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EQUILIBRIA BY THE ITERATIVE METHOD

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APPLICATION OF THE IMPLICIT FOURIER-EXPANSION METHOD
TO THE CALCULATION OF THREE-DIMENSIONAL
EQUILIBRIA BY THE ITERATIVE METHOD*

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ABSTRACT

The iterative method of finding solutions to three-dimensional equilibria is discussed. The implicit Fourier-expansion method is briefly described and applied to the linear problems arising in the iterative loops. The paper shows how to efficiently solve for the magnetic field induced by the plasma.

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1. INTRODUCTION

In calculating equilibria for stellarator or HELIAC geometries, one solves the equation

$$\nabla p = \frac{1}{c} \bar{J} \times \bar{B} \quad , \quad (1)$$

where p is the scalar plasma pressure, $\bar{B}$ the magnetic induction, and $\bar{J}$ the current density. One possible approach to solving Eq. (1) is the iterative method in which successive approximations to $\bar{B}$ are constructed. The algorithm is attractive since no assumptions are made regarding the existence of nested toroidal surfaces. At each stage of the calculation, one solves linear problems. The process constructs a β -correction to a previously computed field, where β is the plasma beta. Thus, after only a few iterations, one obtains a reasonable approximation to the desired field. In particular, the possible break-up of the magnetic surfaces with increasing beta can be observed.

The iterative method has a long history. In the 1950's, Kruskal and Koenig [1] and Grad and Rubin [2] discussed it. More recently, Boozer [3] considered it by expressing the field using magnetic coordinates. Yoshikawa [4] suggested that the Implicit Fourier-expansion method (to be discussed below) might be used to efficiently compute the field induced by a given current distribution. This last is an important step in the iterative process.

The iterative step begins by assuming the existence of a known field $\bar{B}_k$. Next, a magnetic surface label ψ_k is constructed such that

$$\bar{\mathbf{B}}_k \cdot \nabla \psi_k = 0 \quad . \quad (2)$$

Although there is freedom in the choice for ψ_k , we assume that constant ψ surfaces coincide with constant pressure surfaces i.e., $p_k = p(\psi_k)$. Naturally, if a field line fills out a three-dimensional volume, then ψ_k is constant over that volume. The label ψ_k can be equivalenced to the toroidal flux contained by the surface. Solutions to Eq. (2) are discussed in Section 3. The plasma pressure profile, $p(\psi)$, is assumed to be given.

The next step computes a current density which satisfies,

$$\nabla p_k = \frac{1}{c} \bar{\mathbf{J}}_k \times \bar{\mathbf{B}}_k \quad , \quad (3)$$

where $p_k = p(\psi_k)$. The computation of $\bar{\mathbf{J}}_k$ is described in Section 4. The iterative loop is closed when a new field satisfying

$$\nabla \times \bar{\mathbf{B}}_{k+1} = \frac{4\pi}{c} \bar{\mathbf{J}}_k \quad (4)$$

and

$$\nabla \cdot \bar{\mathbf{B}}_{k+1} = 0 \quad (5)$$

is computed. The implicit Fourier-expansion method which is briefly described in Section 2 is used to solve Eq. (4) (see Section 5).

2. IFEM

The Implicit Fourier-Expansion Method (IFEM) is discussed by Shestakov and Mirin [5]. They compute the vacuum magnetic field, $\bar{\mathbf{B}}_{\text{VAC}}$, inside an asymmetric torus which satisfies:

$$\bar{B}_{VAC} \cdot \hat{n} = 0 \quad (6.1)$$

and

$$\int_s \bar{B}_{VAC} \cdot d\bar{S} \neq 0 \quad (6.2)$$

where $\hat{n}$ is the unit surface normal and s is a poloidal cross-section. The authors show that if

$$\chi = \chi_1 + I_p \phi / 2\pi \quad , \quad (7)$$

then

$$\bar{B}_{VAC} = \nabla \chi \quad , \quad (8)$$

where ϕ is the toroidal angle, χ and χ_1 are scalar functions and I_p is the net poloidal current. The potential χ_1 is periodic in ϕ . Furthermore, the assumption $\bar{B}_{VAC} = 0$ exterior to the torus implies that equipotential contours of χ on the surface coincide with surface currents that generate the field [6]. These currents may be discretized and generate modular coil configurations for HELIAC [7]. Since χ and χ_1 are harmonic functions, the relevant equation to solve is Laplace's equation with a Neumann boundary condition.

The principle behind the IFEM is simple. Assume that the equation to solve is $a \chi = b$ where a and b are known functions. If all the functions are periodic, they can be represented by Fourier series. Let $\bar{a}$ and $\bar{b}$ respectively represent the vector of harmonics of a and b . Thus, if $a = a_0 + a_1 \cos \phi + a_2 \sin \phi$, $\bar{a} = (a_0, a_1, a_2)$. A linear system, $A\bar{\chi} = \bar{b}$, is then solved for the harmonics of χ where A is a matrix,

$$A = \begin{bmatrix} a_0 & \frac{1}{2}a_1 & \frac{1}{2}a_2 \\ a_1 & a_0 & 0 \\ a_2 & 0 & a_0 \end{bmatrix} .$$

Details on the generation of the matrix A from the vector $\bar{a}$ are found in [5]. It is straightforward to generalize the above when there is more than one independent angular variable [5].

In the problem considered by Shestakov and Mirin, the three independent variables are (σ, ϕ, ζ) where ϕ and ζ respectively denote toroidal and poloidal angles and σ is a scaled radial distance. Derivatives in σ are approximated by finite differences while derivatives in ϕ or ζ are expressed by means of permutation matrices P_ϕ, P_ζ respectively [5]. Thus,

$$\frac{\partial}{\partial \phi} (a\chi) = P_\phi A \bar{\chi} \quad \text{and} \quad a \frac{\partial \chi}{\partial \phi} = A P_\phi \bar{\chi} . \quad (9)$$

A second order partial differential equation can be expressed in block tridiagonal form; the tridiagonal part arises from the three-point approximation to the differentiation in σ . The resolving linear system is then solved by standard Gaussian elimination [8].

The IFEM may also be used to obtain a global description of $\bar{B}_{VAC}$, the vacuum field due to a known coil configuration [9]. In the geometry of interest, inside the vacuum vessel, $\bar{B}_{VAC}$ may be expressed as in Eqs. (7) and (8). The potential χ is unique up to an additive constant. In equilibrium calculations, the domain of interest, D , will remain inside the vacuum vessel. Once this domain is specified, χ may be arbitrarily initialized (to zero) at some point, a_0 , on the boundary of D , ∂D . The potential is then given on the entire boundary by integrating,

$$\chi(\bar{x}) - \chi(a_0) = \int_{a_0}^{\bar{x}} \bar{B}_{VAC} \cdot d\bar{x} \quad (10)$$

over ∂D . In Eq. (10), $\bar{B}_{VAC}$ is computed by standard methods such as the Biot-Savart law. Once χ is known over the entire ∂D (on a grid), Fourier transforming gives the harmonics of χ (on ∂D). This results in a Dirichlet problem for χ over the domain D . The field $\bar{B}_{VAC}$ is then known everywhere in D by computing $\nabla\chi$. The advantage of this approach is that the time-consuming Biot-Savart integrations need be done only once, at the two-dimensional surface enclosing D .

In the following, the IFEM is applied to the iterative method discussed in Section 1. The coordinates are as in [5] (see Fig. 1). The transformation to cylindrical coordinates (R, ϕ, Z) is given by:

$$\begin{aligned} R &= R_0 + \rho \cos \theta(\phi) + \sigma g(\zeta) \cos(\theta(\phi) + \zeta) \\ \phi &= \phi \\ Z &= \rho \sin \theta(\phi) + \sigma g(\zeta) \sin(\theta(\phi) + \zeta) \end{aligned} \quad (11)$$

where

R_0 = major radius

ρ = winding radius

$g(\zeta)$ = boundary description

$\theta(\phi)$ = winding law

are all assumed to be given. Both HELIAC and stellarator geometries may be described by a proper choice of the above parameters. In the following, the domain D consists of:

$$D = \{(\sigma, \phi, \zeta): 0 \leq \sigma \leq 1, 0 \leq \phi \leq 2\pi/m_h, 0 \leq \zeta \leq 2\pi\} \quad (13)$$

where m_h denotes the number of helical periods in the torus. It is assumed that D is entirely contained in the vacuum vessel and contains the plasma torus surrounded by a vacuum region.

3. CALCULATION OF MAGNETIC SURFACES

In this section, we briefly discuss the solution of Eq. (2) for the scalar function ψ_k . The field $\bar{B}_k$ is a sum of the contributions from the coils and the plasma,

$$\bar{B}_k = \bar{B}_{VAC} + \bar{B}_{k,p} \quad (14)$$

Section 2 describes how to evaluate $\bar{B}_{VAC}$. In Section 5, we discuss how to determine the three Cartesian components of the vector potential that determines $\bar{B}_{k,p}$.

$$\bar{B}_{k,p} = \nabla \times \bar{A}_k \quad (15)$$

Thus, $\bar{B}_k$ can be evaluated at any point in D .

Equation (2) states that $\psi_k = \text{constant}$ defines a field line. It is assumed that $\psi_k = \text{toroidal flux}$. The usual means [3] of obtaining ψ_k is to choose different points in the poloidal plane and integrate along field lines thereby sweeping out the surfaces. The surfaces are then labeled by the enclosed amount of toroidal flux. A splining or averaging process may then be done in order to evaluate ψ_k at a point which does not lie on one of the surfaces.

The IFEM discussed in Section 2 may also be used to compute ψ_k . Recall that $\sigma = \text{constant}$ surfaces are nested tori. The field line integration method is used to compute ψ_k at two neighboring σ surfaces, σ_i, σ_{i-1} . Two calls to a Fourier-transform routine give $\bar{\psi}_{k,i}$ and $\bar{\psi}_{k,i-1}$, the vectors of harmonics (in ϕ, ζ) of ψ_k at the two surfaces. Equation (2) is rewritten in (σ, ϕ, ζ) coordinates:

$$B_k^\sigma \frac{\partial \psi}{\partial \sigma} + B_k^\phi \frac{\partial \psi}{\partial \phi} + B_k^\zeta \frac{\partial \psi}{\partial \zeta} = 0 \quad (16)$$

where $B^\sigma, B^\phi, B^\zeta$ are the contravariant components of B_k , e.g., $B_k^\sigma = \bar{B}_k \cdot \nabla \sigma$. Suppressing the subscript k , the IFEM is applied to Eq. (16):

$$B_i^\sigma (\bar{\psi}_{i+1} - \bar{\psi}_{i-1}) + 2\Delta\sigma_i [B_i^\phi P_\phi + B_i^\zeta P_\zeta] \bar{\psi}_i = 0 \quad (17)$$

This is a linear system of equations; the matrices B_i^σ, B_i^ϕ and B_i^ζ contain the ϕ and ζ harmonics of the contravariant components of $\bar{B}$ at $\sigma = \sigma_i$. The permutation matrices P_ϕ, P_ζ post-multiply B_i^ϕ and B_i^ζ respectively and correspond to the ϕ and ζ derivatives of ψ in Eq. (16) (see [5]). Equation (17) can be easily solved for the unknown vector $\bar{\psi}_{i+1}$. This process is repeated for increasing values of i . After several steps, one obtains ψ over the entire range of σ desired.

Each solution of Eq. (17) involves inverting $B_i^\sigma (= \bar{B} \cdot \nabla \sigma \text{ at } \sigma = \sigma_i)$. The case where $B^\sigma = 0$ implies that σ is a magnetic surface. It remains to calculate $\psi = \psi(\sigma)$, which is done by computing the toroidal flux enclosed by the surface in question.

The procedure monitors the quantity $\bar{B} \cdot \nabla \sigma$. Depending on how much this differs from zero, the next vector $\bar{\psi}_{i+1}$ is computed by solving Eq. (17) or computing ψ at σ_{i+1} by measuring the toroidal flux.

The case where a field line ergodically fills out a three-dimensional volume is not excluded by the above procedure. Ergodicity implies that $\psi =$ constant over the volume. If several σ -surfaces are contained in the volume, each of the vectors $\bar{\psi}$, are equal. Equation (17) still holds; its solution will be $\bar{\psi}_{i+1} = \bar{\psi}_{i-1}$. If the σ -surfaces are not contained in the "ergodic" volume, Eq. (17) can still be solved for the vector $\bar{\psi}_{i+1}$. If enough terms are carried in the Fourier series, evaluation of the series gives the same value for ψ throughout the volume. In this case, $\bar{B} \cdot \nabla \sigma = B^\sigma$ cannot vanish on the entire σ -surface.

4. CALCULATION OF THE CURRENT

Once the surfaces are labeled, the plasma pressure, $p_k = p(\psi_k)$, can be computed. The next step in the iterative loop is to compute the new current which satisfies Eq. (3). This current is given as the sum of the parallel and perpendicular components, $J_{||}$ and $J_{\perp}$, along the field B_k . Equation (4) implies

$$\underline{J}_{\perp} = c(\bar{B}_k \times \nabla p_k) / |B_k|^2 \quad (18)$$

Divergence ($\bar{J}$) = 0 is used to solve for $J_{||}$. If $\hat{b}_k$ is the unit vector along $\bar{B}_k$ and B_k the magnitude, then $\bar{B}_k = B_k \hat{b}_k$ and $\bar{J}_{||} = J_{||} \hat{b}_k$ where $J_{||}$ is a scalar quantity. It follows that

$$\begin{aligned} -\nabla \cdot \underline{J}_{\perp} &= \hat{b}_k \cdot \nabla J_{||} - (J_{||}/B_k) \hat{b}_k \cdot \nabla B_k \\ &= \bar{B}_k \cdot \nabla (J_{||}/B_k) \quad (19) \end{aligned}$$

The left hand side of this equation is computed using Eq. (18):

$$-\nabla \cdot \underline{J}_\perp = -\frac{c}{B_k^2} \nabla p_k \cdot (\nabla \times B_k) - \frac{2}{B_k} \underline{J}_\perp \cdot \nabla B_k \quad . \quad (20)$$

Equation (19) is of the type considered by Newcomb [10]. Applying those results we obtain a necessary and sufficient condition for the existence of $J_{||}$:

$$\oint \frac{1}{B_k} (\nabla \cdot \underline{J}_\perp) d\ell = 0 \quad (21)$$

where the integration is taken over any closed line of force, i.e., on surfaces with rational transform.

The solution of Eqs. (19) and (20) for $J_{||}$ may be obtained by either of two methods. The first uses the fact that Eq. (19) is a first order o.d.e. along a field line. If ℓ is length along B_k ($\hat{b}_k \cdot \nabla B_k = \partial B_k / \partial \ell$), the $J_{||}$ equation is:

$$\frac{\partial J_{||}}{\partial \ell} - \frac{1}{B_k} \frac{\partial B_k}{\partial \ell} J_{||} = -\nabla \cdot \underline{J}_\perp \quad (22)$$

where the r.h.s. is given by Eq. (20). This equation may be solved on each magnetic surface (care is taken to avoid surfaces with rational transform). The integration is begun at $\phi = 0$ and continues until enough transits have been made to sweep out the surface. The result is linear in the initial condition which is left unspecified:

$$J_{||}(\ell) = g_1(\ell) J_{||}(0) + g_2(\ell) \quad . \quad (23)$$

A trivial way to explicitly compute $g_1(\ell)$ and $g_2(\ell)$ is to do the integration twice, once with the initial condition set to zero, the second with it set to one. The arbitrary constant $J_{||}(0)$ is obtained by other constraints.

Commonly, the net toroidal current is assumed to vanish on each magnetic surface:

$$0 = \oint_{\psi_a} (\bar{J} \cdot \nabla \phi) d\ell_p \quad (24)$$

The contour integral is taken along the intersection of the ψ_a surface with the $\phi = 0$ plane and $d\ell_p$ is the incremental length in the poloidal plane.

This constraint defines the initial condition,

$$J_{||}(0) = - \left(\oint_{\psi_a} [\bar{J}_1 + g_2(\ell) \hat{b}] \cdot \nabla \phi d\ell_p \right) / \left(\oint_{\psi_a} g_1(\ell) (\hat{b} \cdot \nabla \phi) d\ell_p \right) \quad (25)$$

The iteration subscript, k , is suppressed in the above equation, $g_1(\ell)$ and $g_2(\ell)$ are the functions in Eq. (23) which are now evaluated at $\phi = 0$.

The above procedure gives $\bar{J}_k$ at any magnetic surface.

The alternate approach is to first compute $J_{||}$ as above but only along two innermost neighboring σ -surfaces, σ_1 and σ_0 ($\sigma_0 = 0$). Two calls to Fourier transform give $\bar{J}_0$ and $\bar{J}_1$ the vector of harmonics of $J_{||}$ at the two surfaces. Equation (19) is then discretized as in Section 3 using IFEM. A linear system results for $\bar{J}_2$ which can be solved if $B_{k,i}^\sigma \neq 0$; $\bar{J}_i$ for larger values of i is obtained inductively. The case $B_{k,i}^\sigma = 0$ is handled as in Section 3. The result of this calculation gives a vector of harmonics of $J_{||}$ in the (σ, ϕ, ζ) coordinates.

5. SOLUTION OF AMPERE'S LAW

Once the current distribution is known, Eq. (4) needs to be solved for the induced field $\bar{B}_{k+1}$. As discussed in Section 3, the field $\bar{B}_{k+1,p}$, due to the plasma currents is expressed in terms of a vector potential,

$$\bar{B}_{k+1,p} = \nabla \times \bar{A}_{k+1} \quad (26)$$

The Cartesian components of $\bar{A}_{k+1}$ are given by the vector equation [11]:

$$\bar{A}_{k+1}(\bar{x}) = \frac{1}{c} \int \frac{\bar{J}_k(\bar{x}')}{|\bar{x} - \bar{x}'|} d^3 x' \quad (27)$$

The integral is taken over the entire plasma torus. Although the integral in Eq. (26) is quite time-consuming to evaluate for all $\bar{x}$ inside D, it needs to be computed only for $\bar{x}$ lying on the outermost σ -surface ($\sigma = 1$). Fourier transforming these values gives a vector of harmonics.

The IFEM is utilized to solve the Poisson equation for the three Cartesian components $A_{k,i}$, $i = x, y, z$. Since the gauge $\nabla \cdot \bar{A} = 0$ is incorporated in Eq. (27), the relevant equations are

$$\nabla^2 A_i = - \frac{4\pi}{c} J_i \quad (28)$$

where the iterative index k has been suppressed. Boundary conditions for Eq. (28) are obtained from Eq. (27).

The numerical solution of Eq. (28) proceeds as in Reference [5]. The Laplacian operator is written in the (σ, ϕ, ζ) coordinates:

$$\begin{aligned} \frac{\partial}{\partial \sigma} \left(a \frac{\partial A}{\partial \sigma} + b \frac{\partial A}{\partial \phi} + c \frac{\partial A}{\partial \zeta} \right) + \frac{\partial}{\partial \phi} \left(b \frac{\partial A}{\partial \sigma} + d \frac{\partial A}{\partial \phi} + e \frac{\partial A}{\partial \zeta} \right) \\ + \frac{\partial}{\partial \zeta} \left[c \frac{\partial A}{\partial \sigma} + e \frac{\partial A}{\partial \phi} + f \frac{\partial A}{\partial \zeta} \right] = -4\pi J / (c \sigma g(\zeta)^2 R) \end{aligned} \quad (29)$$

where

$$\begin{aligned} a &= \sigma \left\{ 1 + \left[\frac{g'(\zeta)}{g(\zeta)} \right]^2 \right\} R + \sigma \left[\frac{V(\sigma, \zeta) \theta'(\phi)}{g(\zeta)} \right]^2 \left(\frac{1}{R} \right) \\ b &= \sigma V(\sigma, \zeta) \theta'(\phi) / R \\ c &= -R g'(\zeta) / g(\zeta) - \left(\sigma + \frac{\rho \cos \zeta}{g(\zeta)} \right) \left(\frac{V(\sigma, \zeta) \theta'(\phi)^2}{R} \right) \\ d &= \sigma g(\zeta)^2 / R \\ e &= -g(\zeta) [\sigma g(\zeta) + \rho \cos \zeta] \theta'(\phi) / R \\ f &= R / \sigma + \theta'(\phi)^2 (\sigma g(\zeta) + \rho \cos \zeta)^2 / (R \sigma) \\ V(\sigma, \zeta) &= \frac{\partial}{\partial \zeta} [\sigma g(\zeta)^2 / 2 + \rho g(\zeta) \cos \zeta] \end{aligned} \quad (30)$$

and R is given by Eq. (11).

Since Eq. (23) is to be solved for $\sigma < 1$, and since the coordinate system (σ, ϕ, ζ) is singular at $\sigma = 0$, another equation is required. If Eq. (29) is multiplied by $\sigma g(\zeta)^2 R$, and integrated over $0 \leq \sigma \leq \delta$, $0 \leq \zeta \leq 2\pi$, an integral relationship is obtained. Using an analysis similar to that in Ref. [5] we obtain:

$$\begin{aligned} \int_0^{2\pi} d\zeta \left\{ \left[R_1 g(\zeta)^2 a - \delta c \frac{\partial R_2}{\partial \zeta} \right] \frac{\partial A}{\partial \sigma} \right. \\ \left. + \left[R_1 g(\zeta)^2 b - \delta e \frac{\partial R_2}{\partial \zeta} \right] \frac{\partial A}{\partial \phi} \right. \\ \left. - \left[R_1 \frac{\partial}{\partial \zeta} (c g(\zeta)^2) - \delta \frac{\partial}{\partial \zeta} (f \frac{\partial R_2}{\partial \zeta}) \right] A \right\} \end{aligned}$$

$$\begin{aligned}
& + \delta R_2 \frac{\partial}{\partial \phi} \left[b \frac{\partial A}{\partial \sigma} + d \frac{\partial A}{\partial \phi} - \frac{\partial e}{\partial \zeta} A \right] - \delta \frac{\partial R_2}{\partial \zeta} \frac{\partial (eA)}{\partial \phi} \Big|_{\sigma=\delta} \\
& = - \frac{8\pi}{c} \int_0^{2\pi} d\zeta J \Big|_{\sigma=\delta/2} \quad (31)
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= R_0 + \rho \cos \theta(\phi), \text{ and} \\
R_2 &= R g(\zeta)^2. \quad (32)
\end{aligned}$$

In Eq. (31), the integrand on the left is evaluated at $\sigma = \delta$, a small non-zero value, while the integrand on the right at $\sigma = \delta/2$. The coefficients $a, b, \dots, f$ are defined in Eq. (30) and R is given by Eq. (11). Note that in Eqs. (29)-(31) and in the following, both k , the iteration index, and i , the Cartesian component index, have been suppressed.

The coordinate σ is discretized into $L + 2$ values, $0 = \sigma_0 < \sigma_1 < \dots < \sigma_{L+1} = 1$. Proceeding as in [5], the IFEM solves for A_ℓ , the vector of harmonics of A at $\sigma = \sigma_\ell$. For $\ell = 1, 2, \dots, L$, Eq. (29) is converted into a block tridiagonal system of equations:

$$Q_\ell A_{\ell-1} + S_\ell A_\ell + T_\ell A_{\ell+1} = - (4\pi/c) J_\ell \quad (33)$$

where J_ℓ denotes the vector of harmonics of the function $J/(\sigma g(\zeta)^2 R)$ at $\sigma = \sigma_\ell$ and where Q_ℓ, S_ℓ, T_ℓ are matrices whose entries stem from the harmonics of the coefficients $a, b, \dots$ (see [5]). At $\ell = L$, the vector $A_{\ell+1}$ is computed from the respective component of Eq. (27) giving the Dirichlet boundary condition. The vector $T_L A_{L+1}$ is then incorporated into the right hand side of the equation.

To close the system, the discrete analogue of Eq. (31) is derived. Note that this equation does not contain ζ derivatives of A. The unknown A (at $\sigma = \delta = \sigma_1/2$) is approximated by an average, e.g., $A = \frac{1}{2} (A_1 + A_0)$, $\frac{\partial A}{\partial \sigma} = (A_1 - A_0)/\sigma_1$. After all the coefficients are properly computed, IFEM gives

$$\int_0^{2\pi} d\zeta [KA_0 + PA_1] = \int_0^{2\pi} d\zeta J_{\delta/2} \quad (34)$$

where K, P are matrices arising from the coefficients multiplying A in Eq. (31) and $J_{\delta/2}$ is the vector of harmonics corresponding to $-(8\pi/c)J$ at $\sigma = \sigma_1/4$. Proceeding as in [5], we consider only the harmonics independent of ζ in Eq. (34). The integration is then trivial. Since the ζ -harmonics of A vanish at $\sigma = 0$ (to insure single-valuedness), we obtain a modified linear equation:

$$\tilde{K} a_0 + \tilde{P} A_1 = \tilde{J}_{\delta/2} \quad (35)$$

Here a_0 and $\tilde{J}_{\delta/2}$ are the subvectors corresponding to A_0 and $J_{\delta/2}$ respectively and contain only the non- ζ harmonics, $\tilde{K}$ is a square matrix, $\tilde{P}$ is rectangular. The above restates that at $\sigma = 0$ the ζ -harmonics of A vanish, hence need not be computed. Equation (35) is properly combined with Eq. (33) (for $\ell = 1$) and the a_0 unknown is eliminated giving the modified equation:

$$(S_1 - \tilde{Q}_1 \tilde{K}^{-1} \tilde{P}) A_1 + T_1 A_2 = - \left(\frac{4\pi}{c}\right) J_1 - \tilde{Q}_1 \tilde{K}^{-1} \tilde{J}_{\delta/2} \quad (36)$$

where $\tilde{Q}_1$ is a rectangular modification of Q_1 .

The above procedure gives a large block-tridiagonal system of equations for the L unknowns A_ℓ , $\ell = 1, 2, \dots, L$. It can be solved by the routines in

Ref. [8]. After this system is solved, for all three Cartesian components of $\bar{A}$, the new field $\bar{B}_{k+1,p}$ is obtained by taking the curl. The total field, $\bar{B}_{k+1}$ also contains the contribution from the coils, $\bar{B}_{VAC}$. This closes the iterative loop, the iterative index k is then updated and a new loop is begun by solving Eqs. (2), (3) and (4). The process continues until Eq. (1) is satisfied to some predetermined tolerance. Using experience from [5], it is estimated that solution of the Poisson equation for the components of $\bar{A}$ take 60 seconds of CRAY-1 computer time once the boundary values are computed.

6. SUMMARY AND CONCLUSION

The most time-consuming part of the iterative step is expected to be the solution of Eq. (4). We note that the apparently formidable computation of Eq. (27) is done only for those $\bar{x}$ enclosing the domain of interest (i.e., for $\sigma \approx 1$). This gives the necessary boundary values for the solution of the Poisson equation (28).

The vacuum field, $\bar{B}_{VAC}$, is inexpensive to evaluate for arbitrary coil configurations once it is expressed as in Eq. (8) and the Dirichlet problem for χ solved. For a complicated coil system, the time-consuming calculation will be Eq. (10) for the boundary values. This, however, needs to be solved only once; $\bar{B}_{VAC}$ is then given by the harmonics of χ which can be kept in memory.

There are other possible time and space-saving procedures. The expensive part of solving Eq. (4) is in computing the matrices in Eqs. (33) and (36). Since those depend only on the coordinate system, they need be computed once,

stored on disc and then read in as required. Only the right hand sides of Eqs. (33) and (36) vary from one iterative step to the next. These and other similar time-saving calculations will significantly reduce the time per iteration. It is expected that the majority of computational time will be spent on computing Eq. (27).

The limitations of the IFEM is in the storage requirements. If M ζ -harmonics and N ϕ -harmonics are carried in the solution, for each σ_L a total of MN harmonics need to be stored. The matrices appearing in Eq. (33) are of order MN . The choice for MN is primarily dictated by the coil configuration. Enough harmonics need to be carried to properly describe the vacuum field. For the cases considered in [5], $M \sim 13$, $N \sim 7$ was sufficient.

It is difficult to predict how many iterations are required to reach equilibrium for a specific configuration. However, if after one iteration, the surfaces are no longer nested, or worse, the field becomes ergodic, one may reasonably assume that no equilibrium exists for the parameters chosen. Thus, the proposed method can be used to make a quick survey of a parameter space.

It is estimated that the necessary initialization will require 3-10 minutes of CRAY-1 computer time. Each iterative loop takes 3×60 seconds to compute the Cartesian components of $\bar{A}$, and 2-5 minutes to obtain the boundary values of the components. The computation of the flux surface function, ψ , and the current density should be inexpensive and take no more than two minutes. Hence each iterative step is expected to require approximately 10 minutes. If an equilibrium exists, two or three iterations should give a

reasonable approximation as each iteration perturbs the existing field by increasing powers of beta. Thus, we estimate that 30 minutes is a reasonable upper limit to solve Eq. (1). The iterative method should, therefore, be considered as a candidate for computing, three-dimensional equilibria.

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Figure 1

Figure 1. Definition of the τ , ϕ , ζ coordinates;
 $\sigma = \tau/g(\zeta)$.