Electron transport in EBT in the low collision
frequency limit*

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ABSTRACT

A variational principle formulation is used to calculate the electron neoclassical transport coefficients in a bumpy torus for the low collisionality regime. The electron radial drift is calculated as a function of the plasma position and the poloidal electric field which is determined self-consistently. A bounce-averaged differential collision operator is used and the results are compared to previous treatments using a BGK operator.

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1. INTRODUCTION

In an asymmetric device like the ELMO Bumpy Torus (EBT), neoclassical diffusion is not automatically ambipolar; hence the diffusion coefficients are functions of the self-consistent electric field. For the observed sign of the electric field in the experiment, the ions are on resonant orbits and their diffusion coefficients are very sensitive functions of the magnitude of the electric field.¹ The electrons are on nonresonant orbits and their loss rate is a relatively insensitive function of the electric field. For this reason, the electron rates have typically been used to obtain the particle lifetime, while the ion rates have been used to obtain the self-consistent ambipolar field. The first work to obtain the electron loss rates was by Kovrizhnykh,² who used a BGK operator and assumed that the electrons were precessing due to an $\underline{E} \times \underline{B}$ drift only. In Ref. 3, this work was generalized to include magnetic drifts by solving for the diffusion coefficients numerically using a Landau collision operator and the full form of the precession drift. In Ref. 4, analytic expressions for the electron loss rates were obtained from a BGK operator with magnetic drifts included. All these analytic expressions neglected both the self-consistent poloidal electric field and the change in the radial drift as the plasma center is moved.

In this paper, we use a variational principle to calculate the electron transport coefficient in a bumpy torus in the limit of low collisionality. First, in Sec. II, we derive the form of the radial drift which is appropriate for an arbitrary choice of the location of the plasma center. Second, in Sec. III, we formulate the variational principle. Then, in Sec. IV, we calculate the self-consistent poloidal electric field

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in the low collision frequency regime. In Sec. V, we bounce average the approximate Fokker-Planck collision operator of Hirshman and Sigmar⁵ to obtain a bounce-averaged differential collision operator. Next, in Sec. VI, we evaluate the fluxes for typical EBT Scale (EBT-S) parameters. Finally, in Sec. VII, we discuss these loss rates and compare them with the previous work of Refs. 2-4.

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II. RADIAL AND POLOIDAL DRIFT VELOCITIES

For the equilibrium in the core plasma we follow the analytic analysis of Ref. 6. Hence, we choose to label the constant pressure surfaces by the contained magnetic flux $2\pi\alpha$. We write the magnetic field $\underline{B}$ as

$$\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta, \quad (1)$$

where β is a generalized poloidal angle.

If we take the magnetic field B as $B = B(\alpha, \beta)$ and the electrostatic potential in the plasma as $\varphi = \varphi(\alpha, \beta)$, then the particle velocity across the pressure surfaces due to the poloidal electric field is

$$\underline{v}_{de} \cdot \underline{\nabla}\alpha = - \frac{\partial\varphi}{\partial\beta}, \quad (2)$$

and the particle velocity across the pressure surfaces due to the toroidal nature of the magnetic field is

$$\underline{v}_{dB} \cdot \underline{\nabla}\alpha = - \frac{(mv^2/2)}{ZeB} (1 + \zeta_p^2) \frac{\partial B}{\partial\beta}, \quad (3)$$

where ζ_p is the pitch angle ($= v_{\parallel}/v$). In the notation of Ref. 6, the magnetic field in the core is

$$\underline{B} = \underline{B}_0 (1 - \epsilon r \cos \vartheta) - \frac{\epsilon}{2} \cos \vartheta \left\{ \underline{e}_r \left[r^2 \frac{\partial^2 \hat{\varphi}_0}{\partial r^2} + \frac{\partial}{\partial r} (r \hat{\varphi}) \right] \right\}$$

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$$+ \frac{e_r}{\zeta} \left(r^2 \frac{\partial^2 \hat{\varphi}_0}{\partial \zeta^2} \right) \} + \frac{e_\theta}{2} \sin \vartheta \frac{\partial}{\partial r} (r \hat{\varphi}_0) \quad , \quad (4)$$

where

$$\underline{B}_{v0} = \underline{e}_r \frac{\partial \hat{\varphi}_0}{\partial r} + \underline{e}_\zeta \left(\bar{B} + \frac{\partial \hat{\varphi}_0}{\partial \zeta} \right) \quad , \quad (5)$$

and we have taken their h function to be zero so that the only distortion of the flux surfaces is caused by toroidicity.

The magnetic potential $\hat{\varphi}_0$ is given by

$$\hat{\varphi}_0 = \sum_k a_k \sin k\zeta I_0(kr) \quad , \quad (6)$$

where

$$k \equiv \epsilon n N \quad , \quad n = 1, 2, \dots \quad , \quad (7)$$

N is the number of mirror sectors, and the inverse aspect ratio ϵ is given by

$$\epsilon \equiv a/R \quad , \quad (8)$$

with a the maximum minor radius of the plasma and R_0 the average major radius of the magnetic axis.

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In Eqs. (4) and (5), r and ϑ are defined by

$$R = R_0(1 + \varepsilon r \cos \vartheta) , \quad (9)$$

$$Z = ar \sin \vartheta , \quad (10)$$

where R is the major radius and Z is the vertical distance from the toroidal midplane. The toroidal angle ζ is defined so that $\zeta = 0$ is the midplane in the mirror sector where the transport is being calculated and $\zeta = \pi/N\varepsilon$ is the coil plane. For the mirror ratios of experimental interest, $\hat{\varphi}_0$ is roughly comparable to the first term in Eq. (6).

We now expand $\hat{\varphi}_0(r, \zeta)$ for $kr \ll 1$, that is, locations near the average magnetic axis, to obtain

$$\hat{\varphi}_0 \approx a_1 \sin(\varepsilon N \zeta) \left(1 + \frac{k_1^2 r^2}{4} + \frac{k_1^4 r^4}{64} + \dots \right) , \quad (11)$$

where

$$k_1 = \varepsilon N . \quad (12)$$

We determine $\bar{B}$ and a_1 by evaluating $\underline{B}$ at the midplane and the coil plane. At the midplane ($r = 0$, $\zeta = 0$), we define B_{m_0} as

$$B_{m_0} \equiv \bar{B} + a_1 \varepsilon N . \quad (13)$$

At the coil plane ($r = 0$, $\zeta = \pi/N\varepsilon$), the magnetic field is

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$$B = \bar{B} - a_1 \varepsilon N ,$$

and near the magnetic axis we can write

$$MB_{m_0} \simeq \bar{B} - a_1 \varepsilon N , \quad (14)$$

where M is the mirror ratio. We define the (small) parameter ε_0 by

$$\varepsilon_0 = (M - 1)/(M + 1) . \quad (15)$$

Then

$$\bar{B} = \frac{B_{m_0}}{1 - \varepsilon_0} , \quad (16)$$

$$a_1 = \frac{\varepsilon_0 B_{m_0}}{1 - \varepsilon_0} \frac{1}{\varepsilon N} . \quad (17)$$

For EBT parameters, where $M \leq 2$, we have $\varepsilon_0 \ll 1$. The magnetic field is given by

$$\begin{aligned} B = \frac{B_{m_0}}{1 - \varepsilon_0} & \left[1 - \varepsilon_0 \cos(N\varepsilon\zeta) \left(1 + \frac{k_1^2 r^2}{4} \right) \right] (1 - \varepsilon r \cos\vartheta) \\ & + \frac{B_{m_0}}{1 - \varepsilon_0} \varepsilon_0 \frac{r\varepsilon}{2} \cos\vartheta \cos(N\varepsilon\zeta) \} \end{aligned}$$

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$$\begin{aligned}
& + \underline{e}_r \left[\frac{B_{m0}}{1 - \varepsilon_0} \varepsilon_0 \frac{1}{2N} \cos \vartheta \sin(N\varepsilon\zeta) \right. \\
& \quad \left. - \frac{B_{m0} \varepsilon_0}{1 - \varepsilon_0} \frac{k_1^2 r}{2} \sin(\varepsilon N\zeta) + (1 - \varepsilon r \cos \vartheta) \right] \\
& + \underline{e}_\vartheta \left[- \frac{B_{m0}}{1 - \varepsilon_0} \varepsilon_0 \frac{1}{N} \sin \vartheta \sin(\varepsilon N\zeta) \right] .
\end{aligned} \tag{18}$$

We work to $O(\varepsilon)$, $O(\varepsilon_0)$; then

$$\begin{aligned}
\underline{B} = \underline{e}_\zeta \left\{ \frac{B_{m0}}{1 - \varepsilon_0} \left[1 - \varepsilon r \cos \vartheta - \varepsilon_0 \cos(N\varepsilon\zeta) \left(1 + \frac{k_1^2 r^2}{4} \right) \right] \right\} \\
+ \underline{e}_r \left[- \frac{B_{m0}}{1 - \varepsilon_0} \varepsilon_0 \frac{k_1^2 r}{2} \sin(\varepsilon N\zeta) \right] ,
\end{aligned} \tag{19}$$

$$B \simeq \frac{B_{m0}}{1 - \varepsilon_0} \left[1 - \varepsilon r \cos \vartheta - \varepsilon_0 \cos(N\varepsilon\zeta) \left(1 + \frac{k_1^2 r^2}{4} \right) \right] . \tag{20}$$

We note that in the midplane, the contours of constant B are circles centered at x_b , $y = 0$, where $x = r \cos \vartheta$, $y = r \sin \vartheta$, and x_b is given by

$$x_b = - \frac{2\varepsilon}{k_1^2 \varepsilon_0} . \tag{21}$$

In the coil plane, the contours of constant B are circles centered at $x = -x_b$, $y = 0$. We note also that the pressure contours are expected to be

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circles of constant U (Ref. 7), where $U = \oint dl/B$. We can calculate the center of the U contours by

$$U = \oint \frac{dl}{B} = \int_0^{\pi/N\epsilon} \epsilon R_0 \frac{d\xi}{E_\xi} \\ = \frac{R_0(1 - \epsilon_0)}{B_{m_0}} \left(\frac{\pi}{N} \right) \left[1 + \epsilon\chi + \frac{\epsilon_0^2 k_1^2}{4} (x^2 + y^2) \right], \quad (22)$$

so that the center is at $y = 0$ and $x = x_{fb}$, where

$$x_{fb} = - \frac{2\epsilon}{k_1^2 \epsilon_0^2}. \quad (23)$$

In the evaluation of $\underline{v}_{dB} \cdot \underline{\nabla} \alpha$ we need $\partial B / \partial \beta$. This is given by

$$\frac{\partial B}{\partial \beta} \Big|_\alpha = \frac{\partial \vartheta}{\partial \beta} \Big|_\alpha \frac{\partial B}{\partial \vartheta} \Big|_r + \frac{\partial r}{\partial \beta} \Big|_\alpha \frac{\partial B}{\partial r} \Big|_\vartheta. \quad (24)$$

Since the shift of the plasma center is $O(\epsilon)$ and $\partial B / \partial \vartheta$ is given from

$$\frac{1}{B} \frac{\partial B}{\partial \vartheta} \Big|_r \simeq \epsilon r \sin \vartheta, \quad (25)$$

we have $\vartheta \simeq \beta$ and

$$\frac{1}{B} \frac{\partial B}{\partial \beta} \Big|_\alpha \simeq \epsilon r \sin \beta \left[1 - \frac{\partial r}{\partial (\cos \beta)} \Big|_\alpha \frac{1}{B} \frac{\partial B}{\partial r} \Big|_\vartheta \frac{1}{\epsilon r} \right]. \quad (26)$$

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We assume that the pressure contours are circles of radius ρ and are offset from the average magnetic axis by Δ . Then with the use of the cosine rule

$$\frac{\partial r}{\partial \beta} \Big|_{\alpha} = \frac{\partial r}{\partial \beta} \Big|_{\rho} = \frac{\rho}{r} \Delta \sin \beta \quad (27)$$

and from Eq. (20)

$$\frac{\partial r}{\partial (\cos \beta)} \Big|_{\alpha} \frac{\partial \ln B}{\partial r} \frac{1}{\epsilon r} = \frac{\Delta \epsilon_0 k_1^2}{2\epsilon} \cos(N\epsilon \zeta) , \quad (28)$$

where in Eq. (27), to 0(1), $\rho = r$. Therefore, the magnetic drift is

$$\underline{v}_{dB} \cdot \underline{\nabla} \alpha = - \frac{(mv^2/2)}{Ze} (1 + \zeta_p^2) \epsilon r \sin \beta \left[1 - \frac{\Delta \epsilon_0 k_1^2}{2\epsilon} \cos(N\epsilon \zeta) \right] . \quad (29)$$

We note that in the midplane ($\zeta = 0$), if the plasma center were at the $|B|$ center then $\Delta \epsilon_0 k_1^2 / 2\epsilon = 1$ and no magnetic drift through the plasma contours would be observed.

Since the bounce time is much smaller than the drift time, we integrate over the bounce time by defining the integral operator

$$\langle \dots \rangle = \oint \frac{d\ell}{v_{\parallel}} \dots \quad (30)$$

The bounce time τ_b is defined as

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$$\tau_b = \oint \frac{d\ell}{v_{\parallel}} , \quad (31)$$

so

$$\langle \underline{v}_{de} \cdot \underline{\nabla} \alpha \rangle = -\tau_b \frac{\partial \psi}{\partial \beta} . \quad (32)$$

For $kr \ll 1$, it is easy to show that

$$v_{\parallel} = v \zeta_m [1 - \kappa^2 \sin^2(N\epsilon \zeta/2)]^{1/2} , \quad (33)$$

where ζ_m is the pitch angle ζ_p in the midplane evaluated for $\epsilon = 0$ and κ^2 is given by

$$\kappa^2 = \frac{1 - \zeta_m^2}{\zeta_m^2} \frac{2\epsilon_0}{1 - \epsilon_0} . \quad (34)$$

The bounce-averaged magnetic drift is given by

$$\langle \underline{v}_{dB} \cdot \underline{\nabla} \alpha \rangle = \frac{mv^2/2}{Ze} \epsilon r \sin \beta \oint \frac{d\ell}{v_{\parallel}} \left(1 + \frac{v_{\parallel}^2}{v^2}\right) \left[1 - \frac{\Delta \epsilon_0 k_{\parallel}^2}{2\epsilon} \cos(n\epsilon \zeta)\right] , \quad (35)$$

where $d\ell \simeq \epsilon R_0 d\zeta + O(\epsilon_0^2)$ and the integration is taken between $\zeta = 0$ and $\zeta = 2/N\epsilon \sin^{-1}(1/\kappa)$ ($v_{\parallel} = 0$).

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The integrals in Eq. (35) can be expressed in terms of elliptic functions, and $\langle \underline{v}_{dB} \cdot \underline{\Delta\alpha} \rangle$ can be written

$$\begin{aligned} \langle \underline{v}_{dB} \cdot \underline{\nabla\alpha} \rangle = & - \frac{1/2mv^2}{Ze} \varepsilon r \sin \beta \\ & \times \tau_b \left[g(\zeta_m) \left(1 - \frac{\Delta\varepsilon_0 k_1^2}{2\varepsilon} \right) + \frac{\Delta\varepsilon_0 k_1^2}{2\varepsilon} g_1(\zeta_m) \right], \end{aligned} \quad (36)$$

where

$$\tau_b = \left(\frac{2}{k_w} \right) \frac{1}{v\zeta_m} \begin{cases} \frac{1}{\kappa} K\left(\frac{1}{\kappa^2}\right), & \kappa^2 > 1 \\ K(\kappa^2), & \kappa^2 < 1 \end{cases} \quad (37)$$

Here

$$k_w = \frac{N}{R_0}, \quad (38)$$

$$g(\zeta_m) = \begin{cases} 1 + \zeta_m^2 \left\{ \kappa^2 \left[\frac{E(1/\kappa^2)}{K(1/\kappa^2)} - \left(1 - \frac{1}{\kappa^2} \right) \right] \right\}, & \kappa^2 > 1 \\ 1 + \zeta_m^2 \frac{E(\kappa^2)}{K(\kappa^2)}, & \kappa^2 < 1 \end{cases}, \quad (39)$$

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and

$$g_1(\zeta_m) = \begin{cases} 2\left[1 - \frac{E(1/\kappa^2)}{K(1/\kappa^2)}\right] + 2\zeta_m^2 \left\{ \frac{1 - \kappa^2}{3\kappa^2} + \frac{2\kappa^2 - 1}{3} \left[\frac{E(1/\kappa^2)}{K(1/\kappa^2)} - \left(1 - \frac{1}{\kappa^2}\right) \right] \right\}, & \kappa > 1 \\ \frac{2}{\kappa^2} \left[1 - \frac{E(\kappa^2)}{K(\kappa^2)}\right] + 2\zeta_m^2 \left[\frac{1 - \kappa^2}{3\kappa^2} + \frac{2\kappa^2 - 1}{3\kappa^2} \frac{E(\kappa^2)}{K(\kappa^2)} \right], & \kappa^2 < 1 \end{cases} \quad (40)$$

We note that if the plasma center is taken to coincide with the average magnetic axis ($\Delta = 0$), then in Eq. (36) the vertical drift reduces to the expression given in Ref. 8. The fact that $\langle \underline{v}_{dB} \cdot \underline{\nabla} \alpha \rangle$ is a function of the position of the plasma center can be understood by noting that $\underline{v}_{dB}$ is determined (in the low- β core) by vacuum magnetic field quantities that are independent of the plasma, so the drift through the pressure surfaces will be a function of the position of the plasma. By contrast, the bounce-averaged electric field drift through the pressure surface, given in Eq. (32), is independent of the plasma position (except insofar as that may determine the poloidal structure of the potential) since we expect that the potential is caused by the plasma and so will center itself near where the plasma is centered. If we choose the plasma center to be the $|B|$ center in the midplane ($\Delta \epsilon k_1^2 / 2\epsilon_0 = 1$), then for $\zeta_m = 0$, $\langle \underline{v}_{dB} \cdot \underline{\nabla} \alpha \rangle = 0$, since in this frame of reference the orbits are perfect circles about $r = 0$. If we choose the plasma center to be the $|B|$ center in the coil plane ($\Delta \epsilon k_1^2 / 2\epsilon_0 = -1$), then at $\kappa^2 = 1$, $\langle \underline{v}_{dB} \cdot \underline{\nabla} \alpha \rangle = 0$. This is because the particles at the transition between being trapped and passing will spend all their time in the coil plane and so will move on the $|B|$ contours there (in the absence

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of electric fields). If we choose the plasma center to be the $\int d\ell/B$ center ($\Delta \varepsilon k_1^2 / 2\varepsilon_0 = 1/\varepsilon_0 \gg 1$), then for $\zeta_m = 1$, $\langle \underline{v}_{dB} \cdot \underline{\nabla} \alpha \rangle = 0$ to $O(1/\varepsilon_0)$ since the highly passing particles move on the $\int d\ell$ contours, whose offset compared to the $|B|$ center is $O(1/\varepsilon_0)$.

In Fig. 1 we plot g and g_1 against ζ_m for the particular choice of $\varepsilon_0 = 1/3$ (this corresponds to EBT-S parameters on axis). We see that $g(\zeta_m = 0) = 1$ and then is almost flat until $\zeta_m = \zeta_c = 1/\sqrt{2}$. At this point it has a cusp and rises sharply to 2 at $\zeta_m = 1$. Similarly, g_1 has a cusp at $\zeta = \zeta_c$ after starting at 0 for $\zeta_m = 0$. The cusp points are obtained because g and g_1 are calculated by assuming that the particles move only along the field lines and do not drift radially or poloidally.

We can write the total drift through the pressure surface as

$$\begin{aligned}
 V &= \langle \underline{v}_d \cdot \underline{\nabla} \alpha \rangle \\
 &= \tau_b \left\{ -\frac{1/2 m v^2}{Ze} \varepsilon r \sin \beta \left[g(\zeta_m) 1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] + g_1(\zeta_m) \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} - \frac{\partial \varphi}{\partial \beta} \right\}.
 \end{aligned}
 \tag{41}$$

In a similar manner the total poloidal drift can be calculated. Here the changes due to the position of the plasma center will be $O(\varepsilon)$; the dominant contribution comes from the $\underline{E} \times \underline{B}$ drift and the bumpy cylinder magnetic field. This calculation was done in Ref. 8, and we give only the results here. For electrons, we have

$$\Omega = \langle \underline{v}_d \cdot \underline{\nabla} \beta \rangle = \tau_b \Omega_{th} [x^2 h(\zeta_m) + w_0]
 \tag{42}$$

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where

$$n_{th} = - \frac{k_b T}{ZeBa^2} \frac{1}{rR_c} , \quad (43)$$

$$x^2 = (1/2mv^2)/k_b T , \quad (44)$$

$$w_o = - \frac{Ze}{k_b T} \frac{\partial \varphi}{\partial \rho} R_c , \quad (45)$$

$$R_c = - \frac{\partial \ln B}{\partial r} \Big|_{\varepsilon=0} , \quad (46)$$

and

$$2\zeta_m^2 \kappa^2 \left[\frac{E(1/\kappa^2)}{K(1/\kappa^2)} - \frac{1}{2} \right] , \quad \kappa^2 > 1$$

$$h(\zeta_m) = \quad (47)$$

$$2\zeta_m^2 \left[\frac{E(\kappa^2)}{K(\kappa^2)} - 1 + \frac{\kappa^2}{2} + \frac{\zeta_m - 1/\sqrt{2}}{5.27\zeta_m^2} \right] , \quad \kappa^2 < 1$$

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In Eq. (47), $h(\zeta_m)$ has been given for $\varepsilon_0 = 1/3$, and k_b is Doltzmann's constant. We also note that in Eqs. (42)-(46) r and ρ have been used interchangeably, since the difference between them is $O(\varepsilon)$. The quantity Ω_{th} is the thermal value of the ∇B particle drift in the bumpy cylinder, and w_0 is a parameter that measures the strength of the $\underline{E} \times \underline{B}$ drift relative to the ∇B drift. In Fig. 1 we have plotted $h(\zeta_m)$ against ζ_m , and we note the cusp that occurs for $\zeta_m = \zeta_c$. As mentioned previously, this cusp occurs because of the neglect of the precessional motion of the particle as it bounces. We can obtain the width of ζ_m space over which the precessional motion cannot be neglected compared with the bounce motion, but first we shall show how toroidal effects can change a particle's midplane pitch angle. This motivates the choice of midplane pitch angle evaluated for $\varepsilon = 0$ as a constant of the motion. The inclusion of the radial motion will "smear" a particle's midplane pitch angle since it will change its local $v_{||}$ and hence the midplane $v_{||}$. We can estimate this in the following manner.

Consider a particle with pitch angle in the midplane of ζ_a . Then we have in the midplane at location (α_0, β_0)

$$\frac{1}{2} m v_{||}^2 = E - \mu B(\alpha_0, \beta_0, \ell = 0) - Ze\varphi(\alpha_0, \beta_0) , \quad (48)$$

where E is the total energy. When the particle gets to its turning point along the field line, its radial and poloidal locations will have changed to $(\alpha_0 + \Delta\alpha, \beta_0 + \Delta\beta)$, where

$$\Delta\alpha \approx \underline{v}_d \cdot \underline{\nabla}\alpha|_{\text{turn}} \tau_b(\zeta_1) , \quad (49)$$

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$$\Delta\beta \simeq \underline{v}_d \cdot \underline{\nabla}\beta|_{\text{turn}} \tau_b(\zeta_1) , \quad (50)$$

and they have been evaluated at the turning point since the particle spends most of its time there. When it returns to the midplane, its location will be $(\alpha_0 + 2\Delta\alpha, \beta_0 + 2\Delta\beta)$, and its midplane pitch angle will be given by

$$\begin{aligned} \frac{1}{2} m \zeta_2^2 v^2 &= E - \mu B(\alpha_0 + 2\Delta\alpha, \beta_0 + 2\Delta\beta, \ell = 0) \\ &- Ze\varphi(\alpha_0 + 2\Delta\alpha, \beta_0 + 2\Delta\beta) . \end{aligned} \quad (51)$$

For small $\Delta\alpha, \Delta\beta$, the change in pitch angle as a result of radial motion is

$$\Delta\zeta_r = |\zeta_2 - \zeta_1| \simeq \frac{\Delta\alpha}{\zeta_1^2} \frac{1}{1/2mv^2} \left[\mu \frac{\partial B}{\partial \alpha} + Ze \frac{\partial \varphi}{\partial \alpha} \right] + \frac{\Delta\beta}{\zeta_1^2} \frac{1}{1/2mv^2} \left[\mu \frac{\partial B}{\partial \beta} + Ze \frac{\partial \varphi}{\partial \beta} \right] . \quad (52)$$

We can estimate the magnitude of a typical term in Eq. (52) as follows: near the cusp, we have $\zeta_1^2 \simeq 1/2$, $\mu B/(1/2mv^2) \simeq 1/2 \partial \ln \mu B / \partial \beta \simeq \varepsilon$, and, for $w_0 \gg 1$, $\Delta\beta \simeq \Omega_{th} w_0 \tau_b(\zeta_1)$. Therefore,

$$\frac{\Delta\beta}{\zeta_1^2} \frac{1}{1/2mv^2} \mu \frac{\partial B}{\partial \beta} \sim \varepsilon \Omega_{th} w_0 \tau_b . \quad (53)$$

For typical EBT-S parameters ($\varepsilon \simeq 0.1$, $T_e = 200$ eV, $w_0 = 3$), $\varepsilon \Omega_{th} w_0 \tau_b \simeq 8 \times 10^{-5}$. Hence, the radial smearing is typically very small, and the only place it will be important is near the cusp point. We can obtain a lower bound on the radial smearing at the cusp point by calculating when $\Delta\zeta_r$ is

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equal to the distance (in pitch angle space) from the cusp point. Near the cusp, we can write

$$\tau_b \approx \frac{d}{\pi} \frac{1}{v_{\zeta_{m_1}}} \frac{1}{2} \ln \left(\frac{16}{|\kappa^2 - 1|} \right), \quad (54)$$

where d is the distance between two adjacent coil planes and ζ_{m_1} is the midplane pitch angle including the $O(\varepsilon)$ structure in β . We want to solve the equation for ζ_{m_1} , given by

$$\frac{1}{\sqrt{2}} - \zeta_{m_1} = \delta \Omega_{th} w_0 \tau_b(\zeta_{m_1}), \quad (55)$$

where $\tau_b(\zeta_{m_1})$ is given by Eq. (54) and $\kappa^2 = 1/\zeta_{m_1}^2 - 1$. This gives a transcendental equation for ζ_{m_1} .

$$\zeta_{m_1} = \frac{1}{\sqrt{2}} - \delta \Omega_{th} w_0 \left(\frac{d}{\pi} \right) \frac{1}{v_{th} \zeta_{m_1}} \frac{1}{2} \left[\ln 16 - \ln \left(\frac{1}{\zeta_{m_1}^2} - 2 \right) \right]. \quad (56)$$

where we have replaced v by $v_{th} = (2k_B T/m)^{1/2}$. For $T = 200$ eV, $d = 40$ cm, $B = 0.7$ T, $rR_C = 1/40$ m², and $w_0 = 3$, then $\Delta \zeta_r = |1/\sqrt{2} - \zeta_m|$ is

$$\Delta \zeta_r \approx 3.3 \times 10^{-4}. \quad (57)$$

An upper bound on $\Delta \zeta_r$ is given by noting that since the drift orbit is periodic in β , the maximum time that a particle can have to drift radially is $1/\Omega_{th} w_0$; hence the radial smearing is $\Delta \zeta_r \approx \varepsilon/2 = 5 \times 10^{-2}$.

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The width of ζ_m space over which the precessional motion is comparable to the bounce motion is given by the condition

$$\Omega_{th} w_0 \tau_b(\zeta_m) \simeq 1.$$

From Eq. (54) we see that we have to come exponentially close to ζ_c for this condition to be satisfied, and therefore the effect of precessional motion at ζ_c will probably be much less important than collisions.

III. VARIATIONAL PRINCIPLE FORMULATION

In this section we review the variational principle for low-collisionality transport in EBT. Our formulation is similar to that of Ref. 9 except for some notational changes.

We start with the steady-state, bounce-averaged, drift kinetic equation,

$$V \frac{\partial f}{\partial \alpha} + \Omega \frac{\partial f}{\partial \beta} = \nu(f) , \quad (58)$$

where

$$V = \langle \underline{v}_d \cdot \underline{\nabla} \alpha \rangle , \quad (59)$$

$$\Omega = \langle \underline{v}_d \cdot \underline{\nabla} \beta \rangle , \quad (60)$$

$$\nu(f) = \langle C(f) \rangle , \quad (61)$$

and $C(f)$ is the Fokker-Planck collision operator. From Eqs. (36) and (42), it is clear that

$$\frac{V}{\Omega} \leq \varepsilon \alpha . \quad (62)$$

Thus to $O(\varepsilon^0)$, $V = 0$ and the drift surfaces and flux surfaces coincide. We expand the distribution function in inverse aspect ratio by writing $[f_1 = O(\varepsilon f_0)]$

$$f = f_0 + f_1 + \dots , \quad (63)$$

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and we make the transport ansatz¹ by choosing f_0 to be a local Maxwellian

$$f_0 = F_m(\alpha, E) = n(\alpha) \exp\left(\frac{Ze\varphi_0}{T}\right) \left[\frac{m}{2\pi k_b T(\alpha)}\right]^{3/2} \exp\left(-\frac{E}{k_b T}\right) . \quad (64)$$

In Eq. (64), we assume that the poloidal structure of φ , the potential, is caused solely by the toroidicity, and we can write

$$\varphi = \varphi_0(\alpha) + \varphi_1(\alpha, \beta) , \quad (65)$$

where

$$\varphi_1 = O(\varepsilon \varphi_0) .$$

To first order in ε , the bounce-averaged drift kinetic equation is

$$\Omega \frac{\partial f_1}{\partial \beta} - \nu \langle f_1 \rangle = -V \frac{\partial F_M}{\partial \alpha} . \quad (66)$$

The partial derivatives are at constant energy and ζ_m . The contribution to the right-hand side of Eq. (66) coming from the β dependence of the midplane pitch angle vanishes since F_M is independent of ζ_m . We now expand in the collisionality by assuming that $\nu_{ei}/\Omega_{ih} \ll 1$, where ν_{ei} is the electron-ion collision frequency, given by

$$\nu_{ei} = \frac{4\pi n Z^2 e^4 \ln \Lambda}{(2T_e)^{3/2} m_e^{1/2}} . \quad (67)$$

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We write $f_1 = f_1^{(0)} + f_1^{(1)} + \dots$, where $f_1^{(1)} = 0[\nu_{ei}/\Omega_{th} f_1^{(0)}]$, and obtain to lowest order in ν_{ei}/Ω_{th}

$$\Omega \frac{\partial f_1^{(0)}}{\partial \beta} = -v \frac{\partial F_M}{\partial \alpha} . \quad (68)$$

Since the electrons are nonresonant, $\Omega \neq 0$ to $O(\epsilon^0)$ is independent of β , so we can solve Eq. (68) as

$$f_1^{(0)} = -\frac{1}{\Omega} \int v \, d\beta \frac{\partial F_M}{\partial \alpha} + g(\alpha, \mu, E) . \quad (69)$$

We can determine the integration constant g by constructing a constraint equation from the first order in collisionality equation. If we integrate this equation around the orbit we obtain

$$\int_0^{2\pi} d\beta \, \nu[f_1^{(0)}] = 0 . \quad (70)$$

To $O(\epsilon^0)$, nothing in the function $\nu(\chi)$ depends on β , and we can write

$$\nu \left[\int_0^{2\pi} d\beta \, f_1^{(0)} \right] = 0 . \quad (71)$$

From Eq. (69) this gives

$$\nu(g) = 0 , \quad (72)$$

which has the solution $g = 0$ or $g = \text{Maxwellian}$. If g is a Maxwellian, it makes no contribution to the transport, so we choose $g = 0$.

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We define the flux surface average of a function F by

$$\langle F \rangle_v = \frac{1}{V_\ell} \oint d\beta \int \frac{d\ell}{B} F, \quad (73)$$

where

$$V_\ell(\alpha) = \oint d\beta \int \frac{d\ell}{B}. \quad (74)$$

The flux-surface-averaged particle flux and energy flux are given by

$$\Gamma = \langle \int d^3v \, \underline{v}_d \cdot \underline{\nabla} a f \rangle_v, \quad (75)$$

$$Q = \langle \int d^3v \left(\frac{1}{2} m v^2 \right) \underline{v}_d \cdot \underline{\nabla} a f \rangle_v. \quad (76)$$

We define the rate of irreversible entropy production

$$\Theta = - \langle \int d^3v \left[\frac{f_1^{(0)}}{F_M} \right] C[f_1^{(0)}] \rangle \quad (77)$$

and the generalized forces

$$A_1 = \frac{n'}{n} + \frac{T'}{T} + \frac{Ze\phi'_0}{T}, \quad (78)$$

$$A_2 = \frac{T'}{T}, \quad (79)$$

and we note that we can write

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$$\frac{\partial F_M}{\partial \alpha} = F'_M = F_M \left[(A_1 - \frac{5}{2} A_2)^2 + x^2 A_2 \right] . \quad (80)$$

If we express θ as a quadratic form in the A_i 's by

$$\theta = L_{11} A_1^2 + 2L_{12} A_1 A_2 + L_{22} A_2^2 , \quad (81)$$

then the fluxes are given by

$$\Gamma = -L_{11} A_1 - L_{12} A_2 , \quad (82)$$

$$(Q - \frac{5}{2} \Gamma T)/T = -L_{12} A_1 - L_{22} A_2 . \quad (83)$$

This formalism is variational since $\delta\theta = 0$ implies

$$\langle \int d^3v \frac{\delta f_I^{(0)}}{F_M} C[f_I^{(0)}] \rangle_v = 0$$

which gives the constraint Eq. (70).

Finally, we express θ in terms of integrals over v and ζ_m . To lowest order in ε ,

$$V_f(\alpha) \simeq 2\pi \int \frac{d\ell}{E} \quad (84)$$

and

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$$\int d^3v = 4\pi \int_0^\infty v^2 dv \int_{v_{||}=0}^1 d\zeta_m \frac{B(l)}{B_m} \frac{\zeta_m}{(v_{||}/v)} , \quad (85)$$

$$\begin{aligned} \Theta &= - \frac{1}{\int d\ell/b} \int_0^{2\pi} \frac{d\beta}{2\pi} \int \frac{d\ell}{B} 4\pi \int_0^\infty dv v^3 \int_{v_{||}=0}^1 d\zeta_m \zeta_m \frac{B(l)}{B_m} \frac{1}{v_{||}} \left[\frac{f_1^{(0)}}{F_M} \right] C[f_1^{(0)}] \\ &= - \frac{1}{\int d\ell/B} \int_0^{2\pi} \frac{d\beta}{2\pi} \int_0^\infty dv 4\pi v^3 \int_0^1 d\zeta_m \frac{\zeta_m}{B_m} \left[\frac{f_1^{(0)}}{F_M} \right] \nu(f_1^{(0)}) . \end{aligned} \quad (86)$$

Here we have interchanged the ℓ and ζ_m integrations and used the fact that $f_1^{(0)}$ is independent of ℓ . In Sec. V, we obtain the bounce-averaged collision operator.

IV. THE SELF-CONSISTENT POLOIDAL ELECTRIC FIELD

In Eq. (65), the zeroth-order part, $\varphi_0(\alpha)$, will be determined by the ambipolarity condition,

$$\Gamma_e = \Gamma_i \quad . \quad (87)$$

The first-order part, $\varphi_1(\alpha, \beta)$, will be determined by the requirement of quasi-neutrality to all orders in the expansion parameters. To zeroth order in ε and ν_{ei}/Ω_{th} , quasi-neutrality is imposed by $n_e = n_i = n$. Hence φ_1 will come from the fact that the first-order densities of ions and electrons must be the same. Since we are looking on a time scale that is long compared to the bounce time, we find φ_1 by imposing the condition

$$\frac{ne\varphi_1}{k_b T} + \frac{1}{\int d\ell/B} \int \frac{d\ell}{B} \bar{n}_3 = \frac{1}{\int d\ell/B} \int \frac{d\ell}{B} \bar{n}_i - \frac{ne\varphi_1}{k_b T} \quad , \quad (88)$$

where

$$\bar{n}_i = \int d^3v f_{ij}^{(0)} \quad . \quad (89)$$

We can motivate Eq. (88) by the following argument. The total number of j particles in the element $(\alpha, \alpha + d\alpha)$ and $(\beta, \beta + d\beta)$ is

$$N_i = \int_{\alpha}^{\alpha+d\alpha} d\alpha \int_{\beta}^{\beta+d\beta} d\beta \int \frac{d\ell}{B} \int d^3v f \quad . \quad (90)$$

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For quasi-neutrality to hold for time scales that are long compared to the bounce time, we must have, for all $d\alpha$, $d\beta$

$$N_e = N_i ,$$

which in the limit $d\alpha, d\beta \rightarrow 0$ gives Eq. (88) to first order in ε and zeroth order in ν_{ei}/Ω_{th} .

The total first-order density is given by

$$n_{1j} = \frac{1}{\int d\ell/B} \int \frac{d\ell}{B} \int d^3v f_1^{(0)} - n \frac{Ze\varphi_1}{k_b T} \quad (91)$$

where the first term arises from the displacement of the particle orbits due to toroidal effects and the second term comes from the density perturbation induced by the first-order potential. To calculate n_{1j} , we need $f_1^{(0)}$ which we can obtain from Eqs. (89), (42), (41), and (65). In order to make analytic progress, we take the large electric field limit ($w_0 \gg 1$) here and in the rest of the paper. We shall keep terms through $O(1/w_0^2)$ in $f_1^{(0)}$. We have

$$\begin{aligned} \frac{f_1^{(0)}}{F_M} &= (\rho R_C B m_0) (\rho \varepsilon a^2) \left\{ \frac{x^2}{w_0} \left[g(\zeta_m) \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) + \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} g_1(\zeta_m) \right] \right. \\ &\quad \left. \cdot \left[1 - \frac{x^2}{w_0} h(\zeta_m) \right] \right\} \cos \beta \left[(A_1 - \frac{5}{2} A_2) + x^2 A_2 \right] \end{aligned}$$

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$$+ \frac{a^2 \varphi_1(\rho, \beta)}{\Omega_E} \left[1 - \frac{x^2}{w_0} h(\zeta_m) \right] \left[(A_1 - \frac{5}{2} A_2) + x^2 A_2 \right], \quad (92)$$

where

$$\Omega_E = + \frac{1}{\rho B_m} \frac{\partial \varphi_0}{\partial \rho}. \quad (93)$$

We use Eq. (85) to obtain

$$\frac{1}{\int d\ell/B} \int \frac{d\ell}{B} \int d^3v \frac{f_1^{(0)}}{F_M} F_M = \frac{4n}{\sqrt{\pi}} \int_0^\infty dx x^2 e^{-x^2} \int_0^1 d\zeta_m \left(\frac{v \zeta_m \tau b}{\int d\ell B_m/B} \right) \frac{f_1^{(0)}}{F_M}. \quad (94)$$

We define sets of numbers a_{kmn} and b_n by

$$a_{kmn} = \int_0^1 d\zeta_m \left(\frac{v \zeta_m \tau b}{\int d\ell B_m/B} \right) h^k(\zeta_m) g^m(\zeta_m) g_l^n(\zeta_m). \quad (95)$$

$$b_n = \int_0^\infty dx e^{-x^2} e^{2(n+1)}; \quad (96)$$

then we have

$$\frac{1}{\int d\ell/B} \int \frac{d\ell}{B} \int d^3v f_1^{(0)} = \frac{4a^2 n}{\sqrt{\pi}} \left[(\rho R_{c B_{m0}}) (\rho \varepsilon) \left(\frac{b_1 a_{010}}{w_0} \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) \right) \right]$$

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$$\begin{aligned}
 & + \frac{b_1 a_{001}}{w_0} \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} - \frac{b_2}{w_0^2} \left[a_{110} \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) + a_{101} \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] \left\{ A_1 - \frac{5}{2} a_2 \right\} \\
 & + A_2 \left\{ \frac{b_2}{w_0} \left[a_{010} \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) + a_{001} \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] - \frac{b_3}{w_0^2} \left[a_{110} \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) \right. \right. \\
 & \left. \left. + a_{101} \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] \right\} \cos \beta + \frac{\varphi_1}{U_E} \left[(b_0 - \frac{b_1 a_{100}}{w_0}) (A_1 - \frac{5}{2} A_2) + A_2 (b_1 - \frac{b_2 a_{100}}{w_0}) \right] \quad (97)
 \end{aligned}$$

The b_n can be integrated to obtain

$$b_n = \frac{(2n+1)!!}{2^n} \frac{\sqrt{\pi}}{4}, \quad (98)$$

which gives the total number density n_{1j} as

$$\begin{aligned}
 n_{1j} = & a^2 n(\rho R_c B_{m_0})(\rho \varepsilon) \cos \beta \left\{ \frac{3}{2} \frac{A_1}{w_0} \left[a_{010} \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) + a_{001} \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] \right. \\
 & + \frac{15}{4} \frac{1}{w_0^2} (A_1 - a_2) \left[a_{110} \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) + a_{101} \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] \left. \right\} \\
 & + \frac{\varphi_1}{U_E} \left[(A_1 - A_2) - \frac{3}{2} \frac{a_{100}}{w_0} \right] - \frac{1}{a^2} \frac{Ze \varphi_1}{k_B T}. \quad (99)
 \end{aligned}$$

We note that from the definition of the generalized forces A_1 and A_2 ,

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$$\frac{A_1 - A_2}{n_E} - \frac{1}{a^2} \frac{Ze}{k_b T} = \frac{1}{n_E} \frac{1}{a^2} \frac{1}{\rho B_{m_0}} \frac{d \ln n}{d\rho} \quad (100)$$

If in Eq. (99) we keep only terms to $O(1/w_0)$ for both the ions and electrons and impose $n_{1e} = n_{1i}$, then all the terms involving φ_1 cancel out and the equation does not determine φ_1 . This is just a statement of the result that if we allow the particles to only $\underline{E} \times \underline{B}$ drift in the azimuthal direction, then they will move on the equipotentials. Since $T_i \ll T_e$ for typical parameters in EBT-S and since we always want to balance the bulk ions against the bulk electrons for quasi-neutrality, we shall keep terms to $O(1/w_{0e}^2)$. We note that for reactor parameters where $T_i \geq T_e$ this formulation would not work, because the bulk ions would have significant resonances and the first-order ion distribution would have to be determined as in Ref. 1.

The first-order electron density n_{1e} is given by

$$n_{1e} \simeq a^2 n \left((\rho, k_c B_{m_0}) (\rho \varepsilon) \cos \beta \left\{ \frac{3}{2} \frac{1}{w_0} A_{1e} \left[a_{010} \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) + a_{001} \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] \right\} \right. \\ \left. + \frac{\varphi_1}{n_E} \left(\frac{1}{a^2} \frac{1}{\rho B_{m_0}} \frac{d \ln n}{d\rho} - \frac{3}{2} \frac{a_{100}}{w_{0e}} A_{1e} \right) \right) ; \quad (101)$$

the first-order ion density is

$$n_{1i} \simeq a^2 n \frac{\varphi_1}{n_E} \left(\frac{1}{a^2} \frac{1}{\rho B_{m_0}} \frac{d \ln n}{d\rho} \right) . \quad (102)$$

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Imposing quasi-neutrality and using the definition of Ω_E gives

$$\varphi_1 = \rho \varepsilon \cos \beta (R_c \frac{\partial \varphi_0}{\partial \rho}) \left[\frac{a_{010}}{a_{100}} \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) + \frac{a_{001}}{a_{100}} \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] . \quad (103)$$

We see that φ_1 is proportional only to $\cos \beta$ and so is symmetric with respect to the equatorial midplane. The numbers a_{010} , a_{001} , and a_{100} have been calculated for the forms of g , g_1 , and h given previously and are

$$\begin{aligned} a_{010} &= 1.33 , \\ a_{001} &= 1.1046 , \\ a_{100} &= 0.1918 . \end{aligned} \quad (104)$$

This gives for $\varphi_1(\rho, \beta)$

$$\varphi_1 = \rho \varepsilon R_c \frac{\partial \varphi_0}{\partial \rho} \left[6.95 \left(1 - \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right) + 5.76 \frac{\Delta \varepsilon_0 k_1^2}{2\varepsilon} \right] \cos \beta . \quad (105)$$

If we define $C_2(\Delta)$ by

$$\varphi_1 = C_2 \varepsilon \cos \beta(\rho, R_c) \frac{\partial \varphi_0}{\partial \rho} \quad (106)$$

and take a parabolic well for φ ,

$$\varphi_0 = \varphi_{00}(1 + \gamma^2 \rho^2) , \quad (107)$$

then

$$\varphi = \varphi_{00}\gamma^2\left(\frac{1}{\gamma^2} + \rho^2 + C_2\varepsilon\rho \cos \beta \ 2\rho R_c\right) . \quad (108)$$

Now, we move to Cartesian coordinates centered at the plasma center; that is, $\rho^2 = \xi^2 + \nu^2$; $\rho \cos \beta = \xi$. Then φ can be written as

$$\varphi = \varphi_{00}\gamma^2\left[\frac{1}{\gamma^2} + (\xi + C_2\varepsilon\rho R_c)^2 + \nu^2 - (C_2\varepsilon R_c)^2\right] . \quad (109)$$

From this we deduce that the potential contours are circles whose offset from $r = 0$ (the average magnetic axis) is given by

$$\Delta_\varphi = C_2\varepsilon(\rho R_c) + \Delta . \quad (110)$$

From Eq. (20) we can express ρR_c to $O(\varepsilon^0)$ as

$$\rho R_c = \frac{2(1 - \varepsilon_0)}{\varepsilon_0 k_1^2} ; \quad (111)$$

and if we define Δ_B as the offset of the magnetic axis in the midplane ($\Delta_B = 2\varepsilon/\varepsilon_0 k_1^2$), then

$$\Delta_\varphi = C_2\Delta_B(1 - \varepsilon_0) + \Delta . \quad (112)$$

Hence, we can write C_2 as

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$$C_2 = \left(\frac{\Delta_\varphi - \Delta}{\Delta_B} \right) \frac{1}{(1 - \varepsilon_0)} . \quad (113)$$

If we choose C_2 as given by Eq. (105),

$$C_2 = 6.95 - 1.19\Delta/\Delta_B , \quad (114)$$

then we can solve Eqs. (113) and (114) for Δ_φ , and we obtain

$$\frac{\Delta_\varphi}{\Delta_B} = 4.63 + 0.21 \frac{\Delta}{\Delta_B} . \quad (115)$$

For EBT-S parameters ($\varepsilon_0 = 1/3$, $\varepsilon = 0.1$, $N = 24$) this gives a potential center offset of 7-9 cm inward from the average magnetic axis. This is somewhat larger than is inferred experimentally,¹⁰ but the value of 7-9 cm is large enough that the near-axis expansion used to obtain B is questionable. In order to allow better comparison of the theoretically derived fluxes with the experimental results we shall define φ_1 as in Eq. (106), where C_2 is given by Eq. (113). In the same spirit, we shall define a parameter $C_1(\Delta)$ as

$$C_1(\Delta) = \frac{\Delta}{\Delta_B} . \quad (116)$$

Then C_1 is a measure of the plasma center relative to the $|B|$ center and C_2 is a measure of offset of the potential center from the plasma center relative to the $|B|$ center. The diffusion theory will be developed in

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terms of C_1 and C_2 and ranges for those numbers will be chosen from the experimental data. Typically, $C_1 = 2/3-5/3$ and $C_2 = 0-1$ (Ref. 10).

V. THE BOUNCE-AVERAGED COLLISION OPERATOR

For use in the variational principle given in Eq. (86) we need the bounce-averaged linearized Fokker-Planck collision operator. In general, this is a complex integrodifferential operator, and in order to make some analytic progress, previous work^{1,2,4} has employed various model operators. In Ref. 1 a Lorentz operator was used, and the resulting integrals were approximated without justification at $\mu = 0$ ($\zeta_m = 1$). In Refs. 2 and 4, a particle- and energy-conserving BGK operator was used. These suffer from the defect that BGK operators do not treat the scattering in velocity space correctly. In Ref. 4, it was pointed out that despite the shortcomings of the BGK model the results agreed very well with the numerical results of Ref. 3. The explanation for this was that for $w_0 \gg 1$, the first-order distribution function is only a weak function of the velocity-space variables and hence velocity-space scattering is not as important as conservation of the integral invariants. We note, however, that in Ref. 3 the Landau operator was not bounce-averaged and the g and h functions were approximated by smoothly varying functions. In addition, the plasma center was assumed to be the geometric center and the poloidal potential was neglected.

In this paper, we shall use the approximate Fokker-Planck operator given in Ref. 5. This operator is analytically tractable and preserves the symmetry properties of the exact collision integral. This implies the physical conservation laws, self-adjointness, and the H theorem. We shall neglect the $\ell = 2$ terms in Eq. (25) of Ref. 5 (i.e., the stress anisotropy terms) and write the collision operator for species (a) colliding with species (b) as

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$$\begin{aligned}
C_{ab}(f_{a_1}, f_{b_1}) = & \nu_D^{ab} L(f_{a_1}) + \frac{1}{v^2} \frac{\partial}{\partial v} \left\{ \frac{1}{2} v^4 \nu_{\parallel}^{ab} F_{M_a} \right. \\
& \cdot \left[\frac{\partial}{\partial v} n_{a_1}(v) - 2q_{ba} \frac{v}{v_{th_a}^2} \right] \} + \Delta \nu^{ab} \frac{v \cdot u_{a1}(v)}{v^2} F_{M_a} \\
& + \left[\frac{2r_{ba} \cdot v}{v_{th}^2} \nu_s^{ab} + \frac{2(h_{ba} \nu_h^{ab} k_{ab} \nu_k^{ab}) \cdot v}{v_{th_a}^2} \left(\frac{v}{v_{th_a}} \right)^2 \right] F_{M_a} , \quad (117)
\end{aligned}$$

where

$$\nu_D^{ab} = \nu_{ab} \left[\operatorname{erf} \left(\frac{v}{v_{th_a}} \right) - G \left(\frac{v}{v_{th_a}} \right) \right] / \left(\frac{v}{v_{th_a}} \right)^3 , \quad (118)$$

$$\nu_{ab} = \frac{4\pi m_b (e_a e_b)^2 \ln \Lambda}{(2T_a)^{3/2} m_a^{1/2}} , \quad (119)$$

$$G(x) = \left[\operatorname{erf}(x) - \frac{2x}{\sqrt{\pi}} e^{-x^2} \right] \frac{1}{2x^2} , \quad (120)$$

$$L(f_{a_1}) = \frac{1}{2} \frac{\partial}{\partial \zeta_p} \left[(1 - \zeta_p^2) \frac{\partial f_{a_1}}{\partial \zeta_p} \right] , \quad (121)$$

$$\nu_{\parallel}^{ab} = 2\nu_{ab} G \left(\frac{v}{v_{th_a}} \right) / \left(\frac{v}{v_{th_a}} \right)^3 , \quad (122)$$

$$n_{a_1}(v) = \frac{1}{2} \int_1^1 d\zeta_p \frac{f_{a_1}}{F_{M_a}} , \quad (123)$$

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and

$$q_{ba} = \int \frac{1}{2} m_b v^2 \nu_E^{ba}(v) f_{b1} d^3v / \int \frac{1}{2} m_a v^2 (v/v_{th_a})^2 \nu_E^{ab} F_{a1} d^3v ; \quad (124)$$

in addition,

$$\nu_E^{ba}(v) = 2\nu_S^{ab}(v) - 2\nu_D^{ab}(v) - \nu_{\parallel}^{ab}(v) \quad (125)$$

with

$$\nu_S^{ab}(v) = \nu_{ab} (2T_a/T_b) (1 + m_b/m_a) G(v/v_{th_b}) / (v/v_{th_a}) . \quad (126)$$

The function $\underline{u}_{a1}(v)$ is an integral whose integrand is odd in $v_{\parallel}$. Since f_{a1} and f_{b1} , as solutions to the bounce-averaged drift kinetic equation, cannot depend on $v_{\parallel}$, then $\underline{u}_{a1} = 0$. Similarly, it can be shown that $\underline{r}_{ba}$, $\underline{h}_{ba}$, and $\underline{k}_{ba}$ also vanish for the same reason. This is a manifestation of the fact that to lowest order in τ_b , momentum conservation is automatic. The first term in Eq. (117) is the pitch angle scattering term, with ν_D^{ab} the pitch angle deflection rate. The second term is the energy scattering experienced by the species (a) colliding with species (b). Note that $\nu_{\parallel}^{ab}$ is the parallel velocity diffusion rate. The third term is the response of species (b) to the perturbation induced by energy scattering of (a), so q_{ba} is the energy-restoring coefficient that gives conservation of energy.

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For electron-ion collisions [(a) = e, (b) = i] it is easy to show that to $O[(m_e/m_i)^{1/2}]$ this reduces to a Lorentz operator,

$$C_{ei}(f_{e1}, f_{i1}) = \nu_D^{ei} L(f_{e1}) , \quad (127)$$

where

$$\nu_D^{ei} \approx \frac{\nu_{ei}}{(v/v_{the})^3} . \quad (128)$$

Therefore, we can write an electron collision operator as

$$\begin{aligned} C_e(f_{e1}) &= C_{ee}(f_{e1}, f_{e1}) + C_{ei}(f_{e1}, f_{i1}) \\ &= \nu_D^{ee} L(f_{e1}) + \frac{1}{v^2} \frac{\partial}{\partial v} \left[\frac{1}{2} v^4 \nu^{eeF} \mu_e \left(\frac{\partial}{\partial v} n_{E1} - 2q_{ee} \frac{v}{v_{the}^2} \right) \right] , \end{aligned} \quad (129)$$

where

$$\nu_D^{ee} = \nu_{ei} \left[1 + \operatorname{erf}\left(\frac{v}{v_{the}}\right) - G\left(\frac{v}{v_{the}}\right) \right] / \left(\frac{v}{v_{the}}\right)^3 \quad (130)$$

and we used quasi-neutrality to write $\nu_{ee} = \nu_{ei}$. We bounce average C_e by applying the operator in Eq. (30). To bounce average the first term we write it as

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$$L(f_{e_1}) = \frac{mv_{\parallel}}{B} \frac{\partial}{\partial \mu} (\mu v_{\parallel} \frac{\partial}{\partial \mu} f_{e_1}) . \quad (131)$$

Then we have

$$\int \frac{d\ell}{v_{\parallel}} L(f_{e_1}) = m \frac{\partial}{\partial \mu} (\mu \int \frac{d\ell}{B} v_{\parallel} \frac{\partial}{\partial \mu} f_{e_1}) .$$

If we transform from μ to ζ_m using

$$\mu = \frac{1}{2} \frac{mv^2}{B_m} (1 - \zeta_m^2) , \quad (132)$$

then

$$\int \frac{d\ell}{v_{\parallel}} L(f_{e_1}) = \frac{B_m}{v \zeta_m} \frac{\partial}{\partial \zeta_m} [(1 - \zeta_m^2) (\frac{Q}{2v \zeta_m}) \frac{\partial f_{e_1}}{\partial \zeta_m}] , \quad (133)$$

where the function $Q(v, \zeta_m)$ is defined by

$$Q = \int \frac{d\ell}{B} v_{\parallel} . \quad (134)$$

We note that we can interpret $Q/(\int d\ell/B)$ as the field-line-averaged value of $v_{\parallel}$. In general, this is not equal to $v_{\parallel}$ in the midplane since the particle spends most of its time at its turning points. With this interpretation of Q , we can interpret $v_{\parallel}^2 Q/(\zeta_m \int d\ell/B)$ as the diffusion rate

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of the average $v_{\parallel}$. Hence, the bounce-averaged pitch angle scattering operator can be understood as giving the change in the first-order distribution function due to the diffusion of its field-line-averaged $v_{\parallel}$. We can now write the full bounce-averaged collision operator as

$$\begin{aligned} \nu(f) = & \frac{\nu_D^e(v)}{v} \frac{B_m}{\zeta_m} \frac{\partial}{\partial \zeta_m} \left[1 - \zeta_m^2 \left(\frac{q}{2v\zeta_m} \right) \frac{\partial f}{\partial \zeta_m} \right] \\ & + \frac{1}{v^3} \frac{\partial}{\partial v} \left[\frac{v^4}{2} \nu_{\parallel}^e F_M \frac{\partial}{\partial v} \pi_{e_1}(v, \zeta_m) \right] - \bar{q}_{ee} \frac{1}{v^3} \frac{\partial}{\partial v} \left(\frac{v^5 \nu_{\parallel}^e}{v_m^2} F_M \right), \end{aligned} \quad (135)$$

with

$$\bar{n}_{e_1}(v, \zeta_m) = \int \frac{d\ell}{v_{\parallel}/v} \int \frac{d\zeta_m \zeta_m}{\gamma(\ell)} \frac{1}{v_{\parallel}/v} (f)_{F_M}, \quad (136)$$

$$\bar{q}_{ee} = \int \frac{d\ell}{v_{\parallel}/v} q_{ee}, \quad (137)$$

and

$$\gamma(\ell) = \frac{B_m}{B(\ell)}. \quad (138)$$

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In Eqs. (136) and (137), we use

$$\frac{v_{\parallel}}{v} = \left[1 - (1 - \zeta_m^2) \frac{1}{\gamma(l)} \right]^{1/2} ; \quad (139)$$

hence the integral over ζ_m goes from a lower limit of $[1 - \gamma(l)]^{1/2}$ to an upper limit of 1. Similarly, the integral over l goes from a lower limit of $l = 0$ to an upper limit of $(d/2\pi)\cos^{-1}[3 - 2/(1 - \zeta_m^2)]$ for $\zeta_m < 1/\sqrt{2}$ and $d/2$ for $\zeta_m > 1/\sqrt{2}$ where we have used Eq. (20) in the limit $\varepsilon = 0$ to obtain $\gamma(l)$.

We define the function $\tilde{Q}(\zeta_m)$ by

$$\tilde{Q}(\zeta_m) = \frac{2}{\sqrt{2}} \frac{Q}{2v\zeta_m} \int \frac{1}{dl/B} . \quad (140)$$

In Fig. 2(a) we plot $\tilde{Q}(\zeta_m)$ against ζ_m . $\tilde{Q}(\zeta_m)$ was determined by integrating numerically the near-axis forms for $v_{\parallel}$ and B . We see that for $\zeta_m \rightarrow 0$, $\tilde{Q}(\zeta_m) = 1/2$, which is necessary so that the bounce-averaged pitch angle scattering operator reduces to the correct pitch angle scattering operator since for $\zeta_m \rightarrow 0$, $\zeta_m \rightarrow \zeta_r$. We can fit $\tilde{Q}$ in Fig. 2 by a series of odd Legendre polynomials. If we keep $P_1(\zeta_m)$ and $P_3(\zeta_m)$, then

$$\tilde{Q}(\zeta_m) \simeq 0.71P_1(\zeta_m) + 0.154P_3(\zeta_m) , \quad (141)$$

where

$$P_1(\zeta_m) = \zeta_m , \quad (142)$$

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$$P_3(\xi_m) = \frac{\xi_m}{2}(5\xi_m^2 - 3) . \quad (143)$$

In Fig. 2(b) we plot $Q(\xi_m)(1 - \xi_m^2)$ against ξ_m and compare it to the Legendre polynomial approximation. We see that the agreement is very good with the greatest discrepancy near the cusp point. We use this approximate form for $Q(1 - \xi_m^2)$ in Sec. VI to determine the fluxes.

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VI. EVALUATION OF THE FLUXES

We use the quantities C_1 and C_2 defined in Eqs. (113) and (116) and we can write the first-order distribution function as

$$\begin{aligned} \frac{f_1^{(0)}}{F_M} = & (\rho R_c B_{m_0}) (\rho \varepsilon a^2) \left\{ \frac{x^2}{w_0} [g(\zeta_m)(1 - C_1) + C_1 g_1(\zeta_m)] + C_2 \right\} \\ & \cdot \left[1 - \frac{x^2}{w_0} h(\zeta_m) \right] \cos \beta \left[(A_1 - \frac{5}{2} A_2) + x^2 A_2 \right] . \end{aligned} \quad (144)$$

From Eqs. (86), (135), and (140) we can write

$$\theta = \theta_1 + \theta_2 + \theta_3 , \quad (145)$$

where

$$\begin{aligned} \theta_1 = & - \frac{4\pi}{\sqrt{\pi}} \int_0^{2\pi} \frac{d\beta}{2\pi} \int_0^\infty dx \, x^2 \nu_D^e(x) \int_0^1 d\zeta_m \left[\frac{f_1^{(0)}}{F_M} \right] \\ & \times \frac{\partial}{\partial \zeta_m} \left\{ (1 - \zeta_m^2) \frac{\sqrt{2}}{2} Q(\zeta_m) \frac{\partial}{\partial \zeta_m} \left[\frac{f_1^{(0)}}{F_M} \right] \right\} . \end{aligned} \quad (146)$$

$$\begin{aligned} \theta_2 = & - \frac{4\pi}{\int \gamma(\ell) d\ell} \int_0^{2\pi} \frac{d\beta}{2\pi} \int_0^\infty dv \, v^3 \int_0^1 d\zeta_m \zeta_m \left[\frac{f_1^{(0)}}{F_M} \right] \\ & \times \frac{1}{v^3} \frac{\partial}{\partial v} \left[\frac{v^4}{2} \nu_D^e(v) F_M \frac{\partial}{\partial v} n_{e_1} \right] . \end{aligned} \quad (147)$$

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$$\theta_3 = \frac{4\pi}{\int \gamma(\ell) d\ell} \int_0^{2\pi} \frac{d\beta}{2\pi} \int_0^\infty dv v^3 \int_0^1 d\zeta_m \zeta_m \left[\frac{f_1^{(0)}}{F_M} \right] \bar{q}_{ee} \frac{1}{v^3} \frac{\partial}{\partial v} \left(\frac{v^5 \nu_{\parallel}^e}{v_{th}^2} F_M \right) . \quad (148)$$

Note that we can integrate by parts in Eq. (146) and use the evenness of $f_1^{(0)}$ in ζ_m to obtain

$$\theta_1 = \frac{4n}{\sqrt{\pi}} \int_0^{2\pi} \frac{d\beta}{2\pi} \int_0^\infty dx x^2 \nu_D^e(x) \int_0^1 d\zeta_m (1 - \zeta_m^2) \frac{\sqrt{2}}{2} q(\zeta_m) \left\{ \frac{\partial}{\partial \zeta_m} \left[\frac{f_1^{(0)}}{F_M} \right] \right\}^2 . \quad (149)$$

If we use $f_1^{(0)}$, as given in Eq. (144), in Eq. (147)-(149) and take the limits of the energy integration to be 0 to infinity, then we are ignoring the fact that for sufficiently low energies the particles may be collisional. We shall assume that the collisionality ν_{ei}/Ω_{ih} is sufficiently small that this collisional contribution can be neglected. Clearly this assumption will break down for $\nu_{ei}/\Omega_{ih} \simeq 0(1)$.

In order to evaluate θ_2 and θ_3 , we define the functions

$$G(\zeta_m) = g(\zeta_m)(1 - C_1) + C_1 g_1(\zeta_m) - C_2 h(\zeta_m) , \quad (150)$$

$$H(\zeta_m) = h(\zeta_m)[g(\zeta_m)(1 - C_1) + C_1 g_1(\zeta_m)] , \quad (151)$$

$$b_{km}(\zeta_m) = \int \frac{d\ell}{v_{\parallel}/v} \int \frac{d\zeta_m \zeta_m}{(v_{\parallel}/v)\gamma(\ell)} H^k(\zeta_m) G^m(\zeta_m) ; \quad (152)$$

then

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$$\begin{aligned} \frac{f(0)}{F_M} = & (\rho R_c B_{m_0}) (\rho \varepsilon a^2) \left\{ C_2 (A_1 - \frac{5}{2} A_2) + x^2 \left[(A_1 - \frac{5}{2} A_2) \frac{G(\zeta_m)}{w_0} + C_2 A_2 \right] \right. \\ & \left. + x^4 \left[\frac{A_2 G(\zeta_m)}{w_0} - (A_1 - \frac{5}{2} A_2) \frac{H(\zeta_m)}{w_0^2} \right] - \frac{x^6 A_2}{w_0^2} H(\zeta_m) \right\} \cos \beta . \end{aligned} \quad (153)$$

We use the definition of π_{e_1} in Eq. (136) to write

$$\begin{aligned} \pi_{e_1}(x, \zeta_m) = & (\rho R_c B_{m_0}) (\rho \varepsilon a^2) \left\{ C_2 (A_1 - \frac{5}{2} A_2) b_{00} \right. \\ & \left. + x^2 \left[(A_1 - \frac{5}{2} A_2) \frac{b_{01}}{w_0} + C_2 A_2 b_{00} \right] \right. \\ & \left. + x^4 \left[\frac{a_2}{w_0} b_{01} - (A_1 - \frac{5}{2} A_2) \frac{b_{10}}{w_0^2} \right] - \frac{x^6 A_2 b_{10}}{w_0^2} \right\} \cos \beta . \end{aligned} \quad (154)$$

We can evaluate Θ_2 from Eqs. (147), (153), and (154), where in the product in the ζ_m integral in Eq. (147) we only keep up to terms of $O(1/w_0^3)$. We define the number B_{kmnp} by

$$B_{kmnp} = \frac{1}{\int d\ell \gamma(\ell)} \int_0^1 d\zeta_m \zeta_m b_{km}(\zeta_m) H^n(\zeta_m) G^p(\zeta_m) . \quad (155)$$

and the integrals K_{nm} by

$$K_{nm} = \int_0^\infty dx \, x^n \frac{\partial}{\partial x} [x^m \nu(x) e^{-x^2}] . \quad (156)$$

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Then θ_2 is given by

$$\begin{aligned}
\theta_2 = & -(\rho R_c B_{m_0})^2 (\rho \varepsilon a^2)^2 \frac{n}{\sqrt{\pi}} \cdot \{ 2K_{25} \left[(A_1 - \frac{5}{2} A_2)^2 \frac{B_{0101}}{w_0^2} \right. \\
& + \frac{C_2 A_2}{w_0} (A_1 - \frac{5}{2} A_2) (B_{0001} + B_{0100}) + C_2^2 A_2^2 B_{0000}] \\
& + 4K_{27} \left[(A_1 - \frac{5}{2} A_2) A_2 \frac{B_{0101}}{w_0^2} - (A_1 - \frac{5}{2} A_2)^2 \frac{B_{1001}}{w_0^3} \right. \\
& + \frac{C_2 A_2^2 B_{0100}}{w_0} - C_2 (A_1 - \frac{5}{2} A_2) A_2 \frac{B_{1000}}{w_0^2}] \\
& - 6K_{29} \left[a_2 (A_1 - \frac{5}{2} A_2) \frac{B_{1001}}{w_0^3} + \frac{C_2 A_2^2 B_{1000}}{w_0^2} \right] \\
& + 2K_{45} \left[A_2 (A_1 - \frac{5}{2} a_2) \frac{B_{0101}}{w_0^2} - (A_1 - \frac{5}{2} A_2)^2 \frac{B_{0110}}{w_0^3} \right. \\
& + \frac{C_2 A_2^2 B_{0001}}{w_0} - C_2 A_2 (A_1 - \frac{5}{2} A_2) \frac{B_{0010}}{w_0^2}] \\
& + 4K_{47} \left[\frac{A_2^2 B_{0101}}{w_0^2} - A_2 (A_1 - \frac{5}{2} A_2) \frac{B_{0110}}{w_0^3} - A_2 (A_1 - \frac{5}{2} A_2) \frac{B_{1001}}{w_0^3} \right] \\
& - \frac{6K_{49} A_2^2 B_{1001}}{w_0^2} - 2K_{65} \left[A_2 (A_1 - \frac{5}{2} A_2) \frac{B_{0110}}{w_0^3} + \frac{C_2 A_2^2 B_{0010}}{w_0^2} \right]
\end{aligned}$$

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$$- \frac{4K_{67}A_2^2B_{0110}}{w_0^3} \} . \quad (157)$$

If we integrate by parts in Eq. (156), we can write

$$K_{nm} = -n\nu_{ei} \bar{K}_{nm} , \quad (158)$$

where we have used Eqs. (120) and (122) to obtain

$$\bar{K}_{nm} = \int_0^\infty x^{m+n-6} e^{-x^2} [\operatorname{erf}(x) - \frac{2}{\sqrt{\pi}} x e^{-x^2}] . \quad (159)$$

The integrals $\bar{K}_{mn}$ can be done exactly and yield

$$\bar{K}_{25} = \frac{1}{4\sqrt{2}} , \quad (160)$$

$$\bar{K}_{27} = \frac{7}{16\sqrt{2}} = \bar{K}_{45} , \quad (161)$$

$$\bar{K}_{65} = \frac{71}{64\sqrt{2}} = \bar{K}_{47} = \bar{K}_{29} , \quad (162)$$

$$\bar{K}_{49} = \left(\frac{957}{256}\right) \frac{1}{\sqrt{2}} = \bar{K}_{67} . \quad (163)$$

We use Eqs. (120) and (125)-(127) to write

$$\nu_E^{ee}(x) = \nu_{ei} \left[\frac{2 \operatorname{erf}(2)}{x^3} - \frac{8}{\sqrt{\pi}} \frac{e^{-x^2}}{x^2} \right] . \quad (164)$$

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We define the numbers d_n by

$$d_n = \int_0^\infty dx x^n e^{-x^2} \left[2 \operatorname{erf}(x) - \frac{8}{\sqrt{\pi}} x e^{-x^2} \right] . \quad (165)$$

We note that $d_1 = 0$. Then we can write q_{ee} as

$$q_{ee} = \frac{1}{nd_3} \frac{\sqrt{\pi}}{4} \int_0^\infty dx x \left[2 \operatorname{erf}(x) - \frac{8}{\sqrt{\pi}} x e^{-x^2} \right] 4\pi v_{th}^3 \int \frac{d\zeta_m d\zeta_m}{(n_{||}/v) \gamma(L)} f_1^{(0)} . \quad (166)$$

We use the definition of $\bar{q}_{ee}$ to obtain

$$\begin{aligned} \bar{q}_{ee} = & (\rho R_c B_{m_0}) (\rho \epsilon a^2) \left\{ C_2 A_2 b_{00} + \frac{1}{w_0} \left[(A_1 - \frac{5}{2} A_2) b_{01} + \frac{d_5}{d_3} d_{01} A_2 \right] \right. \\ & \left. - \frac{1}{w_0^2} \left[(A_1 - \frac{5}{2} A_2) b_{10} + \frac{d_7}{d_3} A_2 b_{10} \right] \right\} \cos \beta . \end{aligned} \quad (167)$$

From $\bar{q}_{ee}$ we can obtain θ_3 as

$$\begin{aligned} \theta_3 = & - \frac{n_0}{\sqrt{\pi}} v_{ei} (\rho R_c B_{m_0})^2 (\rho \epsilon a^2)^2 \left\{ C_2^2 A_2^2 B_{0000} {}^4\bar{K}_{25} \right. \\ & + \frac{1}{w_0} \left[C_2 A_2 (A_1 - \frac{5}{2} A_2) B_{0001} {}^4\bar{K}_{25} + C_2 A_2 (A_1 - \frac{5}{2} A_2) B_{0100} {}^4\bar{K}_{25} \right. \\ & \left. \left. + C_2 a_2^2 \frac{74}{2} \bar{K}_{25} B_{0100} + C_2 A_2^2 B_{0001} {}^8\bar{K}_{45} \right] \right\} \end{aligned}$$

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$$\begin{aligned}
& + \frac{1}{w_0^2} \left\{ \left[\left(A_1 - \frac{5}{2} A_2 \right) + \frac{7}{2} A_2 \right] \left(A_1 - \frac{5}{2} A_2 \right) B_{0101} 4K_{25} - \left[\left(A_1 - \frac{5}{2} A_2 \right) + \frac{213}{16} A_2 \right] \right. \\
& \cdot C_2 A_2 B_{1000} 4K_{25} - \left(A_1 - \frac{5}{2} A_2 \right) A_2 C_2 B_{0010} 8K_{45} \\
& + \left[\left(A_1 - \frac{5}{2} A_2 \right) + \frac{7}{2} A_2 \right] A_2 B_{0101} 8K_{45} - A_2^2 B_{0010} C_2 12K_{65} \} \\
& + \frac{1}{w_0^3} \left\{ - \left[\left(A_1 - \frac{5}{2} A_2 \right) + \frac{213}{16} A_2 \right] \left(A_1 - \frac{5}{2} A_2 \right) B_{1001} 4K_{25} \right. \\
& - \left[\left(A_1 - \frac{5}{2} A_2 \right) + \frac{7}{2} A_2 \right] \left(A_1 - \frac{5}{2} A_2 \right) B_{0110} 8K_{45} \\
& \left. - \left[\left(A_1 - \frac{5}{2} A_2 \right) + \frac{7}{2} A_2 \right] A_2 B_{1001} 8K_{45} - A_2 \left[\left(A_1 - \frac{5}{2} A_2 \right) + \frac{7}{2} A_2 \right] B_{0110} 12K_{65} \right\} , \\
\end{aligned} \tag{168}$$

and to obtain this we have used $d_2/d_3 = 7/2$, $d_7/d_3 = 213/16$.

We now sum θ_2 and θ_3 to obtain

$$\begin{aligned}
\theta_2 + \theta_3 &= (\rho R_c B_{m_0})^2 (\rho \varepsilon a^2)^2 \frac{n \nu e i}{\sqrt{\pi}} \left(\frac{1}{w_0^2} \left[- \frac{5}{2} A_2 \left(A_1 - \frac{5}{2} A_2 \right) C_2 \frac{B_{1000}}{\sqrt{2}} \right. \right. \\
& + \frac{11}{2} \frac{B_{0101}}{\sqrt{2}} A_2^2 \left. \right] + \frac{1}{w_0^3} \left\{ \frac{B_{1001}}{\sqrt{2}} \left[- \frac{5}{2} \left(A_1 - \frac{5}{2} A_2 \right)^2 \right. \right. \\
& \left. \left. - \frac{57}{4} A_2 \left(A_1 - \frac{5}{2} A_2 \right) - \frac{2479}{32} A_2^2 \right] \right. \\
& \left. \left. + \frac{B_{0110}}{\sqrt{2}} \left[- \frac{11}{2} A_2 \left(A_1 - \frac{5}{2} A_2 \right) - \frac{345}{8} A_2^2 \right] \right\} \right) . \\
\end{aligned} \tag{169}$$

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It is interesting to observe that all the terms in θ_2 and θ_3 that were $O(1)$ and $O(1/w_0)$ have cancelled out exactly. This illustrates the importance of having the collision operator conserve energy and shows why a simple Krook model [$C(f) = \nu f$] does not give the correct w_0 scaling.

Finally, we evaluate θ_1 . We use Eq. (153) to obtain

$$\begin{aligned} \frac{\partial}{\partial \zeta_m} \left[\frac{f_1^{(0)}}{F_M} \right] &= (\rho R_c B_{m0}) (\rho \varepsilon a^2) \left\{ x^2 \left(A_1 - \frac{5}{2} A_2 \right) \frac{L_0(\zeta_m)}{w_0} \right. \\ &\quad + x^4 \left[\frac{A_2 L_0(\zeta_m)}{w_0} - \frac{(A_1 - \frac{5}{2} A_2)}{w_0^2} L_2(\zeta_m) \right] \\ &\quad \left. - \frac{x^6 A_2}{w_0^2} L_2(\zeta_m) \right\} \cos \beta, \end{aligned} \quad (170)$$

where

$$L_0 = G'(\zeta_m) = g'(\zeta_m)(1 - C_1) + C_1 g_1'(\zeta_m) - C_2 h'(\zeta_m) \quad (171)$$

$$\begin{aligned} L_2 = H'(\zeta_m) &= h(\zeta_m) [g'(\zeta_m)(1 - C_1) + C_1 g_1'(\zeta_m)] \\ &\quad + [g(\zeta_m)(1 - C_1) + C_1 g_1(\zeta_m)] h'(\zeta_m) \quad (172) \end{aligned}$$

Then to $O(1/w_0^3)$, θ_1 is given by

$$\theta_1 = \frac{2n_{ei}}{\sqrt{\pi}} (\rho R_c B_{m0})^2 (\rho \varepsilon a^2)^2 \left\{ \left(A_1 - \frac{5}{2} A_2 \right)^2 \frac{N_{00}}{w_0^2} I_2 \right.$$

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$$\begin{aligned}
& + I_3 \left[2 \left(A_1 - \frac{5}{2} A_2 \right) \frac{A_2 M_{00}}{w_0^2} - 2 \left(A_1 - \frac{5}{2} A_2 \right)^2 \frac{M_{02}}{w_0^3} \right] \\
& + I_4 \left[-4 \left(A_1 - \frac{5}{2} A_2 \right) A_2 \frac{M_{02}}{w_0^3} + \frac{A_2^2}{w_0^2} M_{00} \right] - I_5 \frac{2 A_2^2}{w_0^3} M_{02} \} , \quad (173)
\end{aligned}$$

where we have used Eq. (130) to define

$$I_p = \int_0^\infty dx \, x^{2p-1} e^{-x^2} \left[1 + \operatorname{erf}(x) \left(1 - \frac{1}{2x^2} \right) + \frac{e^{-x^2}}{x\sqrt{\pi}} \right] , \quad (174)$$

and

$$M_{00} = \int_0^1 d\xi_m (1 - \xi_m^2) \sqrt{2} \tilde{Q}(\xi_m) L_0^2(\xi_m) , \quad (175)$$

$$M_{02} = \int_0^1 d\xi_m (1 - \xi_m^2) \sqrt{2} \tilde{Q}(\xi_m) L_0(\xi_m) L_2(\xi_m) . \quad (176)$$

The integrals I_p can be evaluated exactly and are

$$I_2 = \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}} \right) , \quad (177)$$

$$I_3 = 1 + \frac{9}{8\sqrt{2}} , \quad (178)$$

$$I_4 = 3 + \frac{115}{32} \frac{1}{\sqrt{2}} , \quad (179)$$

$$I_5 = 12 + \frac{3291}{256} \frac{1}{2} . \quad (180)$$

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We now define the functional $N(f)$ on an arbitrary function $f(\zeta_m)$ by

$$N(f) = \int_0^1 d\zeta_m (1 - \zeta_m^2)^{\frac{\sqrt{3}}{3}} Q(\zeta_m) f(\zeta_m) . \quad (181)$$

Then we can write

$$\begin{aligned} M_{00} = & (1 - C_1)^2 N[(g')^2] + C_1^2 N[(g'_1)^2] + C_2^2 N[(h')^2] \\ & - 2C_1 C_2 N(g'_1 h') + 2C_1 (1 - C_1) N(g'_1 g') - C_2 (1 - C_1) N(g' h') \end{aligned} \quad (182)$$

and

$$\begin{aligned} M_{02} = & (1 - C_1)^2 \{ N[(g')^2 h] + N(h' g' g) \} + C_1 (1 - C_1) [2N(g'_1 g_1 h') \\ & + N(g_1 g' h') + N(g'_1 g h')] + C_1^2 \{ N[(g'_1)^2 h] + N(g'_1 g_1 h') \} \\ & - C_2 (1 - C_1) \{ N(g' h' h) + N[(h')^2 g] \} \\ & - C_2 C_1 \{ N(g'_1 h' h) + N[g_1 (h')^2] \} . \end{aligned} \quad (183)$$

In Fig. 3, we plot the integrals of $N[(g')^2]$ and $N[(h')^2]$. We note that they both have a singularity at $\zeta_m = 1/\sqrt{2}$. This arises from the cusp points in Fig. 1, where the derivative is discontinuous. It is not hard to show that the singularity at this point goes like $1/(\zeta_m - 1/\sqrt{2})^2$ and so is not integrable. As we noted in Sec. II, this cusp results from the neglect of collisions and precessional motion for $\zeta_m = 1/\sqrt{2}$; when radial motion is included the derivatives of g , g_1 , and h will go through zero over some small width $\Delta\zeta_{pc}$. For vanishingly small collisionality (e.g., $\bar{\nu}/\omega_0 \leq 1 \times 10^{-6}$) we can make a good estimate of the functional $N(f)$ by writing

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$$\begin{aligned}
N(f) = & \int_0^{1/\sqrt{2}-\Delta\zeta_\ell/2} d\zeta_m (1 - \zeta_m^2) \sqrt{2} Q(\zeta_m) f(\zeta_m) \\
& + \int_{1/\sqrt{2}+\Delta\zeta_\ell/2}^1 d\zeta_m (1 - \zeta_m^2) \sqrt{2} Q(\zeta_m) f(\zeta_m) , \quad (184)
\end{aligned}$$

where $\Delta\zeta_\ell$ is the larger of the collisional boundary layer width and $\Delta\zeta_{pc}$. Since $f(\zeta_m)$ is always proportional to at least one derivative of g , g_1 , or h , we have taken $f(\zeta_m) = 0$ for $\zeta_m \in [(1/\sqrt{2}) - \Delta\zeta_\ell/2, (1/\sqrt{2}) + \Delta\zeta_\ell/2]$. Then $N(f)$ is not infinite for any of the choices of f in Eqs. (182) and (183). In Table 1 we give the value of $N(f)$ for the various choices of f in M_{00} and M_{02} in Eqs. (182) and (183) and for a choice $\Delta\zeta_\ell \simeq 3 \times 10^{-4}$.

If the collisionality is not vanishingly small, then the cusp particles will collide before they have an opportunity to undergo much precessional motion. In this case a collisional boundary layer will form around $\zeta_m = \zeta_c$ that will be wider than the radial smearing width. Another way of looking at this case is to note that where the derivative of $f^{(0)}/F_M$ with respect to ζ_m is large, the pitch angle scattering frequency, which goes like $\nu_D^3 \partial/\partial\zeta_m$, is also large and collisions cannot be ignored. In order to calculate this contribution from the collisional boundary layer at the cusp we return to the drift kinetic equation and proceed as follows. In this analysis we shall follow closely the work of Kovrizhnykh in Ref. 11.

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We assume that only pitch angle scattering across the cusp will be important in the collision operator, and so we can write the first-order bounce-averaged drift kinetic equation as

$$\begin{aligned} \tau_b \Omega_{th} [x^{2h}(\zeta_m) + w_0] \frac{\partial f_1}{\partial \beta} = & -F_M \left[\left(A_1 - \frac{5}{2} A_2 \right) + x^2 A_2 \right] \sin \beta \\ & \cdot \tau_b (\rho E_{m_0} R_c) (\rho \varepsilon a^2) \Omega_{th} \{ x^2 [(1 - C_1) g(\zeta_m) + C_1 g_1(\zeta_m)] + C_2 \} \\ & + \frac{\nu_D^E(v)}{v \zeta_m} \frac{\partial}{\partial \zeta_m} \left[(1 - \zeta_m^2) \left(\frac{Q}{2v \zeta_m} E_m \right) \frac{\partial f_1}{\partial \zeta_m} \right] . \end{aligned} \quad (185)$$

We define $\hat{f}$ by

$$\hat{f} = f_c + i f_s , \quad (186)$$

where

$$f_1 = f_c \cos \beta + f_s \sin \beta . \quad (187)$$

Then we can solve for $\hat{f}$ to $O(1/w_0^2)$ as

$$\begin{aligned} \hat{f} \simeq & F_M (\rho R_c E_{m_0}) (\rho \varepsilon a^2) \left[\left(A_1 - \frac{5}{2} A_2 \right) + x^2 A_2 \right] \frac{1}{w_0} \left[1 - \frac{x^{2h}(\zeta_m)}{w_0} \right] \\ & \cdot \{ x^2 [(1 - C_1) g(\zeta_m) + C_1 g_1(\zeta_m)] + C_2 \} - \frac{1}{i \tau_b \Omega_{th} w_0} \left[1 - \frac{x^{2h}(\zeta_m)}{w_0} \right] \end{aligned}$$

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$$\left[\frac{\nu_D^e(x)}{v \zeta_m} \right] (1 - \zeta_m^2) \left(\frac{Q}{2v \zeta_m} B_m \right) \frac{\partial^2 \hat{f}}{\partial \zeta_m^2}, \quad (188)$$

where in Eq. (188) we have assumed that the collisional boundary layer is sufficiently narrow that the second derivative of $\hat{f}$ with respect to ζ_m will be the largest term arising from the pitch angle scattering operator.

If we define the variable λ by

$$\lambda = \ln \left(\frac{16}{|\kappa^2 - 1|} \right), \quad (189)$$

then from Eqs. (37) and (39)-(41) in the limits $\kappa^2 \rightarrow 1 \pm$

$$\tau_b \simeq \left(\frac{d}{\pi} \right) \frac{1}{v \zeta_m} \left(\frac{\lambda}{2} \right), \quad (190)$$

$$g \simeq 1 + \frac{1}{\lambda}, \quad (191)$$

$$g_1 \simeq 2 - \frac{10}{3\lambda}, \quad (192)$$

$$h \simeq -\frac{1}{2} + \frac{2}{\lambda}. \quad (193)$$

Clearly, as $\kappa^2 \rightarrow 1 \pm$, $\lambda \rightarrow \infty$. We expand $\hat{f}$ near $\zeta_m = 1/\sqrt{2}$ ($\kappa^2 = 1$) and keep terms to $O(1/\lambda)$. We use Eqs. (140) and (141) to write

$$\left[\frac{Q}{2v \zeta_m} B_m \left(\frac{2\pi}{d} \right) \right]_{\zeta_m} = 1/\sqrt{2} = \frac{\pi}{2} \mathfrak{F} \left(\frac{1}{\sqrt{2}} \right);$$

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hence

$$[(1 - \zeta_m^2) \frac{q}{2v\zeta_m} B_m(\frac{2\pi}{d})] \zeta_m + 1/\sqrt{2} = 0.373$$

and $\hat{f}$ is given near $\zeta_m = 1/\sqrt{2}$ by

$$\begin{aligned} \hat{f} \simeq F_M(\rho R_c B_{m_0})(\rho \varepsilon a^2) & \left[(A_1 - \frac{5}{2} A_2) + x^2 A_2 \right] \frac{1}{w_0} \left\{ \left(1 + \frac{x^2}{2w_0} \right) \right. \\ & \cdot [x^2(1 + C_1) + C_2] - \frac{2x^2}{\lambda w_0} [x^2(1 + C_1) + C_2] + \frac{x^2}{\lambda} \left(1 + \frac{x^2}{2w_0} \right) \left(1 - \frac{13}{3} C_1 \right) \} \\ & - \frac{1}{1\Omega_{th} w_0} \frac{1}{\lambda} \nu_D^2(x) (0.373) \left(1 + \frac{x^2}{2w_0} \right) \frac{\partial^2 \hat{f}}{\partial \zeta_m^2} . \end{aligned} \quad (194)$$

For EBT-S parameters ($\varepsilon_0 = 1/3$), $\kappa^2 = 1/\zeta_m^2 - 1$; hence, for $\zeta_m = 1/\sqrt{2}$,

$$\frac{\partial}{\partial \zeta_m} = -4\sqrt{2} \frac{\partial}{\partial (\kappa^2)} . \quad (195)$$

We now define a boundary layer variable z by

$$z = \frac{1 - \kappa^2}{\sqrt{|D|}} , \quad (196)$$

where the boundary layer width $|D|$ will be determined. With these definitions we can write

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$$\frac{\partial}{\partial(\kappa^2)} = \frac{1}{\sqrt{|D|}} \frac{\partial}{\partial z} \quad (197)$$

and

$$\lambda = \lambda_0 - \ln|z|, \quad (198)$$

where

$$\lambda_0 = \ln \frac{16}{\sqrt{|D|}}. \quad (199)$$

For $|z| \simeq 0(1)$ (so that we are just at the edge of the boundary layer) we can write

$$\frac{1}{\lambda} \simeq \frac{1}{\lambda_0} + \frac{1}{2} \frac{\ln z^2}{\lambda_0^2}, \quad (200)$$

and then if we define $|D|$ by

$$|D| = \frac{32(0.373)}{\Omega_{th} w_0 \lambda_0} \nu_D^e(x) \left(1 + \frac{x^2}{2w_0}\right) \quad (201)$$

and $\chi(z)$ by

$$\hat{f} = F_M(\rho R_c B_{m_0})(\rho \varepsilon a^2) \left[(A_1 - (5/2)A_2) + x^2 A_2 \right] \frac{1}{w_0} \left(\left(1 + \frac{x^2}{2w_0}\right) \right)$$

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$$\begin{aligned}
& \cdot \{x^2 [(1 + C_1) + C_2] - \frac{2x^2}{\lambda_0 w_0} [x^2(1 + C_1) + C_2] \\
& + \frac{x^2}{\lambda_0} (1 + \frac{x^2}{2w_0}) (1 - \frac{13}{3} C_1) \} + F_M(\rho R_c B_{m_0})(\rho \varepsilon a^2) [(A_1 - \frac{5}{2} A_2) \\
& + x^2 A_2] \frac{1}{w_0} \{ - \frac{2x^2}{w_0} [x^2(1 + C_1) + C_2] \\
& + x^2 (1 + \frac{x^2}{2w_0}) (1 - \frac{13}{3} C_1) \} \frac{1}{\lambda_0^2} \chi(z) \Big), \tag{202}
\end{aligned}$$

then $\chi(z)$ satisfies the differential equation

$$\frac{d^2 \chi}{dz^2} - q^2 \chi(z) = - \frac{q^2}{2} \ln z^2, \tag{203}$$

where

$$q^2 = -i[q = 1/\sqrt{2}(1 - i)] \tag{204}$$

We note that expression $|D|$ in Eq. (201) is really a transcendental equation for the boundary layer width, since λ_0 depends on $|D|$. However, its dependence is only logarithmic, so it can be taken to be a constant. The function $\chi(z)$ can be interpreted as the collisional contribution to $f_1^{(0)}$ at the cusp point so that the sum $f_1^{(0)} + \chi(z)$ does not have an infinite derivative at $\zeta_m = \zeta_c$.

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Equation (203) is now in the form written down by Kovrizhnykh in Ref. 11 and has the solution, which is regular at $z = 0$ and vanishes for ∞ , given by

$$x(z) = \frac{q}{2} \left\{ \int_{-\infty}^z \exp[q(z' - z)] \ln|z'| dz' + \int_z^{\infty} \exp[q(z - z')] \ln|z'| dz' \right\} . \quad (205)$$

The collisional boundary layer width $\Delta\zeta_c$ at $\zeta_m = 1/\sqrt{2}$ is given by

$$\Delta\zeta_c = \frac{\sqrt{D_1}}{4\sqrt{2}} . \quad (206)$$

Let us interpret the ζ_m integral in Θ_1 as follows:

$$\begin{aligned} & \int_0^1 d\zeta_m \frac{\sqrt{2}}{2} Q(\zeta_m) (1 - \zeta_m^2) \left[\frac{\partial}{\partial \zeta_m} \left(\frac{f}{F_M} \right) \right]^2 \\ &= \int_0^{(1/\sqrt{2}) - (\Delta\zeta_c/2)} d\zeta_m \frac{\sqrt{2}}{2} Q(\zeta_m) (1 - \zeta_m^2) \left[\frac{\partial}{\partial \zeta_m} \left(\frac{f}{F_M} \right) \right]^2 \\ &+ \int_{(1/\sqrt{2}) + (\Delta\zeta_c/2)}^1 d\zeta_m \frac{\sqrt{2}}{2} Q(\zeta_m) (1 - \zeta_m^2) \left[\frac{\partial}{\partial \zeta_m} \left(\frac{f}{F_M} \right) \right]^2 \\ &+ \int_{(1/\sqrt{2}) - (\Delta\zeta_c/2)}^{(1/\sqrt{2}) + (\Delta\zeta_c/2)} d\zeta_m \frac{\sqrt{2}}{2} Q(\zeta_m) (1 - \zeta_m^2) \left[\frac{\partial}{\partial \zeta_m} \left(\frac{f}{F_M} \right) \right]^2 . \end{aligned} \quad (207)$$

Now we write the first integral as

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$$\int_0^{(1/\sqrt{2})-(\Delta\zeta_c/2)} d\zeta_m \frac{\sqrt{2}}{2} Q(\zeta_m)(1 - \zeta_m^2) \left[\frac{\partial}{\partial \zeta_m} \left(\frac{f}{F_M} \right) \right]^2$$

$$= \int_0^{(1/\sqrt{2})} - \frac{\Delta\zeta_r}{2} d\zeta_m \frac{\sqrt{2}}{2} Q(\zeta_m)(1 - \zeta_m^2) \left[\frac{\partial}{\partial \zeta_m} \left(\frac{f}{F_M} \right) \right]^2 \quad (208)$$

$$- \int_{(1/\sqrt{2})-(\Delta\zeta_c/2)}^{(1/\sqrt{2})+(\Delta\zeta_c/2)} d\zeta_m \frac{\sqrt{2}}{2} Q(\zeta_m)(1 - \zeta_m^2) \left[\frac{\partial}{\partial \zeta_m} \left(\frac{f}{F_M} \right) \right]^2 \quad (208)$$

For collisionalities which are sufficiently low so that $\Delta\zeta_c \geq \Delta\zeta_r$, we can ignore the second integral on the right-hand side of Eq. (208). We do a similar analysis for the second integral in Eq. (207), and then the sum of the two integrals will reduce to integrals containing the functionals $N(f)$ given in Eq. (184). When $\Delta\zeta_c \gg \Delta\zeta_r$, we can proceed in the same manner by replacing $\Delta\zeta_r$ in Eq. (208) by some width $\Delta\zeta_m$ such that $\Delta\zeta_c \geq \Delta\zeta_m$. In Table 2, we give the various functionals $N(f)$ for a choice $\Delta\zeta_m = 5 \times 10^{-3}$; in Table 3, for $\Delta\zeta_m = 5 \times 10^{-2}$. Since the integrals in the functionals $N(f)$ vary most quickly near $\zeta = \zeta_c$, we expect that the change in the magnitudes of the $N(f)$ will be greatest going from $\Delta\zeta_m = \Delta\zeta_r$ to $\Delta\zeta_m = 5 \times 10^{-3}$ and will decrease as $\Delta\zeta_m$ is increased. This is indeed what is found from Tables 1-3. Also, we find that for $1 \gg \Delta\zeta_m \geq 5 \times 10^{-2}$, there is relatively little change in the values of the $N(f)$. As we shall calculate later, $\Delta\zeta_m \simeq 5 \times 10^{-2}$ corresponds to a collisionality given by $\nu_{ei}/\Omega_E \simeq 2 \times 10^{-2}$ so for $1 \gg \nu_{ei}/\Omega_E \geq 2 \times 10^{-2}$ we can use the values for the $N(f)$ given in Table 3.

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The third integral in Eq. (207) is the integral over the collisional boundary layer width. We transform this integral from ζ_m to z and extend the range of integration to $z = \pm\infty$, which we can do with negligible error since $\chi(z)$ has its greatest variation around $z = 0$ and $\chi'(z)$ vanishes for $z \rightarrow \pm\infty$. Then we have

$$\begin{aligned} & \int_0^{2\pi} \frac{d\beta}{2\pi} \int_{(1/\sqrt{2})-(\Delta\zeta_c/2)}^{(1/\sqrt{2})+(\Delta\zeta_c/2)} d\zeta_m \frac{\sqrt{2}}{2} \mathcal{Q}(\zeta_m) (1 - \zeta_m^2) \left[\frac{\partial}{\partial \zeta_m} \left(\frac{f}{F_M} \right) \right]^2 \\ & \simeq \frac{\sqrt{2}}{4} \mathcal{Q} \left(\frac{1}{\sqrt{2}} \right) \frac{32}{|D|} \frac{\sqrt{|D|}}{2} \left(\frac{1}{\sqrt{2}} \right)^3 \frac{1}{2} \int_{-\infty}^{\infty} dz \left\{ \left[\operatorname{Re} \frac{\partial}{\partial z} \left(\frac{\hat{f}}{F_M} \right) \right]^2 \right. \\ & \quad \left. + \left[\operatorname{Im} \frac{\partial}{\partial z} \left(\frac{\hat{f}}{F_M} \right) \right]^2 \right\} . \end{aligned} \quad (209)$$

We use Eqs. (198) and (201) and following Ref. 11 write

$$\int_{-\infty}^{\infty} dz \left[\left(\operatorname{Re} \frac{\partial \xi}{\partial z} \right)^2 + \left(\operatorname{Im} \frac{\partial \xi}{\partial z} \right)^2 \right] \simeq \operatorname{Re} \int_{-\infty}^{\infty} \frac{\partial \xi}{\partial z} \frac{dz}{z} = \frac{\pi^2}{2\sqrt{2}}$$

and

$$\frac{1}{\sqrt{|D|}} \simeq 0.289 \Omega_{th}^{1/2} \omega_0^{1/2} \lambda_0^{1/2} [\nu_e^D(x)]^{-1/2} \left(1 - \frac{x^2}{4\omega_0} \right) . \quad (210)$$

Then the contribution to Θ_1 from the collisional boundary layer Θ_1^C is given by

$$\Theta_1^C = n(\rho R_c B_{m_0})^2 (\rho \varepsilon a^2)^2 (1.08) \nu_{ei}^{1/2} \Omega_{th}^{1/2} \omega_0^{-3/2} \lambda_*^{-7/2}$$

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$$\begin{aligned}
& \cdot \left[N_4 \left\{ (A_1 - \frac{5}{2} A_2)^2 \left[(1 - \frac{13C_1}{3})^2 - \frac{2C_2}{w_0} (1 - \frac{13}{3} C_1) \right] \right\} \right. \\
& + N_6 \left(2A_2(A_1 - \frac{5}{2} A_2) \left[(1 - \frac{13}{3} C_1)^2 - \frac{2C_2}{w_0} (1 - \frac{13}{3} C_1) \right] \right. \\
& + \frac{2}{w_0} (A_1 - \frac{5}{2} A_2)^2 \left\{ (1 - \frac{13C_1}{3}) (\frac{1}{2} (1 - \frac{13}{3} C_1) - 2(1 + C_1)) \right\} \Big) \\
& + N_8 \left(A_2^2 \left[(1 - \frac{13C_1}{3})^2 - \frac{2C_2}{w_0} (1 - \frac{13C_1}{3}) \right] + \frac{4A_2}{w_0} (A_1 - \frac{5}{2} A_2) \left\{ (1 - \frac{13C_1}{3}) \right. \right. \\
& \left. \left. [\frac{1}{2} (1 - \frac{13}{3} C_1) - 2(1 + C_1)] \right\} \right) + N_{10} \left(A_2^2 \frac{2}{w_0} \left\{ (1 - \frac{13}{3} C_2) \right. \right. \\
& \cdot \left. \left. [\frac{1}{2} (1 - \frac{13}{3} C_1) - 2(1 + C_1)] \right\} \right) - N_0 \left[\frac{1}{4w_0} (A_1 - \frac{5}{2} A_2)^2 (1 - \frac{13}{3} C_1)^2 \right] \\
& \left. - N_8 \left[\frac{2A_2}{4w_0} (A_1 - \frac{5}{2} A_2) (1 - \frac{13}{3} C_1)^2 \right] - N_{10} \left[\frac{A_2^2}{4w_0} (1 - \frac{13}{3} C_1)^2 \right] \right] . \quad (211)
\end{aligned}$$

In the energy integration that leads to Eq. (211), we have replaced $\lambda_0(\bar{x})$ by λ_* , where

$$\lambda_* = \ln \frac{16}{|D(\bar{x} = 1)|} , \quad (212)$$

since the energy dependence of λ_0 is at most logarithmic. Note that the collisionality scaling of this contribution to Θ_1 goes as $\nu_{e1}^{1/2}$. This is because the collisional boundary layer has a width $\Delta \zeta_c$ which scales as $\nu_{e1}^{1/2}$. The integrals N_k are defined by

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$$N_k = \int_0^\infty dx x^{1/2} \left[1 + \operatorname{erf}(x) \left(1 - \frac{1}{2x^2} \right) + \frac{e^{-x^2}}{x\sqrt{\pi}} \right]^{1/2} x^k e^{-x^2} . \quad (213)$$

These have been evaluated numerically and are

$$N_4 \simeq 1.07 , \quad (214)$$

$$N_6 \simeq 2.99 , \quad (215)$$

$$N_8 \simeq 11.35 , \quad (216)$$

$$N_{10} \simeq 54.25 . \quad (217)$$

Finally, we can use Eqs. (214)-(217), (211), (173), and (169) to write the irreversible entropy production rate as

$$\begin{aligned} \Theta = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \frac{v_{ei}}{\sqrt{\pi}} \left(A_1^2 \left\{ M_{00} \left(1 + \frac{1}{\sqrt{2}} \right) \right. \right. \\ & - \frac{1}{w_0} \left[\left(4 + \frac{9}{2\sqrt{2}} \right) M_{02} + \frac{B_{1001}^5}{2\sqrt{2}} \right] \} + 2A_1 A_2 \left\{ M_{00} \left(\frac{3}{2} + \frac{7}{4\sqrt{2}} \right) - \frac{5}{4} \frac{C_2 B_{1000}}{\sqrt{2}} \right. \\ & - \frac{1}{w_0} \left[M_{02} \left(2 + \frac{25}{8\sqrt{2}} \right) + \frac{7}{8} \frac{B_{1001}}{\sqrt{2}} + \frac{11}{4\sqrt{2}} B_{0110} \right] \} \\ & + A_2^2 \left\{ M_{00} \left(\frac{9}{4} + \frac{35}{16\sqrt{2}} \right) + \frac{25}{4} \frac{B_{1000}}{\sqrt{2}} + \frac{11}{2} \frac{B_{0101}}{\sqrt{2}} \right. \\ & \left. \left. - \frac{1}{w_0} \left[M_{02} \left(13 + \frac{491}{64\sqrt{2}} \right) + \frac{1839}{32\sqrt{2}} B_{1001} + \frac{235}{8\sqrt{2}} B_{0110} \right] \right\} \right) \end{aligned}$$

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$$\begin{aligned}
& + n(\rho B_{m_0})^2 R_c^2 (\rho \varepsilon a^2)^2 (1.08) \frac{(\nu_{ei} \Omega_{th})^{1/2}}{\lambda^{7/2} w_0^{3/2}} \left(A_1^2 \left\{ 1.07 \left(1 - \frac{13}{3} C_1 \right)^2 \right. \right. \\
& + \frac{1}{w_0} \left[2.24 \left(1 - \frac{13}{3} C_1 \right)^2 - 11.96 (1 + C_1) \left(1 - \frac{13}{3} C_1 \right) \right. \right. \\
& - 2.14 C_2 \left(1 - \frac{13}{3} C_1 \right) \left. \right\} + 2 A_1 A_2 \left\{ 0.315 \left(1 - \frac{13}{3} C_1 \right)^2 \right. \\
& + \frac{1}{w_0} \left[2.906 \left(1 - \frac{13}{3} C_1 \right)^2 - 15.5 (1 + C_1) \right. \\
& \cdot \left. \left(1 - \frac{13}{3} C_1 \right) - 0.63 C_2 \left(1 - \frac{13}{3} C_1 \right) \right] \left. \right\} \\
& + A_2^2 \left\{ 3.088 \left(1 - \frac{13}{3} C_1 \right)^2 + \frac{1}{w_0} \left[12.141 \left(1 - \frac{13}{3} C_1 \right)^2 - 64.75 (1 + C_1) \right. \right. \\
& \cdot \left. \left(1 - \frac{13}{3} C_1 \right) - 6.175 C_2 \left(1 - \frac{13}{3} C_1 \right) \right] \left. \right\} \left. \right) . \tag{218}
\end{aligned}$$

We can calculate the boundary layer width $\Delta \zeta_c$ from the equations

$$\lambda_* = 1.44 + \ln \left[\left(\frac{w_0}{\bar{\nu}} \right)^{1/2} \lambda_*^{1/2} \right] , \tag{219}$$

$$\Delta \zeta_c = \left[11.936 \left(\frac{\bar{\nu}}{w_0} \right) \frac{1}{\lambda_*} \left(1 + \frac{1}{6} \right) \right]^{1/2} \frac{1}{4\sqrt{2}} . \tag{220}$$

where we have chosen $w_0 = 3$ since the dependence on w_0 in λ_* is logarithmic and $\bar{\nu} = \nu_{ei}/\Omega_{th}$. In Table 4, we show λ_* and $\Delta \zeta_c$ for several values of the collisionality. We see that the collisional boundary layer width will always exceed the radial boundary layer width even for $\nu/w_0 \geq 2 \times 10^{-5}$ (which is much smaller than is observed experimentally), and when $\Delta \zeta_c = 5 \times$

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10^{-3} , then $\bar{\nu}/w_0 \simeq 5 \times 10^{-4}$; for $x D \zeta_c = 5 \times 10^{-2}$, $\bar{\nu}/w_0 \simeq 2 \times 10^{-2}$. Hence the values for $N(f)$ in Table 1 can be used to estimate the diffusion coefficients for $2 \times 10^{-6} \leq \bar{\nu}/w_0 \ll 5 \times 10^{-4}$; Table 2 can be used for $2 \times 10^{-6} \ll \bar{\nu}/w_0 \ll 2 \times 10^{-2}$; and Table 3 can be used for $5 \times 10^{-4} \ll \bar{\nu}/w_0 \ll 1$. For $\bar{\nu}/w_0 \geq 0.5$ the collisional boundary layer width is large enough so that edge effects may be significant. At this collisionality the entropy production probably cannot be split into a part going like ν_{ei} and a part going like $\nu_{ei}^{1/2}$.

To complete the calculation of ϑ , we need to obtain the numerical values of B_{1001} , B_{1000} , B_{0110} , and B_{0101} . These will tell us the magnitude of the energy scattering terms compared to the pitch angle scattering terms. These coefficients all involve integrals of g , g_1 , and h . For these integrals, we do not need to include the radial smearing or collisional smearing, since by assumption these will both be small. Note that for the integrals in M_{00} and M_{02} this was not true, since the smearing was needed there to resolve a singularity. For a similar reason, the calculation for φ_1 does not need to take into account the correction to the distribution function given in this section.

From Eqs. (150), (151), (152), and (155) we can obtain

$$\begin{aligned} B_{1001} = & (1 - C_1^2)S(b^{hg}_g) + C_1(1 - C_1)S(b^{hg}_{g_1}) - C_2(1 - C_1)S(b^{hg}_h) \\ & + C_1(1 - C_1)S(b^{hg}_{1g}) + C_1^2S(b^{hg}_{1g_1}) - C_1C_2S(b^{hg}_{1h}) , \end{aligned} \quad (221)$$

$$B_{1000} = (1 - C_1)S(b^{hg}) + C_1S(b^{hg}_1) , \quad (222)$$

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$$B_{0110} = (1 - C_1)^2 S(b^g h g) + C_1(1 - C_1) S(b^g h g_1) + C_1(1 - C_1) S(b^g 1 h g) \\ + C_1^2 S(b^g 1 h g_1) - C_2(1 - C_1) S(b^h h g) - C_2 C_1 S(b^h h g_1) , \quad (223)$$

$$B_{0101} = (1 - C_1)^2 S(b^g g) + (1 - C_1) C_1 S(b^g g_1) - C_2(1 - C_1) S(b^g h) \\ + C_1(1 - C_1) S(b^g 1 g) + C_1^2 S(b^g 1 g_1) - C_2 C_1 S(b^g 1 h) \\ - C_2(1 - C_1) S(b^h g) - C_2 C_1 S(b^h g_1) + C_2^2 S(b^h h) , \quad (224)$$

where we define the function

$$b^f(\zeta_m) = \int_0^{l_T} \frac{d\ell}{\gamma(\ell)} \frac{1}{(v_{\parallel}/v)} \int_{(1-\gamma)^{1/2}}^1 \frac{d\zeta_m \zeta_m}{v_{\parallel}/v} f(\zeta_m) \quad (225)$$

and

$$l_T = \begin{cases} (d/2\pi) \cos^{-1} 3 - 2/(1 - \zeta_m^2) , & \zeta_m < 1/\sqrt{2} \\ d/2 , & \zeta_m > 1/\sqrt{2} \end{cases} \quad (226)$$

and the functional $S(f)$ by

$$S(f) = \frac{1}{\int \gamma(\ell) d\ell} \int_0^1 d\zeta_m \zeta_m f(\zeta_m) . \quad (227)$$

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In Table 5, we give the values of $S(f)$ for the various choices of f in Eqs. (217)-(220).

With the use of Tables 1-5, Eqs. (214) for θ , and the formulae in Eqs. (81)-(83), we can write down the diffusion coefficients L_{11} , L_{12} , L_{22} as

$$\begin{aligned}
 L_{11} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ M_{00} \left(1 + \frac{1}{\sqrt{2}}\right) - \frac{1}{w_0} \left[\left(4 + \frac{9}{2\sqrt{2}}\right) M_{02} \right. \right. \\
 & \left. \left. + \frac{B_{10015}}{2\sqrt{2}} \right] \right\} + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \nu_{ei} \frac{1.08}{\lambda_{*}^{7/2}} \left(\frac{w_0}{\nu}\right)^{1/2} \left\{ \left[1.07 \left(1 - \frac{13}{3} C_1\right)^2 \right] \right. \\
 & \left. + \frac{1}{w_0} \left[2.24 \left(1 - \frac{13}{3} C_1\right)^2 - 11.96(1 + C_1)\left(1 - \frac{13}{3} C_1\right) - 2/14 C_2 \left(1 - \frac{13}{3} C_1\right)\right] \right\}
 \end{aligned}
 \tag{228}$$

$$\begin{aligned}
 L_{12} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ M_{00} \left(\frac{3}{2} + \frac{7}{4\sqrt{2}}\right) - \frac{5}{4} C_2 \frac{B_{1000}}{\sqrt{2}} \right. \\
 & \left. - \frac{1}{w_0} \left[M_{02} \left(2 + \frac{25}{8\sqrt{2}}\right) + \frac{7}{8} \frac{B_{1001}}{\sqrt{2}} + \frac{11}{4\sqrt{2}} B_{0110} \right] \right\} \\
 & + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \nu_{ei} \frac{1.08}{\lambda_{*}^{7/2}} \left(\frac{w_0}{\nu}\right)^{1/2} \left\{ 0.315 \left(1 - \frac{13}{3} C_1\right)^2 \right. \\
 & \left. + \frac{1}{w_0} \left[2.906 \left(1 - \frac{13}{3} C_1\right)^2 - 15.5(1 + C_1)\left(1 - \frac{13}{3} C_1\right) \right. \right. \\
 & \left. \left. - 0.63 C_2 \left(1 - \frac{13}{3} C_1\right)\right] \right\} ,
 \end{aligned}
 \tag{229}$$

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$$\begin{aligned}
L_{22} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ M_{00} \left(\frac{9}{4} + \frac{35}{16\sqrt{2}} \right) + \frac{25}{4} C_2 \frac{B_{1000}}{\sqrt{2}} \right. \\
& + \frac{11}{2} \frac{B_{0101}}{\sqrt{2}} - \frac{1}{w_0} \left[M_{02} \left(13 + \frac{491}{64\sqrt{2}} \right) + \frac{1839}{32\sqrt{2}} B_{1001} + \frac{235}{8\rho 2} B_{0110} \right] \{ \\
& + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \nu_{ei} \frac{1.08}{\lambda_d^{1/2}} \left(\frac{w_0}{v} \right)^{1/2} \left\{ 3.088 \left(1 - \frac{13}{3} C_1 \right)^2 \right. \\
& + \frac{1}{w_0} \left[12.141 \left(1 - \frac{13}{3} C_1 \right)^2 - 64.75(1 + C_1) \left(1 - \frac{13}{3} C_1 \right) \right. \\
& \left. \left. - 6.175 C_2 \left(1 - \frac{13}{3} C_1 \right) \right] \right\} \} .
\end{aligned} \tag{230}$$

VII. DISCUSSION

In Refs. 2-4, the particle and energy fluxes (in units of radius rather than units of flux) are given by

$$\Gamma = -D_n \frac{dn}{dp} - D_T \frac{dT}{dp} - \mu_n n \varphi'_0, \quad (231)$$

$$Q = -K_n \frac{dn}{dp} - D_T \frac{dT}{dp} - \mu_T n \varphi'_0, \quad (232)$$

where

$$\mu_n = \frac{Ze}{k_b T} D_n, \quad (233)$$

$$\mu_T = \frac{Ze}{k_b T} D_T. \quad (234)$$

In Ref. 4, the diffusion coefficients are given in the collisionless limit ($\bar{v}/w_0 \ll 1$) by [to $O(1/w_0^3)$]

$$D_n = R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} 4(6.84 \times 10^{-2} + \frac{0.456}{w_0}), \quad (235)$$

$$D_T = \frac{n}{k_b T} \frac{R_c^2 (\rho \varepsilon a)^2}{w_0^2} \nu_{ei} 4(0.137 + \frac{0.906}{w_0}), \quad (236)$$

$$K_n = k_b T \frac{R_c^2 (\rho \varepsilon a)^2}{w_0^2} \nu_{ei} 4(0.2395 + \frac{1.44}{w_0}), \quad (237)$$

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$$K_T = n \frac{R_c^2 (\rho \epsilon a)^2}{w_0^2} \nu_{ei} 4(3.5167) . \quad (238)$$

If we express the fluxes in Eqs. (82) and (83) in the form given by Eqs. (231) and (232), then we can make the identifications

$$D_n = \frac{1}{(\rho B_{m_0} a)^2} \frac{L_{11}}{n} , \quad (239)$$

$$D_T = \frac{1}{(\rho B_{m_0} a)^2} \left(\frac{L_{11} + L_{12}}{k_b T} \right) , \quad (240)$$

$$K_n = \frac{1}{(\rho B_{m_0} a)^2} \left(\frac{5}{2} L_{11} + L_{12} \right) \frac{k_b T}{n} , \quad (241)$$

$$K_T = \frac{1}{(\rho B_{m_0} a)^2} \left(\frac{5}{2} L_{11} + \frac{7}{2} L_{12} + L_{22} \right) . \quad (242)$$

In Ref. 2 it was stated that to $O(1/w_0^2)$ or the large field limit the poloidal electric field makes no contribution to the transport. This is consistent with the expressions here for L_{11} , L_{12} , and L_{22} to $O(1/w_0^2)$ since in the notation of Ref. 2 the number C_2 must be regarded as $O(1/w_0)$. Hence, when all the terms in M_{00} and B_{0101} involving C_2 are moved to higher order in $1/w_0$, we see that D_n , D_T , K_n and K_T are independent of the poloidal electric field to $O(1/w_0^2)$. This result can be understood by noting that the shift of the electrons due to a poloidal electric field and

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just $\underline{E} \times \underline{B}$ drifting is independent of energy and pitch angle, and so in a scattering event the offset of the orbit does not change.

In Ref. 4, it was assumed that the plasma center was at the average magnetic axis and there was no poloidal electric field, that is, $C_1 = 0$, $C_2 = 0$. For this choice of C_1 and C_2 we have

$$\begin{aligned}
 L_{11} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left[0.231 \left(1 + \frac{1}{\sqrt{2}} \right) - \frac{1}{w_0} \left\{ \left(4 + \frac{9}{2\sqrt{2}} \right) 0.1157 \right. \right. \\
 & \left. \left. + \frac{5}{2\sqrt{2}} (0.273) \right\} \right] + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} (1.08) (1.157 \times 10^{-2}) \\
 & \times \left(\frac{w_0}{\nu} \right)^{1/2} \left[1.07 - \frac{9.72}{w_0} \right] , \tag{244}
 \end{aligned}$$

$$\begin{aligned}
 L_{12} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\alpha^2)^2}{w_0^2} \frac{\nu_{ei} R}{\sqrt{\pi}} 0.231 \left(\frac{3}{2} + \frac{7}{4\sqrt{2}} \right) \\
 & - \frac{1}{w_0} \left[\left(2 + \frac{25}{8\sqrt{2}} \right) 0.1157 + \frac{7}{8\sqrt{2}} (0.273) + \frac{11}{4\sqrt{2}} (0.249) \right] \} \\
 & + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \nu_{ei} (1.08) (1.157 \times 10^{-2}) \left(\frac{w_0}{\nu} \right)^{1/2} \left(0.315 - \frac{12.59}{w_0} \right) , \tag{245}
 \end{aligned}$$

$$\begin{aligned}
 L_{22} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ 0.231 \left(\frac{9}{4} + \frac{35}{16\sqrt{2}} \right) \right. \\
 & \left. + \frac{11}{2\sqrt{2}} (1.831) - \frac{1}{w_0} \left[0.1157 \left(13 + \frac{491}{64\sqrt{2}} \right) + \frac{1839}{32\sqrt{2}} (0.273) \right] \right\}
 \end{aligned}$$

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$$\begin{aligned}
& + \frac{235}{8\sqrt{2}} (0.249)] \} + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \epsilon a)^2}{w_0^2} \nu_{ei} (1.08) (1.157 \times 10^{-2}) \\
& \times \left(\frac{w_0}{\bar{\nu}} \right)^{1/2} \left(3.088 - \frac{52.609}{w_0} \right), \quad (246)
\end{aligned}$$

where we have chosen to look at collisionalities in the region of $\bar{\nu}/w_0 \simeq 5 \times 10^{-2}$; hence, we have used Table 3 and taken $\lambda_* \simeq 3.57$ from Table 4. We note a number of points. For the large field limit ($w_0 \gg 1$), energy scattering makes no contribution to D_n , D_T , and K_n but does make a contribution to K_T , the thermal conductivity. This result was first noted by Jaeger et al. in Ref. 12. Even for finite values of w_0 , the dominant contribution to D_n , D_T , and K_n arises from pitch angle scattering, and the only significant energy scattering contribution is in K_T . It is not surprising that energy scattering is important for K_T since K_T measures the diffusivity of the particle energy due to energy flowing down the temperature gradient. In contrast, K_n measures the diffusivity of the particle energy due to particles carrying the energy down the density gradient. We also note that for $\bar{\nu}/w_0 \simeq 5 \times 10^{-2}$ and for $w_0 \gg 1$ the part of D_n that scales as ν_{ei} is only 3.5 times larger than the part that scales as $\nu_{ei}^{1/2}$. More generally, we expect that for $\bar{\nu}/w_0 \sim 0.5$ the two contributions will be equal, and so experimental measurements for $\bar{\nu}/w_0 \geq 5 \times 10^{-2}$ will see a set of diffusion coefficients that scale with a power of ν_{ei} falling somewhere between 1 and 1/2.

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With Eqs. (244)-(246) we obtain D_n , D_T , K_n , and K_T from Eqs. (240)-(243),

$$D_n = R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(0.22 - \frac{0.741}{w_0} \right) + R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{v} \right)^{1/2} \left(1.34 \times 10^{-2} - \frac{0.121}{w_0} \right), \quad (247)$$

$$D_T = \frac{n}{k_b T} \frac{R_c^2 (\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(0.577 - \frac{1.384}{w_0} \right) + \frac{n}{k_b T} T_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{v} \right)^{1/2} \left(1.73 \times 10^{-2} - \frac{0.278}{w_0} \right), \quad (248)$$

$$K_n = k_b T R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(0.907 - \frac{2.49}{w_0} \right) + k_b T R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{v} \right)^{1/2} \left(3.74 \times 10^{-2} - \frac{0.459}{w_0} \right), \quad (249)$$

$$K_T = n R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(6.312 - \frac{14.48}{w_0} \right) + n R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{v} \right)^{1/2} \times \left(8.57 \times 10^{-2} - \frac{1.51}{w_0} \right). \quad (250)$$

We compare these expressions to the ones in Eqs. (235)-(239) from Ref. 4. We see that the expressions from Ref. 4 have no term scaling as $v_{e1}^{1/2}$. This is because this term arises from a careful treatment of the differential scattering process near the cusp point, and the BGK operator is incapable of picking up these fine details in velocity space. This scaling was also not observed in Ref. 3, which used a differential operator. There the reason was that a smoother form for both g and h was assumed, that is, $g(\zeta_m) = 1 + \zeta_m^2$, $h(\zeta_m) = 1 - 3.6\zeta_m^2 + 2.4\zeta_m^4$. In our analysis this would correspond to choosing a fixed size for the boundary layer as the collisionality changed. We also see that when VB effects are included the diffusion coefficients decrease in contrast to what was found in both Refs. 3 and 4. The reason for this is subtle. In Ref. 3, the bounce-averaged collision operator, was approximated by a collision operator, which was obtained by taking the non-bounce-averaged pitch angle scattering operator and replacing ζ_p by ζ_m . This operator then gives the rate at which ζ_m changes. However, if we look at the bounce-averaged pitch angle scattering operator in Eq. (134), it describes the rate at which the field-line-averaged pitch angle $\propto Q(\zeta_m)$ changes. Typically, this field-line-averaged pitch angle changes more slowly than the midplane pitch angle so we have $\langle \zeta_p \rangle_{av} / \zeta_m < 1$, where $\langle \zeta_p \rangle_{av} \propto Q$. Roughly speaking, we can write the diffusion coefficient D_n as

$$D_n \sim \langle \nu \frac{\langle \zeta_p \rangle_{av}}{\zeta_m} \left[\frac{\partial}{\partial \zeta_m} \left(\frac{v}{n} \right) \right]^2 \rangle_v, \quad (251)$$

where $\langle \dots \rangle_v$ means an average over velocity space and all the choices of ζ_m

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and v . The step size term in Eq. (247) is $\partial/\partial\zeta_m(v/\Omega)$, and it occurs as a derivative since the shift of any orbit is V/Ω and only the change in the shift as a particle scatters from one orbit to another leads to diffusion. When magnetic drifts are included in the calculation of D_n , then for $\zeta_m < \zeta_c$, $\partial/\partial\zeta_m(V/\Omega)$ is increased over its value when the magnetic drifts are neglected. Another way of regarding it is to say that, for the trapped particles, including magnetic drifts increases the amount of orbit dispersion in pitch angle space. Conversely, for the passing particles, the inclusion of magnetic drifts decreases the amount of orbit dispersion in pitch angle space [$\partial/\partial\zeta_m(V/\Omega)$ decreases from its value when magnetic drifts are neglected]. Now, $\langle\zeta_p\rangle_{av}/\zeta_m < 1$ and increases with increasing ζ_m , so we take into account the fact that the average ζ_p differs from ζ_m , which tends to weight the particles with the largest values of ζ_m (the passing particles) over the trapped particles. For these particles the orbit dispersion decreases, and this leads to a reduction in the diffusion coefficient. In Refs. 3 and 4 the bounce averaging was approximated in such a way as to correspond to the choice $\langle\zeta_p\rangle_{av}/\zeta_m = 1$. This tends to weight the particles with the greatest change in the orbit dispersion — the trapped particles — and for this case the diffusion coefficient increases when VB drifts are included in the precessional motion.

In the large field limit ($w_0 \gg 1$), if we compare Eqs. (247)-(250) with Eqs. (235)-(239), we see that D_n , D_T , and K_n are slightly smaller in Eqs. (247)-(250) than in the equations from Ref. 4. However, K_n is a factor of 2 times smaller. This arises mainly because of the energy-dependent structure of the collision frequency. In Refs. 2 and 4, the collision frequency was taken to be a constant for all energies whereas

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high energy particles must have fewer collisions than low energy ones. The thermal diffusivity K_T is preferentially caused by high-energy particles, so when the reduced collisionality is taken into account the magnitude of the diffusion coefficient is two times less than is predicted by constant collision frequency results.

We now examine the diffusion coefficients for values of C_1 and C_2 that close to the experimental conditions. We first choose $C_1 = 1$ (plasma center is at the $|B|$ center) and $C_2 = 0$ (potential center is at the $|B|$ center). Since for EBT-S parameters the $|B|$ center is shifted 3 cm inward from the geometric center this choice is for a plasma and potential center shift of 3 cm.

For this choice of C_1 and C_2 and for $\nu/w_0 \simeq 5 \times 10^{-2}$ we have

$$M_{00} = 0.864 , \quad (252)$$

$$M_{02} = 6.4 \times 10^{-2} , \quad (253)$$

$$B_{1001} = 3.62 \times 10^{-2} , \quad (254)$$

$$B_{1000} = 4.71 \times 10^{-2} , \quad (255)$$

$$B_{0110} = -3.15 \times 10^{-3} , \quad (256)$$

$$B_{0101} = 1.401 . \quad (257)$$

These give L_{11} , L_{12} , and L_{22} as follows:

$$L_{11} = n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \epsilon a^2)^2}{w_0^2} \frac{\nu e i}{\sqrt{\pi}} \left\{ 0.864 \left(+\frac{1}{\sqrt{2}} \right) - \frac{1}{w_0} \left[\left(4 + \frac{9}{2\sqrt{2}} \right) (6.4 \times 10^{-2}) \right] \right\}$$

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$$\begin{aligned}
& + \frac{5}{2\sqrt{2}} (3.62 \times 10^{-2}) \} + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \nu_{ei} (1.08) (1.157 \times 10^{-2}) \\
& \times \left(\frac{w_0}{\nu} \right)^{1/2} \left[1.07 (11.11) + \frac{104.62}{w_0} \right] , \quad (257)
\end{aligned}$$

$$\begin{aligned}
L_{12} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ 0.864 \left(\frac{3}{2} + \frac{7}{4\sqrt{2}} \right) - \frac{1}{w_0} \left[(6.42 \times 10^{-2}) \right. \right. \\
& \times \left(2 + \frac{25}{8\sqrt{2}} \right) + \frac{7}{8} \frac{(3.62 \times 10^{-2})}{\sqrt{2}} + \frac{11}{4\sqrt{2}} (-3.15 \times 10^{-3}) \} \\
& + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \nu_{ei} (1.08) \\
& \times (1.157 \times 10^{-2}) \left(\frac{w_0}{\nu} \right)^{1/2} \left[0.315 (11.11) + \frac{135.62}{w_0} \right] , \quad (259)
\end{aligned}$$

$$\begin{aligned}
L_{22} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ 0.864 \left(\frac{9}{4} + \frac{35}{16\sqrt{2}} \right) + \frac{11}{2} \frac{(1.401)}{\sqrt{2}} \right. \\
& - \frac{1}{w_0} \left[(6.42 \times 10^{-2}) \left(13 + \frac{491}{64\sqrt{2}} \right) + \frac{1839}{32\sqrt{2}} (3.626 \times 10^{-2}) \right. \\
& \left. \left. + \frac{235}{8\sqrt{2}} (-3.15 \times 10^{-3}) \right] \right\} + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \nu_{ei} \\
& \times (1.08) (1.157 \times 10^{-2}) \left(\frac{w_0}{\nu} \right)^{1/2} \left[3.088 (11.11) + \frac{566.56}{w_0} \right] . \quad (260)
\end{aligned}$$

If we compare this case ($C_1 = 1$) to the case with $C_1 = 0$, we see that the pitch angle scattering contributions are more important compared to the

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energy scattering contributions than before. In addition, the magnitude of the pitch angle contribution for the large field limit has increased by a factor of about 3.5. Both of these points can be explained by examining the function g_1 compared with g in Fig. 1. We see that g_1 varies much more quickly with ζ_m than g and hence its contribution to pitch angle scattering is much larger. We also note that for $g(\zeta_m)$ the passing particles make the greatest contribution to the transport; for $g_1(\zeta_m)$ the particles near the cusp make the greatest contribution to transport. The diffusion coefficients D_n , D_T , K_n , and K_T are

$$D_n = R_c^2 \frac{(\rho \varepsilon a)^2}{w_o^2} \nu_{ei} \left(0.832 - \frac{0.295}{w_o} \right) + R_c^2 \frac{(\rho \varepsilon a)^2}{w_o^2} \nu_{ei} \left(\frac{w_o}{v} \right)^{1/2} \left(0.149 + \frac{1.31}{w_o} \right) , \quad (261)$$

$$D_T = \frac{n}{k_b T} R_c^2 \frac{(\rho \varepsilon a)^2}{w_o^2} \nu_{ei} \left(2.166 - \frac{0.582}{w_o} \right) + \frac{n}{k_b T} R_c^2 \frac{(\rho \varepsilon a)^2}{w_o^2} \nu_{ei} \left(\frac{w_o}{v} \right)^{1/2} \left(0.193 + \frac{3.006}{w_o} \right) , \quad (262)$$

$$K_n = k_b T R_c^2 \frac{(\rho \varepsilon a)^2}{w_o^2} \nu_{ei} \left(3.414 - \frac{1.024}{w_o} \right) + k_b T R_c^2 \frac{(\rho \varepsilon a)^2}{w_o^2} \nu_{ei} \left(\frac{w_o}{v} \right)^{1/2} \left(0.4165 + \frac{4.97}{w_o} \right) , \quad (263)$$

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$$\begin{aligned}
K_T = nR_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(11.674 - \frac{4.330}{w_0} \right) \\
+ nR_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{\nu} \right)^{1/2} \left(0.955 + \frac{16.29}{w_0} \right) .
\end{aligned} \quad (264)$$

If we compare these diffusion coefficients with those in Eqs. (247)-(250), we see that these are larger due to the larger pitch angle scattering contribution. In Eqs. (261)-(264), in the parts scaling as ν_{ei} , we note that for the range of w_0 of experimental interest ($w_0 \geq 3$) the asymptotic expansion in $1/w_0$ will give a reasonable correction to the lowest-order result. This is not true of the parts scaling as $\nu_{ei}^{1/2}$, and the optimal asymptotic expansion for those will be just the term independent of w_0 . We also note that the collisional correction is much larger here than in Eqs. (247)-(250), so large that for $\bar{\nu}/w_0 \simeq 5 \times 10^{-2}$ the collisional part of D_n is of the same order as the collisionless part. This arises due to the fact that $g_1(\zeta_m)$ approaches the cusp point more sharply than $g(\zeta_m)$, as is clear from Eqs. (187) and (188). We note that even with the increased pitch angle scattering, K_T in Eq. (264) is not as large as K_T in Eq. (239) predicted from the BGK model.

For comparison, we shall now consider the case $C_1 = 1$ and $C_2 = 1$. For this choice we have (for $\bar{\nu}/w_0 \simeq 5 \times 10^{-2}$)

$$M_{00} = 2.158 , \quad (265)$$

$$M_{02} = 1.14 \times 10^{-2} , \quad (266)$$

$$B_{1001} = 1.93 \times 10^{-2} , \quad (267)$$

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$$B_{1000} = 4.71 \times 10^{-2} , \quad (268)$$

$$B_{0110} = -1.87 \times 10^{-2} , \quad (269)$$

$$B_{0101} = 1.192 . \quad (270)$$

These give L_{11} , L_{12} , and L_{22} as

$$\begin{aligned} L_{11} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \epsilon a^2)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ 2.158 \left(1 + \frac{1}{\sqrt{2}} \right) - \frac{1}{w_0} \left[\left(4 + \frac{9}{2\sqrt{2}} \right) (1.14 \times 10^{-2}) \right. \right. \\ & \left. \left. + (1.93 \times 10^{-2}) \frac{5}{2\sqrt{2}} \right] \right\} + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \epsilon a^2)^2}{w_0^2} \nu_{ei} (1.08) (1.157 \times 10^{-2}) \\ & \times \left(\frac{w_0}{\nu} \right)^{1/2} \left[1.07(11.11) + \frac{111.75}{w_0} \right] , \end{aligned} \quad (271)$$

$$\begin{aligned} L_{12} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \epsilon a^2)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ 2.158 \left(\frac{3}{2} + \frac{7}{4\sqrt{2}} \right) - \frac{5}{4\sqrt{2}} (4.71 \times 10^{-2}) \right. \\ & \left. - \frac{1}{w_0} \left[(1.14 \times 10^{-2}) \left(2 + \frac{25}{8\sqrt{2}} \right) + \frac{7}{8} \frac{(1.93 \times 10^{-2})}{\sqrt{2}} - \frac{11}{4\sqrt{2}} (1.87 \times 10^{-2}) \right] \right\} \\ & + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \epsilon a^2)^2}{w_0^2} \nu_{ei} (1.08) (1.157 \times 10^{-2}) \\ & \times \left(\frac{w_0}{\nu} \right)^{1/2} \left[0.315(11.11) + \frac{137.72}{w_0} \right] , \end{aligned} \quad (272)$$

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$$\begin{aligned}
L_{22} = & n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \frac{\nu_{ei}}{\sqrt{\pi}} \left\{ 2.158 \left(\frac{9}{4} + \frac{35}{16\sqrt{2}} \right) + \frac{25}{4} \frac{(4.71 \times 10^{-2})}{\sqrt{2}} \right. \\
& + \frac{11}{2} \frac{(1.192)}{\sqrt{2}} - \frac{1}{w_0} \left[(1.14 \times 10^{-2}) \left(13 + \frac{491}{64\sqrt{2}} \right) + \frac{1839}{32\sqrt{2}} (1.93 \times 10^{-2}) \right. \\
& \left. \left. - \frac{235}{8\sqrt{2}} (1.87 \times 10^{-2}) \right] \right\} + n(\rho B_{m_0})^2 R_c^2 \frac{(\rho \varepsilon a^2)^2}{w_0^2} \nu_{ei} (1.08) (1.157 \times 10^{-2}) \\
& \times \left(\frac{w_0}{\nu} \right)^{1/2} \left[3.088(11.11) + \frac{587.14}{w_0} \right] . \quad (273)
\end{aligned}$$

When we compare Eqs. (271)-(273) with Eqs. (258)-(260) we see that the major effect is the increase of w_{00} . This will increase the diffusion coefficients and arises because the large poloidal electric field coupled with the orbit dispersion introduced by the VB drifts increases the radial drift that the particles see. Note that $C_1 = 1$ corresponds to a potential center that is shifted 5 cm inward from the geometric center and [from Eq. (105)] gives $|e\phi_1/k_b T| = (\rho \varepsilon) w_0$. From Eqs. (271)-(273) we can obtain D_n , D_T , K_n , K_T as

$$\begin{aligned}
D_n = & R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(2.078 - \frac{6.545 \times 10^{-2}}{w_0} \right) \\
& + R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{\nu} \right)^{1/2} \left(0.149 + \frac{1.396}{w_0} \right) , \quad (274)
\end{aligned}$$

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$$D_T = \frac{n}{k_b T} R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(5.39 - \frac{7.875 \times 10^{-2}}{w_0} \right) \\ + \frac{n}{k_b T} R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{v} \right)^{1/2} \left(0.193 + \frac{3.117}{w_0} \right) , \quad (275)$$

$$K_n = k_b T R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(8.507 - \frac{0.177}{w_0} \right) \\ + k_b T R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{v} \right)^{1/2} \left(0.4165 + \frac{5.211}{w_0} \right) , \quad (276)$$

$$K_T = n R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(24.14 - \frac{0.552}{w_0} \right) \\ + n R_c^2 \frac{(\rho \varepsilon a)^2}{w_0^2} \nu_{ei} \left(\frac{w_0}{v} \right)^{1/2} \left(0.955 + \frac{16.85}{w_0} \right) . \quad (277)$$

We see that inclusion of the poloidal electric field increases the term in each diffusion coefficient scaling like ν_{ei} and hardly affects the $\nu_{ei}^{1/2}$ term. This implies that for collisionalities near $v/w_0 \simeq 5 \times 10^{-2}$, the $\nu_{ei}^{1/2}$ term is much smaller than the ν_{ei} term. We conclude by comparison of Eqs. (274)-(277) with Eqs. (261)-(264) that the poloidal electric field can make a significant contribution to the diffusion coefficients and its experimental determination is of importance.

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We speculate that when the assumption $\nu_{ei}/\Omega_E \ll 1$ fails, we can replace ν_{ei}/w_0^2 , at least in the first part of $D_n - K_T$, by $\{\nu_{ei}/\Omega_{th}/[w_0^2 + (\nu_{ei}/\Omega_{th})^2]\}\Omega_{th}$. This possibility will be examined in a further paper. Finally, we note that these nonresonant diffusion coefficients can be used for the ions in EBT if ν_{ei} is replaced by ν_{ii} , the first term in Eqs. (177)-(180) is ignored, Eqs. (214)-(217) is modified appropriately, and L_{11} , L_{12} , and L_{22} in Eqs. (228)-(230) have the obvious numerical coefficients changed.

VIII. CONCLUSIONS

We have obtained the radial particle drift as a function of plasma position and potential position. This has a new function $g_1(\zeta_m)$ in addition to the $g(\zeta_m)$ function which was given before. In the limit of low collisionality we have obtained the self-consistent poloidal electric field with the assumption that the bulk ions and electrons are nonresonant. For EBT-S this is a good assumption. We have followed Ref. 9 and constructed a variational principle that gives the Onsager matrix of diffusion coefficients. The bounce-averaged collision operator was taken to be the approximate one given in Ref. 5. This is a differential operator that conserves particles and energy and includes both pitch angle and energy scattering. The diffusion coefficients have been calculated and compared to the results of Ref. 4. We find that the velocity-space structure of the collision frequency is very important especially in determining K_T , the thermal diffusivity. We also find that the positions of the plasma and potential centers are very important in determining the magnitude of the diffusion coefficients. Finally, we have calculated the $\nu_{ei}^{1/2}$ contribution for EBT-S parameters that was first pointed out by Kovrizhnykh in Ref. 11, and we find for $\bar{v}/w_0 \sim 0(1)$ it can be of the same order as the ν_{ei} contribution.

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FIGURES

Fig. 1. Plot of $g(\zeta_m)$, $g_1(\zeta_m)$, and $h(\zeta_m)$ vs ζ_m , where g and g_1 occur in $\langle \underline{v}_d \cdot \underline{v}_\alpha \rangle$ and h occurs in $\langle \underline{v}_d \cdot \underline{v}_\beta \rangle$.

Fig. 2. Plot of (a) $\tilde{Q}(\zeta_m)$ vs ζ_m and (b) $(1 - \zeta_m^2)\tilde{Q}(\zeta_m)$ and $(1 - \zeta_m^2) \times [0.71P_1(\zeta_m) + 0.154 P_3(\zeta_m)]$ plotted vs ζ_m .

Fig. 3. $(1 - \zeta_m^2)\sqrt{2}\tilde{Q}(\zeta_m)[g'(\zeta_m)]^2$ and $(1 - \zeta_m^2)\sqrt{2}\tilde{Q}(\zeta_m)[h'(\zeta_m)]^2$ plotted vs ζ_m . Note the singularity at $\zeta_m = 1/\sqrt{2}$.

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Table 1. $N(f)$ for the choices of f used in M_{00} and M_{02} for a width $\Delta\zeta_m = \Delta\zeta_r = 3 \times$

f	$N(f)$
$g_1(h')^2$	1.375
$(g')^2$	0.703
$(g_1')^2$	6.088
$(h')^2$	2.226
$g_1'h'$	-3.509
$g_1'g'$	-1.243
$g'h'$	0.933
$(g')^2h$	-0.067
$h'g'g$	0.659
$g_1'g'h$	0.183
$g_1g'h'$	0.886
$g_1'gh'$	-1.320
$g_1'g_1h'$	-1.917
$g'h'h$	-0.121
$(g_1')^2h$	-0.491
$g(h')^2$	0.979
$g_1'h'h$	0.322
$g_1(h')^2$	1.375

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Table 2. $N(f)$ for the choices of f used in M_{00} and M_{02} for a width $\Delta\zeta_m = 5 \times 10^{-3}$.

f	$N(f)$
$(g')^2$	0.341
$(g_1')^2$	2.01
$(h')^2$	0.774
$g_1'h'$	-1.096
$g_1'g'$	-5.123×10^{-2}
$g'h'$	0.212
$(g')^2h$	-1.314×10^{-2}
$h'g'g$	0.244
$g_1'g'h$	2.497×10^{-2}
$g_1'g'h'$	0.285
$g_1'gh'$	-0.115
$g_1'g_1'h'$	-0.1626
$g'h'h$	-2.022×10^{-2}
$(g_1')^2h$	-2.721×10^{-2}
$g(h')^2$	0.211
$g_1'h'h$	2.802×10^{-2}
$g_1(h')^2$	0.2588

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Table 3. $N(f)$ for the choices of f used in M_{00} and M_{02} for a width $\Delta\zeta_m = 5 \times 10^{-2}$.

f	$N(f)$
$(g')^2$	0.231
$(g_1')^2$	0.864
$(h')^2$	0.386
$g_1'h'$	-0.454
$g_1'g'$	-0.1856
$g'h'$	3.729×10^{-2}
$(g')^2h$	1.754×10^{-3}
$h'g'g$	0.1139
$g_1'g'h$	4.90×10^{-3}
$g_1'g'h'$	0.121
$g_1'gh'$	5.76×10^{-2}
$g_1'g_1'h'$	5.93×10^{-2}
$g'h'h$	-1.475×10^{-3}
$(g_1')^2h$	4.759×10^{-3}
$g(h')^2$	4.809×10^{-2}
$g_1'h'h$	7.715×10^{-4}
$g_1(h')^2$	5.19×10^{-2}

Table 4. λ_* and boundary layer width, $\Delta\zeta_c$, vs collisionality ν_{ei}/Ω_E .

$\bar{\nu}/2_0$	λ_*	$\lambda_*^{-7/2}$	$\Delta\zeta_c$
0.5	2.175	6.59×10^{-2}	0.316
0.1	3.168	1.77×10^{-2}	0.117
5×10^{-2}	3.57	1.157×10^{-2}	7.8×10^{-2}
1×10^{-2}	4.494	5.197×10^{-3}	3.1×10^{-2}
5×10^{-3}	4.882	3.889×10^{-3}	2.1×10^{-2}
1×10^{-3}	5.77	2.167×10^{-3}	8.68×10^{-3}
5×10^{-4}	6.148	1.735×10^{-3}	5.95×10^{-3}
1×10^{-4}	7.0195	1.091×10^{-3}	2.49×10^{-3}
5×10^{-5}	7.3914	9.106×10^{-4}	1.7157×10^{-3}
1×10^{-5}	8.252	6.1958×10^{-4}	7.26×10^{-4}
5×10^{-6}	8.62	5.317×10^{-4}	5.024×10^{-4}
1×10^{-6}	9.47	3.82×10^{-4}	2.14×10^{-4}

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Table 5. $S(f)$ for the choices of f used in B_{1001} , B_{1000} , B_{0110} , and B_{0101} .

f	$S(f)$
$b^h g_g$	2.7302×10^{-1}
$b^h g_{g_1}$	1.7501×10^{-1}
$b^h g_h$	7.6384×10^{-2}
$b^h g_1 g$	5.9431×10^{-2}
$b^h g_1 g_1$	3.6157×10^{-2}
$b^h g_1 h$	1.6925×10^{-2}
$b^h g$	2.1660×10^{-1}
$b^h g_1$	4.7064×10^{-2}
$b^3 h g$	2.4936×10^{-1}
$b^g h g_1$	1.9196×10^{-2}
$b^g h g$	1.5273×10^{-1}
$b^{g_1} h g_1$	-3.1554×10^{-3}
$b^h h g$	7.6620×10^{-2}
$b^h h g_1$	1.5584×10^{-2}
$b^g g$	1.8313
$b^g g_1$	1.5655
$b^g h$	2.1800×10^{-1}
$b^{g_1} g$	1.5512
$b^{g_1} g_1$	1.4010
$b^{g_1} h$	1.2965×10^{-1}
$b^h g$	2.3925×10^{-1}
$b^h g_1$	1.4939×10^{-1}
$b^h h$	7.0119×10^{-2}