Dense and Dilute Instanton-Anti-Instanton
Pair Configurations

William F. Palmer and Stephen S. Pinsky
Department of Physics
The Ohio State University
Columbus, Ohio 43210

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Abstract

We present a Yang-Mills gauge field configuration which for large values of a separation parameter is an instanton-anti-instanton pair. At a minimum finite separation the field is a pure gauge. The interaction is calculated for large and small separations. According to an approximation algorithm the configuration is the most important quantum fluctuation about the exact solutions.

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In this note we would like to present a new Yang-Mills gauge field configuration which sheds light on instanton-anti-instanton interactions and the distinction between dense and dilute phases of the topologically neutral instanton gas. The work is essentially a comment on some one dimensional kink calculations by Hietarinta,¹ which we have applied, with some startling results, to Yang-Mills theory in four dimensions. The gauge field configuration we find, while not an exact solution of the field equations, is in the sense formulated by Hietarinta the most important quantum fluctuation about the stationary points of the action.

Let us consider the zero winding number sector. An exact solution is the pure gauge $A_\mu = g\partial_\mu g^{-1}$. What fluctuation about the pure gauge yields the next most important contribution to the sum over paths? Hietarinta has proposed a general algorithm which yields a precise answer to this question for suitable systems. The first step is the conventional gaussian approximation,

$$\begin{aligned}\delta S &= 0 \\ S &\approx S_g\end{aligned}\tag{1}$$

where S_g is the second order expansion of the action S about the classical solutions. In the next step, after approximating the maxima or peaks of e^{-S} by gaussians, one finds the "ridges" dropping off from the peaks.

The ridge is defined in function space by introducing a metric which is given by contours of constant value of the gaussian action S_g . Then the ridge is defined as the extremum of S with the constraint that S_g is held fixed. Introducing a Lagrange multiplier α this gives

$$\delta(S - \alpha S_g) = 0.\tag{2}$$

The solution to this yields a one parameter set of field configurations that continuously connects the extremum of the action with the ridge that gives the next most important contribution to the action.

To apply the formalism to Yang-Mills theory we limit ourselves to the ansatz

$$\begin{aligned} A_\mu &= f(x^2) g^{-1} \partial_\mu g, \\ g &= \frac{x \cdot \bar{\alpha}}{x}, \\ \bar{\alpha}_\mu &= (1, i\vec{\sigma}). \end{aligned} \quad (3)$$

It is obvious that for the general problem of instanton-anti-instanton configurations one needs several parameters to specify the top of the ridge. Our ansatz will, however, effectively fix the relative instanton-anti-instanton gauge orientation, thus allowing us to work with a one parameter family. The action is

$$S = \int d^4x \frac{1}{4g^2} F_{\mu\nu} F_{\mu\nu} = \frac{12\pi^2}{g^2} \int_{-\infty}^{\infty} \left[\left(\frac{df}{du} \right)^2 + 4f^2(f-1)^2 \right] du \quad (4)$$

where $u = \ln x$, yielding

$$S - \alpha S_g = \frac{12\pi^2}{g^2} \int_{-\infty}^{+\infty} \left[(1-\alpha) \left(\frac{df}{du} \right)^2 + 4f^2(f^2 - 2f + 1 - \alpha) \right] du \quad (5)$$

$f=1$ is the classical solution. With $\alpha=0$ we have a well-known one dimensional system with kink solutions; for general α with boundary condition $f(u=\pm\infty)=1$, one finds the following solution

$$f = \frac{x^2/\eta^2 + 4\rho^2/\eta^2 + \rho^2/x^2}{x^2/\eta^2 + 1 + \rho^2/x^2} \quad (6)$$

The dictionary connecting this to the solution of Hietarinta,¹ is

$$\phi = 2f - 1, \quad \eta^2 = 2e^{-T/\sqrt{2}}c, \quad \rho^2 = \frac{e^{-T/\sqrt{2}}}{2c}, \quad \alpha = \frac{1}{2}. \quad (7)$$

The stability condition (ridge not a valley) is

$$\eta^2/\rho^2 \geq 4. \quad (8)$$

We thus find the Yang-Mills field

$$A_\mu(x) = \left(\frac{x^2}{\eta^2} + \frac{4\rho^2}{\eta^2} + \frac{\rho^2}{x^2} \right) \left(\frac{x^2}{\eta^2} + 1 + \frac{\rho^2}{x^2} \right)^{-1} g^{-1} \partial_\mu g \quad (9)$$

as the most important fluctuation about the pure gauge, $A_\mu = g^{-1} \partial_\mu g$. The behavior of Eq.(9) at $x=0$ and $x=\infty$ is

$$A_\mu(x) = \begin{cases} \frac{2i \sigma_{\lambda\mu} x^\lambda \rho^2}{(x^2 + \rho^2)x^2}, & x \rightarrow 0, \text{ anti-instanton in singular gauge} \\ \frac{2i \sigma_{\lambda\mu} x^\lambda}{x^2 + \rho^2}, & x \rightarrow \infty, \text{ instanton in non-singular gauge} \end{cases} \quad (10)$$

We can move the instanton to the position c_μ by the conformal transformation²

$$x'_\mu = \frac{c^2 x_\mu - x^2 c_\mu}{(x - c)^2}, \quad (11)$$

$$A_\mu(x) = f(x') g^{-1}(x') \partial_\mu g(x') = \begin{cases} \frac{2i \sigma_{\lambda\mu} x^\lambda \rho^2}{(x^2 + \rho^2)x^2}, & x \rightarrow 0 \\ \frac{c^2}{\eta^2} \frac{2i \sigma_{\lambda\mu} (x - c)_\lambda}{(x - c)^2 + c^4/\eta^2}, & x \rightarrow c \end{cases} \quad (12)$$

which yields an instanton centered at c_μ , size η , provided $c^2 = \eta^2$. Thus, we have the configuration depicted in Fig. 1, an instanton-anti-instanton pair separated by the size of one member of the pair, with the other size free. Since $\eta^2/\rho^2 = c^2/\rho^2 \geq 4$, there is a minimum finite separation $\eta = c = 2\rho$ at which point the pair melts into the pure gauge.

The action of the configuration is

$$S = \frac{8\pi^2}{g^2} \left[3 - 5 \left(1 - \frac{4\rho^2}{\eta^2} \right) + 4 \left(1 - \frac{4\rho^2}{\eta^2} \right) - \frac{3 \ln \frac{\eta}{2\rho} \left(1 + \sqrt{1 - \frac{4\rho^2}{\eta^2}} \right)}{\sqrt{1 - \frac{4\rho^2}{\eta^2}}} \left(\frac{4\rho}{\eta} \right) \right] \quad (13)$$

which vanishes at $\eta = 2\rho$. To investigate the behavior of S for large separation we fix the product of the sizes, $\rho\eta$, as we let the separation $\eta = c = R$ grow. In this limit

$$S \approx \frac{16\pi^2}{g^2} - \frac{96\pi^2}{g^2} \frac{(\rho\eta)^2}{R^4} \quad (14)$$

in agreement with instanton-anti-instanton interaction obtained by other methods.²

To investigate the behavior for limiting separation we take $\eta = 2\rho + \epsilon$ and expand near $\epsilon = 0$, keeping $\rho\eta$ fixed. The result is

$$S \approx \frac{8\pi^2}{g^2} \frac{12}{5} \left(\frac{\epsilon}{\sqrt{\rho\eta}} \right)^2 \quad (15)$$

Let us consider the interpretation of this configuration in the framework of analog gas ideas, where classical pictures suggest that the cross-over from dilute to dense phases occurs when the instantons and anti-instantons start to overlap,³ and that the dense phase is one of multiply overlapping centers.

Surprisingly, this intuitive conclusion is not sustained by our configuration, which has a finite limiting separation, scaled by the size, at which point the system becomes pure gauge. Thus the cross-over regime is one closer to the conventional vacuum than a dilute gas; correspondingly, systems properly called dilute are probably much more dilute than a classical analog suggests.

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References

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2. D. Föster, Phys. Lett. 66B, 279 (1977).
3. C. G. Callan, R. Dashan, and D. Gross, Phys. Rev. D 17, (1978).

Figure Caption

The instanton-anti-instanton pair configuration, with separation given by one of the sizes.