega} (1+k^2)\right]} \frac{d\hat{\phi}}{dk} + \Omega^2 Q^2(k) \hat{\phi} = 0, \quad (36)$$

where we have introduced

$$Q^2(k) = \frac{1 - \Gamma_0}{b_0 [1 + \tau(1 - \Gamma_0)]}.$$

To solve this equation for a small collision rate we divide the range of k into two overlapping regions. In the outer region $k^2 \gg 1$, and in the inner region $k^2 \ll |\Omega/\lambda|$.

In the inner region we recover the collisionless equation

$$\frac{d}{dk} (1+k^2) \frac{d\hat{\phi}}{dk} + \Omega^2 Q^2(k) \hat{\phi} = 0, \quad (37)$$

and introduce a WKB solution based formally on assuming $\Omega^2 Q^2 \sim \Omega^2/b_0(1+\tau) \gg 1$. The WKB expressions are

$$\frac{1}{(1+k^2)^{1/4}} \frac{1}{[Q(k)]^{1/2}} \exp\left[\pm i\Omega \int^k \frac{Q(k)}{(1+k^2)^{1/2}} dk\right]. \quad (38)$$

If Q^2 were real, the equivalent Schrodinger problem would be one of

penetration and reflection at a nonabsorbing barrier. Using the theory of such reflection⁴ we can readily construct the symmetric and antisymmetric WKB solutions to our problem as

$$\hat{\Phi} = \frac{1}{(1+k)^{1/4} Q^{1/2}} \left[\exp(iQ \int_0^k \frac{Q dk}{(1+k)^{1/2}}) \pm \exp(-iQ \int_0^k \frac{Q dk}{(1+k)^{1/2}}) \right] \quad (39)$$

Since there are no Stokes lines near the real- k axis, these remain the appropriate WKB solutions when Q^2 is complex, provided $\text{Im } Q \ll \text{Re } Q$. At large $|k|$ these solutions take the asymptotic forms

$$\hat{\Phi} \sim R_1 \left(\frac{k}{k_0} \right)^{iQ_\infty - 1/2} \pm \left(\frac{k}{k_0} \right)^{-iQ_\infty - 1/2} \quad (40)$$

where $Q_\infty = \lim Q(k)$ as $|k| \rightarrow \infty$, and k_0 is any convenient value of k sufficiently large that $Q(k_0) = Q_\infty$. The coefficient required for matching to the outer region solution is

$$R_1 = \exp(i2i Q \int_0^{k_0} \frac{Q dk}{(1+k)^{1/2}}) \quad (41)$$

In the outer region, $k^2 \gg 1$, we again set $ik^2/Q = y^2$ to obtain, in the limit $\lambda \rightarrow 0$, the equation

$$\frac{d}{dy} \left(\frac{y^2}{(1+y)^2} \frac{d\hat{\Phi}}{dy} \right) + Q_\infty^2 Q_\infty^2 \hat{\Phi} = 0 \quad (42)$$

This has an exact general solution

$$\hat{\Phi} = \frac{d}{dy} \{ y^{1/2} [A Z_\nu(Q_\infty y) + B Z_{-\nu}(Q_\infty y)] \} \quad (43)$$

where Z_ν are Bessel functions and $\nu^2 = -\Omega^2 Q_\infty^2 + 1/4$. Taking $\text{Re } Q > 0$ the solution which decays as $k \rightarrow \infty$ along the real axis is

$$\hat{\phi} = \frac{d}{dy} \{ y^{1/2} H_\nu^{(1)}(\Omega_\infty y) \} , \quad (44)$$

where $H^{(1)}$ is the Hankel function. For small k this outer solution takes the form

$$\hat{\phi} \sim R_2 \left(\frac{k}{k_0} \right)^{\nu-1/2} + \left(\frac{k}{k_0} \right)^{-\nu-1/2} , \quad (45)$$

where the coefficient R_2 is given by

$$R_2 = \frac{\pi e^{-\nu\pi i}}{\sin \pi \nu} \left(\frac{1}{\Gamma(\nu)\Gamma(1+\nu)} \right) \left(\frac{i\lambda}{\Omega} \right)^\nu \left(\frac{\Omega Q_\infty}{2} \right)^{2\nu} \left(\frac{\nu+(1/2)}{\nu-(1/2)} \right) . \quad (46)$$

Recalling that the inner WKB solution was obtained for $\Omega Q_\infty \gg 1$ we see that the inner limit of the outer region solution [Eq. (45)] and the outer limit of the inner region solution [Eq. (40)] do indeed have the same functional form. The matching condition $R_1 = R_2$ yields the eigenvalue

$$\Omega = n\pi b_s^{1/2} \left\{ \int_0^{2(\Omega/\lambda)^{1/2}} \frac{(\Omega/Q_\infty) dk}{(1+k^2)^{1/2}} + \frac{i\pi}{4} - 1 \right\}^{-1} , \quad (47)$$

where $b_s = b_0(1 + \tau)$. In the limit $\Omega/\lambda \rightarrow \infty$ this result becomes

$$\Omega = n\pi b_s^{1/2} \left(\frac{1}{2} \log \frac{\Omega}{\lambda} \right)^{-1} \left[1 - \frac{i\pi}{4} \left(\frac{1}{2} \log \frac{\Omega}{\lambda} \right)^{-1} \right] . \quad (48)$$

Hence, in the presence of small collisional dissipation the continuous

spectrum of the "MHD plus full ion gyroresponse" model is replaced by a discrete spectrum of low frequency modes whose frequency tends to zero with collision rate as $(\log \nu)^{-1}$.

A special case of the results of this section was obtained by Drake and Kleva⁵ using a model in which $T_i = 0$. Their result, which may be recovered by taking the limit $b_0 \rightarrow 0$ with $b_0 \tau$ finite, also shows a logarithmic dependence on collision rate. It is remarkable that the spectrum of this model with $T_i = 0$ should resemble that with the full gyroradius response rather than that of MHD. This situation arises because the cold-ion model corresponds to a limit

$$\lim_{b_0 \rightarrow 0, \tau \rightarrow \infty} [Q^2(k)] = \frac{(1+k^2)}{1 + b_0 \tau (1+k^2)^2} \quad (49)$$

which, like the full gyroresponse model, is finite as $|k| \rightarrow \infty$, whereas the MHD model corresponds to a limit

$$\lim_{b_0 \rightarrow 0} [Q^2(k)] = (1 + k^2) \quad , \quad (50)$$

which diverges as $|k| \rightarrow \infty$.

The analytic eigenvalue estimates given by Eqs. (47) and (48) are found to be in good agreement with numerical solutions of Eq. (36) over a wide range of parameters.³

V. SPECTRUM WITH COLLISIONLESS (LANDAU) DISSIPATION

With collisionless (Landau) damping and the full ion gyroresponse incorporated, (i.e., the Vlasov model) the eigenvalues are determined by Eq. (15) (see Sec. III); i.e.,

$$x \nabla^2 x \left(1 + \frac{\tau(\Gamma_0 - 1)}{W(x_e/x)} \right) \phi + \frac{\Omega^2}{b_0} (1 - \Gamma_0) \phi = 0 \quad (51)$$

To investigate the spectrum of this equation we again consider two regions; $x \lesssim x_e$ and $x \gtrsim x_e$, where $x_e = \omega L_s / k_y v_{te}$. (Note that it will be shown that $\omega \sim k_y v_{A0} / L_s$ so that $x_e \ll \rho$.) In the outer region $W \sim -1$, and Eq. (51) reduces to

$$x \nabla^2 x [1 + \tau(1 - \Gamma_0)] \phi + \frac{\Omega^2}{b_0} (1 - \Gamma_0) \phi = 0 \quad (52)$$

which is the equation for the nondissipative model treated in Sec. III. The Fourier transform of Eq. (52),

$$\frac{d}{dk} (1 + k^2) \frac{d}{dk} [1 + \tau(1 - \Gamma_0)] \hat{\phi} + \frac{\Omega^2}{b_0} (1 - \Gamma_0) \hat{\phi} = 0 \quad (53)$$

has the WKB solution [cf. Eq. (39)]

$$\begin{aligned} \hat{\phi}(k) = & \frac{1}{[1 + \tau(1 - \Gamma)]} \frac{1}{(1 + k^2)^{1/4}} \frac{1}{Q^{1/2}(k)} \times \left[\exp\left(i\Omega \int_0^k \frac{Q dk}{(1 + k^2)^{1/2}}\right) \right. \\ & \left. + \exp\left(-i\Omega \int_0^k \frac{Q dk}{(1 + k^2)^{1/2}}\right) \right] \quad (54) \end{aligned}$$

As discussed earlier, this solution is applicable for all Q but when transformed into x space, is divergent at $x = 0$. To obtain a valid solution for the present model we must match this diverging solution for small x to the solution valid in the inner region, $x \lesssim x_e$. The form of the diverging solution corresponding to Eq. (54) at small x can be determined as follows. We can write the Fourier transform of Eq. (54) as

$$\phi(x) = \frac{1}{x} \int_{-\infty}^{\infty} \hat{\phi}\left(\frac{\kappa}{x}\right) \exp(i\kappa) d\kappa, \quad (55)$$

so that for small x it is sufficient to know $\hat{\phi}$ for large argument where it takes the asymptotic form

$$\hat{\phi} \sim R_1 \left(\frac{k}{k_0}\right)^{-1/2+is} + \left(\frac{k}{k_0}\right)^{-1/2-is}, \quad (56)$$

with $s = \Omega\omega$ and R_1 is given in Eq. (41). Then the integral in Eq. (55) can be expressed in terms of Gamma functions and

$$\phi(x) \sim R_1 \left(\frac{1}{ik_0 y}\right)^{2is} \frac{\Gamma[(1/2)+is]}{\Gamma[(1/2)-is]} x^{-1/2-is} + x^{-1/2+is}. \quad (57)$$

Turning attention now to the inner region, $x \lesssim x_e$, we note that in this region $k^2 \rho^2 \sim \rho^2/x_e^2 \gg 1$ and $\Gamma_0 \rightarrow 0$. Then, setting $x = \eta x_e$, Eq. (51) reduces to

$$\eta \frac{\partial^2}{\partial \eta^2} \eta \left(\frac{W(1/\eta)-\tau}{W(1/\eta)}\right) \phi + \frac{\Omega^2}{b_0} \phi = 0. \quad (58)$$

We can again construct a WKB solution, based formally on $\Omega^2/b_0 \gg 1$. The equivalent Schrodinger problem is one of reflection from a complex overdense barrier,⁴ and following the theory of reflection at such a barrier the appropriate solution of Eq. (58) in the underdense region, $\eta > 1$, is

$$\begin{aligned} \phi \sim \frac{1}{\eta^{1/2}} \left(\frac{\tau-W}{-W}\right)^{-3/4} \left\{ \exp\left[\frac{i}{b_0^{1/2}} \int_{\eta_1}^{\eta} \left(\frac{-W}{\tau-W}\right)^{1/2} \frac{d\eta}{\eta}\right] \right. \\ \left. + i \exp\left[\frac{-i}{b_0^{1/2}} \int_{\eta_1}^{\eta} \left(\frac{-W}{\tau-W}\right)^{1/2} \frac{d\eta}{\eta}\right] \right\}, \quad (59) \end{aligned}$$

where the lower limit of integration, η_1 , is the zero of W which lies closest to the real axis. When $\eta > 1$, Eq. (59) can be written as

$$\phi = R_3 \left(\frac{\eta}{\eta_0} \right)^{-1/2+is} + \left(\frac{\eta}{\eta_0} \right)^{-1/2-is} , \quad (60)$$

where

$$R_3 = -i \exp \left[\frac{2i}{b_0} \int_{\eta_1}^{\eta_0} \left(\frac{-W}{\tau-W} \right)^{1/2} \frac{d\eta}{\eta} \right] , \quad (61)$$

and η_0 is any value of η sufficiently large that $W \approx -1$. We see that expressions (57) and (60) from the outer and inner regions respectively, have the same functional form, and the matching condition is

$$R_1 R_3 \left(\frac{1}{i \eta_0 k_y x_e} \right)^{2is} \frac{\Gamma(is)}{\Gamma(-is)} = 1 . \quad (62)$$

Introducing $x_A = \Omega/k_y$, this can be expressed as

$$\begin{aligned} \frac{\Omega}{b_s^{1/2}} \left[\int_{\eta_1}^{\eta_0} (1 + \tau)^{1/2} \left(\frac{W}{W-\tau} \right)^{1/2} \frac{d\eta}{\eta} - \log \eta_0 + \int_0^{x_A/x_e b_s^{1/2}} \frac{(Q/Q_\infty) dk}{(1+k)^{2 1/2}} \right. \\ \left. + i \frac{\pi}{2} \right] = n\pi . \end{aligned} \quad (63)$$

When $x_e \ll x_A$, this resembles the eigenvalue condition for the collisional case with collisionless dissipation (represented by x_e) replacing collisional dissipation (represented by λ). Again we see that $\omega \rightarrow 0$ with the logarithm of the effective dissipation.

VI. CONCLUSIONS

We have examined the influence of finite gyroradius and dissipation on the continuous MHD spectrum of the plane slab with magnetic shear. This continuous spectrum, which can lead to a form of dissipation even though the basic equations are conservative, is associated with singularities of the normal modes at $\omega_A(x) = k_{\parallel}(x)V_A = \omega$. Because of its association with singularities, the continuous spectrum is strongly affected by both finite gyroradius and dissipation. In the absence of dissipation, first order gyroradius removes the singularities and the continuum and produces a discrete spectrum whose spacing is related to the gyroradius (as one would expect). However, when the full gyroradius response is incorporated, this discrete spectrum disappears again, and a new continuum appears. This new continuum is associated with singularities at $k_{\parallel}(x) = 0$. It is thus a phenomenon which will always arise in the plane slab but which could be avoided in a cylindrical plasma where, for given mode numbers, the point, $k_{\parallel} = 0$, may not lie within the plasma.

The influence of dissipation has been investigated from two aspects; by consideration of small collisional damping and by consideration of collisionless (Landau) damping. Collisional (resistive) damping removes the continuum of the MHD model (that associated with singularities at $\omega_A(x) = \omega$, and leaves discrete modes whose frequency tends to zero as $\nu^{1/3}$. Collisional damping also removes the continuum of the full gyroradius model (that associated with singularities at $k_{\parallel}(x) = 0$) and leaves discrete eigenvalues whose frequency tends to zero as $(\log \nu)^{-1}$. Collisionless damping has a similar effect.

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FIGURE CAPTION

Fig. 1. Effective potential in Fourier space as given by Eqs. (19), (20), and (21).