ANALYTICAL AND NUMERICAL ANALYSIS OF FINITE AMPLITUDE
RAYLEIGH-TAYLOR INSTABILITY

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"Analytical and Numerical Analysis
of Finite Amplitude Rayleigh-Taylor
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1 October 1987 - 30 September 1988
\$61,718

Gentlemen:

We enclose for your consideration a continuation application (original and 3 copies), prepared by Dr. D. I. Meiron and Dr. P. G. Saffman.

This proposal has not been reviewed and approved by our Faculty Committee on Sponsored Research, but is being mailed today in the interests of timely submission. If you do not hear from us in fifteen days, you may assume that Committee review has been completed and that this proposal has been approved as written.

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Director, Sponsored Research

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Analytical and Numerical Analysis of Finite
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1. Introduction

Funds are requested for continuation of research started June 1st, 1985 on Rayleigh-Taylor Instability. In sections 2 and 3 we summarize the results obtained in the last year. These include a simple model of bubble competition in Rayleigh-Taylor unstable flows which gives results which are in good agreement with experiment. In addition the model has been compared with two dimensional numerical simulations of inviscid Rayleigh-Taylor instability using the cloud-in-cell method. These simulations can now be run into the late time regime and can track the competition of as many as ten bubbles. The improvement in performance over previous applications of the cloud-in-cell approach is due to the application of finite difference techniques designed to handle shock-like structures in the vorticity of the interface which occur at late times.

In section 4 we propose to extend the research carried thus far to Rayleigh-Taylor problems in three dimensional and convergent geometries as well as to two-fluid instabilities in which interface roll-up is observed. Finally in section 5 we present a budget for the fiscal year 1987-1988.

2. Bubble Competition in Rayleigh-Taylor Instability

We have studied the penetration of light fluid into a heavy fluid in a Rayleigh-Taylor unstable flow by modelling the competition of interfacial bubbles. Currently, we have only considered the case in which the lighter fluid is of negligible density relative to the heavier fluid. In addition our studies have been restricted to two dimensional plane-periodic Rayleigh-Taylor instability.

The model is based on an extension of the work of Kull. Kull's model was originally proposed in an attempt to analyze the flow around a single steadily rising bubble. The flow near the bubble is approximated as that arising from a periodic array of sources superposed on a uniform stream. The location and strength of the source array is determined by performing a Taylor series expansion in both the shape of the interface and the potential about the nose of the bubble. By insisting that the kinematic and dynamic boundary conditions be satisfied to a given order in the series, ordinary differential equations may be derived for the position and strength of the source. Details of the procedure may be found in our proposal for renewal for the year 1986-1987 and in Kull's paper (1986).

Zufria has extended the original work of Kull by using sources to model the competition of bubbles in an unstable interface. Rather than use a periodic geometry, a channel geometry was considered instead but this is not crucial to the success of the model. Zufria considers a symmetric bubble rising vertically along the centerline of a channel of width H . The flow is characterized by the velocity at the nose of the bubble U , the location of the bubble X and the radius of curvature at the nose R . These in turn are related to the location A and strength Q of a potential source situated behind the nose of the bubble (see Fig. 1 of Appendix A). The strength and location of the source are adjusted to locally satisfy the kinematic and dynamic boundary conditions at the nose.

The flow near the bubble generated by the moving source appears to be an adequate approximation for the true irrotational flow. It should be noted however that the spikes in the interface are poorly modelled since the sources yield spikes with constant velocities. In reality, the spikes achieve constant acceleration. Thus the success of the model depends

on the limited effect the spikes have on bubble competition once the late stage of the instability is reached.

The velocity potential for a source in a channel is given by

$$\phi + i\psi = Q \log \left[(1 - e^{-(z+A)})(1 - e^{-(z^*+A)}) \right] - Uz,$$

where $z = x + iy$. The kinematic boundary condition is given by

$$\frac{DF}{Dt} = 0,$$

where the shape of the bubble interface is given by

$$F(x, y, t) = y^2 + 2R(t)x = 0.$$

The dynamic condition is given by Bernoulli's equation in a moving frame of reference:

$$\frac{\partial \phi}{\partial t} + 1/2 |\nabla \phi|^2 + x + \frac{dU}{dt} x = 0.$$

Here the gravitational acceleration has been set to 1 and we note that the model calculations are performed in a coordinate system with x as the vertical direction and y the horizontal. Using the form of the source solution and the assumed shape of the interface it is possible to derive ordinary differential equations for U, R, A , and Q . It may be shown that these equations possess a late time solution representing the rise of the bubble at a constant velocity quite close to that measured in numerical simulations. Further details are provided in Appendix A which presents the results in manuscript form.

We next consider the interaction of several bubbles using this model. We place N bubbles at locations $z_k = x_k + iy_k, k = 1, 2, \dots, N$. Each source is again characterized by its strength, velocity, and location and represents a bubble with a corresponding velocity of rise and curvature. In this case the potential is given by

$$\phi + i\psi = \sum_{j=1}^N Q_j \log \left[(1 - e^{z_j - A_j - z_k - z})(1 - e^{z_j^* - A_j - z_k - z}) \right] - U_k z.$$

Again, insisting that the dynamic and kinematic boundary conditions be satisfied locally at each tip we obtain a set of $4N$ first order differential equations which describe the motion of the N bubbles in the channel.

In contrast to the steady state achieved by a single bubble, the steady state consisting of N bubbles in uniform motion is highly unstable. If a bubble is retarded relative to the others it slows down whereas the other bubbles advance and expand at the expense of the slower bubble. This effect is shown in Fig. 3 of Appendix A where we have started with $N = 20$ bubbles. During the period of competition the velocity of the front is not constant. Rather the front advances as $x = \alpha t^2$, where α varies between 0.05 and 0.055. Eventually, a few large bubbles dominate and ultimately one bubble remains and rises with constant velocity. The values of the acceleration constant α are in close agreement with the values observed in the numerical simulations of Youngs.

3. Numerical Verification of the Bubble Model

The interface between two fluids of different densities behaves as a vortex sheet whose strength is determined by the jump in the tangential velocities of both fluids across the interface. For numerical simulation of the motion of the interface, it is typical to assume that this vortex sheet is composed of a set of markers which behave as point vortices. The circulation associated with each marker is the total circulation of the corresponding interval of the vortex sheet (see eq. (2)). The evolution of the circulation at a marker point e_i is given by the following equation

$$\frac{d\Gamma_i}{dt} = 2A \int_{e_-}^{e_+} \frac{dU}{dt} \cdot s \frac{\partial s}{\partial e} de + \left[\left(\alpha + \frac{A}{2} \right) \frac{\gamma^2}{2} + 2y - A\alpha\gamma U \cdot s \right]_{e_-}^{e_+}, \quad (1)$$

where A is the Atwood number defined as $A = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$. Here we have used a standard orientation of axes with the y axis along the vertical. The strength of the vortex sheet $\gamma(e, t)$ is related to Γ by

$$\Gamma(t) = \int_{e_-}^{e_+} \gamma(t, e) \frac{\partial s}{\partial e} de. \quad (2)$$

In equation (1) s is the unit vector tangent to the interface, s is the arc-length, and e is a material marker coordinate along the interface. The limits of integration in (2), e_- and e_+ , represent mid points between markers (see figure 1), and U represents the averaged velocity at the interface, i.e.

$$U = \frac{u_1 + u_2}{2}. \quad (3)$$

The velocities u_1 and u_2 are the velocities obtained as one approaches the vortex sheet from below and above respectively. The time derivative d/dt represents a total derivative following the markers and is related to the Eulerian derivative by the following expression

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(U + \frac{1}{2} \gamma \alpha s \right) \nabla. \quad (4)$$

By choosing $\alpha = -1$ the markers follow the upper fluid and by choosing $\alpha = 1$ the markers follow the lower fluid.

To describe the evolution of the vortex sheet, in addition to equation (1) we need the following kinematic relations that describe the motion of the interface

$$\frac{dx}{dt} = U + \frac{1}{2} \gamma \alpha s. \quad (5)$$

Equations (1) and (5) are similar to the ones obtained by Tryggvason (1987), the only difference is that in our case we have introduced the parameter α which is not in Tryggvason's 1987 formulation.

In order to have a complete system which can be marched forward in time we need to estimate the term dU/dt . From the Biot-Savart law the acceleration dU/dt is coupled to the rate of change with respect to time of Γ . Thus equation (1) is in fact an integral equation. To solve (1) we first compute a guess for the acceleration. We then use this

guess to compute $d\Gamma/dt$. This allows us to compute a guess for the circulation Γ at the next time step. The velocity is then computed and we then can compute a new guess for the acceleration. This process is repeated until convergence, and corresponds to the iterative time-step method outlined by Tryggvason (Tryggvason 1987). To compute the velocity field we use the vortex-in-cell method, i.e. we distribute the circulation of the markers in a fixed regular grid by the so-called area weighting rule, we solve the Poisson equation for the stream-function on the grid and finally we interpolate the velocity field to the interface again.

Currently, we are mainly interested in the case $A = 1$; in this case the interface develops very sharp spikes. At the tip of these spikes the strength of the vortex sheet has a jump, i.e. the strength has a shock like behavior. This behavior could be expected from equation (1) which has some similarity with the Burgers equation. Due to this singularity most of the discretizations of equation (1) based on centered differences are numerically unstable and the integration schemes break down when the shock is formed. This problem is complicated by the fact that the evolution equation is forced and the jump across the shock increases proportionally to t^2 .

One first idea to partially avoid this instability problem of the numerical schemes is to choose $\alpha = -1/2$. This choice causes the term involving γ^2 in equation (1) to disappear. Even though this idea improves the behavior of the integration scheme, better results can be obtained following the ideas that we describe below. Also notice that when $\alpha \neq 0$, another nonlinear term appears in equation (1) and it causes some stability problems too.

Based on the similarity with the Burgers equation we have introduced upwinding differences to define the term in γ^2 of the right hand side of equation (1). We are interested in computing a very sharp shock. To do this the simplest scheme is the one due to Godunov (1959). We have to define the term $F = -\gamma^2$ at the middle points between markers, and the Godunov scheme does it in the following form

$$F = \max[(\gamma_L^+)^2, (\gamma_R^-)^2], \quad (6)$$

where

$$\begin{aligned} \gamma^+ &= \max(-\gamma, 0), \\ \gamma^- &= \min(-\gamma, 0), \end{aligned} \quad (7)$$

The subscripts L and R denote right and left respectively relative to the midpoint between marker i and $i + 1$. Careful examination of rules (6,7) reveals that they are equivalent to insuring the proper upwind difference is performed relative to the position of the shock.

This small change in the way in which the discretization is done, introduces a very big improvement in the power of the methods. With this change we have been able to run the simulations to times one order of magnitude larger than with schemes based on centered differences.

As the time increases, the time step in our method must be decreased due to the CFL condition for stability because the strength of the jump across the shock is increasing. To improve the CFL condition, we have introduced an implicit scheme based on the Engquist-Osher upwinding scheme (Van Leer 1984, Embid *et al.* 1984, Engquist & Osher 1981).

The explicit Engquist-Osher scheme is similar to the Godunov except at the same shock where the discretization is different. In the Engquist-Osher method F is defined as

$$F = (\gamma_L^+)^2 + (\gamma_R^-)^2. \quad (7)$$

We chose this method for the implicit scheme because it gives jacobians that are continuous, and the iterations converge faster (Embid *et al.* 1984).

Several runs have been performed for different initial conditions including more than one perturbation wavelength. In figure 2, we show the time evolution of the interface for the following initial perturbation of the interface

$$y(x) = .05(\cos(8\pi x) + 0.1\cos(2\pi x)). \quad (8)$$

The computation were performed on a grid of 64×256 , and 320 markers were used to describe the interface. A regularization of the markers along the interface based on the Lagrange interpolation polynomials was done at each time step in order to keep the distance between markers as uniform as possible during the computation. The time step was 0.005.

Figure 3 shows Γ versus e along the interface for four different time steps. We can clearly observe the formation of the shocks and how the numerical scheme keeps the shock very sharp. Currently we are comparing the detailed prediction of the bubble model with our simulations. Initial comparisons of the computed values of the Youngs constant α are encouraging. Our simulations give $\alpha = 0.05 \pm 0.01$.

4. Proposed Research

We propose to extend the research described above. Several obvious generalizations of the bubble model will be investigated as described in the following tasks.

1 Extension of the source model to two-fluid Rayleigh-Taylor Instability:

It is natural to attempt to extend the source model described above to the regime in which the Atwood ratio takes on values less than one. The spike region exhibits a roll-up which affects the velocity of the spike. Numerical simulations show a region of almost singular vorticity about the point of rollup. Thus it is natural to add vortices to the source model and to observe their effect on the dynamics of the bubbles.

2 Extension to three dimensions:

The source model can easily be extended to three dimensional flows as well as convergent geometries. The principle of locally adjusting the source strength and location carries over directly to three dimensions. Given the difficulty of obtaining numerical simulations in three dimensional Taylor instability the bubble model may provide a valuable approximation.

3 Impulsively driven Rayleigh-Taylor instability:

In impulsively driven Taylor instability an impulse is instantaneously applied to the interface after which the dynamics are purely inertial. This is relevant to instabilities initiated by shocks. Again an application of the source model to this case is a simple modification of the work discussed above in sections 2 and 3.

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FIGURE 1

FIGURE 2

FIGURE 3

Appendix A

Bubble competition in Rayleigh-Taylor instability.

by

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Abstract

The penetration of a front of light fluid into a heavy fluid in a Rayleigh-Taylor unstable flow is studied by using a model which simulates the competition among the bubbles which are formed in the interface when the density ratio of the two fluids is very large. Several different initial conditions have been considered, and it is found that the front moves with constant acceleration. The values obtained for the acceleration of the front are in very good agreement with experimental results obtained by Read (1984).

Appendix A

1. Introduction.

When at the interface between two fluids of different densities the light fluid pushes the heavy fluid, the interface is Rayleigh-Taylor unstable and develops a fingering instability. This instability appears in many interesting physical situations, such as laser implosion of deuterium-tritium fusion targets, electromagnetic implosion of metal liners, overturn of the outer portion of the collapsed core of massive stars, etc. For a review article giving examples and phenomenology of the Rayleigh-Taylor instability see Sharp (1984). An extensive study of the linear theory of this instability can be found in Chandrasekhar (1961).

One of the physical cases where this instability is present, which has been considered frequently during the last ten years, is laser fusion. When a solid sphere of deuterium-tritium is irradiated with an intense laser light, the sphere is accelerated inwards. The light fluid (vaporized material due to ablation) pushes the heavy fluid (solid material) across a surface which is unstable in the Rayleigh-Taylor sense. The peak of laser power required to reach critical conditions for fusion is significantly reduced when a shell instead of a solid sphere is used as target. It is also predicted that the required laser power decreases as the aspect ratio of the sphere ($r/\Delta r$) is increased. The Rayleigh-Taylor instability disrupts the spherical symmetry and may cause the failure of the fusion process when the aspect ratio is large (McCrory *et al.* 1981). Therefore, it is very important to understand this instability in order to improve the design of the shells.

In the particular case of laser fusion it has been observed that the structure of the interface resembles the classical case in which $\rho_{\text{light}}/\rho_{\text{heavy}} \rightarrow 0$, as no Kelvin-Helmholtz instability is observed and the flow exhibits a laminar structure. Another characteristic which has been observed in this case is that the growth of the heavy fluid spikes, which appear on the interface, tends to decrease significantly at large amplitude and the spikes do not break.

Based on the observed properties of Rayleigh-Taylor instability in the fusion problem, we shall concentrate our attention on the classical Rayleigh-Taylor instability for the case

Appendix A

$\rho_{light}/\rho_{heavy} = 0$. A large amount of work during the last ten years has tried to solve numerically the exact equation for the evolution of the interface between the fluids (e.g. Baker *et al.* 1980; Kerr 1986; Tryggvason 1987). But in most cases the calculations were limited to simple initial conditions like pure sinusoidal perturbations of a straight interface. Only a few authors have considered multifrequency perturbations, and in these cases the computation were performed only for very short periods of time due to the difficulty of following very complex and distorted interfaces (Baker 1987).

When a pure sinusoidal perturbation is considered the interface develops in each wavelength a single bubble going up and a single spike falling down. The bubble rises with constant velocity, whereas the spike falls with constant acceleration. From what has been observed experimentally, it is clear that the phenomenology of the late stages of the instability do not correspond to the single frequency cases which have been numerically considered. The late stages of the instability are characterized by an interface formed by a set of bubbles of light fluid (vacuum) rising with a low constant acceleration through the heavy fluid, while between the bubbles spikes of heavy fluid fall into the vacuum (see Read & Youngs 1983, Read 1984).

To understand how the system ends in a state like this, we analyze in more detail the instability process. When inviscid fluids are considered and the surface tension at the interface is neglected, linear stability analysis of the interface predicts that the surface is unstable to periodic perturbations of all wave numbers k , and the growth rate of the instability is proportional to $k^{1/2}$. Hence, the larger the wave number, the larger the growth rate. But this is not realistic because there exists a large wave number for which surface tension or viscosity become important and introduce a cut-off in the growth rate, stabilizing the surface for larger wave numbers. This means that for all real fluids there exists a wave number for which the linear growth rate is a maximum (Chandrasekhar 1961). For a general initial state in an experiment the real perturbation of the surface will contain all the wave numbers, but the maximum of the growth rate determines the predominant wave number in the instability, and the initial length scale of the process.

Appendix A

The experimental work of Read (1984) shows that during the very early stages of the instability before the bubbles are clearly formed, some drops of heavy fluid are formed and fall freely into the vacuum. These drops could be due to effects of surface tension because they appear in a time scale of the same order as the maximum growth rate given by the linear theory when surface tension is taken into account. A possible explanation for this rain could be that during this first stage the surface becomes highly deformed and the surface tension cuts the continuous surface giving isolated regions of fluid, i.e., drops, similar to the solutions found by Crapper (1957), Schwartz & Vanden-Broeck (1979) and Chen & Saffman (1980) for steady waves. After this first rain, the interface develops bubbles and spikes, with the number of bubbles related to the basic wave number of the instability. These bubbles grow and compete with each other in an unstable process that we will try to understand.

As the number of bubbles is large (in typical experiments between 10 and 20), it is difficult to follow the interface by solving the full equations. Even though the vortex in cell method (Tryggvason 1987) is able to follow interfaces with very complicated shapes, computations following a very large number of bubbles have not been done yet. Also these methods are an expensive way to produce results of interest which are related to averaged properties of the flow, e.g, the penetration of the front of bubbles into the heavy fluid. Further, the generalization to three dimensional or spherical geometries of methods based on the full equations is not always possible.

After these considerations, it seems reasonable to try to develop some simple models which could give some information about some averaged properties of the flow and could be suitable for a generalization to more general geometries.

For the case in which two fluids of comparable density are considered, the Kelvin-Helmholtz instability leads to turbulent mixing between the fluids. In this case some simple models of mixing have been able to give good results about the penetration of the fluids and the growth of the mixing region (Young 1984). But these models cannot be applied to laminar flows where there exists a very clear structure.

Appendix A

The aim of the present work is to construct a very simple two dimensional model, based on the motion of a bubble, in order to compute the penetration of the front of bubbles into the heavy fluid and try to understand the basic mechanisms of the evolution of the bubbles. In the second section we introduce a simple model for a single bubble. Based on the results of this section, in the third section we generalize the model for the case of more than one bubble. We finish with some conclusions and a discussion of the results.

2. Single bubble equations.

The problem of a single axially-symmetric bubble rising in a cylindrical vertical tube of constant cross section was first considered by Davis & Taylor (1950). They gave an approximate theory for the steady state of the rising bubble and found that the velocity of the vertex of the bubble was

$$V = 0.464(gR)^{1/2}, \quad (1)$$

where R is the radius of the tube and g is the acceleration of the gravity. Later Layzer (1955) analyzed the unsteady problem and gave approximate solutions which interpolate the flow between the initial state of the Rayleigh-Taylor instability and the steady state motion of a rising bubble. Layzer applied his study also to the case of a plane bubble rising between two vertical walls. He found that the steady state speed for the vertex of the bubble in this case is

$$V = 0.326(gR)^{1/2}, \quad (2)$$

where now R is half the width of the channel.

The problem of a bubble rising between two parallel walls has been rigorously formulated in terms of nonlinear integral equations by Birkhoff & Carter (1957) and Garabedian (1957).

Appendix A

The approximate theories of Davis & Taylor (1950) and Layzer (1955) are based on the idea of satisfying approximately the equations of motion in the neighborhood of the vertex of the bubble. But in both studies attention is focused on the evolution of the bubble speed and a basically constant size is assumed for the bubble. We will see below that to describe the later states of the Rayleigh-Taylor instability with a model based on bubbles, it is necessary to consider the unstable evolution of the size of the rising bubbles.

In our model we will follow the same ideas as Davis & Taylor (1950) and Layzer (1955), but we will include in our analysis the possibility of the bubbles changing their basic sizes. Following Kull's (1986) work, we will model the bubbles with source flows combined with a uniform stream.

Consider a two-dimensional symmetric empty bubble rising along the central line of a vertical channel of width H filled with an inviscid fluid. Let us characterize the bubble by the following three variables: the velocity of the vertex of the bubble U , the local radius of curvature R of the surface at the vertex and the location X of the vertex of the bubble with respect to a frame of reference attached to the channel (see figure 1). If the flow is assumed to be irrotational there exists a complex velocity potential $f(z) = \phi + i\psi$, where ϕ and ψ are respectively the velocity potential and the streamfunction. This velocity potential in the neighborhood of the vertex of the bubble can be modelled by the velocity potential due to a source singularity of some strength Q located at some distance A below the vertex of the bubble (see figure 1). This assumption is good on the front part of the bubble, but gives a bad description of the flow in the far field behind the bubble. The two spikes of fluid which are formed in the flow because of the motion of the bubble fall with a constant acceleration, and a source singularity gives a constant velocity flow for the far field behind the rising bubble. However, we assume that this error has a negligible effect on the properties of the rising bubble because the spikes do not play an important role in the motion of the bubble. The spikes represent a free fall of the fluid which is displaced by the motion of the bubble, and they have no dynamic effect on the bubble. Support for this hypothesis comes from the good agreement of the velocity predicted by Layzer (1955) and our theory with the limiting velocity obtained by numerical solutions of the full problem

Appendix A

(Baker *et al* 1980, Kerr 1986).

As we will see below, by kinematic considerations it is possible to show that for given values of Q and A , the velocity U is determined. Therefore, the state of the bubble is determined by giving Q , A , R and X . Now our objective is to derive equations which describe approximately the time evolution of these four variables.

In the following we will assume that all the variables have been made dimensionless by using H/π and $\sqrt{gH/\pi}$ as the characteristic length and velocity respectively. Notice that now the width of the channel is π . This choice simplifies some of the following expressions.

Let us consider a non-inertial frame of reference $(\hat{x}, \hat{y})$ with origin at the vertex of the bubble. Locally in the neighborhood of the vertex we assume that the shape of the bubble is given by

$$F(\hat{x}, \hat{y}, t) = \hat{y}^2 + 2R(t)\hat{x} = 0. \quad (3)$$

Using the complex velocity potential due to a source singularity in a channel, we have that the approximate complex velocity potential for the flow is given by

$$f(\hat{z}) = Q \log[(1 - e^{-A-\hat{z}})(1 + e^{-A-\hat{z}})] - U\hat{z}. \quad (4)$$

Now that we have the shape of the bubble and the flow we can satisfy the kinematic and dynamic boundary conditions at the surface of the bubble.

Let us start with the kinematic boundary condition. As the surface of the bubble is a material line, we have

$$\frac{DF(\hat{x}, \hat{y}, t)}{Dt} = 2\frac{dR}{dt}\hat{x} + 2R\hat{u} + 2\hat{y}\hat{v} = 0, \quad (5)$$

where u and v are the two components of the fluid velocity in the frame of reference moving with the vertex of the bubble which are given by

Appendix A

$$u + iv = \frac{df}{dz}. \quad (6)$$

Equation (5) has to be satisfied on the surface of the bubble. If we satisfy the equation up to second order in $\hat{x}$, i.e., order zero and $\hat{x}$, we obtain the two following equations

$$\begin{aligned} U &= \frac{2Q}{e^{2A} - 1}, \\ \frac{dR}{dt} &= \frac{8Qe^{2A}}{(e^{2A} - 1)^2} \left[\frac{3}{2} - \frac{e^{2A} + 1}{e^{2A} - 1} R \right] R. \end{aligned} \quad (7)$$

Notice that the first equation gives the value of U in terms of Q and A , and the second equation gives an ordinary differential equation for the evolution of the local radius of curvature of the surface of the bubble at its vertex. Also from the definition of U and X we have the additional kinematic condition that

$$\frac{dX}{dt} = U. \quad (8)$$

To complete the system of equations we need two more equations which come from the dynamic condition that the pressure has to be constant on the surface of the bubble. This conditions is represented by the following Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + \hat{x} + \frac{dU}{dt}\hat{x} = 0. \quad (9)$$

The first term in $\hat{x}$ is due to gravity and the second to the fact that the frame of reference is non-inertial. Again we have to satisfy this equation on the surface of the bubble. Using equations (3) and (4) we satisfy equation (9) up to third order. In this case the zero order equation determines the constant of the Bernoulli equation, and putting the terms in x and x^2 equal to zero we obtain the following two differential equations

Appendix A

$$(c_1 + c_2 R) \frac{dQ}{dt} - Q(c_2 + c_3 R) \frac{dA}{dt} = Q^2 c_2^2 R - 1,$$

$$(c_2 + c_3 R + c_4 \frac{R^2}{6}) \frac{dQ}{dt} - Q(\frac{c_3}{2} + c_4 R + c_5 \frac{R^2}{6}) \frac{dA}{dt} = Q^2 [c_2^2 - 2c_2 c_3 R + (3c_2^2 - 4c_2 c_4) \frac{R^2}{3}], \quad (10)$$

where

$$\begin{aligned} c_1 &= \frac{2}{e^{2A} - 1}, \\ c_2 &= -\frac{4e^{2A}}{(e^{2A} - 1)^2}, \\ c_3 &= \frac{8e^{2A}(e^{2A} + 1)}{(e^{2A} - 1)^3}, \\ c_4 &= -\frac{16e^{2A}(e^{4A} + 4e^{2A} + 1)}{(e^{2A} - 1)^4}, \\ c_5 &= \frac{32e^{2A}(e^{6A} + 11e^{4A} + 11e^{2A} + 1)}{(e^{2A} - 1)^5}. \end{aligned} \quad (11)$$

Equations (7), (8) and (10) form a system of four first order ordinary differential equations. This system describes the evolution of a single bubble given initial conditions.

This system has an equilibrium solution which corresponds to a steady rising bubble. This equilibrium solution is a very strong attractor. For any initial conditions the system ends up in the steady state solution. This steady state solution has the values

$$\begin{aligned} U^{(1)} &= 0.39332, \\ R^{(1)} &= 0.86612, \\ A^{(1)} &= 0.65850, \\ Q^{(1)} &= 0.53731. \end{aligned} \quad (12)$$

The value given by Layzer's (1955) theory for the limiting velocity is $U^{(1)} = 1/\sqrt{6} \approx 0.40823$. The result predicted by the numerical experiments performed by Kerr (1986) using the full equations is $U^{(1)} \approx 0.35$. Baker *et al.* (1980) found that $U^{(1)} \approx 0.32$.

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Garabedian's (1957) stability argument suggested a lower bound of 0.334 for the value of $U^{(1)}$. Laboratory measurements performed by Emmons (1959) suggested that $U^{(1)}$ is between 0.29 and 0.42.

In figures 3 we show the time evolution of the variables of the bubble for some initial conditions. Figure 3 shows the bubble at six different times.

3. Multiple-bubble equations.

In the previous section we considered the problem of a single bubble rising in a vertical channel. But this is only the basic tool for the more realistic problem of several bubbles interacting with each other while rising in a vertical channel. Rayleigh-Taylor experiments show that the front of the light fluid penetrating into the heavy fluid is generally formed by a set of bubbles. In the present section we extend our model to the case in which there exists more than one bubble in the channel.

Before going into the details of the derivation of the equations, we introduce the following simplification into our model. In accordance with the experimental observations, it seems reasonable to assume that the motion of the bubbles is only in the direction of the channel and the bubbles do not move laterally, i.e., the bubbles move along $y = \text{const.}$ lines. Experiments (Read 1984) show that until a very late state of the instability, when the front is formed by a few bubbles (2, 3 or 4), the lateral motion of the bubbles is negligible. Therefore, in our model we will neglect all lateral effects due to the asymmetry of the flow with respect to a given bubble.

Let us assume that we have a channel which contains N bubbles and these bubbles are located at N different points $Z_k = X_k + iY_k$ where $k = 1, 2, \dots, N$. If for each bubble Y_k is constant, the bubble will be characterized by the four variables Q_k , A_k , R_k and X_k . Now following the ideas of the previous section we can write the velocity potential as the sum of the velocity potentials due to N source singularities located at the points $Z_k - A_k$. The total velocity potential in a frame of reference moving with the vertex of the k -th bubble is

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$$f(z) = \sum_{j=1}^N Q_j \log[(1 - e^{z_j - A, -z_k - i})(1 - e^{z_j^* - A, -z_k - i})] - U_k z, \quad (13)$$

where the star denotes the complex conjugate.

With the velocity potential given by (13) and satisfying equations (5), (8) and (9) for each bubble, we can develop a set of $4N$ first order differential equations which describes the motion of the N bubbles in the channel. To perform a simulation we only need a set of initial conditions.

Before starting with the computations let us analyze some properties of the set of equations. First we notice that confining the bubbles to a channel is equivalent to considering a flow periodic in the y direction with period 2π with the walls as axes of symmetry for the problem. Therefore, the problem of N identical and equally spaced bubbles is equivalent to the problem of one single bubble in a channel of width π/N . Consider N bubbles located at the following points

$$X_k = 0, \quad Y_k = (k-1)/(N-1), \quad k = 1, \dots, N. \quad (14)$$

If the rest of the variables have the following values

$$\begin{aligned} Q_1 &= Q_N = \frac{2Q^{(1)}}{(N-1)^{3/2}}, \\ Q_k &= \frac{Q^{(1)}}{(N-1)^{3/2}}, \quad k \neq 1, k \neq N, \\ A_k &= \frac{A^{(1)}}{N-1}, \\ R_k &= \frac{R^{(1)}}{N-1}, \\ U_k &= \frac{U^{(1)}}{(N-1)^{1/2}}, \end{aligned} \quad (15)$$

then the system of N bubble is in an equilibrium state, corresponding to the steady state for the single bubble case. But in contrast to the single bubble case, this steady state is unstable under any perturbation.

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Whereas in the case of a single bubble we only had one focus for unsteadiness, which was the process of adaptation of the bubble to the size of the channel, in the case of N bubbles there exists another focus. This new focus for unsteady behavior comes from competition between the different bubbles. Each bubble tries to occupy the maximum possible space and competes with the others. A first description of this competition was given by Layzer (1955). Imagine a periodic set of equivalent bubbles and let us assume that one of the bubbles is smaller than its companions. Because the bubble is smaller, its speed will be smaller than the rest of the bubbles. As a consequence, the small bubble will advance less than the others, and the others will gain more space and will grow more, leaving less space for the small bubble. Hence, the small bubble will shrink and slow down, whereas the large bubbles will expand and acquire a higher vertex speed.

This process can be very clearly observed for the three bubbles shown in figures 4 and 5. We have run a case of three bubbles starting at the unstable equilibrium point and reduced by 10% the size of the middle bubble. This initial condition is equivalent to considering a periodic row of bubbles in which one bubble of every two is 10% smaller in size. In figure 4 we can observe the transient behavior of the speeds of the bubbles. After this transient the velocities tend to constant values. The two big bubbles tend to the velocity corresponding to a single bubble. Notice that the final state in which there are only two equal bubbles in the front is equivalent to a single bubble. The difference is a lateral shift of $\pi/2$. On the other hand, the small bubble is pushed back with a constant velocity. We know that the result for the small bubble is not correct as our model does not give a correct description of the flow at the back of the bubbles. But we can observe that the effect of the small bubbles on the motion of the front is completely negligible. We can see in figure 4 that the big bubbles reach the steady final velocity at $t \approx 4$, and at this time the small bubble is still in the front region. What we see is that the small bubble first shrinks, becomes unimportant, and is pushed back afterwards (see figure 5).

Notice also in figure 4 that there is a small initial period of time in which the speed of the big bubbles decreases. This is due to the initial perturbation from the equilibrium state. The bubbles readjust their size and speed to the channel and then they enter the

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competition. When the initial perturbation is reduced, this region disappears and the acceleration of the big bubbles is always positive. Changes in the initial conditions do not affect appreciably the evolution. The perturbation in the size of the bubbles has the same effect as a perturbation in the longitudinal location of the bubbles. In both cases what is changing is the basic space available for each bubble. If one bubble is a little bit ahead of the others, it grows faster because it has more free space.

We have performed computations for different perturbations and we have found that the process is in all cases equivalent and the averaged accelerations α are always between 0.03 and 0.04, where

$$X = \alpha t^2. \quad (16)$$

Now that we understand the basic processes of the transient behavior, we can study more realistic simulations. In the experiments performed by Read (1984), a typical number of bubbles is between 10 and 20. We have performed several runs for different numbers of bubbles and initial conditions. To generate the initial conditions for a given number of bubbles N we construct the equilibrium solution and impose a uniform random perturbation of some given amplitude to the X coordinates of the bubbles, i.e. $X_k = \Delta X \xi_k$, where ξ_k represents a random variable with a uniform distribution in the interval $[0, 1]$.

A typical run for $N = 20$ and $\Delta X = 0.01$ is shown in figure 6. This figure shows the location of the bubbles versus the square of the time. We see that the position of the front of bubbles is practically linear in t^2 . This is a common characteristic of all the runs that we have performed. This is in agreement with the results found experimentally by Read (1984). Hence, equation (16) is a good law to represent the position of the front. This law for the evolution of the front could be expected from direct dimensional considerations. For a front of bubbles the only characteristic length is the size of the bubbles R . Hence, the velocity of the front will be proportional to $(gR)^{1/2}$, and the rate of growth of the bubbles will also be proportional to $(gR)^{1/2}$, from this it follows that the location of the front will be proportional to gt^2 .

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In figure 6 we can observe also that initially all the bubbles have positive velocities, but most of them are progressively stopped and pushed back by the flow due to the largest ones which stay in front. In the final state, the front will contain only one bubble and it will move with constant velocity. Hence, we conclude that the front will move with constant acceleration until the number of bubbles in the front is small, and then the acceleration will decrease until the constant velocity state is reached. But during the phase of active competition among the bubbles, the acceleration is constant.

Figure 7 shows the bubbles at six different time steps. Initially we are very close to the equilibrium state and all the bubbles look similar. As time goes on, we can notice how the number of bubbles at the front decreases.

For the runs performed with $N = 20$ we found the value of α ranges between 0.049 and 0.055 . For the case of $N = 10$ we found that the dispersion was higher and α was between 0.04 and 0.07 . Read and Youngs (1983) and Read (1984) found experimentally that for an instability between alcohol and air the value of α is between 0.058 and 0.065 . The typical number of bubbles which appeared in these experiments was between 10 and 20. They also found that after the constant acceleration region, the acceleration decreases.

Finally we have to mention that numerical experiments performed by Young (1984) using a mixing model gave for α values between 0.04 and 0.05 .

4. Conclusions.

In many cases, when analyzing an process which is Rayleigh-Taylor unstable we are not interested in knowing the exact shape of the interface between the fluids, but rather finding how some average quantities evolve and understanding the basic mechanisms involved in the process.

Following these guide lines, we have constructed a very simple model to simulate the penetration of the light fluid into the heavy fluid in a Rayleigh-Taylor unstable flow. The

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model is based on a bubble competition scheme, and the bubbles are modelled by moving source singularities.

Using the model we have found that the front moves into the heavy fluid with constant acceleration. The values obtained for the acceleration are in very good agreement with values obtained experimentally. This suggests that the acceleration observed in the motion of the interface is due to the competition among the different bubbles which form the front. The largest bubbles move faster than the others, and therefore have more free space to expand. We have also found that, although the detailed structure of the bubbles depends on the initial conditions, the averaged motion of the front is very insensitive to the initial conditions.

We believe that, for the case in which the aspect ratio of the shell is small and spherical effects are important, a similar model could be constructed using three dimensional sources.

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Appendix A

Figure Captions

Figure 1.- Geometrical description of the model flow.

Figure 2.- Results for a single bubble. Initial conditions $X = 0, R = 0.2, Q = 0.1, A = 0.1$. (a). Position of the bubble X versus time. (b). Radius of curvature R at the vertex of the bubble versus time. (c). Speed of the bubble U versus time. (d). Strength Q of the source singularity used to model the bubble versus time.

Figure 3.- Single bubble at six different times.

Figure 4.- Speed of the bubbles versus time for $N = 3$.

Figure 5.- Results for three bubbles. (a). Flow at $t = 0.1$. (b). Flow at $t = 2.0$. (c). Flow at $t = 4.0$. (d). Flow at $t = 6.0$.

Figure 6.- Position of the bubbles versus t^2 for typical run with 20 bubbles.

Figure 7.- Bubbles at six different time steps from a typical run with 20 bubbles.

(a)

(b)

(c)

(d)

$t = 10$

$t = 8$

$t = 6$

$t = 4$

$t = 2$

$t = 0$

(a)

(b)

(c)

(d)

$t = 0.1$

$t = 0.6$

$t = 1.0$

$t = 1.5$

$t = 2.1$

$t = 3.0$