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NUMERICAL SOLUTION OF THE RESISTIVE MAGNETOHYDRODYNAMIC
BOUNDARY LAYER EQUATIONS

By

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BOUNDARY-LAYER EQUATIONS

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ABSTRACT

Three different techniques are presented for numerical solution of the equations governing the boundary layer of resistive magnetohydrodynamic tearing and interchange instabilities in toroidal geometry. Excellent agreement among these methods and with analytical results provides confidence in the correctness of the results. Solutions obtained in regimes where analytical methods fail indicate a new scaling for the tearing mode as well as the existence of a new regime of stability.

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I. INTRODUCTION

Resistive magnetohydrodynamic (MHD) tearing and interchange instabilities are believed to be among the most important and dangerous modes affecting the behavior of toroidal plasma confinement systems, including tokamaks, spheromaks, reversed-field pinches, and stellarators. Most theoretical treatments of these modes are based on an analysis in which an ideal MHD outer region and a narrow resistive boundary layer are matched together to obtain a dispersion relation which determines the complex growth rate.¹⁻⁵ While the ideal region equations are commonly solved numerically, most previous treatments of the inner region equations have been based on analytical solution procedures which depend for their validity on the smallness of interchange driving forces and related parameters. For present and future experiments with relatively high values of β , the ratio of plasma pressure to magnetic field energy density, these parameters are not small, and therefore the analytical results are not valid. Furthermore, the analytical methods are not readily generalizable to more advanced dynamical descriptions which include additional nonideal effects such as viscosity within the boundary layer, nor do they provide solutions for even-parity eigenfunctions. Because the boundary layer equations must be solved many times over a wide range of parameters to give an adequate picture of the stability properties of a family of equilibria, it is essential to develop numerical solution procedures which are fast, accurate, automatic, and general. This is the subject of the present paper.

The remainder of this paper is organized as follows. In Sec. II the basic equations are presented. These are a sixth order system of ordinary differential equations in which the independent variable is the distance x from the center of the singular layer. They have an irregular singular point

at infinity, where matching to the ideal outer region occurs, and boundary conditions are thus imposed at infinity. The next three sections present the different numerical techniques. Section III covers the Series Matching Method, in which a convergent power series about the origin and an asymptotic power series about the point at infinity are matched at an intermediate point where both series attain reasonable accuracy. Section IV presents the Finite Difference Method, in which the solution of a block tridiagonal matrix problem links the origin and the asymptotic series at large x . Section V discusses the Fourier Transform Method, in which the transformed equations are solved as an initial value problem, and the results are then analytically transformed back into configuration space. Section VI presents comparisons of numerical and analytical results and discusses new findings beyond the reach of analytical treatments. The most interesting result is a new regime of stability in which no resistive interchange occurs even if the driving forces are large and positive. Section VII contains a summary and discussion. Appendices A-C contain various algebraic details.

II. STATEMENT OF THE PROBLEM

The resistive inner region equations can be written in vector form

$$\underline{\Psi}'' - \underline{V} \cdot \underline{\Psi} - \underline{U} \cdot \underline{\Psi} = 0, \quad (1a)$$

with the coefficient matrices defined as follows:

$$\underline{U} = \begin{bmatrix} Q & -xQ & 0 \\ -xQ^{-1} & x^2Q^{-1} & -(E+F)Q^{-2} \\ -xQ^{-1} & -(G+KE)Q & x^2Q^{-1} + (G+KF)Q \end{bmatrix}, \quad (1b)$$

$$\underline{V} = \begin{bmatrix} 0 & 0 & H \\ -HQ^{-2} & 0 & 0 \\ HKQ & 0 & 0 \end{bmatrix}, \quad (1c)$$

Here E, F, G, H, K are real constants, defined in Ref. 5, which characterize the equilibrium, Q is the scaled growth rate of the instability, and primes denote derivatives with respect to the independent variable x , which physically measures the scaled distance from the mode rational surface. The components of the vector $\underline{\Psi}(x) = [\psi(x), \xi(x), \tau(x)]$ are the perturbed components of the magnetic field and the fluid displacement in the direction of increasing x , and the component of the perturbed magnetic field in the direction of the equilibrium magnetic field.

Equation (1) is a sixth order system of homogeneous ordinary linear differential equations. The equations conserve parity in x , in that even solutions for $\underline{\Psi}(x)$ couple only to odd solutions for $\xi(x)$ and $\tau(x)$ and vice versa. Since the general solution is a superposition of even and odd solutions, we restrict consideration to positive x and consider separately solutions with the boundary conditions at the origin

$$\{\text{odd modes}\} \quad \Psi'(0) = \Xi(0) = T(0) = 0 \quad , \quad (2a)$$

$$\{\text{even modes}\} \quad \Psi(0) = \Xi'(0) = T'(0) = 0 \quad . \quad (2b)$$

[It is to be noted here that we have defined the parity of a solution according to the behavior of $\Xi(x)$. This is the opposite of the convention in Ref. 5, where parity is defined according to the behavior of $\Psi(x)$]. The equations are homogeneous, so that an additional boundary condition is taken up by an arbitrary normalization constant. The remaining two boundary conditions are determined by the requirement that the solution does not diverge exponentially for large values of x .

The point at infinity is an irregular singular point of Eq. (1), so the solutions there are not analytic and cannot be expressed in terms of convergent power series. Nevertheless, it is still possible to formally represent the solutions by asymptotic expansions in powers of x^{-1} . Since the system is sixth order, there are six independent asymptotic solutions at large x . Four of these have leading order exponential behavior $\sim \exp(\pm x^2/2Q^{1/2})$, and two have powerlike behavior. At large x we have

$$\Psi(x) \sim \sum_{i=1}^6 \alpha_i T_i(x) \quad (3a)$$

with the six asymptotic solutions being given by

$$T_{1,2}(x) = \exp(x^2/2Q^{1/2}) x^{\pm 5/2} s_{1,2}(x) \quad , \quad (3b)$$

$$\underline{T}_{3,4}(x) = x^{\pm} \underline{s}_{3,4}(x), \quad (3c)$$

$$\underline{T}_{5,6}(x) = \exp(-x^2/2Q^{1/2}) x^{\pm} \underline{s}_{5,6}(x). \quad (3d)$$

Here,

$$p_{\pm} = -\frac{1}{2} \pm (-D_I)^{1/2}, \quad (4a)$$

$$\begin{aligned} s_{\pm} = & -\frac{1}{2} + \frac{1}{4} \sigma Q^{3/2} [1 + G + K(F + H^2)] \\ & \pm \frac{1}{4} \{ Q^3 \{ (G+KF-1)^2 + KH^2 [KH^2 + 2(G+KF+1)] \} + 4[(G+KE-1)D_R + H^2] \}^{1/2} \end{aligned} \quad (4b)$$

$\sigma = +1$ for $\underline{T}_{1,2}$ and -1 for $\underline{T}_{5,6}$, $D_I = E + F + H - 1/4$, $D_R = E + F + H^2$. The vectors $\underline{s}_1(x)$ through $\underline{s}_6(x)$ have components expressible as asymptotic series in x^{-1} . Thus, $\underline{s}_3$ and $\underline{s}_4$ take the form

$$\begin{aligned} \underline{s}_{3,4} = & \begin{aligned} & x + \psi_1/x + \psi_2/x^3 + \dots \\ & 1 + \xi_1/x^2 + \xi_2/x^4 + \dots \\ & 1 + \nu_1/x^2 + \nu_2/x^4 + \dots \end{aligned} \end{aligned} \quad (5)$$

while the remaining four vectors have the similar form

$$\begin{aligned} \underline{s}_{1,2}^{(x)} = & \begin{aligned} & 1/x + \psi_1/x^3 + \psi_2/x^5 + \dots \\ & \xi_0 + \xi_1/x^2 + \xi_2/x^4 + \dots \\ & \nu_0 + \nu_1/x^2 + \nu_2/x^4 + \dots \end{aligned} \end{aligned} \quad (6)$$

[We note in passing that each series contains only alternate powers of x^{-1} due to the parity-conserving nature of Eq. (1)]. In each case, the coefficients (ψ_n , ξ_n , v_n) are constants which depend on either $s_{\pm}$ or $p_{\pm}$, and on the growth rate Q and the equilibrium constants E , F , G , H , K . One can obtain a recursion relation for these coefficients by substituting the appropriate T -vector into Eq. (1). This is carried out in Appendices A and B.

The boundary conditions at infinity can now be stated precisely as the condition that two of the coefficients in Eq. (3) vanish, $\alpha_1 = \alpha_2 = 0$, i.e., we seek those solutions that at large enough values of the independent variable x can be decomposed completely into the two powerlike and two exponentially decaying asymptotic series.

The quantity of greatest interest, which we obtain from the calculation, is

$$\Delta \equiv \alpha_3/\alpha_4, \quad (7)$$

the ratio of the coefficients of the two powerlike asymptotic solutions. It is only this quantity (for both the even and the odd parity solutions) that is needed to match onto the outer region ideal MHD equations. In the following three sections we will present three complementary methods for solving for these ratios as functions of the growth rates and the equilibrium parameters, i.e., $\Delta_+(E, F, G, H, K, Q)$ and $\Delta_-(E, F, G, H, K, Q)$.

III. SERIES MATCHING METHOD

The conceptually most straightforward method of solving Eq. (1) is to expand the solution in a series about the origin where the parity boundary conditions are applied, and to equate this to a series expansion about

infinity, where the matching boundary conditions are applied. This method will be shown to be efficient and accurate for $Q < 1$, for which there are overlapping regions of validity of the two series expansions.

Since the system of Eqs. (1) is well-behaved everywhere except at infinity, we can express the solution for finite x as a convergent power series expansion about the origin. We write

$$\underline{\Psi}(x) = \sum_{n=0}^{\infty} \underline{a}_n x^n. \quad (8a)$$

with $\underline{\Psi}$, and $\underline{a}_n$ 3-vectors, or in components,

$$\underline{\Psi}(x) = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} + \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} x + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} x^2 + \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} x^3 + \dots \quad (8b)$$

Substituting Eq. (8a) into Eq. (1) yields recursion relations for the coefficients $\{a_n, b_n, c_n\}$. The first few coefficients are defined as follows:

$$2a_2 = Qa_0 + Hc_1 \quad (9a)$$

$$2Q^2b_2 = -(E + F)c_0 - Ha_1 \quad (9b)$$

$$2Qc_2 = Q^2(G + KF)c_0 - Q^2(G - KE)b_0 + Q^2KH a_1 \quad (9c)$$

$$6a_3 = Qa_1 - Qb_0 + 2Hc_2 \quad (9d)$$

$$6Q^2b_3 = -(E + F)c_1 - Qa_0 - 2Ha_2 \quad (9e)$$

$$6Qc_3 = Q^2(G + KF)c_1 - Q^2(G - KE)b_1 - a_0 + 2Q^2KH a_2. \quad (9f)$$

For $n \geq 4$ we have

$$n(n-1)a_n = Qa_{n-2} - Qb_{n-3} + H(n-1)c_{n-1} \quad (9g)$$

$$Q^2 n(n-1)b_n = Qb_{n-4} - (E + F)c_{n-2} - Qa_{n-3} - H(n-1)a_{n-1} \quad (9h)$$

$$Qn(n-1)c_n = c_{n-4} + Q^2(G + KF)c_{n-2} - Q^2(G - KE)b_{n-2} - a_{n-3} + Q^2KH(n-1)a_{n-1} \quad (9i)$$

It is apparent that the six degrees of freedom of the system reside in the arbitrary values of the initial coefficients $a_0, b_0, c_0, a_1, b_1, c_1$. Once these are chosen, the remaining coefficients are determined by recursion relations (9a) through (9i).

The basic idea of this method is to solve for Δ by evaluating the power series solutions at some large value of x ($x = L$), and then equating this to the asymptotic series solution evaluated at the same point. Since the system is sixth-order, we need to match the three functions $\Psi(x), \Xi(x), T(x)$, and their derivatives $\Psi'(x), \Xi'(x), T'(x)$, at $x = L$.

We now expand our vector notation and write

$$\begin{bmatrix} \Psi(x) \\ \Psi'(x) \end{bmatrix} = \sum_{n=0}^{\infty} \underline{v}_n x^n. \quad (10)$$

Note that the initial vector

$$\underline{v}_0 = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} \Psi(0) \\ \Xi(0) \\ T(0) \\ \Psi'(0) \\ \Xi'(0) \\ T'(0) \end{bmatrix} = \begin{bmatrix} \Psi(0) \\ \Psi'(0) \end{bmatrix}$$

and that all subsequent vectors $\underline{v}_n = \{a_n, b_n, c_n, (n+1)a_{n+1}, (n+1)b_{n+1}, (n+1)c_{n+1}\}$ are by the recursion relations linear functions of the initial

variables $(a_0, b_0, c_0, a_1, b_1, c_1)$. Hence each v_n can be expressed formally as a matrix A_n times the initial vector v_0 . One can thus use the power series to express the solutions at $x = L$ as a propagator matrix P times the solution at the origin:

$$\begin{bmatrix} \Psi(L) \\ \Xi(L) \\ T(L) \\ \Psi'(L) \\ \Xi'(L) \\ T'(L) \end{bmatrix} = \sum_{n=0}^{\infty} v_n L^n = \left[\sum_{n=0}^{\infty} A_n L^n \right] \cdot v_0 = P \cdot v_0 = P \cdot \begin{bmatrix} \Psi(0) \\ \Xi(0) \\ T(0) \\ \Psi'(0) \\ \Xi'(0) \\ T'(0) \end{bmatrix}.$$

It would be prohibitively difficult to determine analytically what P is. However, since a linear transformation is completely defined by its values on a basis, the numerical evaluation of P is straightforward. We compute the first column of P , P_1 , by setting $a_0 = 1, b_0 = c_0 = a_1 = b_1 = c_1 = 0$, solving for the rest of the coefficients $\{a_n, b_n, c_n\}$ from Eq. (9) and evaluating the series expressions for $\{\Psi(L), \Xi(L), T(L), \Psi'(L), \Xi'(L), T'(L)\}$. The other columns of P can be found in a similar fashion.

Thus the power series expression for the solution at $x = L$ is

$$\begin{bmatrix} \Psi(L) \\ \Xi(L) \\ T(L) \\ \Psi'(L) \\ \Xi'(L) \\ T'(L) \end{bmatrix} = \begin{bmatrix} | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad (11)$$

On the other hand, the asymptotic series solution is

$$\begin{bmatrix} \Psi(L) \\ \Xi(L) \\ T(L) \\ \Psi'(L) \\ \Xi'(L) \\ T'(L) \end{bmatrix} = \sum_{i=1}^6 \alpha_i A_i = \begin{bmatrix} | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} \quad (12)$$

where each column $\tilde{A}_i$ is defined by

$$\tilde{A}_i = \begin{bmatrix} T_i(L) \\ T_i'(L) \end{bmatrix}, \quad (13)$$

with T_i defined in Eq. (3).

Equating the two series expressions for $[\psi(L), \psi'(L)]$ yields the following system of equations:

$$\begin{bmatrix} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \tilde{A}_4 & \tilde{A}_5 & \tilde{A}_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix}. \quad (14)$$

This is a set of six homogeneous equations with twelve unknowns: (1) If values are chosen for any six of the variables, the remaining six are then determined. The values we choose are dictated by the boundary conditions:

(1) parity at the origin; for odd modes, set $b_0 = c_0 = a_1 = 0$, and for even modes, set $a_0 = b_1 = c_1 = 0$. (2) No exponential divergence at infinity; set $\alpha_1 = \alpha_2 = 0$. (3) Arbitrary normalization; set $\alpha_4 = 1$. Thus for odd modes the system reduces to

$$\begin{bmatrix} P_1 & P_5 & P_6 & -\tilde{A}_3 & -\tilde{A}_5 & -\tilde{A}_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} a_0 \\ b_1 \\ c_1 \\ \Delta_+ \\ E^- \\ E^- \end{bmatrix} = \begin{bmatrix} \tilde{A}_4 \\ \vdots \end{bmatrix}, \quad (15a)$$

while for even modes we have

$$\begin{bmatrix} R_2 & R_3 & R_4 & -A_3 & -A_5 & -A_6 \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \\ | & | & | & | & | & | \end{bmatrix} \begin{bmatrix} b_0 \\ a_1 \\ \Delta \\ E^+ \\ E^- \\ E \end{bmatrix} = \begin{bmatrix} A_4 \end{bmatrix} \quad (15b)$$

where we have substituted $\Delta = \alpha_3/\alpha_4$, $E^+ = \alpha_5/\alpha_4$, $E^- = \alpha_6/\alpha_4$.

Equations (15) are of the form

$$\underline{A}(L) \cdot \underline{X} = \underline{b}(L) \quad (16)$$

and can be solved for the unknown vector $\underline{X}$ by standard routines for solving linear systems of equations. The notation underscores the fact that every element of the matrix $\underline{A}$ and the vector $\underline{b}$ are functions of the matching point L , but that the solution vector $\underline{X}$ is asymptotically independent of this point. In reality, the numerical value of $\underline{X}$ does depend upon the value of L through errors introduced by two competing effects. When L is small, the asymptotic series of Eqs. (5) and (7) cannot be computed accurately, thus leading to large errors in $\underline{A}$ and $\underline{b}$. When L is large, the exponentially large solutions will tend to dominate each of the power series expansions of Eq. (13), making the first three columns of the matrix $\underline{A}$ become linearly dependent. This causes $\underline{A}$ to become singular and eventually makes solving for $\underline{X}$ in Eq. (16) impossible.

To be more quantitative, we note from the diagonal terms in Eq. (B-14) from Appendix B that the relative size of the $2N^{\text{th}}$ term in the asymptotic series Eq. (5) is, for $Q < 1$,

$$t_{2N} = \frac{4^{2N} Q^N (N!)^2}{L^{4N}} \quad , \quad (17)$$

or, by using Stirling's approximation for the factorial,

$$t_{2N} \sim \left(\frac{16QN^2}{L^4 e^2} \right)^N \quad . \quad (18)$$

To apply the optimal truncation rule,⁷ we note that the smallest term in this series occurs for

$$N \sim \frac{L^2}{4Q^{1/2}} \quad , \quad (19)$$

with the relative error introduced by truncating at this term

$$\epsilon_{2N} \sim \exp(-L^2/2Q^{1/2}) \quad . \quad (20)$$

This is equivalent to saying that $L^2/(2Q^{1/2} \ln 10)$ digits are computed accurately in b and in the last column of A .

The exponential problem, on the other hand, causes the first three columns of the matrix A to have added a multiple of the same vector which is of order $\exp(L^2/2Q^{1/2})$. This will cause a loss of $L^2/(2Q^{1/2} \ln 10)$ significant digits in these columns, which otherwise could have been computed to arbitrary accuracy. If D is the number of significant digits available on the computer, then a balance of these two effects for optimal overall precision requires

$$D = \frac{L^2}{2Q^{1/2} \ln 10} = \frac{L^2}{2Q^{1/2} \ln 10} \quad ,$$

or

$$L = RQ^{1/4} \quad , \quad R = (D \ln 10)^{1/2} \quad . \quad (21)$$

For the CRAY-1 with $D = 15$ for single precision and $D = 30$ for double precision, Eq. (21) predicts the optimal matching points to be at $R \approx 5.8$ and $R \approx 8.3$, respectively. This estimate is consistent with Tables I and II, which give the values of Δ for the odd and the even solution for $Q = 0.01$, $E = 0.1$, $F = G = H = K = 0$, and for different values of the matching parameter R .

Finally, we remark that for $Q > 1$ this method does not work because the two series expansions no longer have overlapping regions of validity. This can be seen from the leading-order Q behavior of Eq. (B-13), which indicates that L must now scale linearly with Q to compute the asymptotic series accurately. In fact, we find that an adequate empirical criterion is

$$L \geq 2QZ^{1/2} \quad , \quad \text{where } Z \text{ is a constant defined by} \quad (21b)$$

$$Z = \max \{ |G+KF|, |KH^2|, 1 \} \quad .$$

However, the power series still has degenerate terms of order

$$\exp(L^2/2Q^{1/2}) \sim \exp(2Q^{3/2}Z)$$

which can be prohibitively large for $Q > 1$.

IV. FINITE DIFFERENCE METHOD

The series matching method of Sec. III was based on finding a matching point $x = L$ such that a power series representation of the solution is valid for $x \leq L$ and an asymptotic series representation is valid for $x \geq L$. We have shown that in practice, the exponential contamination problem makes this impossible for a finite word length computer in certain regions of parameter space, in particular for values of the scaled growth rate Q sufficiently large.

To overcome this exponential problem for the $x \leq L$ solution, we have utilized a boundary value method which replaces the power series expansion of Sec. III by a finite difference solution of the differential equations for $x \leq L$, subject to the boundary condition that it match onto the asymptotic series representation valid for $x \geq L$.

It is known that⁶ an exceedingly stable method for solving the system of finite difference equations arising from an even order boundary value problem with exponentially diverging solutions is to cast the system as a block tridiagonal matrix equation and solve using Gauss elimination. The elimination and the back substitution steps each correspond to first order matrix recursion relations, each being stepped in its stable direction.

We divide the interval $(0, L)$ into $N - 1$ intervals with the notation that $x_i = (i-1)h$, where $h = L/(N-1)$. The solution vector at location x_i is denoted by $\underline{\psi}_i \equiv (\psi_i, \xi_i, v_i)$. A 3×3 block tridiagonal system with a leading error term proportional to h^4 is obtained by Taylor expanding $\underline{\psi}(x)$ about x_i , and using Eq. (1) to eliminate higher derivatives of $\underline{\psi}$ in favor of $\underline{\psi}$ and $\underline{\psi}'$. Thus, we have

$$\underline{\psi}'' = \underline{U} \cdot \underline{\psi} + \underline{V} \cdot \underline{\psi}' \quad , \quad (22a)$$

$$\Psi''' = (\underline{U}' + \underline{V} \cdot \underline{U}) \cdot \underline{\Psi} + (\underline{V}^2 + \underline{U}) \cdot \underline{\Psi}' \quad , \quad (22b)$$

etc., where we have used $\underline{V}' = 0$.

The fourth order accurate approximation to Eq. (1) is thus given by

$$\underline{A}_i \cdot \underline{\Psi}_{i+1} + \underline{B}_i \cdot \underline{\Psi}_i + \underline{C}_i \cdot \underline{\Psi}_{i-1} = 0 \quad , \quad i = 2, N-1 \quad . \quad (23)$$

Here, the coefficient matrices are given by

$$\underline{A}_i = \underline{I} - \frac{h}{2} \underline{V} - \frac{h^3}{24} (2\underline{U}'_i + \underline{U}_i \cdot \underline{V} - \underline{V} \cdot \underline{U}_i - \underline{V}^3) \quad , \quad (24a)$$

$$\underline{B}_i = -2\underline{I} - h^2 \underline{U}_i - \frac{h^4}{12} (\underline{U}''_i - \underline{V} \cdot \underline{U}'_i - \underline{V}^2 \cdot \underline{U}_i + \underline{U}_i^2) \quad , \quad (24b)$$

$$\underline{C}_i = \underline{I} + \frac{h}{2} \underline{V} + \frac{h^3}{24} (2\underline{U}'_i + \underline{U}_i \cdot \underline{V} - \underline{V} \cdot \underline{U}_i - \underline{V}^3) \quad . \quad (24c)$$

Gauss elimination on the system, Eq. (23), corresponds to defining the block vectors $\underline{E}_i$ by the recursion relation

$$\underline{E}_i = (\underline{B}_i + \underline{C}_i \cdot \underline{E}_{i-1})^{-1} \cdot \underline{A}_i \quad ; \quad i = 2, N-1 \quad , \quad (25a)$$

with the back substitution step given by

$$\underline{\Psi}_i = \underline{E}_i \cdot \underline{\Psi}_{i+1} \quad ; \quad i = N-1, N-2, \dots, 1 \quad . \quad (25b)$$

The left-hand (parity) boundary condition is imposed by initiating the recursion relation Eq. (25a) for $\underline{E}_i$ with

$$\underline{E}_1 = -\underline{g}^{-1} \cdot \underline{B}_1, \quad (26a)$$

where the elements σ_{lj} of the 3×3 matrix $\underline{g}$ consist of sums and differences of a_{lj} and c_{lj} , the elements of the 3×3 matrices $\underline{A}_1$ and $\underline{C}_1$, thus

$$\begin{aligned} \sigma_{l1} &= a_{l1} \pm c_{l1}, \\ \sigma_{l2} &= a_{l2} \mp c_{l2}, \\ \sigma_{l3} &= a_{l3} \mp c_{l3}, \end{aligned} \quad (26b)$$

for $l = 1, 2, 3$, the top sign corresponding to the odd parity solution, Eq. (2a), and the bottom sign to the even, Eq. (2b).

The back-substitution step, Eq. (25b), needs values of the vector $\underline{\psi}_N$ at $x_N = L$. From Eq. (3), this will be a linear combination of the four asymptotic vectors

$$\underline{\psi}(L) = \alpha_3 \underline{t}_3 + \alpha_4 \underline{t}_4 + \alpha_5 \underline{t}_5 + \alpha_6 \underline{t}_6. \quad (27)$$

Using Eqs. (22) and (25b), we compute from a Taylor series expansion a fourth order accurate finite difference approximation to $\underline{\psi}'(L)$ which involves only $\underline{\psi}(L)$, the coefficient matrices $\underline{U}$ and $\underline{V}$, and the (3×3) matrix $\underline{E}_{N-1}$ from Eq. (25a)

$$\begin{aligned} \underline{\psi}'(L) &= \underline{D} \cdot \underline{\psi}(L) \\ &= \alpha_3 \underline{D} \cdot \underline{t}_3 + \alpha_4 \underline{D} \cdot \underline{t}_4 + \alpha_5 \underline{D} \cdot \underline{t}_5 + \alpha_6 \underline{D} \cdot \underline{t}_6. \end{aligned} \quad (28a)$$

Here,

$$D = h^{-1} \left[I_N - \frac{1}{2} h V + \frac{1}{6} h^2 (V^2 + U_N) - \frac{1}{24} h^3 (2U_N' + U_N \cdot V + V \cdot U_N + V^3) \right]^{-1} \\ \cdot \left[I_N - E_{N-1} + \frac{1}{2} h^2 U_N - \frac{1}{6} h^3 (U_N' + V \cdot U_N) + \frac{1}{24} h^4 (U_N'' + V \cdot U_N' + V^2 \cdot U_N + U_N^2) \right].$$

The ratios of the coefficients, $\Delta \equiv \alpha_3/\alpha_4$, $E^+ \equiv \alpha_5/\alpha_4$, and $E^- \equiv \alpha_6/\alpha_4$, are determined by the requirement that $\Psi'(L)$ as obtained from Eq. (27) by term differentiation of the asymptotic series agrees with Eq. (28a). Thus

$$(T_4' - D \cdot T_4) = -\Delta (T_3' - D \cdot T_3) - E^+ (T_5' - D \cdot T_5) - E^- (T_6' - D \cdot T_6). \quad (28b)$$

Equation (28b) is a (3×3) matrix equation to be inverted for the Δ , E^+ , E^- .

We remark here that computer time and storage can be economized by observing that the back-substitution steps, Eq. (25a,b) need not be performed in order to determine Δ from Eq. (28b). In addition, the matrices A_i , B_i , C_i , V_i , V_i' need not be stored, but instead can be computed as needed, thus eliminating the need for storing any large vectors of dimension N . Timing tests indicate that a typical set of parameters with $Q < 1$ will require approximately 40 points and take 40 ms total on the CRAY-1 computer to obtain Δ for the even and odd solutions with fully complex arithmetic used throughout. Parameters with $Q > 1$ can require substantially more mesh points as explained below. The computer time scales linearly with the number of finite difference points.

To implement this method, we need to choose a distance L at which to evaluate the asymptotic series, and a number of intervals N . From Sec. III we have the result that an adequate computation (to machine accuracy) of the

asymptotic series is possible at

$$L = \begin{cases} 6Q^{1/4} & Q < 1 ; \\ 2QZ^{1/2} & Q > 1 ; \end{cases}$$

with Z being defined in Eq. (21b).

Tables III and IV present the results of varying the number of intervals N for two sets of parameters. Also displayed are the results obtained by Richardson's extrapolation using the values at N and $N/2$ assuming N^{-4} convergence, i.e.,

$$\Delta_{\text{extrapolated}} = [16\Delta(N) - \Delta(N/2)]/15 .$$

We infer that extrapolation can be used to improve the accuracy of the results for $N \leq 400$ when the truncation error is dominant over the machine roundoff error (results performed on single-precision CRAY-1 computer).

The finite difference method allows computation with the scaled growth rate Q substantially greater than unity, but will eventually fail for Q large enough. To estimate the maximum computable value of Q , we note that the finite difference method has a truncation error proportional to h^4 and thus requires $h \sim 0.01$ to obtain a relative accuracy of 10^{-4} . Since for $Q > 1$ the interval length is $L \sim 2QZ^{1/2}$ with Z defined by Eq. (21b), this will require

$$K = h^{-1} L \sim 20 QZ^{1/2}$$

points to obtain an accurate solution. We have empirically determined that the roundoff error associated with the finite word size of the computer (single precision CRAY-1) begins to dominate for $N \geq 2 \times 10^4$, thus setting an effective limit of $Q < 1000 Z^{-1/2}$.

V. FOURIER TRANSFORM METHOD

The third solution procedure is based on the set of equations obtained by Fourier transforming Eq. (1), and is similar to the analytical method presented in Ref. 5. Fourier transformation has two beneficial effects on this system of equations. First, it reduces the order of the system from six to four. Second, it analytically separates powerlike asymptotic behavior to a different singular point in Fourier space from that at which exponential behavior is found, eliminating the need to separate these solutions numerically. The transformed equations have a regular singular point at the origin, where only powerlike solutions are found, as well as an irregular singular point at infinity where only exponential solutions are found.

The numerical method consists of integrating the transformed equations from the large-argument asymptotic region to the small-argument asymptotic region, using an adaptive numerical integration routine and fitting the solutions to power series solutions near the origin to extract asymptotic coefficients. These coefficients are combined analytically into the even and odd asymptotic ratios in configuration space.

The Fourier-transformed equations are obtained by introducing Laplace integral representations,

$$\Psi(x) = i \int_c dz \exp(izx) \psi(z) \quad , \quad (29a)$$

$$\Xi(x) = \int_c dz \exp(izx) \xi(z) \quad , \quad (29b)$$

$$T(x) = \int_c dz \exp(izx) u(z) \quad , \quad (29c)$$

where c is a contour to be specified later into Eq. (1), or equivalently Eqs.

(9), (22), and (23) of Ref. 5. This yields

$$Q(\psi + \xi_z) + z^2 \psi - Hzv = 0 \quad , \quad (30a)$$

$$Q^2 z^2 \xi - Q(\psi + \xi_z)_z - (E + F)v = 0 \quad , \quad (30b)$$

$$Qz^2 v - Q(\psi + v_z)_z + Q^2 [G(v - \xi) + K(E\xi + Fv + Hz\psi)] = 0 \quad , \quad (30c)$$

as well as conditions which must be satisfied at the end points of the contours.

Equations (30) are cast in matrix form by introducing the column 4-vector $\underline{x} = \{x_i\}$, where $x_1 = \xi$ is the Fourier transform of the normal component of the fluid displacement, $x_2 = \xi - v$ is simply related to the fluid compression, $x_3 = \psi + \xi_z$ is proportional to the Fourier transform of the parallel electric field, and $x_4 = -(\xi - v)_z / (1 + KQ^3)$ is proportional to the parallel derivative of x_2 . With these definitions, Eq. (30) becomes

$$\frac{dx}{dz} = \underline{K} \cdot \underline{x} \quad , \quad (31)$$

where the 4×4 coefficient matrix $\underline{K}$ has coefficients

$$\underline{K} = \begin{bmatrix} -H/Z & H/Z & 1+Q/Z^2 & 0 \\ 0 & 0 & 0 & -(1+KQ^3) \\ Qz^2 - D_R/Q & D_R/Q & H/Z & 0 \\ D_R/Q & K_{42} & -H/Z & 0 \end{bmatrix} \quad (32)$$

with $K_{42} = -D_R/Q - [(G - KE)Q^2 + Qz^2]/(1 + KQ^3)$.

The K matrix has an important symmetry property. Let E be the 4×4 skew-symmetric matrix,

$$E = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} , \quad (33)$$

where I is the 2×2 identity matrix and 0 is the 2×2 null matrix. Then K is Hamiltonian,⁸ i.e.,

$$E \cdot K \cdot E = K^t , \quad (34)$$

where K^t is the transpose of K . This can also be expressed by writing K in 2×2 block form

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} , \quad (35)$$

and noting that

$$K_{22} = -K_{11}^t , \quad K_{12}^t = K_{12} , \quad K_{21}^t = K_{21} . \quad (36)$$

A brief summary of the properties of Hamiltonian and symplectic matrices is presented in Appendix C.

To further elucidate the structure of Eq. (31), we make the shearing transformation,⁹

$$x = P \cdot y , \quad L = P^{-1} \cdot K \cdot P - P^{-1} \cdot \frac{dP}{dz} , \quad (37)$$

where

$$p_{1j} = z^{p_1} \delta_{1j}, \quad p_1 = p_2 = -p_3 = -p_4 = -1/2. \quad (38)$$

This converts Eq. (31) to the equivalent system

$$z \frac{dy}{dz} = L \cdot y, \quad (39)$$

where

$$L = L_0 + L_1 z^2, \quad (40)$$

and L_0 and L_1 are the constant Hamiltonian matrices,

$$L_0 = \begin{bmatrix} 1/2 - H & H & Q & 0 \\ 0 & 1/2 & 0 & 0 \\ -D_R/Q & D_R/Q & H - 1/2 & 0 \\ D_R/Q & -D_R/Q - a_1 & -H & -1/2 \end{bmatrix}, \quad (41)$$

with

$$a_1 = (G - KE)Q^2/(1 + KQ^3), \quad (42)$$

and

$$L_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(1+KQ^3) \\ 0 & 0 & 0 & 0 \\ 0 & -Q/(1+KQ^3) & 0 & 0 \end{bmatrix}. \quad (43)$$

It is clear from the structure of Eqs. (39) and (40) that there are singular points at the origin and at infinity. We need series solutions about both singular points. To obtain a convergent series solution about the origin, we seek an ascending power series of the form

$$u(z) = z^p \sum_{n=0}^{\infty} u_n z^{2n}, \quad (44)$$

where the u_n are constant 4-vectors, and the series has only even powers of z because Eqs. (39) and (40) have only even powers of z . Inserting Eq. (50) into Eqs. (39) and (40), we obtain

$$z^p \sum_{n=0}^{\infty} z^{2n} \{ [L_{\pi 0} - (p + 2n)I] u_n + L_{\pi 1} u_{n-1} \} = 0. \quad (45)$$

The vanishing of the individual coefficients implies

$$[L_{\pi 0} - pI] u_0 = 0, \quad (46)$$

$$[L_{\pi 0} - (p + 2n)I] u_n = -L_{\pi 1} u_{n-1}, \quad n > 0. \quad (47)$$

Equation (46) shows that p must be an eigenvalue of $L_{\pi 0}$,

$$\text{Det}(L_{\pi 0} - pI) = (p^2 + D_I)(p^2 - 1/4) = 0 \quad (48)$$

$$p = \pm p_1, \pm 1/2, \quad (49)$$

where

$$p_1 = (-D_I)^{1/2} \quad (50)$$

and

$$D_I = E + F + H - 1/4 \quad , \quad (51)$$

and that $\underline{u}_0$ must be the corresponding eigenvector. The powers in Eq. (50) correspond to those in Eq. (4a) in configuration space. The four solutions for the column 4-vector $\underline{u}_0$ can be arranged into the 4×4 symplectic diagonalizing matrix

$$\underline{u}_0 = \begin{bmatrix} Qa_2 & 1 & -a_2 Q & 0 \\ 0 & 1 & 0 & 0 \\ a_+ a_2 & 0 & -a_2 a_- & 0 \\ -a_+ a_2 & -a_1 & a_2 a_- & 1 \end{bmatrix} \quad , \quad (52)$$

where

$$a_{\pm} \equiv H - \frac{1}{2} \pm p_1 \quad , \quad (53)$$

and

$$a_2 \equiv \frac{1}{2} p_1 Q \quad . \quad (54)$$

Since $\underline{u}_0$ is symplectic, its inverse is given by

$$\underline{u}_0^{-1} = -\underline{E} \cdot \underline{u}_0^+ \cdot \underline{E} \quad . \quad (55)$$

Equation (47) is used iteratively to compute the coefficients in Eq. (44) to arbitrarily high order. It is most efficient to do this numerically in the representation in which $L_{\infty 0}$ is diagonalized to

$$A_{\infty 0} \equiv U_{\infty 0}^{-1} L_{\infty 0} U_{\infty 0}$$

$$= \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -p_1 & 0 \\ 0 & 0 & 0 & -1/2 \end{bmatrix} \quad (56)$$

and L_1 is transformed to

$$A_1 = U_{\infty 0}^{-1} L_1 U_{\infty 0} \quad (57)$$

In this representation, the series solutions begin with the identity matrix I_n . Multiplication on the left by $U_{\infty 0}$ transforms back to the original representation. The representation in which $L_{\infty 0}$ is diagonalized to $A_{\infty 0}$ is also the most convenient and efficient one in which to integrate the equations numerically.

Equation (47) is singular if two roots differ by $2n$, with n an integer. This happens for $D_1 = -1/4$, such as when the plasma pressure gradient vanishes. Such cases are of limited practical importance and can be treated as limits.

The behavior at large z is determined by seeking a formal descending power series solution,

$$\tilde{v}(z) \sim \exp(1/2 \lambda z^2) z^q \sum_{n=0}^N \tilde{v}_n z^{-2n} \quad , \quad (58)$$

where the $\tilde{v}_n$ are constant 4-vectors. The finite upper limit N in the series is used because the series is asymptotic rather than convergent. Inserting Eq. (58) into Eqs. (39) and (40), we obtain

$$\exp(1/2 \lambda z^2) z^{q+2} \sum_{n=0}^N z^{-2n} \{ (L_{n1} - \lambda I) \tilde{v}_n + [L_{n0} - (q - 2n)I] \} = 0. \quad (59)$$

The vanishing of individual coefficients yields

$$(L_{n1} - \lambda I) \tilde{v}_0 = 0 \quad , \quad (60)$$

$$(L_{n1} - \lambda I) \tilde{v}_n = - [L_{n0} - (q - 2n)I] \tilde{v}_{n-1}, \quad n > 1 \quad . \quad (61)$$

Equation (60) shows that λ must be an eigenvalue of L_{n1} ,

$$\text{Det}(L_{n1} - \lambda I) = (\lambda^2 - Q)^2 = 0 \quad , \quad (62)$$

$$\lambda = \pm \sqrt{Q} \quad , \quad (63)$$

and that $\tilde{v}_0$ must be a corresponding eigenvector. It can be seen from Eqs. (58) and (63) that for $|\arg(Q)| < \pi$ and for a quadrant of z including the real axis, solutions with $\lambda = -\sqrt{Q}$ are exponentially small as $|z| \rightarrow \infty$, while solutions with $\lambda = \sqrt{Q}$ are exponentially large.

Since each root of λ is doubly degenerate, the eigenvectors $\tilde{v}_0$ span two-dimensional invariant subspaces. There is thus some indeterminacy in the form of the $\tilde{v}_0$'s, which is resolved at higher order. A particular set of four

eigenvectors $\underline{w}_i$ can be arranged into the symplectic matrix

$$\underline{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1/\sqrt{Q} & 0 \\ 0 & 1 & 0 & (1+KQ^3)/\sqrt{Q} \\ -\sqrt{Q} & 0 & 1 & 0 \\ 0 & \sqrt{Q}/(1+KQ^3) & 0 & 1 \end{bmatrix}, \quad (64)$$

where the first two columns, $\underline{w}_1$ and $\underline{w}_2$, correspond to $\lambda = -\sqrt{Q}$, while the last two, $\underline{w}_3$ and $\underline{w}_4$, correspond to $\lambda = \sqrt{Q}$. An arbitrary vector is obtained by multiplying $\underline{W}$ on the right by a column 4-vector of coefficients $\underline{d}$,

$$\underline{v} = \underline{W} \underline{d}. \quad (65)$$

An eigenvector for $\lambda = -\sqrt{Q}$ is obtained if $d_3 = d_4 = 0$ while an eigenvector for $\lambda = \sqrt{Q}$ is obtained if $d_1 = d_2 = 0$.

Equation (61) is an inhomogeneous equation for $\underline{v}_n$ in terms of $\underline{v}_{n-1}$. Since the matrix on the left is singular, the inhomogeneity must satisfy a constraint: it must be orthogonal to the null space of the adjoint of the operator on the left. At first order, this constraint determines the power q in Eq. (52) and resolves the indeterminacy in $\underline{v}_0$. At higher order, two components of $\underline{v}_n$ are determined as the inhomogeneous solution to Eq. (61), while the other two are determined by the constraint equations.

To obtain an explicit form of the constraint on Eq. (61), we multiply on the left by $\underline{w}_i^t \cdot \underline{E}$, where $\underline{w}_i^t$ is the transpose of an eigenvector of $-\lambda$ and $\underline{E}$ is the skew-symmetric matrix defined in Eq. (33). Because of the Hamiltonian property of $\underline{L}_1$, Eq. (34), this annihilates the left-hand side. Since each value of λ is doubly degenerate, there are two such equations. We express $\underline{v}_n$ in the form

$$\underline{v}_n = \underline{w}_n \underline{d}_n \quad (66)$$

with

$$\underline{d}_n = \begin{bmatrix} \underline{d}_n^- \\ \underline{d}_n^+ \end{bmatrix}, \quad (67)$$

and the $\underline{d}_n^\pm$ are 2-vector of coefficient associated with $\lambda = \pm\sqrt{Q}$, respectively.

Then the constraint equations become

$$[M - (q - 2n)I] \underline{d}_n^\sigma = -N \underline{d}_n^{-\sigma} \quad (68)$$

where

$$M_{11} = \lambda Q(1 - D_R/Q^3), \quad (69a)$$

$$M_{22} = \lambda Q[G + K(F + H^2) + D_R/Q^3], \quad (69b)$$

$$M_{12} = H + D_R/\lambda Q, \quad (69c)$$

$$M_{21} = (1 + KQ^3)(H - D_R/\lambda Q), \quad (69d)$$

$$N_{11} = \lambda Q(1 + D_R/Q^3) + 1 - 2H, \quad (70a)$$

$$N_{22} = 1 - \lambda Q[G + K(F + H^2) + D_R/Q^3] \quad (70b)$$

$$N_{12} = H - D_R/\lambda Q, \quad (71a)$$

$$N_{21} = -(1 + KQ^3)(H - D_R/\lambda Q), \quad (71b)$$

and $\sigma = \pm 1$.

The lowest-order constraint, Eq. (68) with $n = 0$ and with a vanishing right-hand side, shows that the power q is an eigenvalue of $\underline{M}$,

$$\text{Det} [\underline{M} - q \underline{I}] = 0, \quad (72)$$

$$q_{\pm} = \frac{1}{4} (B \pm D), \quad (73)$$

where

$$B = \lambda Q[1 + G + K(F + H^2)] \quad (74)$$

and

$$D^2 = Q^3 \{ (G + KF - 1)^2 + KH^2 [KH^2 + 2(G + KF + 1)] \} + 4[(G - KE - 1)D_R + H^2]. \quad (75)$$

The powers in Eq. (73) correspond to those in Eq. (4b) in configuration space. The solutions for d_0^σ are the corresponding eigenvectors. Higher order terms in the asymptotic series, Eq. (58), are obtained by solving Eq. (61) for $d_n^{-\sigma}$ in terms of d_{n-1} , and then solving Eq. (68) for d_n^σ in terms of $d_n^{-\sigma}$. This is best accomplished in the representation in which $\underline{L}_1$ and $\underline{M}$ are diagonalized simultaneously.

We have found that the solutions to the Fourier-transformed equations,

Eqs. (30) or (31) or (39), can be expressed as linear combinations of four powerlike solutions as $z \rightarrow 0$, and as a linear combination of two exponentially large and two exponentially small solutions as $z \rightarrow \infty$, and we have obtained equations for generating series solutions in both limits to arbitrarily high order. We must now find contours in the complex z plane and particular linear combinations of solutions which satisfy two types of boundary conditions. First, the integrated terms obtained when Fourier transforming Eqs. (1) must vanish at the ends of the contours. This is essentially equivalent to requiring that the integrands in Eqs. (29) vanish at the ends of the contours. Second, we must construct solutions which are even and odd, respectively, about the point $x = 0$ in configuration space, and which are not exponentially large as $x \rightarrow \pm \infty$.

When both x and Q are real and positive, we start with the contours introduced in Ref. 5 and shown here in Fig. 1. Contour c_1 starts at $|z| = \infty$ on the ray $\arg z = \frac{3}{4}\pi$ and goes to $|z| = \infty$ on the ray $\arg z = \frac{1}{4}\pi$. Contour c_2 starts at $|z| = \infty$ on the ray $\arg z = -\frac{5}{4}\pi$ and goes to $|z| = \infty$ on the ray $\arg z = -\frac{1}{4}\pi$. Two other contours, c_3 and c_4 , are shown, which begin at $|z| = \infty$ in the quadrant $-\frac{3}{4}\pi < \arg z < -\frac{1}{4}\pi$. These contours always yield solutions which are exponentially large as $x \rightarrow \pm \infty$, and will not be used. The rays on which contours c_1 and c_2 begin and end are antistokes lines for the factors $\exp(\pm \sqrt{Q} z^2/2)$, across which the large and small solutions exchange roles, and on which both types of solutions are oscillatory rather than exponentially large or small. The contributions from the ends of the contours vanish because of the factor e^{ixz} in Eqs. (29). These contours thus yield well-defined solutions for $x > 0$, but not for $x < 0$. For $x < 0$ we must use contours obtained by reflecting c_1 and c_2 about the real axis. If Q is not real, the antistokes lines rotate by $1/4 \arg Q$. If a branch cut is introduced

into the complex Q plane at $\arg Q = \pm \pi$, then the real z axis remains in the same quadrant as for $\arg Q = 0$.

In order to construct a solution for $\Xi(x)$ which has even or odd symmetry about $x = 0$, it is necessary to continue analytically the solutions from positive to negative values of x . This can be accomplished by deforming contours c_1 and c_2 into the quadrants on the left and right between the antistokes lines and finding solutions which converge for both signs of x . In these quadrants, the large- z behavior of the factor $\exp(\pm \frac{1}{2} \sqrt{Q} z^2)$ dominates over that of $\exp(ixz)$, and we must find solutions for $\xi(z) = z^{-1/2} y_1(z)$ which are exponentially small in both quadrants. It is not generally the case that the solutions which are small on the right are also small on the left. However, it is possible to construct solutions for $\Xi(z)$ from a linear combination of integrals over c_1 and c_2 such that the sum of the two integrands is small on both sides.

To obtain a solution for $\Xi(x)$ with a given parity, let c'_1 and c'_2 be contours obtained by deforming c_1 and c_2 so that they go along the real z axis, with c'_1 starting from $\arg z = \pi$ and going above the real axis to $\arg z = 0$, and c'_2 starting from $\arg z = -\pi$ and going below the real axis to $\arg z = 0$. Let

$$\Xi(x) = \int_{c'_1} dz e^{ixz} \xi_1(z) + \int_{c'_2} dz e^{ixz} \xi_2(z). \quad (76)$$

Then

$$\Xi(-x) = \int_{c'_1} dz e^{-ixz} \xi_1(z) + \int_{c'_2} dz e^{-ixz} \xi_2(z). \quad (77)$$

If we now change the variable of integration in the first integral to $z' =$

$ze^{-\pi i}$ and in the second to $z'' = ze^{\pi i}$, the two contours exchange places, and

$$\Xi(-x) = \int_{C'_2} dz' e^{ixz'} \xi_1(z'e^{\pi i}) + \int_{C'_1} dz'' e^{ixz''} \xi_2(z''e^{-\pi i}) \quad (78)$$

Then Eqs. (76) and (78) show that

$$\Xi(-x) = \pm \Xi(x) \text{ if } \xi_2(z) = \pm \xi_1(ze^{\pi i}). \quad (79)$$

To construct a solution for $\xi_1(z)$ such that the sum of the integrands is small on both sides as $|z| \rightarrow \infty$, we note that Eq. (C-6) implies that such a solution must be orthogonal, in the appropriate sense, to any two independent small solutions. To apply this condition, let v_1 and v_2 be two independent solutions which are exponentially small as $|z| \rightarrow \infty$ in the quadrant containing the positive real z axis, and let χ be a solution corresponding to an even solution for $\Xi(x)$, so that $\xi_1(z) = z^{-1/2} y_1(z)$ and $\xi_2(z) = \xi_1(ze^{\pi i})$. The orthogonality condition in the right quadrant, Eq. (42), can be written

$$(v_1, \{z^{-1/2} y_1(z) + (ze^{\pi i})^{-1/2} y_1(ze^{\pi i})\}) = 0. \quad (80)$$

Let U be the symplectic fundamental matrix of solutions constructed from the four independent column vectors u defined in Eq. (44), i.e., the four power-series solutions about the origin arranged in the same order as in Eqs. (52) and (56). Then v_1 , v_2 , and χ can be expressed as

$$v_1 = U \cdot c_1, \quad \chi = U \cdot \zeta, \quad (81)$$

where the c 's are constant 4-vectors, the coefficients of the four powerlike

components. Since U is symplectic, and thus satisfies Eq. (C-2), Eq. (80) can be expressed as an orthogonality condition on the c 's. In the sum over the two integrands for the even solution on the right, the contribution from c_4 , corresponding to the eigenvalue $p = -1/2$, vanishes, and the equations become

$$\{1 + \exp[-(p_1 + 1/2)\pi i]\}c_{i1}c_3 - \{1 + \exp[(p_1 - 1/2)\pi i]\}c_{i3}c_1 = 2c_{i4}c_2. \quad (82)$$

The two equations of this form, with $i = 1$ and 2 , are then solved for the ratio c_1/c_3 for even $\Xi(x)$,

$$\frac{c_1}{c_3} = \exp(p_1\pi i) \tan[(1/2 - p_1)\pi] \left[\frac{c_{13}c_{24} - c_{23}c_{14}}{c_{11}c_{24} - c_{21}c_{14}} \right]. \quad (83)$$

A similar calculation for odd $\Xi(x)$ yields

$$\frac{c_1}{c_3} = \exp(p_1\pi i) \cot[(1/2 - p_1)\pi] \left[\frac{c_{13}c_{22} - c_{23}c_{12}}{c_{11}c_{22} - c_{21}c_{12}} \right]. \quad (84)$$

Equations (80) - (82) represent the condition that the sum of the integrands, $\xi_1(z) + \xi_2(z)$, be small in the right quadrant for an even solution $\Xi(x)$. The condition that the sum be small on the left is that $\xi_1(ze^{\pi i}) + \xi_2(ze^{-\pi i})$ satisfy a similar orthogonality condition. Since $\xi_1(ze^{\pi i}) = \xi_2(z)$ and $\xi_2(ze^{-\pi i}) = \xi_1(z)$, this condition is identical to the first one, and is automatically satisfied if the first one is. The same result holds for odd $\Xi(x)$. Therefore, the construction of even and odd solutions is now complete.

The next step is to find the behavior of $\Xi(x)$ as $|x| \rightarrow \infty$. Since the function has known parity, it is sufficient to find the behavior as $x \rightarrow +\infty$.

and use the original contours c_1 and c_2 . The dominant behavior is powerlike, and can be determined by integrating the power series solutions about $z = 0$ term by term to obtain an asymptotic series in decreasing powers of x^2 . To integrate a term of the form

$$I = \int_{c_2} dz e^{ixz} z^p, \quad (85)$$

we first deform the contour so that it starts from $\arg z = -3/2 \pi$ on the positive imaginary axis, loops around the origin in the positive sense, and returns to the positive imaginary axis with $\arg z = 1/2 \pi$. We then change the variable of integration to $t \equiv e^{3/2\pi i} xz$ and write the integral as

$$I = e^{-1/2\pi i} x^{-(p+1)} i \int_{c_2''} dt e^{-t} (e^{-\pi i} t)^p, \quad (86)$$

where the contour c_2'' starts on the positive real axis with $\arg t = 0$, circles the origin, and returns to the positive real axis with $\arg t = 2\pi$. The result is a contour integral representation of the gamma function¹⁰ and yields

$$I = 2\pi \exp(-1/2\pi i) x^{-(p+1)} / \Gamma(-p). \quad (87)$$

If p is zero or a positive integer, there is no singularity inside the contour, the gamma function in the denominator has a pole, and the integral vanishes. As a result, all terms associated with the eigenvalues $p = \pm 1/2$ vanish, leaving only the contributions from $p = \pm p_1$. Furthermore, since the contour c_1 does not circle the origin, it contains no singularities and contributes no powerlike terms. Its contribution does not vanish, but it is exponentially small as $x \rightarrow \infty$. It could be evaluated by using the saddle point

method and the large $|z|$ asymptotic behavior of $\xi_1(z)$. Since these small terms play no role in matching to the outer region, this is not necessary.

The large- x limit can thus be written

$$\begin{aligned} \Xi(x) = & Ax^{-1/2-p_1} \left[1 + \frac{a_1}{x^2} + \frac{a_2}{x^4} + \dots \right] \\ & + Bx^{-1/2+p_1} \left[1 + \frac{b_1}{x^2} + \frac{b_2}{x^4} + \dots \right]. \end{aligned} \quad (88)$$

We define the asymptotic ratio

$$\Delta \equiv B/A. \quad (89)$$

Using Eqs. (52), (83), (84), and (87), we find that for the even solution $\Xi_+(x)$,

$$\Delta_+ = \tan[(p_1 - \frac{1}{2})\pi] \frac{\Gamma(\frac{1}{2} - p_1)}{\Gamma(\frac{1}{2} + p_1)} \left[\frac{c_{13}c_{24} - c_{23}c_{14}}{c_{11}c_{24} - c_{21}c_{14}} \right], \quad (90)$$

while for the odd solution $\Xi_-(x)$,

$$\Delta_- = \cot[(p_1 - \frac{1}{2})\pi] \frac{\Gamma(\frac{1}{2} - p_1)}{\Gamma(\frac{1}{2} + p_1)} \left[\frac{c_{13}c_{22} - c_{23}c_{12}}{c_{11}c_{22} - c_{21}c_{12}} \right]. \quad (91)$$

We can now describe the numerical method. We first evaluate the coefficients in the power series solutions about zero and infinity to an arbitrary number of terms by diagonalizing and solving Eqs. (46), (47), (60), and (61). We then initialize to each of the two exponentially small solutions, v_1 and v_2 , in Eq. (58) at a large value of z and integrate Eq. (39), in the representation in which $L_{\infty 0}$ is diagonalized, to a small value of

z , using an adaptive numerical integrator.¹⁰ The constant vectors $\underline{c}_1$ and $\underline{c}_2$ are obtained by inverting Eq. (81). Finally, the components c_{1j} are used to construct $\Delta_{\pm}$ according to Eqs. (90) and (91).

Since both the origin and the point at infinity are singular points, it is necessary to choose the maximum and minimum values of z on the bases of convergence criteria. Since the origin is a regular singular point and the associated power series about the origin are convergent, there is a simple procedure for choosing $z_{\min}$ to give a desired relative precision ϵ . Since the leading terms in Eq. (44) are of order unity, the highest retained terms are of order ϵ if

$$z_{\min} = 1/2 \left| \epsilon / u_{\max} \right|^{1/2N}, \quad (92)$$

where N is the number of retained terms and $u_{\max}$ is the largest element of y_N . In practice, we take $N=8$ and find that the computation time is insensitive to small changes in N .

Because the point at infinity is an irregular singular point and the series solutions there are asymptotic rather than convergent, the choice of $z_{\max}$ is more difficult. An examination of the solution in Eq. (58) reveals that there are two factors which determine the behavior at large z , a power and a Gaussian factor. If the different powers q given by Eqs. (73) through (75) are not substantially different from each other, such as when neither Q nor G nor K is very large, then the Gaussian factor dominates the behavior. In this case, the boundary condition at large z , exclusion of the two exponentially large solutions, can be imposed by choosing $z_{\max}$ such that

$$\exp \left(- \frac{1}{2} z_{\max}^2 \right) \sim \epsilon. \quad (93)$$

It is possible to impose boundary conditions simply by taking a sufficiently large value of $z_{\max}$, initializing to two arbitrary independent solutions, and relying on the exponential factor to annihilate the two large solutions during the inward integration. There is an advantage, however, to initializing to the two small solutions and retaining higher order terms in the series, since this permits a reduction in $z_{\max}$ and hence in computation time for a given accuracy. Since the series are asymptotic, there is an optimal number of terms at a fixed value of z beyond which the error terms stop decreasing and start increasing. This optimal number is in general different for different components of V , and each component may be truncated at its optimal number of terms.

If D^2 in Eq. (75) is large and positive, there is a large discrepancy in the different value of the power q in Eq. (73). There is then a large intermediate region between $z_{\min}$ and $z_{\max}$ where the powerlike behavior is dominant. For sufficiently large values of D^2 , this behavior causes the solution with the most negative value of $\text{Re } q$ to overwhelm numerically all other solutions before getting through this intermediate region. This makes it impossible to obtain two independent small solutions, and the method fails.

There are cases in which the powerlike behavior can be exploited to impose the boundary conditions. If the dominant powerlike factors happen to be correlated with the dominant exponential factors, then convergence to the correct result is found to occur for values of $z_{\max}$ much less than those given in Eq. (93). For example, for $G = K = 0$, the choice $z_{\max} = 10 z_{\min}$ gives good convergence to a result which agrees with the results of the finite difference method for arbitrarily large values of Q . On the other hand, as G , K , and Q all became large, there is a window of convergence in $z_{\max}$ which becomes successively smaller. Eventually this window closes up and the method fails.

Our understanding of the convergence properties of this system is limited by the complexity of the problem. A full analytical understanding seems to be beyond our abilities at this time. Fortunately, the cases described here include all situations of known physical importance.

This method has been incorporated into a computer code called DELTAR on a CRAY-I computer. For a single pair of values Δ_+ and Δ_- evaluated to a relative precision of 10^{-4} , computation times are typically about 20 ms. This precision can be maintained for values of $D^2 \leq 10^3$ if G and K are finite, and for all values of D^2 if $G = K = 0$.

VI. RESULTS

The numerical methods presented in Secs. III-IV enable us to compare values for the odd Ξ ratio Δ_- more accurately than is possible with previous analytical methods,⁵ and they also yield solutions for the even Ξ ratio Δ_+ for the first time. Results for these quantities are presented in this section for a wide range of values of the parameters E , F , G , H , K , and Q . Since the results are presented in graphic form and the differences among the values obtained from the different numerical methods are far too small to show up on a graph, it is not necessary to distinguish which method is used to obtain the results. We do present comparisons, where possible, to the analytical results of Ref. 5, hereafter referred to as GGJ.

Before presenting the results, it is useful to understand the context in which these quantities occur. A dispersion relation is obtained by matching the different powerlike terms on either side of the boundary layer to the corresponding terms in the outer ideal MHD regions. For a mode with one singular surface between the magnetic axis and a conducting wall, the dispersion relation takes the form

$$2\Delta_+\Delta_- - \epsilon(\Delta_+ + \Delta_-)(\Delta_L + \Delta_R) + 2\epsilon^2\Delta_L\Delta_R = 0 \quad (94)$$

The quantities Δ_+ and Δ_- are the ratios defined in Sec. II and computed by the numerical methods of Secs. III-V, with Δ_+ referring to the even Ξ solution and Δ_- to the odd Ξ solution. The quantities Δ_L and Δ_R are the corresponding ratios for the ideal region solutions to the left and right of the singular surface, respectively. They contain information about the global properties of the current profile away from the singular surface and are primarily responsive to the parallel current forces which drive kink and tearing modes. While Δ_+ and Δ_- are computed with the scaled variable x appropriate to the inner region, the quantities Δ_L and Δ_R are computed with an unscaled global coordinate appropriate to the outer region. The difference in scales is accounted for by the small parameter

$$\epsilon = S^{-2/3}p_1 \ll 1 \quad (95)$$

in which $S = \tau_R/\tau_A \gg 1$ is the magnetic Reynold's number, the ratio of the local resistive diffusion time to the local Alfvén transit time at the singular layer, and p_1 is given in Eq. (50). For modes with multiple singular surfaces, Eq. (94) is replaced by a more complicated expression obtained as the determinant of a large matrix involving many more Δ 's and ϵ 's.

The dispersion relations in Eqs. (86)-(88) and (A-9) of GGJ are obtained as special cases of Eq. (94) of the present work. To recover Eq. (86), we first drop the third term in Eq. (94) of this paper because it is proportional to ϵ^2 . The second term is retained even though it is of order ϵ . This is because the quantity $\Delta_L + \Delta_R$, which corresponds to Δ in GGJ and Δ' in Refs. 1-

4, gets large as the mode approaches ideal marginal stability, and also because the quantity Δ_- in the first term is small near marginal interchange stability. This last consideration also allows us to neglect Δ_- compared to Δ_+ in the second term. This approximation is not explicit in GGJ, but is implicit in the assumption that Ψ is even about the singular surface, and is closely related to the constant- Ψ approximation of Refs. 1-4. The numerical solutions for Δ_+ and Δ_- obtained here enable us to evaluate the validity of this approximation. As a consequence of these simplifications, Δ_+ is a common factor in the remaining terms of Eq. (94). If Δ_+ has no roots, we may divide it out and recover Eq. (86) of GGJ. Then Δ_- is seen to be equivalent to the quantity $1/2(x_0/v_g)^{1-2H} \Delta'(Q)$ of GGJ. Equations (87), (88), and (A-9) of GGJ represent analytical approximations to Δ_- for $G = K = 0$, $G = H = K = 0$, and $H = 0$, respectively, with $|D_R| \sim |Q|^{3/2} \ll 1$ in all cases.

As a first example of results, Fig. 2 shows numerical values for Δ_- as a function of $\log_{10} Q$ for the scaled growth rate Q real and positive and for a range of values of $D_R = E$ between -1 and +1, with $F = G = H = K = 0$. Also shown are analytical results from Eq. (88) of GGJ. The analytical results are in closest agreement with the numerical values for $|D_R| \sim |Q|^{3/2} \ll 1$, and become progressively more inaccurate as $|D_R|$ gets large and as $|Q|$ deviates from $|D_R|^{2/3}$. Fortunately, this implies that the analytical results give excellent values for the root of Δ_- when $D_R > 0$, which corresponds to the growth rate of the resistive interchange or G-mode, and for the minimum of Δ_- when $D_R < 0$, which is related to the threshold value Δ_c in GGJ.

The most surprising numerical result is the existence of a pole in Δ_- near $Q = 1$. There is no corresponding behavior in the analytical results. Equation (88) of GGJ shows that $\Delta_- \sim Q^{5/4} + \infty$ as $Q \rightarrow \infty$, but Δ_- remains finite for finite values of Q . This discrepancy does not require an explanation,

since the analytical results are based on the approximation $|Q| \ll 1$. In fact, this limitation may now be interpreted as a requirement that Q be sufficiently far below the pole.

The presence of this pole in Δ_- has an important effect on the scaling of the tearing mode. The outer region quantity $\Delta' = \Delta_L + \Delta_R$ is represented in Fig. 2 by a horizontal line in the upper half of the figure. The point of intersection of this line with the Δ_- curve gives the scaled growth rate Q of the tearing mode. For the analytical solution, the behavior $\Delta_- \sim Q^{5/4}$ as $Q \rightarrow \infty$ gives the well-known result that the unscaled growth rate of the tearing mode varies at $S^{-3/5}$ as $S \rightarrow \infty$. The pole in the numerical solution has the effect of changing this scaling to $S^{-1/3}$, since Q approaches a constant for large values of Δ' . This vindicates the ordering assumptions of GGJ.

Figure 2 also shows values of Δ_+ as a function of $\log_{10} Q$. It is clear that there are no zeroes of Δ_+ and that, with the exception of the immediate vicinity of the pole in Δ_- , the assumption $|\Delta_+| \gg |\Delta_-|$ required to obtain Eq. (86) of GGJ is well-satisfied. This indicates that the mode is indeed primarily even Ψ and odd Ξ .

After passing through the pole, Δ_- exhibits totally different behavior from that indicated by the analytical results. As $|Q| \rightarrow \infty$, the scaling of Δ_- and Δ_+ with Q can be extracted from the numerical results. For $G = K = 0$ and $|Q| \rightarrow \infty$, the following scaling is found.

$$\Delta_{\pm} = D_{\pm} Q^{-2p}, \quad (96)$$

$$D_- = -\frac{\pi}{2}, \quad D_+ = -\frac{\pi}{5D_1}, \quad (97)$$

$$D_1 = D_I + 1/4 = E + F + H. \quad (98)$$

Combining this with Eqs. (94) and (95), we find

$$2\Delta_L \Delta_R (S^{-1/3} Q)^{4P_1} - (\Delta_L + \Delta_R)(D_+ + D_-)(S^{-1/3} Q)^{2P_1} + 2D_+ D_- = 0. \quad (99)$$

This is a quadratic equation for a power of the unscaled growth rate $s = S^{-1/3} Q$, with coefficients independent of S . The solutions are ideal growth rates, in the sense that they are independent of resistivity. It would therefore appear that these results match smoothly onto ideal MHD stability theory. While the expressions for $D_{\pm}$ in Eq. (97) are valid only for $G = K = 0$, the scaling with Q in Eq. (96), and hence this conclusion, are found to be universal.

However appealing this may be, there are two reasons why it is questionable. First, since the unscaled growth rate s obtained from this analysis is independent of resistivity, it is not formally small. A similar statement applies to the boundary layer thickness. Since the smallness of these quantities is a necessary condition for the validity of the boundary layer analysis, this validity comes into question. The condition for validity is that there be a region of overlap where both the outer and inner asymptotic solutions are good approximations. As $Q \rightarrow \infty$, the distance over which the inner solution approaches its asymptotic form moves out, and eventually the method must fail. Exactly when it fails depends upon the rate of convergence of the outer solution as it approaches the singular surface, and this is beyond the scope of the present study.

The second reason for questioning this result is that it leads to a contradiction with Newcomb's well-known ideal MHD stability analysis of a cylindrical plasma column. To see this, consider the case $G = K = 0$ and $(\Delta_L + \Delta_R)^2 (D_+ + D_-)^2 \gg 16 \Delta_L \Delta_R D_+ D_-$. The larger of the two roots is then

given approximately by

$$s = \frac{(\Delta_L + \Delta_R)(D_+ + D_-)}{2 \Delta_L \Delta_R}^{1/p_1} \quad (100)$$

By Eq. (97), both D_+ and D_- are negative. If Δ_L and Δ_R have opposite signs and $\Delta' \equiv \Delta_2 + \Delta_R > 0$, then $s > 0$ and the mode is unstable. This can occur without having Newcomb's test function pass through zero before reaching the singular surface from either side. This procedure thus predicts an ideal MHD instability in a system which is stable by Newcomb's criterion. We believe this discrepancy is due to the breakdown of the matching method as discussed above.

A principal result of GGJ is the effect of finite H on Δ_- . It is emphasized there that the approximations used limit the validity of these results to the case $H < 1/2$. Since our numerical methods make no such approximations, they enable us to explore the transition which occurs as H passes through $1/2$. The most interesting result is a new type of stabilization of the resistive interchange mode.

Since the ideal and resistive interchange criteria of GGJ are related by

$$D_I = D_R - \left(H - \frac{1}{2}\right)^2, \quad (101)$$

and we wish to examine the case $D_R > 0$ but $D_I < 0$ to ensure ideal stability, it is necessary to avoid a region

$$\frac{1}{2} - \sqrt{D_R} < H < \frac{1}{2} + \sqrt{D_R} \quad (102)$$

in which the system is also Mercier unstable. Figure 3 shows values of Δ_-

vs. $\log_{10} Q$ for $D_R = 2.5 \times 10^{-3}$ and for several values of H approaching $1/2 - \sqrt{D_R} = 0.45$ from below. Also shown are analytical results from Eq. (87) of GGJ. Again there is fairly good agreement between the analytical and numerical results for H sufficiently below the limiting value, but the agreement becomes progressively worse as $H \rightarrow 0.45$. The pole and zero seen previously in Fig. 2 approach each other as $H \rightarrow 0.45$. Figure 4 shows values of Δ_- as $H \rightarrow 0.55$ from above. Here there is neither a zero nor a pole. The pair seem to annihilate each other as H passes through the forbidden region. Since there is no zero, there is no resistive interchange even though $D_R < 0$. The analytical results totally fail in this region, as expected.

Figures 3 and 4 also show values of Δ_+ for the same values of D_R and H as for Δ_- . These values appear to be symmetric in that $|\Delta_+| \gg |\Delta_-|$ for $H < 1/2$, except near the pole, but $|\Delta_+| \ll |\Delta_-|$ for $H > 1/2$. This may indicate that the mode changes character from primarily odd Ξ to primarily even Ξ as H passes through $1/2$.

It is not clear how the relatively large values $H > 1/2$ might be achieved in practice. An examination of H in Eq. (13) of GGJ reveals that low shear, high pressure, and strong asymmetry all contribute to making H large. Together with the observation that this has the effect of stabilizing pressure-driven interchange modes, this suggests that the effect of large H may be related to the stabilizing influences which give rise to the high- β second region of stability for ballooning modes.

Equation (A-9) of GGJ gives the effects of G and K on Δ_- with $H = 0$. Figure 5 shows numerical and analytical results for $D_R = 2.5 \times 10^{-3}$, $F = H = K = 0$, and a range of values of G . The pole described in Eqs. (A-13) and (A-14) of GGJ is evident for $G = 4000$. Agreement with analytical results is excellent for all cases except $G = 4000$. Figure 5 also shows values of Δ_+ for

the same values of D_R and G . A new feature here is the zero of Δ_+ , which indicates the existence of additional though not more unstable roots.

Figure 6 shows the effects of K for $D_R = 2.5 \times 10^{-3}$ and $F = G = H = 0$. Even for these very large values of K , the effects are rather weak.

The large number of parameters in this problem preclude the possibility of presenting exhaustive studies of the results. The results presented here represent just a sample of the variability possible in this rich set of equations. In practice, the codes themselves are used to study realistic families of equilibria with all values of E , F , G , H , and K chosen consistently.

VII. SUMMARY AND DISCUSSION

We have developed three numerical methods for solving the equations of Glasser, Greene, and Johnson governing the boundary layer of resistive instabilities in a torus. All three methods give results which agree with each other to a high degree of accuracy over a wide range of parameters. Two of the methods, the Finite Difference Method and the Fourier Transform Method, are valid over a larger range of parameter space than the other one, the Series Matching Method. The excellent agreement of these results with each other and with analytical results in suitably restricted regimes leads us to place great confidence in their correctness. All three methods have been carefully tuned to provide fast, accurate, and automatic procedures for obtaining their results.

Using these methods, we have explored regimes of parameter space inaccessible to analytical methods and revealed several novel features. The asymptotic ratio Δ_- , which primarily determines the growth rate of tearing and resistive interchange instabilities, is found to have a pole where the scaled

growth rate $Q = 1$. This pole modifies the scaling of the tearing mode growth rate from the well-known $S^{-3/5}$ behavior to the $S^{-1/3}$ behavior more commonly associated with interchange modes. Beyond this pole lies a region which appears to merge with ideal MHD stability theory, but which may lead to spurious results because of a failure of the boundary layer analysis there. A new region of stability to both tearing and interchange modes appears to lie in the range $H > 1/2$, but it is not clear how this promised land can be reached.

Many of the numerical difficulties addressed in this paper were anticipated by Coppi, Greene, and Johnson in Appendix A of Ref. 3, and many of the techniques developed here may be considered elaborations on the methods discussed there. The main improvements in our methods can be summarized as follows. The Finite Difference Method of Sec. IV provides a particularly effective procedure for eliminating the exponentially large solutions referred to there as the exponential problem. The Fourier Transform Method permits the exploitation of the simplifications referred to there in their discussion of their Eqs. (133) and (134) by showing how to obtain results numerically in Fourier space and transform them analytically back to configuration space. Finally, the equations treated here incorporate toroidal effects derived since the publication of Ref. 3.

It is useful to note a distinction between the fourth-order system of Eqs. (60), (61), and (69) of Ref. 3 and the sixth-order system treated here. Formally, our system passes over to a fourth-order system when $G \rightarrow \infty$. Since G is given in GGJ as $\langle B^2 \rangle / M\gamma p$, this is equivalent to the limit $\gamma p \rightarrow 0$, or to the limit in which the growth rate s is large compared to sound frequencies. This is often referred to as the incompressible limit, although this is a misnomer because the incompressible limit is the one in which $\gamma p \rightarrow \infty$. Since we find the limit $G \rightarrow \infty$ the most troublesome case to treat numerically, the exact

relationship between our sixth-order results and the simpler and hence more popular fourth-order system remains problematic.

Two main problems remain before these techniques can be effectively used to describe the stability properties of toroidal plasmas in collision dominated regimes. One is to couple these calculations of the inner regions to a global numerical treatment of the outer region in full toroidal geometry. The other is to extend these methods to include all the additional physical effects described by a complete fluid model of the plasma, such as that of Braginskii. Even then, the treatment of the trapped-particle and gyrokinetic effects characteristic of less collisional plasmas will remain a more difficult problem for the future.

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APPENDIX A: ASYMPTOTIC SERIES AT LARGE X

To obtain recursion relations for the coefficients of the asymptotic series solutions [Eqs. (5) and (6)], we start with the basic set of Eqs. (1a), which can be written in component form as

$$\frac{d^2 \Psi}{dx^2} - Q\Psi + Qx\Xi - H \frac{dT}{dx} = 0 \quad (\text{A-1})$$

$$Q^2 \frac{d^2 \Xi}{dx^2} - Qx^2 \Xi + Qx\Psi + H \frac{d\Psi}{dx} + (E + F)T = 0 \quad (\text{A-2})$$

$$Q \frac{d^2 T}{dx^2} - [x^2 + Q^2(G + KF)]T + Q^2(G - KE)\Xi + x\Psi - Q^2KH \frac{d\Psi}{dx} = 0 \quad (\text{A-3})$$

For the powerlike solution vectors $T_3(x)$, $T_4(x)$, we make the following substitutions into the above set of equations:

$$\begin{bmatrix} \Psi \\ \Xi \\ T \end{bmatrix} = x^{\frac{p_{\pm}}{2}} \sum_{j=0}^{\infty} \begin{bmatrix} \psi_j x^{1-2j} \\ \xi_j x^{-2j} \\ v_j x^{-2j} \end{bmatrix} \quad (\text{A-4})$$

$$\begin{bmatrix} \Psi' \\ \Xi' \\ T' \end{bmatrix} = x^{\frac{p_{\pm}}{2}} \sum_{j=0}^{\infty} \begin{bmatrix} \psi_j (p_{\pm} + 1 - 2j) x^{-2j} \\ \xi_j (p_{\pm} - 2j) x^{-1-2j} \\ v_j (p_{\pm} - 2j) x^{-1-2j} \end{bmatrix} \quad (\text{A-5})$$

$$\begin{bmatrix} \Psi'' \\ \Xi'' \\ T'' \end{bmatrix} = x^{\pm} \sum_{j=0}^{\infty} \begin{bmatrix} \psi_j (p_{\pm} + 1 - 2j)(p_{\pm} - 2j)x^{-1-2j} \\ \xi_j (p_{\pm} - 2j)(p_{\pm} - 1 - 2j)x^{-2-2j} \\ u_j (p_{\pm} - 2j)(p_{\pm} - 1 - 2j)x^{-2-2j} \end{bmatrix}$$

[Henceforth we shall drop the subscript $\pm$ from $p_{\pm}$, understanding that $p = p_+$ gives $T_3(x)$, while $p = p_-$ gives $T_4(x)$.] We then require that in each Eq. (A-1), (A-2), and (A-3) the coefficients of each power of x vanish separately. Equation (A-1) thus gives:

$$Q\psi_0 - Q\xi_0 = 0 \quad (\text{A-7a})$$

$$Q\psi_1 - Q\xi_1 = \psi_0(p+1)(p) - Hu_0p \quad (\text{A-7b})$$

$$Q\psi_j - Q\xi_j = \psi_{j-1}(p+3-2j)(p+2-2j) - Hu_{j-1}(p+2-2j), \quad j \geq 2 \quad (\text{A-7c})$$

Equation (A-2) yields:

$$Q\psi_0 - Q\xi_0 = 0 \quad (\text{A-8a})$$

$$Q\psi_1 - Q\xi_1 = -11\psi_0(p+1) - (E+F)u_0 \quad (\text{A-8b})$$

$$Q\psi_j - Q\xi_j = -11\psi_{j-1}(p+3-2j) - (E+F)u_{j-1} - Q^2\xi_{j-2}(p+4-2j)(p+3-2j), \quad j \geq 2 \quad (\text{A-8c})$$

And Eq. (A-3) gives us:

$$-u_0 + \psi_0 = 0 \quad (\text{A-9a})$$

$$-u_1 + \psi_1 = Q^2(G+KF)u_0 - Q^2(G-KE)\xi_0 + Q^2KH\psi_0(p+1) \quad (\text{A-9b})$$

$$\begin{aligned} -u_j + \psi_j &= Q^2(G+KF)u_{j-1} - Q^2(G-KE)\xi_{j-1} + Q^2KH\psi_{j-1}(p+3-2j) \\ &\quad - Qu_{j-2}(p+4-2j)(p+3-2j), \quad j \geq 2 \end{aligned} \quad (\text{A-9c})$$

Equations (A-7b) and (A-8b) contain degenerate combinations of ψ_1 and ξ_1 . There is a similar problem with Eqs. (A-7c) and (A-8c). The degeneracy can be removed by subtracting (A-8c) from (A-7c) and relabeling indices. The new equation then obtained is

$$\psi_j(p+1-2j)(p-2j+H) + u_j[E+F-H(p-2j)] = -Q^2(p+2-2j)(p+1-2j)\xi_{j-1}, \quad (\text{A-10})$$

which is valid for $j \geq 1$.

Equations (A-7a), (A-8a), and (A-9a) taken together imply that $\psi_0 = \xi_0 = u_0 = 1$ (arbitrary normalization).

Equations (A-8b) and (A-9b), taken together with the $j = 1$ version of Eq. (A-10), imply that

$$\psi_1 = \frac{1}{2(1-2p)} [-Q^2p(p-1) - \delta_0(E+F-H(p-2))] \quad (\text{A-11a})$$

$$\xi_1 = \psi_1 + \frac{H}{Q}(p+1) + \frac{E+F}{Q} \quad (\text{A-11b})$$

$$v_1 = \psi_1 + \delta_0, \quad (A-11c)$$

where we have defined a quantity

$$\delta_0 = -Q^2(G+KF) + Q^2(G-KE) - Q^2KH(p+1). \quad (A-11d)$$

Finally for general $j \geq 2$, Eqs. (A-8c), (A-9c), and (A-10) may be solved simultaneously to obtain

$$\psi_j = \frac{1}{2j(2j-1-2p)} \{-Q^2(p+2-2j)(p+1-2j)\xi_{j-1} - [E+F-H(p-2j)]\delta_{j-1}\} \quad (A-12a)$$

$$\xi_j = \psi_j + \frac{H}{Q}(p+3-2j)\psi_{j-1} + \frac{E+F}{Q}\gamma_{j-1} + Q(p+4-2j)(p+3-2j)\xi_{j-2} \quad (A-12b)$$

$$v_j = \psi_j + \delta_{j-1}, \quad (A-12c)$$

where

$$\delta_j = -Q^2(G+KF)v_j + Q^2(G-KE)\xi_j - Q^2KH(p+1-2j)\psi_j + Q(p+2-2j)(p+1-2j)v_{j-1}. \quad (A-12d)$$

For the exponentially decaying solutions $T_5(x)$ and $T_6(x)$, the appropriate substitutions are

$$\begin{bmatrix} \Psi \\ \Xi \\ \Upsilon \end{bmatrix} = \exp\left(\frac{-x^2}{2Q^{1/2}}\right) x^s \sum_{j=0}^{\infty} \begin{bmatrix} \psi_j x^{-1-2j} \\ \xi_j x^{-2j} \\ \upsilon_j x^{-2j} \end{bmatrix} \quad (\text{A-13})$$

$$\begin{bmatrix} \Psi' \\ \Xi' \\ \Upsilon' \end{bmatrix} = \exp\left(\frac{-x^2}{2Q^{1/2}}\right) x^s \sum_{j=0}^{\infty} \begin{bmatrix} \psi_j (s-1-2j) x^{-2-2j} - \frac{1}{\sqrt{Q}} \psi_j x^{-2j} \\ \xi_j (s-2j) x^{-1-2j} - \frac{1}{\sqrt{Q}} \xi_j x^{1-2j} \\ \upsilon_j (s-2j) x^{-1-2j} - \frac{1}{\sqrt{Q}} \upsilon_j x^{1-2j} \end{bmatrix} \quad (\text{A-14})$$

etc. (Again we have dropped the $\dagger$ subscript on s for convenience.) Requiring each power of x to vanish separately as before, we obtain from Eq. (A-1):

$$\frac{1}{Q} \psi_0 + Q\xi_0 + \frac{H}{\sqrt{Q}} \upsilon_0 = 0 \quad (\text{A-15a})$$

$$\frac{1}{Q} \psi_2 + Q\xi_2 + \frac{H}{\sqrt{Q}} \upsilon_2 = \left(Q + \frac{1}{\sqrt{Q}}(2s-1)\right) \psi_0 \quad (\text{A-15b})$$

$$\begin{aligned} \frac{1}{Q} \psi_j + Q\xi_j + \frac{H}{\sqrt{Q}} \upsilon_j &= \left(Q + \frac{1}{\sqrt{Q}}(2s-\psi_j+3)\right) \psi_{j-1} + H(s+2-2j) \upsilon_{j-1} \\ &\quad - (s+3-2j)(s+2-2j) \psi_{j-2}, \quad j \geq 2. \end{aligned} \quad (\text{A-15c})$$

From Eq. (A-2):

$$\left(Q - \frac{H}{\sqrt{Q}}\right) \psi_0 - Q^{3/2}(2s+1) \xi_0 + (E+F) \upsilon_0 = 0 \quad (\text{A-16a})$$

$$\left(Q - \frac{H}{\sqrt{Q}}\right)\psi_j - Q^{3/2}(2s-\psi_j+i)\xi_j + (E+F)v_j = -H(s+1-2j)\psi_{j-1} \\ - Q^2(s+2-2j)(s+1-2j)\xi_{j-1}, \quad j \geq 1 \quad . \quad (A-16b)$$

And from Eq. (A-3):

$$\left(1 + Q^{3/2}KH\right)\psi_0 + Q^2(G-KE)\xi_0 - \left\{Q^2(G+KF) + Q^{1/2}(2s+1)\right\}v_0 = 0 \quad (A-17a)$$

$$\left(1 + Q^{3/2}KH\right)\psi_j + Q^2(G-KE)\xi_j - \left\{Q^2(G+KF) + Q^{1/2}(2s-\psi_j+1)\right\}v_j \\ = Q^2KH(s+1-2j)\psi_{j-1} - Q(s+2-2j)(s+1-2j)v_{j-1}, \quad j \geq 1 \quad . \quad (A-17b)$$

The solution to these is straightforward, but lengthy.

Finally, we remark that the series expressions for $T_1(x)$ and $T_2(x)$ are not needed, since we are only considering solutions which do not diverge exponentially at infinity.

APPENDIX B: MATRIX FORMS

In this appendix we exhibit a matrix form for the recursion relations which generate the coefficients for the powers series and asymptotic series in Eqs. (5), (6), and (8) in the text. While not as efficient for numerical computation, the matrix forms are useful for deriving the convergence properties of the series.

Let us first consider the power series expansion about the origin $x = 0$, Eq. (8). We represent the even $\Xi(x)$ powers series solution, Eq. (2a), by the expansion

$$\Psi(x) = \psi_1^e x + \psi_2^e x^3 + \dots + \psi_n^e x^{2n-1}$$

$$\Xi(x) = \xi_1^e + \xi_2^e x^2 + \dots + \xi_n^e x^{2n-2} \quad (B-1)$$

$$T(x) = v_1^e + v_2^e x^2 + \dots + v_n^e x^{2n-2} .$$

If we introduce the n^{th} coefficient vector by the notation

$$z_n^e \equiv (\psi_n^e, \xi_n^e, v_n^e) , \quad (B-2)$$

we can show that it obeys the recursion relation

$$z_{n+1}^e = C_n^e \cdot z_n^e + D_n^e \cdot z_{n-1}^e , \quad (B-3)$$

$$C_n^e = \frac{1}{2n} \begin{bmatrix} \frac{Q(1+KH^2)}{(2n+1)} & \frac{-Q[(2n-1)-H(KE-G)]}{(2n+1)(2n-1)} & \frac{QH(G+KF)}{(2n+1)(2n-1)} \\ -\frac{H}{Q^2} & 0 & \frac{-(E+F)}{(2n-1)Q^2} \\ QKH & \frac{Q(KE-G)}{(2n-1)} & \frac{Q(G+KF)}{(2n-1)} \end{bmatrix} \quad (B-4)$$

$$D_n^e = \frac{1}{(2n)(2n-1)Q} \begin{bmatrix} \frac{-H}{(2n+1)} & 0 & \frac{H}{(2n+1)} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} . \quad (B-5)$$

Equation (B-3) is valid for $n \geq 1$ if we set $\psi_0 = \xi_0 = v_0 = 0$.

Similarly, we represent the odd $\Xi(x)$ powers series solution Eq. (2b), by the expansion

$$\Psi(x) = \psi_1^0 + \psi_2^0 x^2 + \dots + \psi_n^0 x^{2n-2}$$

$$\Xi(x) = \xi_1^0 x + \xi_2^0 x^3 + \dots + \xi_n^0 x^{2n-1} \quad (B-6)$$

$$U(x) = u_1^0 x + u_2^0 x^3 + \dots + u_n^0 x^{2n-1}$$

with the coefficient vectors

$$\Psi_n^0 \equiv (\psi_n^0, \xi_n^0, u_n^0) \quad (B-7)$$

obeying the recursion relation

$$\Psi_{n+1}^0 = C_n^0 \cdot \Psi_n^0 + D_n^0 \cdot \Psi_{n-1}^0, \quad (B-8)$$

$$C_n^0 = \frac{1}{2n} \begin{bmatrix} \frac{Q}{2n-1} & 0 & H \\ \frac{-Q^{-1}}{2n+1} \left(1 + \frac{H}{2n-1}\right) & 0 & \frac{-Q^{-2}}{2n+1} D_R \\ \frac{1}{2n+1} \left(\frac{Q^2 KH}{2n-1} - Q^{-1}\right) & \frac{-Q(G-KE)}{2n+1} & \frac{Q}{2n+1} (G-KE+KD_R) \end{bmatrix}, \quad (B-9)$$

$$D_n^0 = \frac{1}{2n(2n-1)Q} \begin{bmatrix} 0 & Q^2 & 0 \\ 0 & \frac{1}{2n+1} (H+2n-1) & 0 \\ 0 & \frac{-Q^3 KH}{2n+1} & \frac{2n-1}{2n+1} \end{bmatrix}. \quad (B-10)$$

Equation (B-8) is valid for $n \geq 1$ if we set $\psi_0 = \xi_0 = u_0 = 0$.

As for the powers series coefficients, the asymptotic series, Eq. (3), can be expressed in terms of matrix recursion relations. The asymptotic series for the vectors $\underline{s}_3$ and $\underline{s}_4$, Eq. (5), takes the form

$$\begin{aligned} x + \psi_1/x + \psi_2/x^3 + \dots + \psi_n/x^{2n-1} \\ \underline{s}_3 = 1 + \xi_1/x^2 + \xi_2/x^4 + \dots + \xi_n/x^{2n} \\ 1 + u_1/x^2 + u_2/x^4 + \dots + u_n/x^{2n} \end{aligned} \quad (B-11)$$

The n^{th} coefficient vector $\underline{\psi}_n \equiv (\psi_n, \xi_n, u_n)$ obeys the recursion relations

$$\underline{\psi}_{n+1} = M_n^1 \cdot \underline{\psi}_n + M_n^2 \cdot \underline{\psi}_{n-1}, \quad (B-12)$$

$$M_n^1 = \begin{bmatrix} \frac{Q^2 b_n^{KH}}{b_n + a_n} (s_n + 1) & \frac{-Q^2}{b_n + a_n} [s_n (s_n - 1) + b_n (G - KE)] & \frac{Q^2 b_n}{b_n + a_n} (G + KF) \\ H(s_n + 1) \left(\frac{1}{Q} + \frac{Q^2 b_n^K}{b_n + a_n} \right) & \frac{-Q^2}{b_n + a_n} (s_n (s_n - 1) + b_n (G - KE)) & \frac{E + F}{Q} + \frac{b_n Q^2}{b_n + a_n} (G + KF) \\ -\frac{Q^2 a_n^{KH}}{b_n + a_n} (s_n + 1) & \frac{-Q^2}{b_n + a_n} [s_n (s_n - 1) - a_n (G - KE)] & \frac{-Q^2 a_n}{b_n + a_n} (G + KF) \end{bmatrix}, \quad (B-13)$$

$$M_n^2 = \begin{bmatrix} 0 & 0 & \frac{-Q b_n}{b_n + a_n} (s_n + 2) (s_n + 1) \\ 0 & Q (s_n + 2) (s_n + 1) & -\frac{Q b_n}{b_n + a_n} (s_n + 2) (s_n + 1) \\ 0 & 0 & \frac{Q a_n}{b_n + a_n} (s_n + 2) (s_n + 1) \end{bmatrix}. \quad (B-14)$$

Here $p = -1/2 \pm (-D_1)^{1/2}$, $a_n = (H+p-2n-2)(p-1-2n)$, $b_n = E+F-H(p-2n-2)$, $s_n = p-2n$. The series (B-12) is initiated by setting $\psi_{-1} = 0$.

APPENDIX C: PROPERTIES OF HAMILTONIAN & SYMPLECTIC MATRICES

A brief summary of the properties of Hamiltonian and symplectic matrices is presented here. Proofs are found in Ref. 8. If H is Hamiltonian, and if (λ, ψ_λ) is an eigenvalue-eigenvector pair, i.e., $H \psi_\lambda = \lambda \psi_\lambda$, then $-\lambda$ is also an eigenvalue of H . Any two pairs $(\lambda_1, \psi_{\lambda_1})$, $(\lambda_2, \psi_{\lambda_2})$ satisfy the equation

$$(\lambda_1 + \lambda_2)(\psi_{\lambda_1}, E \psi_{\lambda_2}) = 0 \quad (C-1)$$

where (ψ_1, ψ_2) is the scalar product. This shows that ψ_1 is orthogonal to $E \psi_2$ unless $\lambda_1 = -\lambda_2$.

A matrix u satisfying the equation

$$u^t E u = E \quad (C-2)$$

is called symplectic. If u satisfies the equations

$$\frac{du}{dz} = H u, \quad \text{Det } u \neq 0, \quad (C-3)$$

i.e., if it is a fundamental matrix of solutions for a Hamiltonian matrix H , then

$$\frac{d}{dz} (u^t E u) = 0. \quad (C-4)$$

Thus, if u is initially symplectic, it remains symplectic. A Hamiltonian

matrix is therefore the generator of a symplectic matrix. If $(e^\lambda, \underline{u}_\lambda)$ is an eigenvalue-eigenvector pair for a symplectic matrix $\underline{U}$, i.e.,

$$\underline{U} \underline{u}_\lambda = e^\lambda \underline{u}_\lambda, \quad (C-5)$$

then $E^{-\lambda}$ is also an eigenvalue, and the orthogonality relation

$$(e^{\lambda_1} - e^{-\lambda_2})(\underline{u}_{\lambda_1} \cdot \underline{U} \underline{u}_{-\lambda_2}) = 0 \quad (C-6)$$

is satisfied. Finally, if $\underline{H}$ is Hamiltonian and $\underline{u}$ is symplectic, then the similarity transformation

$$\underline{H}' = \underline{u}^{-1} \underline{H} \underline{u}$$

leaves $\underline{H}'$ Hamiltonian. Symplectic matrices thus play a role similar to that of canonical transformations in classical mechanics.

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TABLE I

R	Single Precision	Double Precision
3.0	-0.26572 3	-0.26572 3
4.0	-0.26572 0	-0.26572 0
5.0	-0.26572 1221	-0.26572 1220
6.0	-0.26572 124	-0.26572 12228 3
7.0	-0.2656	-0.26572 12228 52048
8.0	-0.3	-0.26572 12228 52056
9.0	*	-0.26572 12228 51
10.0	*	-0.26572 121
11.0	*	-0.2658
12.0	*	-0.3

Table I. Values of Δ for the odd solution using the Series Matching Method with the matching point at $L = RQ^{1/4}$. These solutions are for $Q = 0.01$, $E = 0.1$, $F = G = H = K = 0$.

TABLE II

R	Single Precision	Double Precision
3.0	-95.0	-95.0
4.0	-94.92	-94.92
5.0	-94.92 38	-94.91 38
6.0	-94.91 364	-94.91 3637
7.0	-96.0	-94.91 36365 283
8.0	*	-94.91 36365 275
9.0	*	-94.91 36364
10.0	*	-94.91 33
11.0	*	*
12.0	*	*

Table II. Values of Δ for the even solution using the Series Matching Method with the matching point at $L = RQ^{1/4}$. These solutions are for $Q = 0.01$, $E = 0.1$, $F = G = H = K = 0$.

TABLE III

N	Δ	Δ Extrapolated
Even Solution		
25	-0. <u>26557</u> 17657	*
50	-0. <u>26571</u> 32951	-0.26572 2
100	-0. <u>26572</u> 07275	-0.26572 1223
200	-0. <u>26572</u> 11918	-0.26572 12228 25
400	-0. <u>26572</u> 12220	-0.26572 12226
800	-0. <u>26572</u> 12227	-0.26572 12229
Odd Solution		
25	- <u>94.86</u> 30758	*
50	- <u>94.91</u> 09500	-94.91 4
100	- <u>94.91</u> 34684	-94.91 36363
200	- <u>94.91</u> 36250	-94.91 36354
400	- <u>94.91</u> 36378	-94.91 3632
800	- <u>94.91</u> 36201	-94.91 361

TABLE III. Values of Δ for the even and odd solution using the Boundary Values Method, without and with applying h^4 Richardson's extrapolation. Parameters are $Q = 0.01$, $E = 0.1$, $F = G = H = K = 0$, $L = 3.1623$.

TABLE IV.

N	Δ	Δ extrapolated
Even Solution		
50	-. <u>32016</u> 39283	
100	-. <u>32057</u> 80313	-.32060 5
200	-. <u>32059</u> 89035	-.32060 02
400	-. <u>32059</u> <u>99367</u>	-.32060 0005
800	-. <u>32059</u> <u>99919</u>	-.32059 9995
1600	-. <u>32059</u> <u>99966</u>	-.32059 99969
Odd Solution		
50	-. <u>36260</u> 75725	
100	-. <u>36257</u> 94371	-.36257 75
200	-. <u>36257</u> <u>77121</u>	-.36257 7597
400	-. <u>36257</u> <u>75998</u>	-.36257 75923
800	-. <u>36257</u> <u>75930</u>	-.36257 75925
1600	-. <u>36257</u> <u>75946</u>	-.36257 7594

TABLE IV. Values of Δ for the even and odd solution using the Boundary Value Method, without and with applying Richardson's h^4 extrapolation. Parameters are $Q = 3.0$, $E = 0.1$, $F = .005$, $G = 40.0$, $H = .05$, $K = 500.$, $L = 39.11$.

FIGURE CAPTIONS

- FIG. 1. Contours of integration in the complex z plane used to define the integral of transforms in Eq. (29). 742098
- FIG. 2. Numerical (solid) and analytical (dotted) results for $F = G = H = K = 0$. 82T0177
- FIG. 3. Numerical (solid) and analytical (dotted) results as H approaches $1/2 - \sqrt{D_R}$ from below. 82T0181
- FIG. 4. Numerical (solid) and analytical (dotted) results as H approaches $1/2 + \sqrt{D_R}$ from above. 82T0180
- FIG. 5. Numerical (solid) and analytical (dotted) results for $F = H = K = 0$, $D_R = 0.0025$, and a range of values of G . 82T0179
- FIG. 6. Numerical (solid) and analytical (dotted) results for $F = G = H = 0$, $D_R = 0.0025$, and a range of values of K . 82T0178

Fig. 1

Fig. 2

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Fig. 3

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Fig. 4

82T0179

Fig. 5

#82T0178

Fig. 6

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