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**An Eikonal Expansion of
the Vlasov-Maxwell
Equations Valid Near
Cyclotron Resonance**

H. Weitzner
D. B. Batchelor

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FOR THE UNITED STATES
DEPARTMENT OF ENERGY

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AN EIKONAL EXPANSION OF THE VLASOV-MAXWELL EQUATIONS
VALID NEAR CYCLOTRON RESONANCE

H. Weitzner
D. B. Batchelor

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CONTENTS

ABSTRACT	1
I. INTRODUCTION	2
II. FORMAL EXPANSION OF THE LINEARIZED VLASOV-MAXWELL SYSTEM	6
III. APPLICATION TO THE ELECTRON CYCLOTRON RESONANCE	12
IV. APPLICATION TO ION CYCLOTRON RESONANCE	25
V. SUMMARY	28
ACKNOWLEDGMENTS	29
APPENDIX	30

ABSTRACT

In the usual formulations of geometrical optics, the physics of the medium enters the equations through a conductivity tensor operator $\underline{g}$. An essential assumption in the subsequent expansion is that the magnitude of $\underline{g}^H$, the Hermitian part of $\underline{g}$, is much smaller than $\underline{g}^A$, the anti-Hermitian part. In a finite temperature plasma with $\omega_{pe} \sim |\Omega_e|$, this condition is always violated sufficiently close to cyclotron resonance, even though in many cases the waves are weakly damped and $\underline{k}$ is slowly varying. Simultaneously expanding the Vlasov equation and Maxwell equations and taking explicit account of the relative magnitude of the electric field components in the ordering scheme yields a formalism in terms of real rays, real eikonal function, and slowly varying amplitude that is valid at cyclotron resonance. It is assumed that $\omega_{pe} \sim |\Omega_e| \sim \omega$ are large, that $\omega \cong |\Omega_e|$, and that $v_e k/\omega$ is small. It is shown that when the waves are weakly damped at cyclotron resonance, the ray trajectories are to leading order exactly those of cold plasma theory. A Poynting theorem that can be put into the standard form is obtained by taking the expansion to first order. Care must be taken, however, to include finite temperature effects in calculating the zero order electric field. The theory is also applied to ion cyclotron resonance.

I. INTRODUCTION

The geometrical optics approximation is an extensively used device for the approximate solution of linear wave propagation problems. Geometrical optics is currently proving to be of great value in studying wave heating processes in the complicated geometry of modern fusion plasma experiments.¹⁻⁶ The assumptions for the approximation are that the wavelength is much shorter than the gradient scale lengths of the medium and that it is shorter than the characteristic growth or decay lengths of the wave. In the usual formulation of geometrical optics for weakly dispersive and dissipative inhomogeneous media, the physics of the medium enters the equations through a nonlocal conductivity tensor operator $\underline{\underline{g}}(\underline{x}, \underline{x}', t)$,

$$\underline{\underline{J}}(\underline{x}, t) = \int d^3 \underline{x}' dt' \underline{\underline{g}}(\underline{x}, \underline{x}', t - t') \cdot \underline{\underline{E}}(\underline{x}', t'), \quad (1)$$

where $\underline{\underline{J}}(\underline{x}, t)$ is the local plasma current and $\underline{\underline{E}}(\underline{x}, t)$ is the local electric field.^{7,8} An essential assumption in the subsequent derivation of the equations describing ray trajectories and field amplitudes is that the Hermitian part of $\underline{\underline{g}}$ is much smaller than the anti-Hermitian part,

$$||\underline{\underline{g}}^H|| \ll ||\underline{\underline{g}}^A||, \quad (2)$$

where $\underline{\underline{g}}^H = 1/2 (\underline{\underline{g}} + \underline{\underline{g}}^\dagger)$, $\underline{\underline{g}}^A = 1/2i (\underline{\underline{g}} - \underline{\underline{g}}^\dagger)$, and $\underline{\underline{g}}^\dagger$ denotes the conjugate transpose of $\underline{\underline{g}}$. This assumption ensures that the dispersion relation for the waves is manifestly real; hence, the ray trajectories $\underline{r}(t)$ and

$k(t)$ remain in real space. Because the presence of $\tilde{g}^H$ generally results in absorption or growth of the waves, this assumption also ensures that the waves are weakly damped. However, as we shall see, it is not at all necessary that $\tilde{g}^H$ be small in order to have weakly damped waves.

For application to problems of plasma heating, propagation of waves to and through cyclotron resonances is a matter of considerable importance. However, sufficiently near cyclotron resonance $[(\omega - \Omega_j)/\omega \ll 1]$, where $\Omega_j = e_j |B|/m_j c$ is the cyclotron frequency of the j th species, we find that $\tilde{g}^H$ is not small. In fact, at resonance, $\omega = \Omega_j$, the inequality in Eq. (2) is reversed no matter how low the plasma temperature is. Physically, this is because particles need not travel at the wave-phase velocity in order to be in cyclotron resonance. (Resonant denominators of the form $\omega - \ell\Omega_j - k_z v_z$ are small for the bulk of the distribution in v_z when $\omega = \ell\Omega_j$.) Despite the large number of resonant particles, investigation of the finite temperature dispersion relation for an infinite homogeneous plasma indicates that for a wide range of parameters, electromagnetic waves are only weakly damped near cyclotron resonance. It is the wave polarization that actually determines the strength of the resonance interaction.

Because weakly damped waves such as this satisfy the physical requirements of geometrical optics, a geometrical-optics-type expansion in terms of a real, rapidly varying phase function (eikonal function) and a slowly varying amplitude should be possible. It is the purpose of this paper to present such an approximation that is valid near cyclotron resonance and to provide an energy flux or Poynting theorem that describes the variations in field amplitude.

In Sec. II we expand in a formal small parameter $\eta \sim (Lk)^{-1}$, where k is the wave number and L is a scale length for variations in the plasma equilibrium or in the amplitude function. In our procedure the linearized Vlasov equation and the Maxwell equations are expanded. In Sec. III we specialize for the case of electron cyclotron waves by assuming that the wave frequency ω , the electron plasma ω_{pe} , and the electron cyclotron frequency Ω_e are all large and comparable. We assume ω to be near cyclotron resonance $(\omega - |\Omega_e|)/\omega \sim 0(\eta)$ and the electron thermal speed v_e to be much smaller than the wave-phase velocity $v_e k/\omega \sim 0(\eta)$. We also assume that the plasma beta is low enough to make equilibrium plasma currents negligible, that the equilibrium distribution functions are isotropic in velocity space, and that no equilibrium electric fields are present. Any of the latter assumptions could be dropped, but this would cause much unnecessary complication.

Results show that when the waves are weakly damped near cyclotron resonance, the ray trajectories are to leading order exactly those of the cold plasma theory. When the expansion is taken to first order in η , we obtain an energy flux relation or Poynting theorem that can ultimately be put into the standard form. However, care must be taken to properly include finite temperature effects in evaluating the zero order electric field.

In Sec. IV the procedure is applied to waves near the ion cyclotron frequency. We find that the weakly damped fast wave or magnetosonic wave can be readily treated by our method. The slow wave is heavily damped at cyclotron resonance, and no geometrical optics description is possible. In Sec. V the results are summarized. Derivation of the warm

plasma conductivity tensor in the $(+,-,\parallel)$ representation is outlined in the Appendix, and expansions appropriate for $\omega \cong |\Omega_e|$ and $\omega \cong \Omega_i$ are given.

II. FORMAL EXPANSION OF THE LINEARIZED VLASOV-MAXWELL SYSTEM

For a spatially inhomogeneous but time-invariant plasma, the linearized Vlasov-Maxwell equations can be expressed in the form

$$\begin{aligned} \nabla \times \tilde{\mathbf{E}} &= -\frac{1}{c} \frac{\partial \tilde{\mathbf{B}}}{\partial t}, \\ \nabla \times \tilde{\mathbf{B}} &= \frac{4\pi}{c} \tilde{\mathbf{J}} + \frac{1}{c} \frac{\partial \tilde{\mathbf{E}}}{\partial t}, \\ \frac{\partial \tilde{f}_j}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f}_j + \frac{e_j}{m_j c} \mathbf{v} \times \tilde{\mathbf{B}}_0 \cdot \frac{\partial \tilde{f}_j}{\partial \mathbf{v}} &= -\frac{e_j}{m_j} \tilde{\mathbf{E}} \cdot \frac{\partial \tilde{f}_j}{\partial \mathbf{v}}, \\ \tilde{\mathbf{J}} &= \sum_j \int d^3v \, e_j \mathbf{v} \tilde{f}_j, \end{aligned} \tag{3}$$

where each species j satisfies its own Vlasov equation, $\tilde{G}_j(\mathbf{v}, \mathbf{x})$ is the equilibrium distribution function, $\tilde{\mathbf{B}}_0$ is the equilibrium magnetic field, $\tilde{f}_j$ is the perturbed distribution function, $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are the perturbed electric and magnetic fields, and $\tilde{\omega}$ is the wave frequency. For simplicity the equilibrium distribution function is assumed to be isotropic in velocity space, $\tilde{G}_j(\mathbf{v}, \mathbf{x}) = \tilde{G}_j(v^2, \mathbf{x})$. We introduce the formal small parameter η and define scaled variables by the equations

$$\begin{aligned} \tilde{\mathbf{B}}(\mathbf{x}, t) &= \tilde{\mathbf{B}}(\mathbf{x}) e^{i[\Phi(\mathbf{x}) - \omega t]/\eta}, \\ \tilde{\mathbf{E}}(\mathbf{x}, t) &= \tilde{\mathbf{E}}(\mathbf{x}) e^{i[\Phi(\mathbf{x}) - \omega t]/\eta}, \\ \tilde{f}(\mathbf{x}, t) &= f(\mathbf{x}, t) e^{i[\Phi(\mathbf{x}) - \omega t]/\eta}, \end{aligned}$$

$$\tilde{G}_j(\tilde{v}, \tilde{x}) = \frac{G_j(\tilde{v}, \tilde{x})}{\eta^2},$$

$$\tilde{\omega} = \frac{\omega}{\eta}, \quad (4)$$

$$\frac{e_j |B_0(\tilde{x})|}{m_j c} = \frac{\Omega_j(\tilde{x})}{\eta},$$

where $\phi(\tilde{x})$ is the eikonal or phase function.

Our scaling assumptions are that the wave frequency $\tilde{\omega}$, cyclotron frequency $\tilde{\Omega}_j$, and plasma frequency $\tilde{\omega}_{pj}$ are all large and comparable. The scaling of $\tilde{G}_j$ is determined by the requirement that $\tilde{\omega}_{pj}^2 = 4\pi e_j^2/m_j \int dv^3 \tilde{G}_j(\tilde{v})$ be of order $\tilde{\omega}^2 = \omega^2/\eta^2$. The equilibrium spatial gradients and gradients in the amplitude functions $\tilde{E}(\tilde{x})$ and $\tilde{B}(\tilde{x})$ are assumed to be of order one.

In terms of the scaled variables, Eq. (3) becomes

$$i\tilde{k} \times \tilde{E} + \eta \tilde{\nabla} \times \tilde{E} = \frac{i\omega}{c} \tilde{B},$$

$$i\tilde{k} \times \tilde{B} + \eta \tilde{\nabla} \times \tilde{B} = \frac{4\pi}{c} \tilde{J} - \frac{i\omega}{c} \tilde{E}, \quad (5)$$

$$-i(\omega - \tilde{k} \cdot \tilde{v}) \tilde{f}_j - \Omega_j \frac{\partial \tilde{f}_j}{\partial \phi} + \eta \tilde{v} \cdot \tilde{\nabla} \tilde{f}_j = -\frac{e_j}{m_j} \tilde{E} \cdot \frac{\partial \tilde{G}_j}{\partial \tilde{v}},$$

where $\tilde{k}(\tilde{x}) = \tilde{\nabla} \phi(\tilde{x})$. We can see that the expansion parameter η appears explicitly in the scaled Vlasov equation. However, for the application to weakly damped waves near cyclotron resonance, there are other small parameters that appear in the Vlasov equation, namely the ratio of

thermal speed to wave phase speed $\omega v_j/k$ and the frequency deviation from resonance $(\omega - |\Omega_j|)/\omega$. These we also assume to be of order η . It is this complexity that leads to the rather intricate arguments employed to obtain a reasonable expansion.

Following the usual procedure, Eq. (5) is now expanded in an asymptotic power series in η ,

$$\tilde{E} = \tilde{E}^0 + \eta \tilde{E}^1 + \dots, \quad (6)$$

and similarly for the other variables. To zero order in η we obtain

$$\begin{aligned} i\tilde{k} \times \tilde{E}^0 &= \frac{i\omega}{c} \tilde{B}^0, \\ i\tilde{k} \times \tilde{B}^0 + \frac{i\omega}{c} \tilde{E}^0 &= \frac{4\pi}{c} \tilde{J}^0, \\ i(\omega - \tilde{k} \cdot \tilde{v})f_j^0 + \Omega_j \frac{\partial f_j^0}{\partial \phi} &= \frac{e_j}{m_j} \tilde{E}^0(\tilde{x}) \cdot \frac{\partial G_j}{\partial \tilde{v}}, \end{aligned} \quad (7)$$

$$\tilde{J}^0 = \underline{\underline{g}} \cdot \tilde{E}^0 \equiv \sum_j e_j \int d^3v \tilde{v} f_j^0(\tilde{v}).$$

To this order the equations are just those for waves in an infinite uniform plasma. The conductivity tensor $\underline{\underline{g}}$ is the standard result for a finite temperature plasma;^{9,10} its properties are described in the Appendix. We will find that the zero order equations contain terms that, although determined in zero order, have magnitude of order $\omega v_j/k$. These terms are subsequently moved to first order, thereby considerably modifying the form of the equations from the usual warm plasma formulation.

It is convenient to introduce a local coordinate system in which $\underline{B}$ is in the z direction and to define the quantities:

$$E_{\pm} = E_x \pm iE_y, \quad E_{\parallel} = E_z,$$

$$\underline{k} = (k_{\perp} \cos \psi, k_{\perp} \sin \psi, k_{\parallel}) ,$$

$$k_{\pm} = k_x \pm ik_y = k_{\perp} e^{\pm i\psi} ,$$

$$v_{\pm} = v_x \pm iv_y = v_{\perp} e^{\pm i\phi} , \quad (8)$$

$$J_{\pm}^0 = J_x^0 \pm iJ_y^0 = \sum_j e_j \int_0^{\infty} dv_{\perp} v_{\perp} \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{2\pi} d\phi v_{\perp} e^{\pm i\phi} f_j^0(v_{\perp}, v_{\parallel}, \phi, \underline{x})$$

$$\equiv \sigma_{\pm+} E_+^0 + \sigma_{\pm-} E_-^0 + \sigma_{\pm\parallel} E_{\parallel}^0 ,$$

$$J_{\parallel}^0 = \sum_j e_j \int d^3 v v_{\parallel} f_j^0(\underline{v}, \underline{x}) \equiv \sigma_{\parallel+} E_+^0 + \sigma_{\parallel-} E_-^0 + \sigma_{\parallel\parallel} E_{\parallel}^0 .$$

In terms of these quantities, Eq. (7) becomes

$$\left(1 - \frac{n_{\parallel}^2}{2} - n_{\parallel}^2 + \frac{4\pi i}{\omega} \sigma_{++}\right) E_+^0 + \left(\frac{n_{\perp}^2}{2} e^{2i\psi} + \frac{4\pi i}{\omega} \sigma_{+-}\right) E_-^0$$

$$+ \left(n_{\parallel} n_{\perp} e^{i\psi} + \frac{4\pi i}{\omega} \sigma_{+\parallel}\right) E_{\parallel}^0 = 0 ,$$

$$\left(\frac{n_{\perp}^2}{2} e^{-2i\psi} + \frac{4\pi i}{\omega} \sigma_{-+}\right) E_+^0 + \left(1 - \frac{n_{\perp}^2}{2} - n_{\parallel}^2 + \frac{4\pi i}{\omega} \sigma_{--}\right) E_-^0$$

$$+ \left(n_{\parallel} n_{\perp} e^{-i\psi} + \frac{4\pi i}{\omega} \sigma_{-\parallel}\right) E_{\parallel}^0 = 0 , \quad (9)$$

$$\left(\frac{n_{\parallel} n_{\perp}}{2} e^{-i\psi} + \frac{4\pi i}{\omega} \sigma_{\parallel+} \right) E_{+}^0 + \left(\frac{n_{\parallel} n_{\perp}}{2} e^{i\psi} + \frac{4\pi i}{\omega} \sigma_{\parallel-} \right) E_{-}^0 + \left(1 - n_{\parallel}^2 + \frac{4\pi i}{\omega} \sigma_{\parallel\parallel} \right) E_{\parallel}^0 = 0 ,$$

$$\begin{aligned} \frac{\partial f_j^0}{\partial \phi} + i \left[\frac{\omega - k_{\parallel} v_{\parallel}}{\Omega_j} - \frac{k_{\perp} v_{\perp}}{\Omega_j} \cos(\psi - \phi) \right] f_j^0 \\ = \frac{e_j}{m_j \Omega_j} \left[\frac{1}{2} (E_{-} e^{i\phi} + E_{+} e^{-i\phi}) \frac{\partial G_j}{\partial v_{\perp}} + E_{\parallel} \frac{\partial G_j}{\partial v_{\parallel}} \right] , \end{aligned}$$

where $\tilde{n} = \frac{c}{\omega} k$ is the wave refractive index.

From Eq. (5) the first order Maxwell equations are

$$i \tilde{k} \times \tilde{E}^1 + \nabla \times \tilde{E}^0 = \frac{i\omega}{c} \tilde{B}^1 \quad (10a)$$

and

$$i \tilde{k} \times \tilde{B}^1 + \nabla \times \tilde{B}^0 = - \frac{i\omega}{c} \tilde{E}^1 + \frac{4\pi}{c_1} \tilde{J}^1 , \quad (10b)$$

and the linearized Vlasov equation is

$$-i(\omega - \tilde{k} \cdot \tilde{v}) f_j^1 + \Omega_j \frac{\partial f_j^1}{\partial \phi} = -\tilde{v} \cdot \nabla f_j^0 - \frac{e_j}{m_j} \tilde{E}^1 \cdot \frac{\partial G_j}{\partial \tilde{v}} . \quad (11)$$

It is clear that the last term on the right of Eq. (11) is just $\tilde{g} \cdot \tilde{E}^1$, where $\tilde{g}$ is defined in Eq. (8), while the first term on the right is the contribution to the first order current due to plasma inhomogeneity and variations in field amplitude. We separate off the part of f^1 due to inhomogeneity and write

$$\tilde{J}^1 = \sum_j e_j \int d^3v \, \tilde{v} f_j^1 = \tilde{g} \cdot \tilde{E}^1 + \sum_j e_j \int d^3v \, \tilde{v} h_j(\tilde{v}, \tilde{x}) \equiv \tilde{g} \cdot \tilde{E}^1, \quad (12)$$

where $h_j(\tilde{v}, \tilde{x})$ satisfies

$$-i(\omega - \tilde{k} \cdot \tilde{v})h_j + \Omega_j \frac{\partial h_j}{\partial \phi} = -\tilde{v} \cdot \tilde{\nabla} f_j^0. \quad (13)$$

III. APPLICATION TO THE ELECTRON CYCLOTRON RESONANCE

To this point the results have been rather general, but now we specialize for the case of wave propagation near the electron cyclotron resonance $(\omega - |\Omega_e|)/\omega = O(\eta)$ and for situations in which the electron thermal velocity v_e is small in comparison to the wave-phase velocity $v_e k/|\Omega_e| = O(\eta)$. The ion current is neglected throughout. The electron conductivity tensor, accurate to order η , is given in Eq. (A.6). These relations indicate that all components of $\underline{g}$ are $O(1)$ or smaller, except for σ_{--} , which is $O(1/\eta)$. We conclude that if $n = O(1)$, then any solution of the system Eq. (9) must have $E_-^0 = O(\eta)$. This breaks down if the refractive index becomes large enough that $n^2 = O(1/\eta)$. Because a large refractive index is generally associated with strong absorption, in which case a geometrical optics expansion is invalid, we restrict consideration to the case $n^2 \ll 1/\eta$. Then E_-^0 is indeed $O(\eta)$. To carry the expansion to higher order, it will be necessary to solve systems such as Eq. (9) with inhomogeneous terms determined in lower order. Therefore, we expect that in any order, $E_-^{(n)}$ is $O(\eta)$ so that E_- has asymptotic expansion:

$$E_- = \eta[E_-^0 + \eta E_-^1 + \eta^2 E_-^2 + \dots] \quad (14)$$

We still identify ηE_-^0 as zero order because it is determined by the zero order equations.

Making use of Eq. (A.7), Eq. (9) can now be written accurate to order η^0 ,

$$\left(1 - \frac{n_{\perp}^2}{2} - n_{\parallel}^2 + \frac{4\pi i}{\omega} \sigma_{++}^0\right) E_{+}^0 + n_{\perp} n_{\parallel} e^{i\psi} E_{\parallel}^0 = 0, \quad (15a)$$

$$\frac{n_{\perp}^2}{2} e^{-2i\psi} E_{+}^0 + \frac{4\pi i}{\omega} \sigma_{--}^0 E_{-}^0 + \left(n_{\perp} n_{\parallel} e^{-i\psi} + \frac{4\pi i}{\omega} \sigma_{-||}^0\right) E_{\parallel}^0 = 0, \quad (15b)$$

and

$$\frac{n_{\perp} n_{\parallel}}{2} e^{-i\psi} E_{+}^0 + \left(1 - n_{\perp}^2 + \frac{4\pi i}{\omega} \sigma_{||||}^0\right) E_{\parallel}^0 = 0, \quad (15c)$$

where we have assumed $E_{-}^0 = 0(\eta)$. Equations (15a) and (15c) determine the dispersion relation,

$$D = n_{\perp}^2 n^2 - \left[2 \left(1 - \frac{\omega^2}{\Omega_e^2}\right) n^2 + n_{\perp}^2\right] + \left(1 - \frac{\omega^2}{\Omega_e^2}\right) \left(2 - \frac{\omega^2}{\Omega_e^2}\right) = 0, \quad (16)$$

which is precisely the cold plasma dispersion relation evaluated at $\omega = |\Omega_e|$. Because the first order partial differential equations for the ray trajectories follow directly from the dispersion relation^{9,11}

$$\frac{dx}{dt} = v_g = \frac{\partial \omega}{\partial k} = - \frac{\partial D / \partial k}{\partial D / \partial \omega},$$

$$\frac{dk}{dt} = - \frac{\partial \omega}{\partial x} = \frac{\partial D / \partial x}{\partial D / \partial \omega},$$

we have established that near electron cyclotron resonance, the rays for the warm plasma being considered here are exactly the same as the rays in cold plasma theory. In particular, the rays and the eikonal function $\phi(\underline{x})$ are real. It is also easy to show that away from resonances, the

Vlasov theory reduces to the cold plasma theory provided the wave-phase velocity is much greater than the electron thermal velocity. Equation (15b) serves to determine $E_-^{(0)}$:

$$E_-^0 e^{i\psi} = \frac{1}{(\omega_{pe}^2 / \Omega_e^2) Z(\xi)} \left(\frac{n_1^2}{2} e^{-i\psi} E_+^0 + \left\{ n_1 n_{\parallel} + \frac{\omega_{pe}^2}{\Omega_e^2} \frac{k_{\perp}}{k_{\parallel}} [1 + \xi Z(\xi)] \right\} E_{\parallel}^0 \right), \quad (17)$$

where ξ and $Z(\xi)$ are defined in Eq. (A.6). At resonance, this is a considerable modification from the cold plasma result, where $E_-^0 = 0$.

Because Eq. (15) determines E_-^0 only to within a complex multiplicative factor, it is necessary to go to the first order equations to obtain the amplitude or energy flux relation. To obtain Eq. (15) we dropped from Eq. (9) all terms containing E_-^0 except the $\sigma_{--}^0 E_-^0$ terms. These must be included in the first order equations. In addition, the small $[O(\eta)]$ terms of $\underline{g}$ that were dropped from the zero order current are grouped into $\hat{j}^0$ and included in first order [see Eq. (A.7)]:

$$\hat{j}_+^0 = 0,$$

$$\hat{j}_-^0 = \sigma_{-+}^1 E_+^0,$$

$$\hat{j}_{\parallel}^0 = \sigma_{\parallel-}^0 E_-^0 + \sigma_{\parallel\parallel}^1 E_{\parallel}^0.$$

With the vector $\underline{E}_-^0$ defined as $\underline{E}_-^0 = E_-/2 (1, i, 0)$, the corrected first order equations [Eq. (10)] can be written in the form

$$ik \times (\underline{E}^1 + \underline{E}_-^0) = \frac{i\omega}{c} \underline{B}^1, \quad (18a)$$

$$\mathbf{k} \times \mathbf{B}^1 + \nabla \times \mathbf{B}^0 = -\frac{i\omega}{c} (\mathbf{E}^1 + \mathbf{E}^0) + \frac{4\pi}{c} \hat{\mathbf{J}}^0 + \frac{4\pi}{c} \mathbf{J}^1. \quad (18b)$$

In order that Eq. (18) have a solution, there is a consistency condition on $\mathbf{E}^0$ and $\mathbf{B}^0$ that provides the multiplicative amplitude factor. Dotting $\mathbf{B}^{0*}$ into Eq. (18a), defining $\hat{\mathbf{E}}^0 = \mathbf{E}^0 - \mathbf{E}^0_-$, dotting $\mathbf{E}^{0*}$ into Eq. (18b), and subtracting yields

$$\begin{aligned} \hat{\mathbf{E}}^{0*} \cdot (\nabla \times \mathbf{B}^0) - \mathbf{B}^{0*} \cdot (\nabla \times \hat{\mathbf{E}}^0) &= \frac{4\pi}{c} (\hat{\mathbf{E}}^{0*} \cdot \hat{\mathbf{J}}_0 + \mathbf{J}^{0*} \cdot \mathbf{E}^0_-) \\ &+ \frac{4\pi}{c} (\mathbf{J}^{0*} \cdot \mathbf{E}^1 + \hat{\mathbf{E}}^{0*} \cdot \mathbf{J}^1). \end{aligned} \quad (19)$$

Using Eq. (12) and exploiting the symmetries of $\mathbf{g}$, we find that

$$\begin{aligned} \mathbf{J}^{0*} \cdot \mathbf{E}_1 + \hat{\mathbf{E}}^{0*} \cdot \mathbf{J}_1 &= \frac{1}{2} \sigma_{--}^{0*} \mathbf{E}^0_- \mathbf{E}^1_- + \left(\frac{1}{2} \sigma_{-||}^{0*} + \sigma_{||-}^0 \right) \mathbf{E}_{||}^{0*} \mathbf{E}^1_- \\ &- e \hat{\mathbf{E}}^0 \cdot \int d^3v \mathbf{v} h_j(\mathbf{v}, \mathbf{x}). \end{aligned} \quad (20)$$

In view of the form of the asymptotic expansion for $\mathbf{E}_-$ [i.e., $\mathbf{E}^1_- = O(n^2)$], the first two terms of Eq. (20) can be dropped from the first order equations. We must now evaluate the more involved quantity $\hat{\mathbf{E}}^{0*} \cdot \int d^3v \mathbf{v} h_e(\mathbf{v}, \mathbf{x})$. Since we shall finally show that to leading order this quantity is zero, largely because the wave-phase velocity is much greater than v_t , we shall only outline the derivation, carrying the structure of $h_e(\mathbf{v}, \mathbf{x})$. From Eq. (A.4) we see that, consistent with our ordering,

$$\begin{aligned}
-4\pi e f^0 = \frac{\omega_{pe}^2}{\Omega_e v_e} \left\{ e^{-i(\phi-\psi)} \left[\frac{E_-^0}{(\omega_* - 1)} \frac{1}{2} \frac{\partial F}{\partial u_{\perp}} + \frac{\epsilon}{\omega_* - 1} \frac{\partial F}{\partial u_{\parallel}} E_{\parallel}^0 + O(\eta) \right] \right. \\
\left. + \sum_{\substack{n=-\infty \\ n \neq 1}}^{\infty} e^{-in(\phi-\psi)} A_n(u_{\perp}, u_{\parallel}, \tilde{x}) \right\}, \quad (21)
\end{aligned}$$

where the leading factor of $1/v_e$ in Eq. (21) comes from the replacement of the $v_{\perp}, v_{\parallel}$ derivatives by $u_{\perp}, u_{\parallel}$ derivatives. We see that both $E_-^0/(\omega_* - 1)$ and $\epsilon/(\omega_* - 1)$ are $O(1)$ in η . We now apply the differential operator $\tilde{v} \cdot \tilde{\nabla} = v_e (\tilde{u} \cdot \tilde{\nabla})$ to f^0 to obtain the inhomogeneous term in Eq. (13),

$$\begin{aligned}
-4\pi e \tilde{v} \cdot \tilde{\nabla} f^0 = \mathcal{D} \frac{\omega_{pe}^2}{\Omega_e} \left\{ \frac{e^{-i(\phi-\psi)}}{[\omega - \Omega(\tilde{x})/k_{\parallel} v_e] - u_{\parallel}} c(u_{\perp}, u_{\parallel}, \tilde{x}) \right. \\
\left. + \sum_{\ell \neq 1} e^{-i\ell(\phi-\psi)} A_{\ell}(u_{\perp}, u_{\parallel}, \tilde{x}) \right\}, \quad (22)
\end{aligned}$$

where $\mathcal{D}$ is a spatial derivative operator. It is now possible to determine $h(\tilde{v}, \tilde{x})$ by applying Eqs. (A.2) and (A.3) to Eq. (22). It is significant to note that spatial derivatives do not change the order of magnitude of terms in Eq. (22) even when applied to a resonance denominator $\{[\omega - \Omega_e(\tilde{x})]/k_{\parallel} v_e - u_{\parallel}\}^{-1}$. Assuming that ω has a small imaginary part, integrals of the form

$$\int d\tilde{u} \, \phi(\tilde{u}) \tilde{\nabla} \frac{1}{[\omega - \Omega_e(\tilde{x})/k_z v_e] - u_{\parallel}}$$

are well defined and of order one, provided that $\int d\tilde{u} \, \phi(\tilde{u})$ is order one. Such integrals are the moments of $h(\tilde{v}, \tilde{x})$ needed to determine J_0^1 . We

are, in effect, treating the distribution function $f(\underline{x}, \underline{v})$ as a distribution or generalized function in the sense of Dirac or Schwartz. Such an approach is consistent with other experience with the Vlasov equation. Using Eq. (22) to solve for $h(\underline{v}, \underline{x})$, we find

$$-4\pi e h(\underline{x}, \underline{v}) = \frac{1}{n} e^{-i(\phi-\psi)} D_1(u_\perp, u_\parallel, \underline{x}) + \sum_{n \neq 1} e^{-in(\phi-\psi)} D_n(u_\perp, u_\parallel, \underline{x}) , \quad (23)$$

where $D_1(u_\perp, u_\parallel, \underline{x})$ are all $O(1)$ in n , in the generalized function sense [i.e., resonance denominators are now $O(1)$]. Taking moments, we find that

$$[\int d^3v \underline{v} h(\underline{v}, \underline{x})]_+ = v_e \int du_\perp du_\parallel u_\perp^2 e^{i\phi} h(\underline{x}, u_\perp, u_\parallel) = O(n) ,$$

$$[\int dv \underline{v} h(\underline{v}, \underline{x})]_- = v_e \int du_\perp u_\perp u_\parallel^2 e^{-i\phi} h(\underline{x}, u_\perp, u_\parallel) = O(1) ,$$

and

$$\int dv v_\parallel h(\underline{v}, \underline{x}) = v_e \int du_\perp du_\parallel u_\perp u_\parallel h(\underline{x}, u_\perp, u_\parallel) = O(n)$$

so that even though the minus component is $O(1)$, the dot product $\hat{E}^{0*} \cdot J^1 = O(n)$ and drops out of the first order equations.

With the above results Eq. (19) reduces to

$$\begin{aligned} \underline{\nabla} \cdot (\hat{E}^{0*} \times \underline{B}^0) - 2i \operatorname{Im} \{ \underline{B}^{0*} \cdot (\underline{\nabla} \times \hat{E}^{0*}) \} \\ + \frac{4\pi}{c} (\hat{E}^{0*} \cdot \hat{J}_0 + \underline{J}^{0*} \cdot \underline{E}^0) = 0 . \quad (24) \end{aligned}$$

The real part of Eq. (24) gives the equation of energy flux or the Poynting theorem for the system, whereas the imaginary part contains information on the variation of the phase of $\hat{E}^0$ and B^0 along a ray. This information is not needed for energy flux and absorption calculations but could be used to calculate a small correction to the effective wave vector k due to thermal or inhomogeneity effects.

The real part of Eq. (24) gives the equation

$$\nabla \cdot \underline{S} + \text{Re} \{ \hat{E}^{0*} \cdot \underline{J}_0 + J^{0*} \cdot \underline{E}^0 \} = 0, \quad (25)$$

where

$$\underline{S} = \frac{c}{4\pi} \text{Re} \{ \hat{E}^{0*} \cdot \underline{B}^0 \}$$

is the electromagnetic energy flux with the $\underline{E}^0$ contribution excluded.

It should be noted that the contribution to $\underline{S}$ of energy flux due to plasma thermal motion (i.e., terms of the form $\underline{E}^* \cdot \partial \underline{g} / \partial \underline{k} \cdot \underline{E}$) is higher order in η . Using Eq. (17), Eq. (25) can be expressed

$$\nabla \cdot \underline{S} + \text{Re} \left\{ \frac{1}{2} \sigma_{\perp\perp}^{0*} |\underline{E}_{\perp}^0|^2 + \sigma_{\parallel\parallel}^1 |\underline{E}_{\parallel}^0|^2 + 2 \text{Re} \{ \sigma_{\parallel\perp}^0 \} \underline{E}_{\perp}^0 \underline{E}_{\parallel}^{0*} \right\}. \quad (26)$$

It is easy to show that to order η , this is identical to

$$\nabla \cdot \underline{S} = -\text{Re} \{ \hat{E}^{0*} \cdot \underline{J}^0 \} = -\text{Re} \{ \hat{E}^{0*} \cdot \underline{g} \cdot \underline{E}^0 \}, \quad (27)$$

where the $\underline{E}_{\perp}^0$ contribution is included in $\underline{E}^0$ and $\sigma_{\perp\perp}^1$ is included in $\underline{g}$.

This is the standard form for the Poynting theorem except that $\underline{E}_{\perp}^0$ must be calculated from Eqs. (7) and (8). If the plasma is homogeneous

or nearly so and if the wave source is at a great distance from the observation point, we can assume that the wave fields locally have the form of an inhomogeneous plane wave,

$$\tilde{E}(\tilde{x}, t) = E_0 e^{i[k_{\tilde{r}} + ik_{\tilde{i}}] \cdot \tilde{x} - \omega t},$$

where $|k_{\tilde{i}}| \ll |k_{\tilde{r}}|$ (i.e., we have made an ansatz on the slowly varying amplitude function). Using this in Eq. (27) gives

$$2k_{\tilde{i}} \cdot \tilde{S} = \text{Re} \left\{ \frac{1}{2} \sigma_{--}^0 |E_-^0|^2 + \sigma_{||}^1 |E_{||}^0|^2 + 2 \text{Re} \{ \sigma_{||-}^0 \} E_-^0 E_{||}^{0*} \right\}. \quad (28)$$

We see that the component of $k_{\tilde{i}}$ along the direction of the group velocity is independent of the direction chosen for $k_{\tilde{i}}$. We choose $k_{\tilde{i}}$ to be parallel to $\tilde{S}$, use Eq. (15b) to express E_-^0 in terms of $E_{||}^0$, and obtain an expression for the imaginary part of the refractive index,

$$n_i = - \frac{n_{\perp}^2 \left[1 - n^2 - 2(\omega_{pe}^2 / \omega^2) \right]^2 |E_{||}|^2}{4n_{||} 4\pi |S|} \frac{\omega^2}{\omega_{pe}^2} \frac{v_c}{c} \text{Im} \left\{ \frac{1}{Z(\xi)} \right\}. \quad (29)$$

Using Eq. (15c) to express $|S|$ in terms of $E_{||}$,

$$|S| = \frac{c |E_{||}|^2}{4\pi} \frac{\left[1 - n^2 - (\omega_{pe}^2 / \Omega_e^2) \right]}{n_{\perp}^2 n_{||}^2} \left\{ n_{\perp}^2 \left[2 \left(1 - n_{\perp}^2 - \frac{\omega_{pe}^2}{\Omega_e^2} \right) + (1 - n_{||}^2) \right]^2 + n_{||}^2 \left[2 \left(1 - n_{\perp}^2 - \frac{\omega_{pe}^2}{\Omega_e^2} \right) + n_{\perp}^2 \right]^2 \right\}^{1/2}, \quad (30)$$

we obtain for n_i

$$n_{\perp} = \frac{n_{\parallel} n_{\perp}^4 \left(1 - n_{\perp}^2 - 2 \frac{\omega_{pe}^2}{\Omega_e^2}\right)^2}{4 \left(1 - n_{\perp}^2 - \frac{\omega_{pe}^2}{\Omega_e^2}\right)}$$

$$\times \frac{\frac{\Omega_e^2}{\omega_{pe}^2} \frac{v_e}{c} \operatorname{Im} \left\{ \frac{1}{Z(\xi)} \right\}}{\left\{ n_{\perp}^2 \left[2 \left(1 - n_{\perp}^2 - \frac{\omega_{pe}^2}{\Omega_e^2}\right)^2 + (1 - n_{\parallel}^2) \right]^2 + n_{\parallel}^2 \left[2 \left(1 - n_{\perp}^2 - \frac{\omega_{pe}^2}{\Omega_e^2}\right) + n_{\perp}^2 \right]^2 \right\}^{1/2}} .$$

(31)

We have verified the analysis by comparing damping rates obtained from Eq. (31) with those determined by numerically solving the full finite temperature dispersion relation. In the numerical solution we let $\tilde{k} = \tilde{k}_r + i\tilde{k}_i$, where $\tilde{k}_i$ was chosen in the direction of the group velocity $\tilde{s}$ for the given values of ω_{pe} , Ω_e , and $\tilde{k}_r$. The agreement was virtually exact for weakly damped waves (e.g., $|\tilde{k}_i|/|\tilde{k}_r| \sim 10^{-3}$) such as ordinary mode and was within a few percent even for fairly heavily damped waves ($|\tilde{k}_i|/|\tilde{k}_r| \sim 0.2$).

It should be mentioned that even if the plasma is not homogeneous, Eqs. (28) or (31) can be used to calculate that portion of $\nabla \cdot \tilde{s}$ that is due to energy absorption. In that case, another term is present that describes the field variations due to focusing and defocusing of the rays.

We now return to consideration of the validity of the expansion procedure. The key assumptions are that $kv_e/\omega = nv_e/c = 0(\eta)$ and that

$|\sigma_{--}| \sim (\omega_{pe}^2/\Omega_e^2)(c/v_e)/n_z \gg n^2$. For plasmas of moderate temperature, $v_e/c \ll 1$, and relatively high density, $\omega_{pe}^2/\Omega_e^2 \sim 0(1)$, this requires that the refractive index remain order one at resonance. Now the magnitude of n at resonance depends sensitively on the angle of propagation. Solving the zero order dispersion relation, Eq. (16), for n^2 as a function of θ (the angle between $\underline{n}$ and $\underline{B}$) yields

$$n_{\pm}^2 = \frac{2\left(1 - \frac{\omega_{pe}^2}{\Omega_e^2}\right) + \sin^2 \theta \pm \left\{ \left[2\left(1 - \frac{\omega_{pe}^2}{\Omega_e^2}\right) + \sin^2 \theta \right]^2 - 4\left(1 - \frac{\omega_{pe}^2}{\Omega_e^2}\right) \sin^2 \theta \right\}^{1/2}}{2 \sin^2 \theta}, \quad (32)$$

where + and - refer to the extraordinary and ordinary modes, respectively. This is $0(1)$ except for $\theta \cong 0$, in which case

$$n_{\pm}^2 \cong \begin{cases} 2 \frac{[1 - (\omega_{pe}^2/\Omega_e^2)]}{\sin^2 \theta} & \text{extraordinary mode} \\ 1 - \frac{1}{2} \frac{\omega_{pe}^2}{\Omega_e^2} & \text{ordinary mode} \end{cases}$$

The most stringent condition is imposed by the requirement that

$|\sigma_{--}| \gg n^2$ for the extraordinary mode, which means that

$$\sin^3 \theta > \frac{\Omega_e^2}{\omega_{pe}^2} \frac{v_e}{c} \left[2 \left(1 - \frac{\omega_{pe}^2}{\Omega_e^2} \right) \right]^{3/2}.$$

On the other hand, for propagation angles near $\theta = \pi/2$ ($n_{\parallel} \rightarrow 0$), Eq. (16) can be solved for $n_{\perp}^2$ to give

$$n_{\perp}^2 \pm \cong \begin{cases} 2 - \frac{\omega_{pe}^2}{\Omega_e^2} (1 - n_{\parallel}^2) & \text{extraordinary mode} \\ \left(1 - \frac{\omega_{pe}^2}{\Omega_e^2}\right) (1 - n_{\parallel}^2) & \text{ordinary mode} \end{cases}$$

For the ordinary mode branch, we see from Eq. (31) that $n_{\perp}$, which is proportional to the factor $n_{\parallel} / [1 - n_{\perp}^2 - (\omega_{pe}^2 / \Omega_e^2)]$, diverges as $1/n_{\parallel}$. Our ordering, therefore, breaks down for sufficiently small $n_{\parallel}$. Actually, the formulation of cyclotron absorption in terms of a homogeneous plasma, nonrelativistic dispersion relation also breaks down for waves propagating nearly perpendicular to the magnetic field ($\theta \cong \pi/2$). In that case, resonance-broadening processes other than the thermal Doppler broadening described by $Z(\xi)$ must be included. Relativistic broadening is important for $n_z \lesssim v_e/c$,¹² and broadening due to magnetic field inhomogeneity becomes important for $n_z < (c/v)(k_{\perp} L)^{-1}$, where $L = (d \ln B/dx)^{-1}$ is the magnetic field scale length.¹³ In either case the effect is to give nonzero absorption of the extraordinary mode and to limit the local damping rate of the ordinary mode at perpendicular propagation to a finite value.

The angle between $\mathbf{k}$ and $\mathbf{B}$, which is actually obtained at resonance, depends entirely on the plasma geometry. If, for example, the gradients in density and $\mathbf{B}$ are nearly perpendicular to the field lines (perpendicular

stratification), then $n_{\parallel}$ is nearly constant whereas all variation is in $n_{\perp}$. In this case the plasma resonance for the extraordinary mode occurs at the upper hybrid frequency $\omega = (\omega_{pe}^2 + \Omega_e^2)^{1/2}$ rather than at $\omega = |\Omega_e|$, and there is no dramatic change in angle for either mode at the cyclotron resonance. As determined from Eq. (16), the ray paths in this geometry should be accurate in all cases except those involving extraordinary mode rays launched nearly parallel to the magnetic field and ordinary mode rays launched nearly perpendicular to $\tilde{B}$. In either of these cases the waves are heavily damped so that the wave energy is completely absorbed at resonance and the concept of ray paths is of little significance.

In plasmas having strong gradients in $\tilde{B}$ along the field lines, such as mirror machines or ELMO Bumpy Torus (EBT), the ray behavior is substantially different. In this geometry, which corresponds to the magnetic beach of Stix,¹⁴ the components of $\tilde{n}$ perpendicular to $\tilde{B}$ are nearly constant and all variation is in $n_{\parallel}$. The plasma resonance is at $\omega = |\Omega_e|$ so that $n_{\parallel}$ becomes large as the ray approaches resonance from the accessible high field side. Thus, $\tilde{n}$ turns parallel to $\tilde{B}$, the group velocity $\tilde{v}_g$ turns parallel to the resonance surface, and the parallel phase velocity $c/n_{\parallel}$ becomes slow. The large number of resonant particles produces very strong cyclotron damping even for relatively low density, low temperature plasmas. In this situation our formalism will always break down sufficiently near resonance. Again, this is not really a serious difficulty because the waves are so heavily damped that the ray paths are of little importance. Ray tracing studies of magnetic geometry in EBT, which does have strong gradients in $\tilde{B}$ along field lines, confirm

that the energy in a ray is completely absorbed before reaching the resonance surface when the plasma density is sufficiently high that

$$\omega_{pe}^2 / \Omega_e^2 (c/v_e) \gtrsim 1.3$$

In concluding this section we consider higher harmonic electron cyclotron resonances. In particular, it is easy to see from Eq. (A.5) that for $\omega \cong 2|\Omega_e|$, the zero order conductivity tensor is anti-Hermitian, coinciding with the results of cold plasma theory. The Hermitian, resonant part is $O(n)$ or higher. Ray tracing near this resonance can thus be treated by the standard methods or those given here. Again, the rays are given by cold plasma theory and an energy absorption formula of the form Eq. (26) is obtained. All components of $\underline{E}$ are $O(1)$ and enter in the cold plasma dispersion relation. The absorption enters in order n as it does for the fundamental resonance. Clearly, all other electron cyclotron resonances are much weaker in energy absorption.

IV. APPLICATION TO ION CYCLOTRON RESONANCE

It is of interest to apply the methods of the previous section to the propagation of waves near the ion cyclotron frequency. For the ions, we make assumptions similar to those for electrons in previous sections; namely, $v_i k / \omega \sim 0(\eta)$ and $(\omega - \Omega_i) / \omega = 0(\eta)$. The scaling is again given by Eq. (4), although we retain the assumption that $\omega_{pe}^2 / \Omega_e^2 \sim 0(1)$, which implies $\omega_{pi}^2 / \Omega_i^2 \sim 0(m_i / m_e)$. With these assumptions the ion conductivity tensor accurate to $\omega_{pi}^2 / \Omega_i \times 0(\eta)$ is given in Eq. (A.8). In this case we have $\sigma_{++} = \omega_{pi}^2 / \Omega_i \times 0(1/\eta)$, although all other elements are $\omega_{pi}^2 / \Omega_i \times 0(1)$ or smaller. For simplicity we assume that the electrons are sufficiently cold that electron Landau damping does not play a role, $v_e k / \omega = 0(\eta)$, in which case the electron conductivity is just the cold plasma result,

$$4\pi\sigma_{++}^e = -4\pi\sigma_{--}^e = -\frac{i\omega_{pe}^2}{\Omega_e}, \quad (33)$$

$$4\pi\sigma_{|||}^e = \frac{i\omega_{pe}^2}{\omega}.$$

This is a fairly stringent requirement on v_e because k tends to be large, $|\tilde{n}| \sim \sqrt{m_i / m_e}$; however, it does not modify the ion resonant behavior that is of primary interest here.

When these results are used in the zero order equations [Eq. (9)], we find that the parallel electron conductivity is quite large, $4\pi i \sigma_{|||}^e / \omega = -\omega_{pe}^2 / \omega^2 \sim m_i / m_e \times \omega_{pi}^2 / \Omega_i \omega$. Therefore if $|\tilde{n}|$ is order ω_{pi}^2 / ω^2 or smaller, we must have $E_{||}^0 = 0(m_e / m_i)$. We move all $E_{||}^0$ terms to higher order and obtain the system

$$\left(1 - \frac{n_{\perp}^2}{2} - n_{\parallel}^2 + \frac{4\pi i}{\omega} \sigma_{++}^0\right) E_+^0 + \frac{n_{\perp}^2}{2} e^{2i\psi} E_-^0 = 0, \quad (34a)$$

$$\frac{n_{\perp}^2}{2} e^{-i\psi} E_+^0 + \left(1 - \frac{n_{\perp}^2}{2} - n_{\parallel}^2 + \frac{4\pi i}{\omega} \sigma_{--}^0\right) E_-^0 = 0. \quad (34b)$$

Again, for $n^2 \sim 0(\omega_{pi}^2/\omega^2)$, any solution of Eq. (34) must have $E_+^0 = 0(\eta)$ so that the dispersion relation reduces to

$$1 - \frac{n_{\perp}^2}{2} - n_{\parallel}^2 + \frac{4\pi i}{\omega} \sigma_{--}^0 = 1 - \frac{n_{\perp}^2}{2} - n_{\parallel}^2 + \frac{\omega_{pi}^2}{2\Omega_i^2} = 0. \quad (35)$$

This coincides with the cold plasma dispersion relation for the fast wave or magnetosonic wave evaluated at $\omega = \Omega_i$.^{9,10} E_+^0 is $0(\eta)$ and is determined from Eq. (34a), whereas $E_{\parallel}^0$ is determined from the third equation in Eq. (9). For this wave the ray paths are real at resonance and are again determined by the cold plasma dispersion relation.

If $n_{\parallel}^2$ is allowed to become large, $n_{\parallel}^2 \sim \omega_{pi}^2/\omega^2 \times 1/\eta$ (as, for example, in a magnetic beach), another solution of the system of Eq. (34) is possible with $E_+^0 = 0(1)$. In this instance the dispersion relation is complex,

$$n_{\perp}^2 n_{\parallel}^2 + n_{\parallel}^4 = -\frac{1}{2} \frac{\omega_{pi}^4}{\Omega_i^4} \frac{c}{v_i} \frac{1}{n_{\parallel}} Z\left(\frac{\omega - \Omega_i}{k_{\parallel} v_i}\right), \quad (36)$$

and the waves are heavily damped in zero order. This solution corresponds to the ion cyclotron slow wave or shear Alfvén wave at ion cyclotron resonance. Clearly, our geometrical optics formulation breaks down at this resonance.

The expansion can be carried to first order in η for the fast wave in exactly the same way as for electron cyclotron waves. Again we find that the contribution of $h_1(\underline{v}, \underline{x})$ [Eq. (13)] to the energy flux is higher order in η . Because both E_+^0 and $E_{||}^0$ are small, the zero order part of the zero order field is just $\hat{E}^0 = \underline{E}^0 = E_-^0/2 (1, i, 0)$ and $\hat{J}_0$ is higher order in η . The Poynting theorem analogous to Eq. (25), therefore, reduces to

$$\underline{\nabla} \cdot \underline{S} = -\text{Re} \{ \underline{J}^{0*} \cdot \underline{E}_+^0 \} = -\text{Re} \{ 1/2 \sigma_{++}^{0*} |E_+^0|^2 \} . \quad (37)$$

As in the case of the fundamental electron cyclotron resonance, the absorption is first order in η .

V. SUMMARY

Despite the fact that waves can be weakly damped and k can be slowly varying near a cyclotron resonance, the standard formulations of geometrical optics break down near cyclotron resonances because of the large Hermitian part of the conductivity. By performing a formal expansion of the Vlasov-Maxwell equations in the small parameter $\eta = (Lk)^{-1}$ and taking explicit account of the relative magnitudes of the electric field components, we have obtained a formulation in terms of a real eikonal function and slowly varying amplitude that is valid at cyclotron resonance. The key assumptions are that $v_j k / \omega = O(\eta)$ and that $|\sigma_{--}| \sim (\omega_{pe}^2 / \Omega_e^2)(c/v_e)(1/\eta_{||}) = O(n^2/\eta)$. [For ion cyclotron waves, the latter condition is $|\sigma_{++}^i| \sim (\omega_{pi}^2 / \Omega_i^2)(c/v_i)(1/\eta_{||}) = O(n^2/\eta)$.] When these conditions are met, we find that the ray trajectories at cyclotron resonance are given by the cold plasma dispersion. Taking the expansion to first order, we obtain a Poynting theorem that can ultimately be put into the standard form, although care must be taken to include finite temperature effects in calculating the zero order electric field [Eq. (17)].

For fairly dense plasmas of moderate temperature, $(\omega_{pe}^2 / \Omega_e^2)(c/v_e) \gg 1$, the theory is valid unless η becomes large at resonance. On the other hand, the magnitude of η at resonance depends on the plasma geometry. In a parallel stratified plasma (∇B parallel to B), the extraordinary mode and slow ion cyclotron wave have cold plasma resonance in $\eta_{||}$ at the cyclotron frequency (i.e., $\eta_{||} \rightarrow \infty$ as $\Omega_j \rightarrow \omega$ independent of $\eta_{\perp}$). In this geometry, these waves are always heavily damped, and our theory always breaks down at cyclotron resonance regardless of the injected angle. In a perpendicularly stratified plasma (∇B perpendicular to B),

there are no cold plasma resonances in $n_{\perp}$ at the cyclotron frequency. The theory will therefore be valid except when the extraordinary mode or the slow ion cyclotron mode is injected with such large $n_{\parallel}$ as to violate the above conditions on $|\sigma_{--}|$ and $|\sigma_{++}|$. The ordinary mode has a warm plasma resonance at $\omega = |\Omega_e|$ for nearly perpendicular propagation (i.e., $n_{\perp i} \rightarrow \infty$ at $\omega = |\Omega_e|$ as $n_{\parallel} \rightarrow 0$). The formalism can break down for the ordinary mode at $\theta \cong 90^\circ$ unless the resonance is broadened by relativistic or ∇B effects.

In summary: When electromagnetic waves are weakly damped at cyclotron resonance, ray trajectories can be defined and can be traced through resonance using the cold plasma dispersion relation. The absorption can be calculated using the Poynting theorem [Eq. (25)]. In other cases rays can be traced to the cyclotron resonance using cold plasma theory, but there the waves are heavily damped and a geometrical optics description is meaningless.

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APPENDIX: PROPERTIES OF THE CONDUCTIVITY TENSOR

We present various properties of the conductivity tensor and distribution function in an infinite homogeneous plasma described by the linearized Vlasov equation. The results are standard, but they are presented here in a form appropriate for our applications. Dropping species subscripts, the linearized Vlasov equation in Eq. (9) can be written

$$\frac{\partial f}{\partial \phi} + i[\omega_* - \epsilon \cos(\phi - \psi)] f = \frac{e}{m\Omega} \left[\frac{1}{2} (E_+ e^{-i\phi} + E_- e^{i\phi}) \frac{\partial G}{\partial v_\perp} + E_\parallel \frac{\partial G}{\partial v_\parallel} \right], \quad (\text{A.1})$$

where $\omega_* = \frac{\omega - k_\parallel v_\parallel}{\Omega}$ and $E = \frac{k_\perp v_\perp}{\Omega}$. It is readily verified that the unique periodic solution of

$$\frac{\partial f_m}{\partial \phi} + i[\omega_* - \epsilon \cos(\phi - \psi)] f_m = e^{im\phi} \quad (\text{A.2})$$

is

$$f_m(\phi) = \sum_{\ell} J_\ell(\epsilon) \frac{e^{i[\epsilon \sin(\phi - \psi) + (m - \ell)\phi + \ell\psi]}}{i(\omega_* + m - \ell)}, \quad (\text{A.3})$$

where the $J_\ell(\epsilon)$ are Bessel equations of order ℓ . The solution to (A.1) is therefore

$$f = \frac{e}{m\Omega} e^{i\epsilon \sin(\phi-\psi)} \sum_{\ell} J_{\ell}(\epsilon) \left[\frac{E_+}{2} e^{-i\psi} \frac{\partial G}{\partial v_{\perp}} \frac{e^{-i(\ell+1)(\phi-\psi)}}{i(\omega_* - \ell - 1)} + \frac{E_-}{2} e^{i\psi} \frac{\partial G}{\partial v_{\perp}} \frac{e^{-i(\ell-1)(\phi-\psi)}}{i(\omega_* - \ell + 1)} + E_{\parallel} \frac{\partial F}{\partial v_{\parallel}} \frac{e^{-i\ell(\phi-\psi)}}{i(\omega_* - \ell)} \right]. \quad (\text{A.4})$$

We now evaluate the perturbed plasma current and conductivity tensor $\underline{\underline{g}}$ as defined in Eq. (8). In order to facilitate the expansion procedure, it is convenient to introduce dimensionless variables. We therefore introduce the thermal speed for the j th species v_j , a dimensionless velocity variable $\underline{u} = \underline{v}/v_j$ and the dimensionless equilibrium distribution function $F(\underline{u})$ such that

$$2\pi G(\underline{v}) v_{\perp} dv_{\perp} dv_{\parallel} = n F(\underline{u}) u_{\perp} du_{\perp} du_{\parallel},$$

where n is the plasma density and $\int F(\underline{u}) u_{\perp} du_{\perp} du_{\parallel} = 1$. Using Eq. (A.4) in Eq. (8) gives

$$\begin{aligned} \sigma_{++} &= \frac{1}{4\pi} \frac{\omega_p^2}{\Omega} \int du_{\parallel} du_{\perp} u_{\perp}^2 \sum_{\ell} \frac{J_{\ell-1}^2(\epsilon)}{i(\omega_* - \ell)} \frac{1}{2} \frac{\partial F}{\partial u_{\perp}}, \\ \sigma_{+-} e^{-2i\psi} &= \sigma_{-+} e^{2i\psi} = \frac{1}{4\pi} \frac{\omega_p^2}{\Omega} \int du_{\parallel} du_{\perp} u_{\perp}^2 \sum_{\ell} \frac{J_{\ell-1}(\epsilon) J_{\ell+1}(\epsilon)}{i(\omega_* - \ell)} \frac{1}{2} \frac{\partial F}{\partial u_{\perp}}, \\ \sigma_{+||} e^{-i\psi} &= 2\sigma_{||+} e^{i\psi} = \frac{1}{4\pi} \frac{\omega_p^2}{\Omega} \int du_{\parallel} du_{\perp} u_{\perp}^2 \sum_{\ell} \frac{J_{\ell}(\epsilon) J_{\ell-1}(\epsilon)}{i(\omega_* - \ell)} \frac{\partial F}{\partial u_{\parallel}}, \\ \sigma_{--} &= \frac{1}{4\pi} \frac{\omega_p^2}{\Omega} \int du_{\parallel} du_{\perp} u_{\perp}^2 \sum_{\ell} \frac{J_{\ell+1}^2(\epsilon)}{i(\omega_* - \ell)} \frac{1}{2} \frac{\partial F}{\partial u_{\perp}}, \end{aligned} \quad (\text{A.5})$$

$$\sigma_{-||} e^{i\psi} = 2\sigma_{||-} e^{-i\psi} = \frac{1}{4\pi} \frac{\omega_p^2}{\Omega} \int du_{||} du_{\perp} u_{\perp}^2 \sum_{\ell} \frac{J_{\ell}(\epsilon) J_{\ell+1}(\epsilon)}{i(\omega_* - \ell)} \frac{\partial F}{\partial u_{||}} ,$$

$$\sigma_{||||} = \frac{1}{4\pi} \frac{\omega_p^2}{\Omega} \int du_{||} u_{||} du_{\perp} u_{\perp} \sum_{\ell} \frac{J_{\ell}^2(\epsilon)}{i(\omega_* - \ell)} \frac{\partial F}{\partial u_{||}} .$$

We need various expansion of the above relations appropriate to the ion or electron species in the electron cyclotron or ion cyclotron frequency regime. First, addressing the electron cyclotron frequency range, we note that the ion response is very small and can be neglected throughout. Assuming that the wave frequency is near resonance $(\omega - |\Omega_e|)/\omega = 0(\eta)$ and that the electron thermal speed is small in comparison to the wave-phase velocity $v_e k/|\Omega_e| = 0(\eta)$ yields the following expansion for σ accurate to first order in η ,

$$4\pi\sigma_{++} = - \frac{i\omega_p^2}{2\Omega_e} [1 + 0(\eta^2)] ,$$

$$4\pi\sigma_{+-} e^{-2i\psi} = - \frac{i\omega_p^2}{4\Omega_e} \left[\frac{k_{\perp}}{k_{||}} \frac{k_{\perp} v_e}{\Omega_e} \tilde{Z}(\xi) + 0(\eta^2) \right] = - \frac{i\omega_p^2}{\Omega_e} \times 0(\eta) ,$$

$$4\pi\sigma_{--} = - \frac{i\omega_p^2}{\Omega_e} \left[\frac{\Omega_e}{k_{||} v_e} Z(\xi) + 0(\eta^2) \right] = - \frac{i\omega_p^2}{\Omega_e} \times 0(1/\eta) ,$$

(A.6)

$$4\pi\sigma_{+||} e^{-i\psi} = \frac{i\omega_p^2}{\Omega_e} \times 0(\eta^2) ,$$

$$4\pi\sigma_{-||} e^{i\psi} = \frac{i\omega_p^2}{\Omega_e} \left\{ \frac{k_{\perp}}{k_z} [1 + \xi Z(\xi)] + 0(\eta^2) \right\} = \frac{i\omega_p^2}{\Omega_e} \times 0(1) ,$$

$$4\pi\sigma_{||} = \frac{i\omega_{pe}^2}{\Omega_e} \left\{ \frac{\Omega_e}{\omega} - \frac{k_{\perp}}{\omega} \frac{k_{\perp} v_e}{2\Omega_e} \xi [1 + \xi Z(\xi)] \right\} = \frac{i\omega_{pe}^2}{\Omega_e} \times 0(1) ,$$

where

$$\xi = \frac{\omega - |\Omega_e|}{k_{||} v_e} ,$$

$$Z(\xi) = \int du_{||} du_{\perp} u_{\perp} \frac{G(u)}{u_{||} - \xi} ,$$

$$\tilde{Z}(\xi) = \int du_{||} du_{\perp} u_{\perp}^3 \frac{G}{u_z - \xi} .$$

The parallel velocity integrals in $Z(\xi)$ and $\tilde{Z}(\xi)$ are defined in terms of the usual Landau contour. For a Maxwellian equilibrium distribution, $Z(\xi)$ is the standard plasma dispersion function and $\tilde{Z}(\xi) = Z(\xi)/2$. It is significant to note that the apparent singularities at $k_{||} \rightarrow 0$ disappear upon expansion of the dispersion function integrals except precisely at $\omega = |\Omega_e|$.

It is convenient to divide $\underline{g}$ into a part of order η^0 or larger and a part of order η

$$\underline{g} = \underline{g}^0 + \underline{g}^1 ,$$

where

$$4\pi\underline{g}^0 = - \frac{i\omega_{pe}^2}{\Omega_e} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & \frac{\Omega_e}{k_z v_e} Z(\xi) & - \frac{k_{\perp}}{k_{||}} [1 + \xi Z(\xi)] e^{-i\psi} \\ 0 & -2 \frac{k_{\perp}}{k_{||}} [1 + \xi Z(\xi)] e^{i\psi} & - \frac{\Omega_e}{\omega} \end{bmatrix}$$

(A.7)

and

$$4\pi\tilde{g}^1 = -\frac{i\omega_{pe}^2}{\Omega_e} \frac{k_{\perp} v_e}{\Omega_e} \begin{bmatrix} 0 & \frac{k_{\perp}}{4k_{\parallel}} \tilde{Z}(\xi) e^{2i\psi} & 0 \\ \frac{k_{\perp}}{4k_{\parallel}} \tilde{Z}(\xi) e^{-2i\psi} & 0 & 0 \\ 0 & 0 & \frac{k_{\perp}}{2k_{\parallel}} \xi [1 + \xi Z(\xi)] \end{bmatrix}.$$

For application to the ion cyclotron resonance, we assume that $(\omega - \Omega_i)/\Omega_i = 0(\eta)$ and $kv_i/\Omega_i = 0(\eta)$. In this case, only the zero order terms are required. For ions, we obtain from Eq. (A.5),

$$4\pi\tilde{g}^0 = -\frac{i\omega_{pi}^2}{\Omega_i} \begin{bmatrix} \frac{\Omega_i}{k_{\parallel} v_i} Z(\xi_i) & 0 & \frac{k_{\perp}}{k_{\parallel}} [1 + \xi_i Z(\xi_i)] e^{i\psi} \\ 0 & 1/2 & 0 \\ \frac{k_{\perp}}{k_{\parallel}} [1 + \xi_i Z(\xi_i)] e^{-i\psi} & 0 & -\frac{\Omega_i}{\omega} \end{bmatrix}, \quad (A.8)$$

where $\xi_i = (\omega - \Omega_i)/k_{\parallel} v_i$ and $Z(\xi)$ is defined below Eq. (A.6).

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