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Computational Models of Beam Emittance Growth

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Two fast running computer codes have been developed to calculate and verify the predictions of phenomenological models of beam emittance growth and radial oscillation damping. This report describes the codes - a point model, EMY, and a particle simulation, DESTIN. Comparison of model predictions and simulation results is made for the cases of a beam launched near self-pinch equilibrium and for a cold beam launched in near ballistic condition.

There is a growing consensus that the United States will be able to pay off its \$1.5 trillion foreign debt by 2010, and that the United States will be able to pay off its \$1.5 trillion foreign debt by 2010, and that the United States will be able to pay off its \$1.5 trillion foreign debt by 2010.

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EMY - A Phenomenological Point Model of Emittance Growth

The simplest description of beam behavior during propagation obtains through the use of a beam envelope equation. Recent advances ¹ in understanding the behavior of beams launched at conditions close to equilibrium have led to phenomenological descriptions of the evolution of the beam envelope and emittance as the beam propagates. The code EMY computes the prediction of these models.

EMY explicitly solves the envelope equation,

$$R''(z) = \frac{E^2}{R^3(z)} - \frac{\overline{k_B^2 r^2}}{R(z)} \quad (1)$$

where the prime denotes differentiation with respect to z , the propagation direction. The emittance, E , changes from its initial value according to the model dependent equation,

$$E^{2'} = -2\alpha \left(\frac{\overline{k_B^2 r^2} R^2 R''}{E/R + \overline{k_B^2 r^2} R/E} \right) . \quad (2)$$

In equations (1) and (2), the quantity,

$$\overline{k_B^2 r^2} \propto I_{\text{BEAM}} , \quad (3)$$

is held constant. The quantity, α , is a profile dependent constant which ranges from 0.0 for a beam with flat profile to ~ 0.5 for beams with rounded

¹ L. Lee and S. Yu, "Model of Emittance Growth in a Self-pinchd Beam."

profiles. For small perturbations away from equilibrium the beam envelope will oscillate with variations in scale radius, R , being determined by ;

$$\delta R \propto e^{-i\Omega z} \quad (4)$$

where

$$\Omega = \left[-i \sqrt{\frac{\alpha^2}{8}} \pm \sqrt{1 - \frac{\alpha^2}{8}} \right] \Omega_0 \quad (5)$$

$$\Omega_0 = \sqrt{2 \frac{k_\beta^2 r^2}{R_0}} \quad (6)$$

Thus α will set the frequency of radial oscillation, the damping rate, and the degree of macroscopic emittance growth. For the model described in Ref. 1 the values of α for various beam profiles is given in Table 1.

TABLE 1	
Beam Spatial Profile	α
Flat	0.00
Parabolic	0.15
Gaussian	0.38
Truncated Bennett (at 2 Bennett radii)	0.42

The code uses an iterative center differencing scheme to solve a set of coupled finite difference equations:

$$E_{i+1}^2 = E_i^2 - \frac{2\alpha \left(\overline{k^2 r^2} \right) R_{i+1/2}'' R_{i+1/2}^2}{\frac{E_i + 1/2}{R_i + 1/2} + \frac{R_i + 1/2}{E_i + 1/2} \left(\overline{k_\beta^2 r^2} \right)} \Delta z \quad (7)$$

$$R'_{i+1/2} = R'_{i-1/2} + \left(\frac{E_i^2}{R_i^3} - \frac{k_\beta^2 r^2}{R_i} \right) \Delta z \quad (8)$$

$$R''_i = (R'_{i+1/2} - R'_{i-1/2}) \Delta z \quad (9)$$

where Δz is the z-step size.

The user is asked to specify the number of steps and the final value of z from which Δz is calculated. Initial values of R , R' , and E as well as the constants $(k_\beta^2 r^2)$ and α are user specified.

The code produces output in two forms: an ASCII file, RJETT, which can be sent out to a printer and a file for the LLL graphics package FR80. The values of R^2 , R' , R'' , and E^2 are written in RJETT at each z -step. The FR80 file contains plots of R^2 , R' , R'' , and E^2 vs. z . Quantities in both files are labeled as follows:

RAV2 = R^2
 DR/DZ = R'
 DZR/DZ2 = R''
 EMIT2 = E^2
 K**2*R**2 = $k_\beta^2 r^2$
 V0 = $R'(z = 0)$
 ZZ = Z stepvalue

DESTIN - A Particle Pushing Simulation.

A model independent history of beam profile and emittance evolution is computed by a fast-running particle simulation code, DESTIN. The simulation results provide the basis for verifying the phenomenological predictions as computed in EMY.

The particles occupy a disk of infinitesimal thickness, being distributed according to a specified radial probability density function, $n(r)$. The disk is propagated forward along the longitudinal direction, z , at a constant velocity. The only force acting on the particles is that arising from the particles' self-fields. This force is given in terms of a betatron frequency, k_β^2 , in a quasi-frozen field approximation.

The code follows the motion of $N(\leq 5000)$ particles (index j) according to the equations of motion.

$$\frac{dr_j}{dz} = v_j, \quad (10a)$$

$$\frac{dv_j}{dz} = -k_\beta^2(r_j, z) r_j. \quad (10b)$$

In practice the particle positions r_j and velocity v_j are handled in Cartesian coordinates. Thus the assumed azimuthal symmetry is not exploited in the particle pusher. The force, $-k_\beta^2 r_j$, is a function of particle position and beam size whose functional form and initial strength and scale is specified by the user. According to user choice k_β^2 may be strictly frozen or may scale with changes in an appropriate beam scale radius. Generally k_β^2 is chosen by integrating Ampere's law,

$$\left(\frac{1}{r}\right) \frac{d}{dr} r F_r = \frac{4\pi}{c} J_B \quad (11a)$$

with

$$F_r = -k_\beta^2 r = -\frac{q}{\beta \gamma m c^2} \frac{2I_r}{rc} \quad (11b)$$

where I_r is the current within a radius r for the chosen initial radial current distribution. The user is free, however, to choose forces not derived from Eq. (11). Abandoning the self-consistent field approach, which would require the integration of Eq. (2) at each z -step, leads to the major saving in computational time. Instead, the user can scale the magnitude of the originally chosen force by using the appropriate function of the mean square radius in place of the initial scale radius appearing in the integral of Eq. (2).

The validity of this approach may be checked in two ways: first, by comparison of the results of the frozen field calculation with those from a self-consistent field calculation; second, by computing a diagnostic which measures the deviation of the actual profile from the assumed profile. The code employs the second method, although a self-consistent field version, DESTINSF, has been written for use in a limited number of cases.

The radial profile of the beam is described by a probability density function $n(r)$ where r is a radial coordinate. The user may select the initial form of $n(r)$ from four choices: They are flat, parabolic, Gaussian, and Bennett (cut off at 2 Bennett radii), distributions. The scale radii of the distributions, A , are the cutoff radii for the flat and parabolic cases; the rms radius for the Gaussian case, and the Bennett radius for the Bennett profile. Specifically, the density functions are

(Flat)

$$\begin{aligned} n(r) &= \frac{1}{\pi A^2} & (r \leq A) \\ &= 0 & (r > A) \end{aligned} \quad (12a)$$

(Parabolic)

$$n(r) = \frac{2}{\pi A^2} \left(1 - \frac{r^2}{A^2} \right), \quad (12b)$$

(Gaussian)

$$n(r) = \frac{1}{\pi A^2} e^{-r^2/A^2}, \quad (12c)$$

(Bennett)

$$n(r) = \left(\frac{1}{\pi A^2} \right) \frac{1}{(1 + r^2/A^2)^2}. \quad (12d)$$

The particles are distributed uniformly in azimuth around the beam axis.

The particles are given initial directed velocities so as to converge ballistically at some specified value of z . The magnitude of the initial random velocities may be chosen in three different ways: (1) isothermal with V_{thermal} set by the user; (2) consistent with the radial distribution so as to give a beam in equilibrium; (3) in the form of case 2 with the magnitude of all velocities scaled by a user-selected multiplier. The procedure for computing the velocity distribution consistent with a given radial profile has been described by E. Lee² and is computed in the subroutine VDIST. This

² E. P. Lee, "Velocity Distribution in a Pinched Beam" (UCID-18303)

routine assigns to a particle at radial position r a thermal velocity of magnitude given by the following expressions:

(Flat)

$$v = \left[2 \overline{k_{\beta}^2} r^2 (1 - r^2/A^2) \right]^{1/2}, \quad (13)$$

(Parabolic)

$$v = \left[2 \overline{k_{\beta}^2} r^2 \left[\frac{3}{2} \left(\frac{s^4}{A^4} - \frac{r^4}{A^4} \right) + \pi (s^2 n(s) - r^2 n(r)) \right] \right]^{1/2}, \quad (14a)$$

with

$$s = A \left(\rho + (1-\rho) \frac{r^2}{A^2} \right)^{1/2}, \quad (14b)$$

(Gaussian)

$$v = \left[2 \overline{k_{\beta}^2} r^2 \left[\ln \left(1 + \frac{A^2}{r^2} \ln(1-\rho) \right) + \text{Ei} \left(-\frac{r^2}{A^2} \right) - \text{Ei} \left(-\frac{r^2}{A^2} + \ln(1-\rho) \right) \right] \right]^{1/2} \quad (15a)$$

where

$$\text{Ei}(x) = \int_{-\infty}^x dz \frac{e^z}{z} \quad (15b)$$

for $x < 0$,

(Bennett)

$$v = \left[-\overline{k_{\beta}^2} r^2 \ln(1-\rho) \right]^{1/2}. \quad (16)$$

In equations 14, 15, and 16 ρ is a random variable uniformly distributed in the interval $[0,1]$. The direction of the thermal velocities is assigned uniformly distributed over $[0,2\pi]$.

At each z step the code computes the following diagnostics; mean square radius, R^2 ; mean square velocity, V^2 ; emittance squared, E^2 ; the average value of $k_\beta^2 r^2$. The emittance is defined by the macroscopic quantity,

$$E^2 = R^2 \left(V^2 - \left(\frac{dR}{dz} \right)^2 \right) . \quad (17)$$

The emittance so defined is not identical with the phase space area of the beam, which by Liouville's theorem is conserved in the absence of scattering. In fact, this macroscopic quantity can decrease as the beam undergoes radial oscillation. Every n_{st} step, the code computes the radial probability density function in histogram form. If the force has been chosen consistent with the radial distribution,

$$\left. \overline{k_\beta^2 r^2} \right|_{z=0} = \left(\frac{I}{I_A} \right) = k_0^2 \quad (18)$$

where k_0^2 is a user specified quantity. If the force selected varies in scale with the beam scale radius as the disk advances in z , then the quantity

$$\overline{k_\beta^2 r^2} - k_0^2 = S^2 \quad (19)$$

is a measure of the variation of beam profile as the beam propagates.

The code produces output in two forms: an ASCII file, RJETT, which can be sent out to a printer and a FR80 graphics file. The diagnostics are written in RJETT at each zstep along with initial and final particle distributions. In the FR80 file plots of the diagnostics along with the intermediate distribution functions and phase space plots are given. Quantities in both output files are labeled as follows:

ZMAX = Maximum value of Z

RAV2 = Mean square radius = R^2

VAV2 = Mean square velocity = V^2

EMIT2 = Emittance squared = E^2

EFBAR = $\overline{k_{\beta}^2 r^2}$

ZZ = Z step value

Figure 1

Equilibrium Beam Behaviour

As a first example of the output of the simulation, consider the case of beam with Gaussian radial profile injected with an equilibrium velocity distribution. Initial beam parameters are

$$\begin{aligned} R_0^2 &= 2 \\ \frac{R_0^2}{k_\beta^2 r^2} &= 1. \\ dR/dz &= 0. \end{aligned} \tag{20}$$

Equilibrium requires that the emittance be given by

$$E_0^2 = R_0^2 \overline{k_\beta^2 r^2} \tag{21}$$

The simulation used 2000 particles and 2000 Z-steps and followed the particles many betatron wavelengths. Figure 1a shows the computed value of R^2 vs z while Fig. 1b shows E^2 vs z . Deviations from constancy in E^2 and R^2 are most likely the effect of the few particles at large distances upon the diagnostics. For the case of a beam with a flat profile such fluctuations do not appear.

Beams Launched Near Equilibrium

Comparison of the point model and the simulation can be made by considering a beam launched with an emittance slightly greater than that required by the equilibrium condition, Eqn. (21). Again the beam initial conditions are given by Eqn. (20). The initial emittance is 20% larger than that for equilibrium, $E^2 = 3$. For a beam injected at a neck with Gaussian profile, $\alpha = 0.38$. The predictions of the point model as computed by EMY appear in Fig. 2a, b.

The particle simulation of this case considers 800 particles with a Gaussian radial profile. Each particle is assigned a thermal velocity 20% greater than that given by Eqn. (15). The beam is followed for several betatron wavelengths with 200 Z steps. The fields at each step are assumed to be quasi-frozen; the force always has the form.

$$F_r = \frac{2 k_B^2 r^2}{r} \left(e^{-r^2/R^2} - 1 \right) \quad (22)$$

where R^2 is the actual mean square radius. The results of the DESTIN calculation, which appear in Fig. 2c, d, are in remarkable agreement with those of the point model. The deviation of the value of $k_B^2(r,z) r^2(z)$ from its initial value of unity was less than a few percent in agreement with the assumption of approximate profile invariance.

Nonetheless, one cannot have strict profile invariance, since self-similar beam oscillations do not damp or give rise to emittance growth. As a final check one can repeat the simulation with a self-consistent field calculation at each Z-step. Running DESTINSF with the same initial conditions as DESTIN

yielded the results of Fig. 2e, f. Again agreement among the three sets of computations is excellent although noticeable differences are present especially in the emittance diagnostic.

An estimate of the final values of R^2 and E^2 also obtains from the conservation of the total transverse energy of the self-pinch beam.³ The conserved energy integral at a position z is given by

$$W = \frac{V^2(z)}{2} - \overline{k_B^2(r,z) r^2(z)} \left[\log \left(\frac{b}{R(z)} \right) - \frac{C(z)}{2} \right] . \quad (23)$$

In Eqn. (23), b is an arbitrary radial integration limit and C is a profile dependent, dimensionless parameter. At equilibrium, $R' = 0$ and Eqn. (23) is supplemented by

$$E^2 = R^2 V^2 \quad (24a)$$

and

$$E^2 = R^2 \overline{k_B^2 r^2} . \quad (24b)$$

Thus Eqn. (23) becomes at equilibrium

$$W = \overline{k_B^2 r^2} \left[\frac{1}{2} + \frac{C}{2} - \log \frac{b}{R} \right] . \quad (25)$$

Consistent with the simulation results, one may assume that C is nearly constant for the example under consideration. Then, the final radius, R_f , is given by

³E. Lee, "Determination of the Radius of a Self-Charged Particle Beam from its Energy Integral," (UCID 18495).

Figure 2

$$\log \frac{R_f}{R_0} = \frac{E_0^2}{2R_0^2 k_B^2 r^2} - \frac{1}{2} . \quad (26)$$

For the example $E_0^2 = 2.97$, $R_0^2 = 2$ and $k_B^2 r^2 = 1$; therefore

$$R_f^2 = 1.65 R_0^2 = 3.25; \quad (27a)$$

The final emittance would be

$$E_f^2 = R_f^2 k_B^2 r^2 = 3.25. \quad (27b)$$

Not surprisingly, the results of the frozen field calculation give the best agreement with Eqn. (27) which follow from the assumption of an invariant profile form.

Cold Converging Beam

As a final example consider the case of a very cold, converging beam such as one might have in a heavy ion fusion application. The initial beam mean square radius is $R_0^2 = 200$. Initial directed velocities are chosen so that the beam would converge to a point at $Z = 1000$ in the absence of self-forces. The initial beam profile is Gaussian. In the case of initial emittance equal to zero, the analysis of Ref. 1 predicts that the emittance will grow according to

$$E^2 = \left(\frac{1}{\alpha k_B^2 r^2} \right)^2 z . \quad (28)$$

To avoid computational difficulties DESTIN starts the beam with a finite

but small emittance. In this example DESTIN followed the motion of 2000 particles in 400 Z-steps. The initial values were

$$\begin{aligned} R_0^2 &= 200 \\ E_0^2 &= 2 \times 10^{-8} \\ \frac{E_0^2}{k_B^2 r^2} &= 10^{-4} \end{aligned} \quad (29)$$

Initial thermal velocities were loaded with a distribution of magnitude given by Eqn. (15) scaled by a factor of 10^{-3} . As seen in Fig. 3a, the beam reaches a minimum radius of ~ 1.14 at $z = 730$. However, the beam profile becomes significantly different from Gaussian for z greater than 5.5 as Fig. 3b indicates. Hence the results of the quasi-frozen field computation must be considered suspect for $z > 550$. The emittance evolution is pictured in Fig. 4. The points are sample values from DESTIN and the line is the prediction of Eqn. (28). After a region of initial disagreement due to the finite starting emittance in the simulation, the simple cold beam model predicts the growth in E^2 over three orders of magnitude.

Conclusion

Simulations of the behaviour of self-pinch beams based upon elementary physical assumptions supports the predictions of the phenomenological models Ref. 1. That heuristic relationships such as Eqn. (2) describe the simulations suggest that they derive from a few fundamentals. Conversely, the agreement suggests applicability of "stripped-down" simulations such as DESTIN to a variety of interesting regimes of intense beam behaviour.

Figure 3

Figure 4