Measurement of the η Parameter in μ^+ Decay

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April 1989

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This work was supported by the Director, Office of Energy Research, Division of Nuclear Physics of the office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contracts DE-FG03-87ER40323 and DE-AC03-76SF00098.

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Acknowledgements

Long gone is the time when a physics graduate student dug deep into his pocket, bought and built equipment to set up in a corner and emerged a couple of years later with his thesis in hand. Similar circumstances still exist in many other areas of graduate study (to which many cab drivers and restaurant workers can attest), but no longer in physics in the United States. Credit for this must be given to the taxpayers of this country who, through their representatives, have generously allocated support to the sciences for the last several decades; credit must also be given to future taxpayers, who will repay the fraction appearing as debt.

Yet, one is forced to observe that the bargain which science has struck with the government is essentially a Faustian one, for we have been allowed to pursue our Truth, in part, because of the hope by the funding sources that the work will further the purposes of the military-industrial complex. Even those who have no intention of working on overtly military projects advance its goals by instructing and interacting with those who will; by design, more information flows toward military research than away from it. Thus, my acknowledgement of the government-derived money which has kept me above the “poverty line” is given gratefully, but with the grimness of realizing that Faust knew better what bargain he had struck.

The number of remaining acknowledgements reflects upon the extent to which physics has become a collective undertaking, as well as upon the length of this particular project: I regret any omissions.

The following people have all participated in the planning or execution of the experiment

on which this thesis was based: J. A. Bistirlich, J. H. Brewer, K. M. Crowe, J. A. Jansen, R. Keitel, C. J. Martoff, J. P. Miller, L. Oddi, C. J. Oram, M. Salomon, U. Straumann, E. Yoo and W. A. Zajc; all deserve credit for their hard work. I will expand upon some specific contributions.

- Jim Bistirlich worked on this experiment throughout its development, including engineering, machining and installation. He has made many helpful comments on my work and has been supportive throughout.
- Jesse Brewer performed the earliest off-line analysis of the data. His program provided the nucleus for some of my later analysis code.
- Ken Crowe, as experiment spokesperson, has obtained funding and beam time for this experiment, guiding it through its development. His work on the hardware, in design and fruition, has been essential.
- Johan Jansen performed studies of experimental effects and designed several refinements to the hardware. He also wrote the original, experiment-specific code for data acquisition and on-line analysis.
- Jeff Martoff originally conceived the experiment and did the early design work on it. It would not have occurred without his creative imagination and effort.
- Chris Oram's knowledge of the M13 beam line at TRIUMF was of great value, and he was consistently helpful in his capacity as M13 beam-line coordinator.
- Martin Salomon worked on the early data analysis and made several helpful suggestions regarding the magnetic measurements problem for the experiment.
- Uli Straumann designed an improved spectrometer configuration and designed several counters for the highly-constrained target area.
- Bill Zajc has been a frequent source of good advice on the analysis of the experiment and a frequent source of encouragement that it be completed. He is also to be thanked

for having bequeathed his personal stock of thesis paper in support of this effort.

In addition, several other members of the research group have contributed to the experiment in various ways:

- Dean Chacon ran several early Monte Carlo studies of the spectrometer using the EGS code, writing the user code for these.
- Carl Clawson is to be thanked for his unique contribution to the philosophy of thesis writing, as well as the necessary hardware. I appreciate his patience in answering my questions on muons, detectors and computation.
- Jim Kurck and Keith Wong undertook some of the drudgery of tape copying and pruning.

I wish to thank several theorists for helpful discussions: Robert Shrock who discussed massive neutrinos; Alberto Sirlin who discussed radiative corrections; Mahiko Suzuki who explained an assortment of theoretical points; and Mary K. Gaillard for her information on Fierz transformations and other matters. Thanks are also due to J. F. Ziegler for information regarding the calculation of particle stopping distributions, as related to the TRIM85 computer code.

My grateful acknowledgements also go to my candidacy committee consisting of Professors K. M. Crowe, J. O. Rasmussen and H. A. Shugart. I appreciate their willingness to read and comment on this thesis.

Following preliminary tests at the Los Alamos National Laboratory, the experiment was conducted at the TRIUMF meson factory in Vancouver, Canada: thanks are due to both institutions for the beam time and resources provided. Of the TRIUMF staff, I especially want to thank Robert Openshaw for his help with the NMR and MWPC apparatus, and Heinz Biegenzien for his help with the mechanical preparation of the experiment.

Given the importance of software to modern physics experiments, non-proprietary contributions should be acknowledged when possible. For this experiment they were numerous, some of them essentially transparent to the user, as the best software is. Thus, this list is undoubtedly, though not intentionally, incomplete.

- Data acquisition and on-line analysis were done with a combination of TRIUMF's DA and Fermilab's MULTI. DA was written by, and the package maintained by, Tim Miles.
- Preliminary off-line analysis was done using MOLLI, which was written at TRIUMF by Anne Bennett and John Lloyd.
- The program IO, and its subroutines MAGTA and MAGTASS, has been useful for tape manipulation. The original code was written at TRIUMF by Anne Bennett and Renee Poutissou.
- The histogram/plotting package, HBOOK/HPLOT, produced the working graphics during the analysis, as well as many of the figures for this thesis. The packages were written by Howard Watkins of the European Organization for Nuclear Research (CERN). Other routines from the CERN libraries have also been used extensively.
- DISPLAY, an interactive shell for HPLOT, has been used during the data analysis. It was written by B. Gabioud, H. L. Videau and M. A. Garnjost of the TPC Detector group at Lawrence Berkeley Laboratory (LBL).
- The Unified Graphics System (UGS), written at the Stanford Linear Accelerator Center (SLAC) by Robert C. Beach, has been the principal graphics interface used—both with my own and with borrowed software.
- The EGS/PEGS code system, discussed elsewhere in this thesis, has been of great value in doing Monte Carlo studies. The excellent documentation has also provided a helpful review article on $e^{\pm}, \gamma$ physics and Monte Carlo techniques. The early

version, EGS3, was written at SLAC by Richard L. Ford and Walter R. Nelson, while the updated version, EGS4, was written by Walter R. Nelson, Hideo Hirayama and David W. O. Rogers. Discussions of this code with Walter Nelson of SLAC and Alex Bielajew of the National Research Council of Canada (NRCC) were very valuable.

- Magnet studies were done with POISSON, PANDIRA and related programs, as provided by Mary Mendel and Helen Stokes of the Los Alamos National Laboratory (LANL).
- A program for tracking trajectories in a magnetic field, TRACK, was kindly provided by Don Lobb; he and D. E. Henin were the authors.
- Werner Koellner of LBL has provided documentation on, help with, and access to, a variety of software.

Much of the computation done for this experiment was on VAXen independent of those operated by the LBL Computing Division. The BEVAX cluster, operated by the LBL Nuclear Science Division, has provided CPU time and tape drive access for the data analysis and some of the Monte Carlos which were run. This has been managed by Mark Bronson, Chuck McParland and Russ Wright. The PHYSIX cluster, operated by the LBL Physics Division, provided CPU time and routinely made up-to-date software available. The advice and documentation provided by Edgar Whipple and John Waters, in addition to system management, have been invaluable. The JANUS VAX, operated by Ken Crowe and John Rasmussen at LBL has also provided valuable CPU and tape drive access. Dean Chacon's responsive and dedicated management is gratefully acknowledged.

On a more personal level, I must credit my wife, Carol Wickersham, with a great deal during my time as a graduate student—not least patience. Most, I thank her for the discussions which have helped to clarify the place that physics occupies in the general scheme of things, and for glimpses of aspects of the world that I would not have otherwise had.

This work was supported by the Director, Office of Energy Research, Division of Nuclear Physics of the office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contracts DE-FG03-87ER40323 and DE-AC03-76SF00098.

Chapter 1

Introduction

1.1 Unpolarized Muon Decay Spectrum

The usual, unpolarized decay $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$ is well-known. While the energies involved (the muon mass is $M_\mu = 105.66$ MeV) are typically higher than those in nuclear beta decay, they are still far below the mass of the W^+ ($M_W = 81,000$ MeV). Therefore, Fermi's theory of beta decay with its four-point interaction view of the weak force, together with the V-A rule, gives an excellent approximation to the spectrum, although, being non-renormalizable, it cannot be strictly correct. Calculations based on this approximation are easily found elsewhere and will not be reproduced here. One excellent discussion of muon decay can be found in the work by E. D. Commins and P. H. Bucksbaum.¹

The differential transition probability for muon decay is the product of three basic terms. The first term is the total transition rate, $-(A/16) \cdot (M_\mu^5/192\pi^3) \cdot (1 + 4\eta m_e/M_\mu)$, which contains the usual fifth-power dependence on the decay energy (Sargent rule). The constant $A/16$ is extracted from the squared transition-matrix element and, aside from correction factors of order unity, equals $G_F^2/2$, where G_F is the Fermi coupling constant. The quantity $(1 + 4\eta m_e/M_\mu)$ appears here only to maintain the normalization and does not

¹E. D. Commins and P. H. Bucksbaum, *Weak Interactions of Leptons and Quarks* (Cambridge University Press, Cambridge, 1983) pp. 93-108.

deviate far from unity; m_e is the positron mass and η is discussed below. The second term is an x^2 factor due to the positron's available phase space. Here, $x = E_e/E_e(max)$, with E_e being the positron energy and $E_e(max) = \frac{M_\mu^2 + m_e^2}{2M_\mu}$ being the spectrum endpoint energy. At $x = 1$, the massless particles are ejected in one direction and the positron in the opposite one, with about half of the energy being carried off in each direction. The third term contains the x -dependence of the squared transition-matrix element, integrated over the momenta of the neutrinos, which escape undetected. This term contains two parameters, ρ and η , which are determined by the spin(s) and coupling(s) of the exchanged boson(s). A more complete discussion appears in Chapter 2.

The differential transition probability, neglecting m_e in most terms for clarity, is

$$\frac{d\Gamma^{(0)}(x)}{dx} = -\frac{A}{16} \frac{M_\mu^5}{192\pi^3} x^2 \left[12(1-x) + \frac{8}{3}\rho(4x-3) + 24\eta \frac{m_e}{M_\mu} \frac{1-x}{x} \right]. \quad (1.1)$$

Figure 1 shows the x -dependence of this expression for $\rho \equiv \frac{3}{4}$ and $\eta \equiv 0$, which is predicted theoretically by the two-component neutrino hypothesis; the total transition-rate factor has been dropped to display the energy spectrum, normalized to 1.

That η only appears in Equation 1.1 multiplied by $\frac{m_e}{M_\mu} \approx 0.00967$ means that the spectrum shape is dominated by ρ . However, because of the x^{-1} factor, η becomes increasingly important at low energies. This is shown in Figure 2, which plots the fractional effect of η on the spectrum shape. Note that the divergence, predicted by Equation 1.1 at $x = 0$, does not appear due to radiative corrections. These will be discussed in Sections 2.4 and 2.5.

The relative insensitivity of the unpolarized spectrum shape to η is reflected in the previously extracted experimental values:

$$\rho = 0.7518 \pm 0.0026^2 \quad \text{and} \quad \eta = -0.12 \pm 0.21^3.$$

While these values are consistent with the predictions of the two-component neutrino theory, the accuracy of η from this direct measurement is markedly less than that of ρ .

²G. P. Yost et al., Review of Particle Properties, Phys. Lett. **204B**, 1 (1988).

³S. E. Derenzo, Phys. Rev. **181**, 1854 (1969).

1.2 Spectrum Measurement Procedure

In concept, the measurement of the η parameter is very straightforward. Muons (μ^+ to avoid atomic or nuclear capture) are stopped in a thin target, where they decay; a narrow line-width spectrometer is used to measure the momentum spectrum intensity at several points in each of three regimes:

- The kinematic endpoint provides a momentum calibration for the spectrometer at $P_e = P_e(max) = \frac{M_\mu^2 - m_e^2}{2M_\mu}$. Also, because of the extremely sharp drop in intensity at the endpoint, as well as the fact that the spectrum shape here is well-known and almost independent of η , it is possible to verify the calculated line shape of the spectrometer: one simply folds the calculated line shape with the theoretical spectrum and compares to the measurement in this region.
- The midrange of the spectrum is also fairly insensitive to η and, therefore, when data are fit to the theory spectrum with two free parameters, η and amplitude, provides amplitude normalization. This allows the lower-energy data to influence mostly the determination η . While the event rate in the midrange is not as high as that nearer the endpoint, the beam time required for adequate normalization is not excessive, and the non-statistical uncertainties are smaller. One of these is the effect of ρ , which largely determines the shape of the upper half of the unpolarized spectrum; while the amplitude of the lower half is also sensitive to ρ , its *shape* is much less so. This will be discussed further in Section 2.7.
- The region between $x = 0.1$ and $x = 0.4$ holds the maximum statistical sensitivity to η for a narrow line-width spectrometer. This is shown in Figure 3 in terms of the beam time required for the spectrometer used in this experiment to achieve a statistical precision in η of ± 0.088 , for a fixed amplitude normalization and a muon flux of 40K/s. One point to be made is that the lower statistical sensitivity for $x < 0.1$ is partly the result of the spectrometer type. If P_{tune} is the mean of the momentum acceptance at a given magnetic field, and ΔP_{accept} is the FWHM of that acceptance,

then $\Delta P_{\text{accept}} \propto P_{\text{peak}} \approx E_{\text{peak}}$). Thus, the data rate for small x varies nearly as x^3 , rather than as x^2 like the spectrum intensity.

An important thing to note about the procedure as outlined above is that the experiment is self-normalizing. The analysis is complicated by various energy-dependent corrections (for scattering, detection efficiency and so forth), but the experiment is fundamentally simple.

1.3 Muon Beam Characteristics

In an experiment such as this, one needs both very large numbers of μ^+ (to allow adequate statistics, despite the few accepted e^+ at any given, η -sensitive P_{tune}) and a beam with a minimum of range straggling (so that the thickness of the stopping target and, consequently, spectrum distortions can be minimized). The FWHM range straggling, ΔR , for muons in the momentum range of interest, has been approximated as⁴

$$\Delta R = \sqrt{(0.10)^2 + \left(\frac{7}{2} \frac{\Delta P}{P}\right)^2} R, \quad (1.2)$$

where

$$R \propto P^{7/2}.$$

P is the muon momentum and ΔP is the FWHM momentum spread of the beam; it is assumed that $\Delta P/P \ll 1$. The first term of Equation 1.2 then gives the intrinsic straggling, while the second term is due to the momentum spread of the beam; the contributions are equal in this formula when $\Delta P/P = 2.9\%$.

Since the absolute range straggling will decrease for muon beams with low momentum and small spread in momenta, achieving small ΔR requires a low-momentum beam with enough intensity that adequate rate resides in a small momentum slice. Thus, it is the meson factories such as TRIUMF that made this experiment practical with their "surface muon beams," produced by pions decaying at rest near the production target surface.

⁴A. E. Pifer, T. Bowen and K. R. Kendall, Nucl. Instrum. Methods **135**, 39 (1976).

These beams have $P_\mu \approx \frac{M_\pi^2 - M_\mu^2}{2M_\pi} = 29.80 \text{ MeV}/c$ and, for a momentum bite of 2% in the M13 beam line at TRIUMF, muon rates in excess of $10^5/s$.

In actuality, the intrinsic straggling is not as high as that given by Equation 1.2, at least in low- Z materials. It is also more complicated, with scattering giving rise to both material and geometrical dependence in stopping distributions, as well as a long tail toward short ranges. The distribution is best determined by Monte Carlo calculation, as discussed in Section 6.1.1.

1.4 Studies with a Positron Beam

In addition to μ^+ , the M13 beam line at TRIUMF delivers several other particle species. None of these presented a significant difficulty in this experiment, and the e^+ content was of some value for various checks and calibrations.

One important calibration is that of the target counter scintillator in terms of energy deposition by e^+ 's. By comparing the counter's output from accepted decay e^+ with that from e^+ of known energy passing through a known thickness, it is possible to measure the thickness through which the accepted decay e^+ pass.

Another procedure performed was the search for contamination of the low-energy portion of the spectrum by high-energy e^+ . For this, the spectrometer was positioned with the beam e^+ 's entering the angular acceptance; data were taken at various settings of the magnetic field. While it is not practical to use this information directly to correct the spectrum, it provides a useful check on the existence of contaminations and their approximate size.

Finally, the e^+ data provided a check on the spectrometer momentum calibration. The beam-line tune at the surface muon momentum is known, and positrons of this momentum define a secondary calibration point.

1.5 Systematic Effects

The systematic errors are displayed in Table 6.8 on page 92, and a discussion of the error estimates appears in Appendix E. As shown in the table, the primary systematic effects derive from decay positron interactions in the spectrometer: in the plastic scintillator used to stop the muons, in the counters which detect the decay positrons and in the intervening material of the spectrometer. Measurements of the low-energy portion of the spectrum are very vulnerable to contamination by positrons of higher energy which lose energy by various processes.

Energy losses in the muon-stop target, including the large ones due to Bhabha scattering and bremsstrahlung, make a major contribution. While correction for this can be made in principle, uncertainties in the muon depth in the target, in the relevant cross-sections, in counter calibrations and in the Monte Carlo calculations leave some error.

The detection efficiency for positrons is energy dependent due to scattering and annihilation, which is not completely corrected due to uncertainties in counter calibrations and geometry, as well as approximations in its calculation. The major interactions in other parts of the spectrometer appear in Table 6.8 under the entries for “back plate,” “cables,” “K²” and “C1.” The relevant geometry is shown in Figure 8. Estimation of these processes is affected by several things, including approximations and statistics in Monte Carlos, counter calibrations, geometrical errors and uncertainties in the cross sections for the physical processes. The “back plate” correction, made for contamination due to showering in the spectrometer back plate, has an empirically-determined component in addition to the Monte Carlo result, and is limited in accuracy mostly by statistics on the measurement.

The other significant error, listed in Table 6.8 as “ μ^+ spin angle,” reflects one mechanism for the muon polarization not being completely canceled in the measurement. This would occur for the partially-polarized muons of this experiment if their spins were not, on the average, perpendicular to the spectrometer axis.

Chapter 2

The Muon Decay Spectrum

2.1 Spectrum for General Interactions

Within the framework of the Weinberg-Salaam theory, muon decay has been calculated^{1,2} to second order. The conclusion is that, for reasonable masses of the Higgs boson, the spectrum shape differs only negligibly from that found using the four-point Fermi interaction, provided that the Fermi coupling constant is suitably redefined:

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} [1 + O(\alpha)] .$$

There is then no difficulty in retaining the four-point interaction formalism, and we shall do so.

There are several conventions which are currently used in writing a general Hamiltonian for muon decay. For historical reasons relating to the non-detection of decay neutrinos, the most common form is the charge-retention ordering, and it is for this form that results are

¹D. A. Ross, Nucl. Phys. **B51**, 116 (1973).

²A. Donnachie and J. Mohammad, CERN Report TH-2132 (1976).

usually specified:

$$\mathcal{H}_{cr} = \sum_j \left(\bar{\Psi}_e O_j \Psi_\mu \right) \bar{\Psi}_{\nu_\mu} O_j (C_j + C'_j \gamma_5) \Psi_{\nu_e} + h.c.$$

$$j = S, V, T, A, P$$

$$O_S = 1, \quad O_V = \gamma^\mu, \quad O_T = \frac{1}{\sqrt{2}} \sigma^{\mu\nu}, \quad O_A = \gamma^\mu \gamma_5, \quad O_P = i\gamma_5.$$

However, another form often appears in theoretical calculations; this charge-exchange order is more physical, given the success of the Weinberg-Salaam theory:

$$\mathcal{H}_{ce} = \sum_j \left(\bar{\Psi}_{\nu_\mu} O_j \Psi_\mu \right) \bar{\Psi}_e O_j (\hat{C}_j + \hat{C}'_j \gamma_5) \Psi_{\nu_e} + h.c.$$

The O_j are defined as before.

In either of these forms there are, most generally, 10 complex constants, C_j, C'_j or $\hat{C}_j, \hat{C}'_j$. Dismissing one variable for the arbitrary, overall phase, 19 real parameters remain, in general, to be determined by experiment. Since $\mathcal{H}_{cr}$ and $\mathcal{H}_{ce}$ must be physically equivalent, it is clear that the parameters of the two representations are closely related. Conversions between C_j and $\hat{C}_j$, or between C'_j and $\hat{C}'_j$, are given by linear Fierz³ transformations:

$$C_i = \sum_j \Lambda_{ij} \hat{C}_j, \quad \hat{C}_i = \sum_j \Lambda_{ij} C_j,$$

$$C'_i = \sum_j \Lambda_{ij} \hat{C}'_j, \quad \hat{C}'_i = \sum_j \Lambda_{ij} C'_j.$$

The explicit form of (Λ_{ij}) is

$$(\Lambda_{ij}) = \frac{1}{4} \begin{pmatrix} -1 & -4 & -6 & 4 & 1 \\ -1 & 2 & 0 & 2 & -1 \\ -1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & -4 & 6 & 4 & -1 \end{pmatrix}.$$

Naturally, (Λ_{ij}) would be different for another definition of the O_j operators; the lack of a standard definition amongst authors (and their frequent failure to specify the convention used) makes the literature a hazardous place in which to compare theoretical discussions.

³M. Fierz, Z. Phys. **104**, 553 (1937).

There is also another notation which has been developed by Scheck.^{4,5} This is the helicity-projection form in which terms correspond to states of definite helicity for massless particles, simplifying expressions in many treatments. This form will not be used in this thesis and is mentioned only for completeness.

Independent of the convention used, the $\mu^\pm$ decay spectrum without radiative corrections is given below for the case when all aspects of the $e^\pm$ are measured and neither of the neutrinos is detected.^{6,7} Note that only 10 real parameters involving the interaction coupling constants appear in the spectrum formula, meaning that nine parameters are undetermined in the most general case, when neutrinos are not observed:

$$\begin{aligned} \frac{d^2\Gamma^{(0)}(x,\theta,\phi,\psi)}{dx d(\cos\theta)} = & 6 \frac{A}{16} \frac{M_\mu^5}{192\pi^3} \left(1 + \frac{m_e^2}{M_\mu^2}\right)^4 \sqrt{x^2 - x_0^2} \cdot \\ & \left\{ \left[x(1-x) + \frac{2}{3}\rho(4x^2 - 3x - x_0^2) + \eta x_0(1-x) \right] \right. \\ & \pm \frac{1}{3}\xi\sqrt{x^2 - x_0^2} \cos\theta \left[1 - x + \frac{2}{3}\delta \left(4x - 3 - \frac{m_e}{M_\mu}x_0 \right) \right] \\ & \pm \xi'\sqrt{x^2 - x_0^2} \cos\phi \left[1 - x + \frac{2}{3}\delta' \left(4x - 3 - \frac{m_e}{M_\mu}x_0 \right) \right] \\ & + \frac{1}{3}\xi'' \cos\theta \cos\phi [x(1-x) + \frac{2}{3}\rho'(4x^2 - 3x - x_0^2) + \eta'x_0(1-x)] \\ & + \sin\theta \sin\phi \cos\psi \left[(1-x)x_0 \frac{3a-2b-2c}{3A} + x(1-x) \frac{\alpha}{A} + (x-x_0^2) \frac{2\beta}{3A} \right] \\ & \left. + \sin\theta \sin\phi \sin\psi \sqrt{x^2 - x_0^2} \left[(1-x) \frac{\alpha'}{A} + \frac{2}{3} \left(1 - \frac{m_e}{M_\mu}x_0 \right) \frac{\beta'}{A} \right] \right\} . \end{aligned}$$

As before, $x = E_e/E_e(max)$, where $E_e(max) = (M_\mu^2 + m_e^2)/2M_\mu$; $x_0 = m_e/E_e(max)$. Obviously, one has

$$x_0 \leq x \leq 1 .$$

In order to specify the angles, $\vec{\zeta}_\mu$ is defined as the direction of the muon spin, $\vec{P}_e$ as the momentum of the emitted $e^\pm$ and $\vec{\zeta}_e$ as its spin direction. Then, θ is the angle between $\vec{P}_e$ and $\vec{\zeta}_\mu$; ϕ is the angle between $\vec{P}_e$ and $\vec{\zeta}_e$; and ψ is the azimuthal angle by which $\vec{\zeta}_e$ is rotated from $\vec{\zeta}_\mu$, around $\vec{P}_e$.

⁴F. Scheck, in *Leptons, Hadrons and Nuclei* (North Holland, Amsterdam, 1983), Chap. 5, sec. 6.2.2.

⁵K. Mursula and F. Scheck, Nucl. Phys. **B253**, 189 (1985).

⁶T. Kinoshita and A. Sirlin, Phys. Rev. **108**, 844 (1957).

⁷F. Scheck, Phys. Reports **44**, 187 (1978).

The remaining variables depend only upon the coupling constants and are given here for the charge-retention form; the following real, bilinear combinations are defined as

$$\begin{aligned}
 a &= |C_S|^2 + |C'_S|^2 + |C_P|^2 + |C'_P|^2 \\
 \alpha &= |C_S|^2 + |C'_S|^2 - |C_P|^2 - |C'_P|^2 \\
 b &= |C_V|^2 + |C'_V|^2 + |C_A|^2 + |C'_A|^2 \\
 \beta &= |C_V|^2 + |C'_V|^2 - |C_A|^2 - |C'_A|^2 \\
 c &= |C_T|^2 + |C'_T|^2 \\
 a' &= 2\text{Re}(C_S C'_P{}^* + C'_S C_P{}^*) \\
 b' &= 2\text{Re}(C_V C'_A{}^* + C'_V C_A{}^*) \\
 c' &= -2\text{Re}(C_T C'_T{}^*) \\
 \alpha' &= 2\text{Im}(C_S C'_P{}^* + C'_S C_P{}^*) \\
 \beta' &= 2\text{Im}(C_V C'_A{}^* + C'_V C_A{}^*)
 \end{aligned}$$

or, using the Fierz transformation, as

$$\begin{aligned}
 a &= 2 \left(|\hat{C}_V - \hat{C}_A|^2 + |\hat{C}'_V - \hat{C}'_A|^2 \right) + \frac{1}{8} \left(|\hat{C}_S + 6\hat{C}_T - \hat{C}_P|^2 + |\hat{C}'_S + 6\hat{C}'_T - \hat{C}'_P|^2 \right) \\
 \alpha &= \text{Re} \left[(\hat{C}_V - \hat{C}_A)(\hat{C}_S + 6\hat{C}_T - \hat{C}_P)^* + (\hat{C}'_V - \hat{C}'_A)(\hat{C}'_S + 6\hat{C}'_T - \hat{C}'_P)^* \right] \\
 b &= \frac{1}{2} \left(|\hat{C}_V + \hat{C}_A|^2 + |\hat{C}'_V + \hat{C}'_A|^2 \right) + \frac{1}{8} \left(|\hat{C}_S + \hat{C}_P|^2 + |\hat{C}'_S + \hat{C}'_P|^2 \right) \\
 \beta &= -\frac{1}{2} \text{Re} \left[(\hat{C}_V + \hat{C}_A)(\hat{C}_S + \hat{C}_P)^* + (\hat{C}'_V + \hat{C}'_A)(\hat{C}'_S + \hat{C}'_P)^* \right] \\
 c &= \frac{1}{4} \left(|\hat{C}_S - 2\hat{C}_T - \hat{C}_P|^2 + |\hat{C}'_S - 2\hat{C}'_T - \hat{C}'_P|^2 \right) \\
 a' &= \text{Re} \left[4(\hat{C}_V - \hat{C}_A)(\hat{C}'_V - \hat{C}'_A)^* - \frac{1}{4}(\hat{C}_S + 6\hat{C}_T - \hat{C}_P)(\hat{C}'_S + 6\hat{C}'_T - \hat{C}'_P)^* \right] \\
 b' &= \text{Re} \left[(\hat{C}_V + \hat{C}_A)(\hat{C}'_V + \hat{C}'_A)^* - \frac{1}{4}(\hat{C}_S + \hat{C}_P)(\hat{C}'_S + \hat{C}'_P)^* \right] \\
 c' &= -\frac{1}{8} \text{Re} \left[(\hat{C}_S - 2\hat{C}_T - \hat{C}_P)(\hat{C}'_S - 2\hat{C}'_T - \hat{C}'_P)^* \right] \\
 \alpha' &= -\text{Im} \left[(\hat{C}_V - \hat{C}_A)(\hat{C}'_S + 6\hat{C}'_T - \hat{C}'_P)^* + (\hat{C}'_V - \hat{C}'_A)(\hat{C}_S + 6\hat{C}_T - \hat{C}_P)^* \right] \\
 \beta' &= \frac{1}{2} \text{Im} \left[(\hat{C}_V + \hat{C}_A)(\hat{C}'_S + \hat{C}'_P)^* + (\hat{C}'_V + \hat{C}'_A)(\hat{C}_S + \hat{C}_P)^* \right] .
 \end{aligned}$$

From these, the muon decay parameters are defined as

$$\begin{aligned}
 A &= a + 4b + 6c \\
 \rho &= \frac{1}{A}(3b + 6c) \\
 \eta &= \frac{1}{A}(\alpha - 2\beta) \\
 \xi &= -\frac{1}{A}(3a' + 4b' - 14c') \\
 \delta &= \frac{1}{A\xi}(-3b' + 6c') \\
 \xi' &= -\frac{1}{A}(a' + 4b' + 6c') \\
 \delta' &= -\frac{1}{A\xi'}(b' + 2c') \\
 \xi'' &= \frac{1}{A}(3a + 4b - 14c) \\
 \rho' &= \frac{1}{A\xi''}(3b - 6c) \\
 \eta' &= \frac{1}{A\xi''}(3\alpha + 2\beta) .
 \end{aligned}$$

Substitution into the above expressions gives the explicit definition of η ; in the charge-retention form it is

$$\eta = \frac{1}{A} \left[|C_S|^2 + |C'_S|^2 - 2 \left(|C_V|^2 + |C'_V|^2 - |C_A|^2 - |C'_A|^2 \right) - |C_P|^2 - |C'_P|^2 \right] ,$$

where

$$A = |C_S|^2 + 4|C_V|^2 + 6|C_T|^2 + 4|C_A|^2 + |C_P|^2 + (\text{primed terms}) ;$$

and, in the charge-exchange form, it is

$$\eta = \frac{2}{A} \text{Re} \left[\hat{C}_V \hat{C}_S^* + \hat{C}'_V \hat{C}'_S{}^* + \hat{C}_A \hat{C}_P^* + \hat{C}'_A \hat{C}'_P{}^* + 3(\hat{C}_V - \hat{C}_A) \hat{C}_T^* + 3(\hat{C}'_V - \hat{C}'_A) \hat{C}'_T{}^* \right] ,$$

where

$$A = |\hat{C}_S|^2 + 4|\hat{C}_V|^2 + 6|\hat{C}_T|^2 + 4|\hat{C}_A|^2 + |\hat{C}_P|^2 + (\text{primed terms}) .$$

2.2 Physical Motivations

There is presently no compelling evidence of inconsistency with a universal V-A interaction given by

$$(\hat{C}_j) = (C_j) = \frac{G_F}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (\hat{C}'_j) = (C'_j) = \frac{G_F}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

There is also no evidence of neutral decay products aside from a massless (or near massless) neutrino, anti-neutrino and photons. Other possibilities are not excluded, however, and some of these will be discussed below.

2.2.1 Limitations of Positron-Inclusive Measurements

It was previously noted that only 10 of the 19 parameters allowed in the most general Hamiltonian can be measured in muon decay, if neutrinos are not detected. One can, however, ask how much ambiguity would remain in an ideal world of completely accurate measurements on the $e^\pm$, if the commonly accepted V-A description is correct. The answer is that only two, rather than nine, degrees of freedom would remain. Looking at the charge-retention formalism on page 10, one can see that this is because a showing that $a = c = a' = c' = \alpha' = 0$ eliminates 12 degrees of freedom. The two remaining degrees of freedom are illustrated below by expressions for $(\hat{C})$ and $(\hat{C}')$; the positron spectrum is completely independent of the value of ϵ , which is not necessarily either small or real:

$$(\hat{C}_j) = \frac{G_F}{\sqrt{2}} \begin{pmatrix} \epsilon \\ 1 \\ 0 \\ 1 \\ \epsilon \end{pmatrix}, \quad (\hat{C}'_j) = \frac{G_F}{\sqrt{2}} \begin{pmatrix} \epsilon \\ -1 \\ 0 \\ -1 \\ \epsilon \end{pmatrix}.$$

The Hamiltonian corresponding to this would be

$$\begin{aligned}\mathcal{H}_{ce} &= \frac{G_F}{\sqrt{2}} \bar{\Psi}_{\nu_\mu} \gamma^\nu (1 - \gamma_5) \Psi_\mu \bar{\Psi}_e \gamma_\nu (1 - \gamma_5) \Psi_{\nu_e} \\ &+ \epsilon \frac{G_F}{\sqrt{2}} \bar{\Psi}_{\nu_\mu} (1 - \gamma_5) \Psi_\mu \bar{\Psi}_e (1 + \gamma_5) \Psi_{\nu_e} + h.c. ,\end{aligned}$$

which shows that this remaining ambiguity would be eliminated by an helicity measurement upon either of the neutrinos; no such measurement has yet been made, and it is upon faith that “ ν_e ” and “ ν_μ ” are taken to be the same particles as those found in nuclear beta decay and π^+ decay, respectively.

A possible source of this type of Hamiltonian is the exchange of a charged Higgs boson ϕ'' , in addition to the usual W_L^+ , when there are (possibly) massive neutrinos of opposite helicity to the usual ones. A paper by Fayet⁸ develops this possibility. The *effective* Hamiltonian for the decay positrons would be as above, with

$$\epsilon = \frac{m_{\nu'_\mu} m_{\nu'_e}}{m_{\phi''_\pm}^2} .$$

Fayet claims that the positron spectrum would be altered in this case (specifically, that ρ would deviate from 3/4), though this is contradicted by the above discussion. One could obtain Fayet’s result if one were to calculate $\hat{C}, \hat{C}'$ and then inadvertently use them in formulae intended for C, C' .

2.2.2 Lorentz-Structure Physics

Next we will consider what physics might be found as measurements of the $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$ parameters are improved. The following discussion draws heavily upon a paper by Mursula, Roos and Scheck.⁹ The notation, however, has been changed for consistency here.

If one assumes that muon decay is mediated by heavy, charged bosons with spins of 0, 1 and/or 2, the effective four-fermion interaction becomes

$$\mathcal{H}_{ce} = \frac{G_F}{\sqrt{2}} \left[\sum_i K^{(i)\dagger} K^{(i)} + \sum_i J_\alpha^{(i)\dagger} J^{(i)\alpha} + \sum_i T_{\alpha\beta}^{(i)\dagger} T^{(i)\alpha\beta} \right] ,$$

⁸P. Fayet, Nucl. Phys. **B78**, 14 (1974).

⁹K. Mursula, M. Roos and F. Scheck, Nucl. Phys. **B219**, 321 (1983).

where the index i refers to the different charged bosons of a given spin and where

$$\begin{aligned} K^{(i)} &= g_{S,i}^e (\bar{e} 1 \nu_e) + g_{S,i}^\mu (\bar{\mu} 1 \nu_e) + g_{P,i}^e (\bar{e} \gamma_5 \nu_e) + g_{P,i}^\mu (\bar{\mu} \gamma_5 \nu_e) , \\ J_\alpha^{(i)} &= g_{V,i}^e (\bar{e} \gamma_\alpha \nu_e) + g_{V,i}^\mu (\bar{\mu} \gamma_\alpha \nu_e) + g_{A,i}^e (\bar{e} \gamma_\alpha \gamma_5 \nu_e) + g_{A,i}^\mu (\bar{\mu} \gamma_\alpha \gamma_5 \nu_e) , \\ T_{\alpha\beta}^{(i)} &= g_{T,i}^e (\bar{e} \sigma_{\alpha\beta} \nu_e) + g_{T,i}^\mu (\bar{\mu} \sigma_{\alpha\beta} \nu_e) + g_{T',i}^e (\bar{e} \sigma_{\alpha\beta} \gamma_5 \nu_e) + g_{T',i}^\mu (\bar{\mu} \sigma_{\alpha\beta} \gamma_5 \nu_e) . \end{aligned}$$

The constants $g_{i,i}^e$ and $g_{i,i}^\mu$ are the coupling constants for the heavy boson of mass m_i , except for a factor $2^{1/4}/m_i\sqrt{G_F}$. These constants are related to our previous notation by

$$\begin{aligned} \hat{C}_S &= \sum_i g_{S,i}^e g_{S,i}^{\mu*} \\ \hat{C}_V &= \sum_i g_{V,i}^e g_{V,i}^{\mu*} \\ \hat{C}_T &= \sum_i (g_{T,i}^e g_{T,i}^{\mu*} - g_{T',i}^e g_{T',i}^{\mu*}) \\ \hat{C}_A &= \sum_i g_{A,i}^e g_{A,i}^{\mu*} \\ \hat{C}_P &= \sum_i g_{P,i}^e g_{P,i}^{\mu*} \\ \hat{C}'_S &= \sum_i g_{P,i}^e g_{S,i}^{\mu*} \\ \hat{C}'_V &= \sum_i g_{A,i}^e g_{V,i}^{\mu*} \\ \hat{C}'_T &= \sum_i (g_{T,i}^e g_{T',i}^{\mu*} - g_{T',i}^e g_{T,i}^{\mu*}) \\ \hat{C}'_A &= \sum_i g_{V,i}^e g_{A,i}^{\mu*} \\ \hat{C}'_P &= \sum_i g_{S,i}^e g_{P,i}^{\mu*} . \end{aligned}$$

These formulae can be seen to predict relationships between the $\hat{C}_l$ constants, under rather weak assumptions: if there is no more than one exchanged boson with zero spin, one obtains

$$\hat{C}'_S \hat{C}'_P = \hat{C}_S \hat{C}_P$$

or, if there is no more than one exchanged boson with unit spin, one obtains

$$\hat{C}'_V \hat{C}'_A = \hat{C}_V \hat{C}_A .$$

There are further relationships in the event that lepton universality ($g_{i,i}^e = g_{i,i}^\mu$) or weak universality ($g_{i,i}^e = (M_\mu/m_e)g_{i,i}^\mu$) holds.

(Pseudo)scalar Exchange

One class of theories affecting η contains scalar or pseudoscalar exchange in addition to the usual V-A structure. If strict lepton universality holds, the expression for η becomes

$$\eta = -2\frac{\beta}{A} = \frac{2|g_S|^2 + 2|g_P|^2 - 4\text{Re}(g_S g_P^*)}{16|g_V|^2 + (|g_S|^2 + 2|g_P|^2)^2}.$$

However, a significant pseudoscalar coupling is almost excluded by existing measurements¹⁰ of the $\Gamma(\pi^+ \rightarrow e^+ \nu_e)/\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu)$ reaction. While it can be argued that this reaction is not purely leptonic, and that this causes some theoretical uncertainty in its interpretation, the more intriguing case evades these limits through weak universality. This would be the case for a charged Higgs boson, which exists in some non-minimal models. Thus, η becomes

$$\eta = -2\frac{\beta}{A} = \frac{M_\mu}{m_e} \frac{2|g_S^e|^2 + 2|g_P^e|^2 - 4\text{Re}(g_S^e g_P^{e*})}{16|g_V^e|^4 + \frac{M^2}{m_e^2}(|g_S^e|^2 + 2|g_P^e|^2)^2}.$$

These physics are slightly clarified through yet another change of notation, as used by Mursula.¹¹ If we define

$$g_i^e = \frac{M_W}{M_H} \frac{M_e}{M} c_i, \quad$$

where M and c_i ($i = S, P$) are an unknown mass scale and coupling constants, respectively, we can further define

$$\lambda_i = \frac{M_\mu}{m_e} \frac{|g_i^e|}{g_V} = \frac{m_e m_\mu}{M^2} \frac{M_W^2}{M_H^2} \left| \frac{c_i}{g_V} \right|^2, \quad \alpha_s = \arg(c_S c_P^*).$$

Expressions for η and $1 - \xi$ then become

$$\eta = \frac{\lambda_S + \lambda_P - 2\sqrt{\lambda_S \lambda_P} \cos \alpha_s}{16 + (\lambda_S + \lambda_P)^2}$$

and

$$1 - \xi = (\lambda_S + \lambda_P) \eta.$$

Clearly, the sensitivity of these parameters to the hypothetical Higgs physics depends strongly upon the unknown mass scale M . If, Mursula points out, $c_i \approx g_V$, $M \approx M_H$ and

¹⁰D. Bryman *et al.*, Phys. Rev. Lett. **50**, 7 (1983).

¹¹K. Mursula, Univ. Bern Report BUTP-84/25 (1984).

$M_H > 20$ GeV, it would be doubtful that one could ever observe the effect on muon decay, whether or not it existed. Also, the sensitivity would vanish for $\alpha_S = 0$ and $\lambda_S = \lambda_P$. Nonetheless, the possibility of observing these physics in muon decay should not be rejected.

Spin-2 Exchange

In the most general case, inspection of the formula for η shows that there is possible sensitivity to tensor currents. However, $\hat{C}_T^*, \hat{C}_T'^*$ appear with the factors $\hat{C}_V - \hat{C}_A$ and $\hat{C}_V' - \hat{C}_A'$, respectively, so the sensitivity is reduced or eliminated, due to the small upper limit on these factors.

Alternatively, it might be said that η is sensitive to right-left (broken) symmetries only in the presence of fairly strong tensor currents. Thus, at least two hypothetical extensions to the standard model must exist for measurements of η to detect either one. This makes any search for these possibilities, using η , very speculative.

2.2.3 Massive Mixed Neutrinos

It has been pointed out^{12,13,14} that the weak-interaction eigenstates of neutrinos are not necessarily mass eigenstates, so that many of the conventional mass limits on neutrinos do not exclude the possibility that ν_e and/or ν_μ might contain a small admixture of a heavy neutrino. Such an admixture would manifest itself by the incoherent addition of one or more muon decay spectra to the main one with endpoints at

$$(x_{\max})_{i,j} = 1 - \frac{(m_{\nu_i} + m_{\nu_j})^2}{M_\mu^2 + m_e^2},$$

where m_{ν_i} and m_{ν_j} are the masses of the emitted neutrinos. This being so, one should consider the possible effects of massive neutrino mixing in an analysis of the muon decay spectrum; the Lorentz structure cannot be assumed to be the sole determining factor.

¹²A. Sirlin, Proceedings of the TRIUMF Muon Physics/Facility Workshop, p.81 (1980).

¹³R. E. Shrock, Phys. Rev. D **24**, 1275 (1981).

¹⁴P. Kalyniak and J. N. Ng, Phys. Rev. D **24**, 1874 (1981).

In particular, massive neutrinos could result in a measurement of $\eta \neq 0$, even if weak interactions were purely $V - A$. In the paper by Shrock referenced above, Fig. 25 graphs an effective value of η when there is a massive component to one the neutrinos and ρ, η are allowed to vary freely to provide the best fit to the unpolarized muon decay spectrum. However, one must be careful in interpreting this figure, because it has been calculated for one specific experiment. Even then, it is only accurate if one attempts to extract the massive neutrino information from a value of η_{eff} ; this is not the best approach because the effect of massive, mixed neutrinos on the spectrum has an entirely different energy dependence than does a truly non-zero value of η .

To make this explicit, consider a muon decay with one neutrino of mass m_{ν_i} . We define $d = (m_{\nu_i}/M_\mu)^2$, drop terms in m_e and retain our usual definition of x . Also, $\Theta(s) \equiv 1$ for $s \geq 0$ and $\Theta(s) \equiv 0$ for $s < 0$. Then, the spectrum supplement is proportional to

$$2x^2 \left(1 - \frac{d}{1-x}\right)^2 \left(3 - 2x + d \frac{3-x}{1-x}\right) \Theta(x_{max} - x),$$

which for $x \ll 1$ becomes

$$6x^2(1-d)^2(1+d)\Theta(x_{max} - x).$$

This clearly has little in common with the usual term proportional to η :

$$12\eta x_0 x(1-x).$$

An additional problem exists when considering the effect upon the low-energy part of the decay spectrum: the radiative corrections for massive neutrinos differ from those for the massless case: the correction terms are of order $(m_{\nu_i}/M_\mu)^2$ and higher, where ν_i is the massive neutrino component. A calculation has been performed¹⁵ in which terms proportional to powers of m_e were dropped, but which allows for massive neutrinos. The main effect is to reduce the spectrum at low energies and near the endpoint. This is just as one would expect: the effect on the bulk of the spectrum is small since the neutrinos are

¹⁵P. Kalyniak and J. N. Ng, Phys. Rev. D **25**, 1305 (1982).

not involved in the photon diagrams directly. However, the spectrum near the endpoint is reduced by the emission of soft photons and, since fewer high-energy positrons are kinematically allowed, hard photon emission does not increase the low-energy spectrum as much as in the massless neutrino case.

In general, pseudoscalar decays are intrinsically more sensitive to massive neutrinos than the 3-body decay of muons: the decay $\pi^+ \rightarrow \mu^+ \nu_\mu$ has been used to establish¹⁶ $|U_{\mu i}|^2 < 2 \times 10^{-4}$ for $10 \text{ MeV} < m_{\nu_i} < 30 \text{ MeV}$, and the decay $K^+ \rightarrow \mu^+ \nu_\mu$ has been used to set¹⁷ the limit of $|U_{\mu i}|^2 < 10^{-4}$ for $70 \text{ MeV} < m_{\nu_i} < 335 \text{ MeV}$. For m_{ν_i} between 30 MeV and 70 MeV, however, the current limits are relatively weak; Shrock¹⁸ has used the ρ parameter of muon decay to set the limit $|U_{\mu i}|^2 < 10^{-2}$ for $12 \text{ MeV} < m_{\nu_i} < 63 \text{ MeV}$. This limit is tightest near $m_{\nu_i} = 40 \text{ MeV}$, where $|U_{\mu i}|^2 < 2 \times 10^{-3}$. Above this range, the best limits come from an analysis¹⁹ of existing data on tritium recoil in the reaction $\mu^- + {}^3\text{He} \rightarrow \nu_\mu + {}^3\text{H}$. The limit set is $|U_{\mu i}|^2 < 10^{-2}$ for $60 \text{ MeV} < m_{\nu_i} < 72 \text{ MeV}$.

For $30 \text{ MeV} < m_{\nu_i} < 70 \text{ MeV}$, especially, it would be desirable to improve the limits on $|U_{\mu i}|^2$, although the current limit on the tau neutrino mass ($m_{\nu_\tau} < 35 \text{ MeV}$ at the 95% confidence level²⁰) reduces the motivation. High-statistics muon decay spectrum measurements may improve limits on $|U_{\mu i}|^2$ for m_{ν_i} in, at least, part of this range. In line with comments earlier in this section, it is possible to obtain more accurate limits on $|U_{\mu i}|^2$ than a casual reading of Shrock's paper might lead one to believe.

2.2.4 Light Scalar Neutrinos

As Buchmüller and Scheck²¹ have pointed out, if neutrinos and the W have supersymmetric partners ($\tilde{\nu}_e$, $\tilde{\nu}_\mu$ and $\tilde{W}$) and the scalar neutrinos are light enough for the decay to be

¹⁶R. Abela *et al.*, Phys. Lett. **105B**, 263 (1981).

¹⁷R. S. Hayano, Phys. Rev. Lett. **49**, 1305 (1982).

¹⁸R. E. Shrock, Phys. Rev. D **24**, 1275 (1981).

¹⁹J. P. Deutsch, M. Lebrun and R. Prieels, Phys. Rev. D **27**, 1644 (1983).

²⁰G. P. Yost *et al.*, Review of Particle Properties, Phys. Lett. **204B**, 1 (1988).

²¹W. Buchmüller and F. Scheck, Phys. Lett. **145B**, 121 (1984).

kinematically allowed, the positron spectrum from μ^+ decay would be affected. For very light scalar neutrinos, the effect on the spectrum mimics changes in ρ , δ and ξ . In this case, the current values of ρ and δ set similar limits on the mass of the $\tilde{W}$, and the combined limit stated by Buchmüller and Scheck is $M_{\tilde{W}}/M_W > 4$. Polarization measurements are not sensitive to these physics, since the decay positrons are still completely polarized (in the $m_e = 0$ approximation). Also, the combination ξ_S^P is only very weakly sensitive.

The limits from ρ and δ do not apply when the $\tilde{\nu}_e, \tilde{\nu}_\mu$ masses are large enough to place the endpoint below the range over which the most accurate ρ and δ measurements were done; one must then look to the lower part of the spectrum. In the limit that $m_e = 0$ and $m_{\tilde{\nu}_e} \ll M_\mu$, the comparison to the usual unpolarized spectrum is given by

$$\frac{d\tilde{\Gamma}/dx}{d\Gamma/dx} = \epsilon \Theta(1 - r - x) \left(1 - \frac{r}{1-x}\right)^3 \frac{3-x}{3-2x},$$

where

$$\epsilon = \left(\frac{M_W}{M_{\tilde{W}}}\right)^4, \quad r = \left(\frac{m_{\tilde{\nu}_\mu}}{M_\mu}\right)^2.$$

As with the massive Dirac neutrinos, this spectrum effect does not mimic a term proportional to η , so a direct fit to the measured spectrum should be used to set limits.

2.2.5 Familon

If the lepton family symmetry is global, its breaking leads to a massless Goldstone boson, G . (By contrast, if the symmetry is local, the breaking leads to mirror fermions.) It has been noted²² that a measurement of the ratio $\Gamma(\mu \rightarrow eG)/\Gamma(\mu \rightarrow e\nu\bar{\nu})$ would establish, or place limits on, the breaking scale.

Since the Goldstone boson is massless, the effect for which one is searching is a small spike in the positron energy spectrum at $x = 1$. For this search one would prefer an instrument of moderate acceptance and extremely high resolution, so as to resolve a narrow spike at the endpoint of the conventional spectrum. The Comus spectrometer is, unfortunately,

²²G. B. Gelmini, S. Nussinov and T. Yanagida, Nucl. Phys. **B219**, 31 (1983).

not well-matched to this search due to its 2.8% FWHM resolution; the width of the spike, by contrast, would be of order $10^{-13}\%$, being spread only by radiative corrections.

2.2.6 Majorons

The spectrum for $\mu \rightarrow eMM$, where the Majoron M is a hypothetical Goldstone boson coupling to neutrinos, has been calculated.²³ The same calculation applies to any light scalar or pseudoscalar particle that couples to neutrinos. The comparison to the usual unpolarized spectrum, in the $m_e = 0$ limit, is given by

$$\frac{d\Gamma_M/dx}{d\Gamma/dx} = \frac{24\alpha_M^2}{\pi^2} \left(\ln \frac{M_W}{M_\mu} \right)^2 \frac{1}{3-2x} \approx \frac{107\alpha_M^2}{3-2x} ,$$

where

$$\alpha_M = \sum_i \frac{g_{\mu i}^* g_{ie}}{4\pi}$$

and the g_{ij} are defined by the Majoron-neutrino couplings

$$-ig_{ij}\bar{\nu}_i^c\gamma_5\nu_j M .$$

Clearly, the limits on this model are best provided by measurements of the high-energy part of the spectrum.

Another related effect on the spectrum is Majoron bremsstrahlung, the rate of which was also calculated by Goldman, Kolb and Stephenson in the same work:

$$\begin{aligned} \frac{d\Gamma_B/dx}{d\Gamma/dx} &= \frac{\alpha_B}{8\pi} \left[2 \ln \frac{M_\mu}{m_e} - 1 - \ln \frac{x}{2} + \frac{x+\ln(1-x)}{x^2(3-2x)} \right] \\ &\approx \frac{\alpha_B}{8\pi} \left[9.663 - \ln \frac{x}{2} + \frac{x+\ln(1-x)}{x^2(3-2x)} \right] , \end{aligned}$$

where

$$\alpha_B = \left(|g_{ee}|^2 + |g_{\mu e}|^2 + |g_{\tau e}|^2 + |g_{e\mu}|^2 + |g_{\mu\mu}|^2 + |g_{\tau\mu}|^2 \right) / 4\pi .$$

While the relative effect on the spectrum would be larger at low energies, better limits would probably be provided by high-energy measurements, where high accuracy can be more easily achieved. The most accurate limit in muon decay would probably not use the

²³T. Goldman, E. W. Kolb and G. J. Stephenson, Jr., Phys. Rev. D **26**, 2503 (1982).

unpolarized spectrum at all, but, rather, would take advantage of the fact that $d\Gamma_B/dx$ is isotropic; one would derive the limit from measurements near $x = 1$ in the direction opposite to the muon spin (for μ^+) where the rate is almost zero for the conventional decay.

2.2.7 Impact upon ρ Measurements

While the coupling between fitted values of η and ρ does not constitute a physical effect within the meaning of this section, it does provide another motivation to obtain an accurate measurement of η . Derenzo²⁴ discusses the correlation between the combined measurements of ρ and a measurement of η . His result could be reasonably summarized as

$$\rho = [0.7523 + 0.0044\eta] \pm [0.0024^2 + (0.0044\Delta\eta)^2]^{1/2}.$$

Thus, typical measurements of ρ must either assume that $\eta = 0$ or be limited in their accuracy to about $0.0044\Delta\eta$. This degree of correlation varies, of course, depending upon how a particular measurement of ρ distributes the statistical weight over the spectrum.

2.3 Unpolarized Spectrum

We now specialize the decay rate to a spectrometer in which the acceptance is nearly azimuthally symmetric around an axis at $\theta = 90^\circ$ and the $e^\pm$ spin is not detected. This averages ϕ , the angle between the positron momentum and its spin direction, between 0 and π ; ψ , the angle by which the spin is rotated around the momentum, is averaged between 0 and 2π :

$$\frac{d^2\Gamma^{(0)}(x)}{dx d(\cos\theta)} = \frac{A}{16} \frac{M_\mu^5}{192\pi^3} \left(1 + \frac{m_e^2}{M_\mu^2}\right)^4 x \sqrt{x^2 - x_0^2} \left[6(1-x) + \frac{4}{3}\rho(4x-3 - \frac{x_0^2}{x}) + 6\eta x_0 \frac{1-x}{x}\right].$$

Integrating over θ , we obtain the unpolarized rate at x :

$$\frac{d\Gamma^{(0)}(x)}{dx} = \frac{A}{16} \frac{M_\mu^5}{192\pi^3} \left(1 + \frac{m_e^2}{M_\mu^2}\right)^4 \mathcal{A}\left(\frac{x}{B}\right) 2x \sqrt{x^2 - x_0^2} \left[6(1-x) + \frac{4}{3}\rho(4x-3 - \frac{x_0^2}{x}) + 6\eta x_0 \frac{1-x}{x}\right].$$

²⁴S. E. Derenzo, Phys. Rev. **181**, 1854 (1969).

Here we have defined

$$\mathcal{A}\left(\frac{x}{B}\right) = \frac{1}{2} \int_0^\pi a(\theta, x/B) \sin \theta \, d\theta ,$$

where B is the magnetic field at some reference point in the spectrometer and $a(\theta, x/B)$ is the probability that a particle emitted at θ with energy corresponding to x will be accepted.

This recovers Equation 1.1, except for the factor $\mathcal{A}(x/B)$ and a few previously-ignored terms in x_0 . Thus, providing measurements are corrected for $\mathcal{A}(x/B)$, one has a measurement of the unpolarized decay spectrum.

2.4 First-Order Radiative Corrections

While the calculation of the muon decay spectrum seems straightforward, a complication exists: the charged particles in the decay couple to photons, so that diagrams with internal or external photon lines must be included in serious calculations. The fractional effect on the spectrum can be much larger than naive estimates of $O(\alpha)$; at $x = 0.1$, for example, the rate is increased by about 25%, and by even more at smaller x . The corrected spectrum, calculated for the V-A interaction and normalized so that the integral over x is unity, is shown in Figure 4, while the fractional effect of the first-order radiative corrections is shown in Figure 5. There is a logarithmic divergence to $-\infty$ at $x = 1$.

Because of the large size of the radiative corrections, a natural concern is the extent to which they are model-dependent. This concern is deepened by what appears to be the only explicit, published calculation of these corrections in the standard model, that of Fukuda and Sasaki.²⁵ They find a term proportional to $\log(m_e/M_W)$ in the radiative corrections, which causes a divergent deviation from the four-point Fermi interaction as $M_W \rightarrow \infty$. This contradicts experiment, as well as expectation, and there can be little doubt that an error exists in their work.

²⁵R. Fukuda and R. Sasaki, *Lett. Nuovo Cimento* 10, 17 (1974).

Ross²⁶ and Sirlin²⁷ more reasonably conclude that the spectrum corrections are terms of order $\alpha(M_\mu^2/M_W^2)$, $\alpha(q^2/M_W^2)$ and $\alpha(m_e^2/M_W^2)$ (q is the momentum transfer in the diagram), which are completely negligible for any likely experiment. Elsewhere²⁸ Sirlin estimates the corrections to the parameters ρ and ξ as 5.8×10^{-7} and 1.0×10^{-6} , respectively. He does not, however, give values for the corrections to the other parameters, or an explicit form for the corrections.

Because corrections due to the finite mass of the W are so small for a V-A interaction, one expects to be able to do general radiative calculations in the four-point Fermi interaction model. This is, however, not possible for general weak-interaction Lagrangians, as Sirlin discusses:

For S , P , T , S' , P' , and T' interactions in the charge-retention order the corrections are divergent, which is a reflection of the non-renormalizability of the local theory. In comparing experiments with the general four-component theory, it has become customary to describe the radiative corrections by means of the finite expressions obtained for the V , A , V' , and A' interactions. The justification for this procedure is that the experimental information is consistent with pure V , A , V' , and A' interactions and, therefore, terms of order $\alpha/2\pi$ times ($|C_S|^2$, $|C'_S|^2$, $|C_T|^2$, $|C'_T|^2$, $|C_P|^2$, and $|C'_P|^2$) are regarded as being of second order in the small quantities. To the extent that measurements are consistent with the existence of these interactions only, the procedure is rational to check the consistency. From a theorist's point of view, it is more satisfactory to restrict oneself to the two-component theory (in which case the corrections are finite), and attempt to ... verify the quality of the fit.

Thus, should it be determined that experiment is inconsistent with the inclusion of only vector and axial-vector couplings, the traditional formulae for radiative corrections would

²⁶D. A. Ross, Nucl. Phys. **B51**, 116 (1973).

²⁷A. Sirlin, Nucl. Phys. **B71**, 29 (1974).

²⁸A. Sirlin, Proceedings of the TRIUMF Muon Physics/Facility Workshop, p.81 (1980).

need to be replaced before one could be said to have determined the level of deviation.

Radiative corrections to the spectrum were first studied in the late fifties.^{29,30} In these early treatments, m_e was set to zero wherever this would not cause a spurious divergence. For the low positron energies with which we are concerned, the calculation by Grotch³¹ is more accurate, as it does not make this approximation. The remaining approximations are that it, like the previous works, considers only single photon diagrams with a four-point Fermi interaction involving only $C'_V = C_V$ and $C'_A = C_A$, with all other $C_j = C'_j = 0$. This implies that $\rho = 3/4$, among other things.

Grotch also provides an approximate formula which he says is "good to a few percent down to $E_e = 3m_e$." The approximation is much better than this modest statement implies, as Figure 6 shows in the region below $x = 0.1$. The accuracy is even better in the region used for this η parameter measurement. Nonetheless, the exact first-order formula was used in the calculations for this experiment (since it was already computed to check the approximation). It is given below in a different notation, which allows one to calculate the spectrum as an explicit, linear function of η . This simplifies the fitting of the data to the theory, compared to Grotch's representation.

$$\frac{d\Gamma(x, \eta)}{dx} = \frac{|C_A|^2 + |C_V|^2}{2} \frac{M_\mu^5}{192\pi^3} 2x(x^2 - x_0^2)^{\frac{1}{2}} [f_1(x) + \eta f_2(x)] ,$$

where

$$\begin{aligned} \eta &= \frac{1}{2} \frac{|C_A|^2 - |C_V|^2}{|C_A|^2 + |C_V|^2} , \\ f_1(x) &= A(x) + \frac{e^2}{2\pi} [A(x)B(x) + C(x) + E(x)F(x) + G(x)H(x)] , \\ f_2(x) &= F(x) + \frac{e^2}{2\pi} \left[4A(x)E(x) + B(x)F(x) + 2H(x) \frac{\theta}{\sinh \theta} \right] , \end{aligned}$$

²⁹R. E. Behrends, R. J. Finkelstein and A. Sirlin, Phys. Rev. **101**, 866 (1956).

³⁰T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959).

³¹H. Grotch, Phys. Rev. **168**, 1872 (1968).

and where

$$A(x) = 3 - 2x - \frac{x_0^2}{x}$$

$$B(x) = B_1(x) + 2B_2(x) \coth \theta$$

$$B_1(x) = -2\omega + V(x) \left[\omega + \theta - Z(x) - \frac{e^{-\omega}}{\cosh \theta} \right] + 2 \left[V(x) + \frac{\sinh \omega}{\sinh \theta} - 1 \right] \ln (1 - e^{-\theta-\omega}) \\ + 2 \left[V(x) - \frac{\sinh \omega}{\sinh \theta} - 1 \right] \ln (1 - e^{\theta-\omega}) + R(x) \left(\cosh \omega - \frac{2}{3} \cosh \theta \right)$$

$$B_2(x) = (\theta - \omega) \ln \left(\frac{e^{\omega} - e^{\theta}}{e^{\omega+\theta} - 1} \right) - \frac{\pi^2}{12} + \left[\theta + \frac{1}{2} \ln 2 - Z(x) \right] (2\theta + \ln 2) + Z(x) [Z(x) - \theta] \\ + L \left(\frac{2 \sinh \theta}{e^{\theta} - e^{-\omega}} \right) - L \left(\frac{2 \sinh \theta}{e^{\omega} - e^{-\theta}} \right) + L(e^{-\theta-\omega}) - L(e^{\theta-\omega}) + L(\tanh \theta) \\ + L \left(\frac{1 - e^{-2\theta}}{2} \right) + L \left(\frac{1}{1 + e^{2\theta}} \right)$$

$$C(x) = 4 \frac{x_0^2}{x} \left[\frac{\theta}{\tanh \theta} - 1 \right] \sinh^2 \omega$$

$$E(x) = \frac{1}{6} R(x) - \frac{\theta}{\sinh \theta}$$

$$F(x) = 6 \frac{x_0}{x} (1 - x)$$

$$G(x) = \frac{5}{3} \left[(2 \cosh \theta + \cosh \omega) \frac{\theta}{\sinh \theta} - 2 \right]$$

$$H(x) = \frac{x_0^2}{x} (\cosh \omega - \cosh \theta)^2$$

$$L(t) = \int_0^y [\ln(1 - t')/t'] dt'$$

$$R(x) = \frac{\omega \sinh \omega - \frac{2}{3} \theta \sinh \theta}{(\cosh \omega - \frac{2}{3} \cosh \theta)^2 - \frac{1}{9}}$$

$$V(x) = 2\theta \coth \theta$$

$$Z(x) = \ln (1 + e^{2\theta}) .$$

These expressions use two new variables:

$$\cosh \theta = x \quad \text{and} \quad \omega = \ln \left(\frac{M_\mu}{m_e} \right) .$$

2.5 Higher-Order Radiative Corrections

Because of the size of the first-order radiative corrections, it is not obvious that higher-order diagrams will not also have significant effects. Indeed, at the spectrum endpoint, higher-order terms constitute an infinite correction, eliminating the logarithmic, infrared-

photon divergence in the first-order terms.^{32,33,34} While theorists agree that these infrared corrections are to be handled by exponentiating terms in the first-order correction, they disagree on the specifics of the calculation. Fortunately, these discrepancies are not very large and have no significant effect on this measurement of η . The fractional effect of the exponentiation, according to the prescription of Ross, is shown in Figure 7.

Higher-order terms are much more difficult to calculate away from the endpoint since the photons are no longer necessarily soft; a comprehensive calculation has not been done. However, a calculation³⁵ for $x < 0.1$ implies that the effect is small for this measurement of η . Though the relative effect on the spectrum becomes substantial at lower energies, the spectrum is increased by only about 0.09% at $x_e = 0.1$. This corresponds to an effect on a measurement of η of about 0.007—were the measurement to be done with a single data point at $x_e = 0.1$ and an absolute normalization (the effect on the spectrum shape in no way simulates a value of η). The effective value of η fit with this procedure is almost proportional to $1/x$ at $x = 0.1$.

The conclusion is that it is adequate to use only the first-order radiative corrections for this particular measurement of the η parameter. Measurements of higher accuracy ($\Delta\eta < 0.01$) or which use very low-energy positron data ($x < 0.1$) may need to include higher-order terms.

2.6 Radiative Decay

Closely related to radiative corrections is radiative decay, $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu \gamma$. The literature contains references to this process as having a relative branching ratio of about 10^{-4} , which is surprisingly small in view of the size of the radiative corrections, even though virtual photon processes add to the latter. The resolution of this paradox is that only decays with

³²L. Mattson, Nucl. Phys. **12B**, 647 (1969).

³³K. A. Edin, K. E. Eriksson, V. Gerdjikov and L. Matsson, Physica Scripta **2**, 237 (1970).

³⁴D. A. Ross, Nuovo Cim. **10A**, N. 3, 475 (1972).

³⁵A. V. Kuznetsov and N. V. Mikheev, Sov. J. Nucl. Phys. **31**(1), (1980).

γ 's more energetic than about 45 MeV are included in this 10^{-4} figure; this is presumably because actual measurements of the photon energy spectrum are done at high energies to avoid bremsstrahlung contamination.

Calculations of radiative decay have been done for both the V-A interaction^{36,37} and general Lorentz structures.³⁸ Results show that if the arbitrary low-energy cut is made at $2m_e$, the branching ratio rises to 4.9% and that, for low positron energies, the radiative decay constitutes an even larger relative rate. By way of example, one may calculate the relative cross section at $x = 0.1$ to be about 27% for $E_\gamma > 2m_e$. This is about what one expects from the size of the radiative corrections and is large enough to indicate that the photons from radiative decay are not necessarily a negligible problem. The process, as it relates to this experiment, will be discussed in Section 5.1.6.

2.7 Effect of ρ on the Spectrum

Let us recall the uncorrected formula for unpolarized muon decay:

$$\frac{d\Gamma^{(0)}(x)}{dx} = -\frac{A}{16} \frac{M_\mu^5}{192\pi^3} \left(1 + \frac{m_e^2}{M_\mu^2}\right)^4 2x\sqrt{x^2 - x_0^2} \left[6(1-x) + \frac{4}{3}\rho \left(4x - 3 - \frac{x_0^2}{x}\right) + 6\eta x_0 \frac{(1-x)}{x}\right].$$

Since ρ is dominant in determining the spectrum shape, there is a very real danger that uncertainty in its value will dominate the other errors in measuring η (or, to phrase it differently, there is a tendency for any measurement of the unpolarized spectrum to become a measurement of ρ). Neglecting x_0 terms and using the experimental fact that $\rho \approx \frac{3}{4}$, we estimate the fractional uncertainty in the spectrum relating to ρ as

$$\left. \frac{\frac{\partial(d\Gamma(x)/dx)}{\partial\rho}}{d\Gamma(x)/dx} \Delta\rho \right|_{\rho=\frac{3}{4}} = -\left(\frac{4}{3}\right) \left(\frac{3-4x}{3-2x}\right) \Delta\rho.$$

This is encouraging since the *shape* of the spectrum at small enough x becomes independent of ρ , but it is also a warning that amplitude normalization points should not be concentrated

³⁶T. Kinoshita and A. Sirlin, Phys. Rev. Lett. **2**, 177 (1959).

³⁷S. G. Eckstein and R. H. Pratt, Ann. Phys. **8**, 297 (1959).

³⁸C. Fronsdal and H. Überall, Phys. Rev. **113**, 654 (1959).

at high values of x , to minimize the effect of ρ .

Fortunately, the present knowledge of ρ is good, compared to the accuracy likely to be achieved in η ; the current world average is³⁹

$$\rho = 0.7518 \pm 0.0026 .$$

Inserting $\Delta\rho = 0.0026$ in the above equation, the fractional uncertainty in the spectrum shape for $0.1 \leq x \leq 0.5$ is less than 0.15%. When radiative corrections to the spectrum are considered, this is reduced to 0.10%.

In order to estimate the uncertainty in η resulting from that in ρ , an idealized experiment will be considered in which only two points on the spectrum are measured, x_1 and x_2 . The radiative corrections can be ignored for this purpose since they reduce the spectrum sensitivity to η and ρ similarly, and one finds

$$\Delta\eta = \left| -\frac{4}{3} \cdot \frac{\Delta\rho}{x_0} \cdot \frac{x_1 x_2 (x_2 - x_1)}{(3 - 2x_2)(1 - x_1)x_2 - (3 - 2x_1)(1 - x_2)x_1} \right| .$$

If one uses $x_1 = 0.2$, $x_2 = 0.5$ and $\Delta\rho = 0.0026$, then $\Delta\eta = 0.02$. This does not constitute the limit of accuracy in this experiment, then, though it is not insignificant. However, were one to try to normalize the spectrum using mostly the region near $x = 1$, the conclusion would be different: using $x_1 = 0.2$, $x_2 = 1.0$ yields $\Delta\eta = 0.07$. Thus, one must be careful to not rely upon the “cheap” amplitude normalization available near the endpoint.

³⁹G. P. Yost et al., Review of Particle Properties, Phys. Lett. **204B**, 1 (1988).

Chapter 3

Previous Determinations of Eta

3.1 Direct Measurements

In the 1960's there were several measurements of the upper part of the muon decay spectrum which, although they were basically measurements of ρ , also quoted a value for η by fixing $\rho = 3/4$; all had uncertainties of ± 0.5 or greater.^{1,2,3} There was also an early measurement by Plano⁴ which did a two parameter fit to ρ and η based on a measurement of the whole spectrum. His result was $\eta = -2.0 \pm 0.9$, which he discounted.

However, aside from the present one, the only *direct* η measurement of significant precision was that of Derenzo.⁵ The method used was very different: a beam of π^+ and μ^+ was stopped in a 10-liter hydrogen bubble chamber. Observed were 2,070,000 μ^+ decay positrons, 6346 of which had $P_e \in (0.03, 0.13)\frac{1}{2}M_\mu c$, the region taken by Derenzo as sensitive to η . The result quoted was $\eta = -0.12 \pm 0.21$.

The use of a 4π visual detector filled with liquid hydrogen let Derenzo examine the extremely low-energy portion of the spectrum with less uncertainty than would have resulted

¹J. Peoples, Nevis-147 (unpublished) (1966).

²B. A. Sherwood, Phys. Rev. **156**, 1475 (1967).

³D. Fryberger, Phys. Rev. **166**, 1379 (1968).

⁴R. J. Plano, Phys. Rev. **119**, 1400 (1960).

⁵S. E. Derenzo, Phys. Rev. **181**, 1854 (1969).

with the current apparatus for several reasons:

- Correction for Bhabha scattering was not needed when both secondaries had energies above 0.35 MeV (because their energies could be added together).
- Since there were no veto counters, there was no uncertainty in calculating the veto probability.
- Nuclear bremsstrahlung, scaling as Z^2 , was less of a problem in hydrogen ($Z = 1$) than it is in carbon ($Z = 6$).
- The low density of liquid H_2 helped to reduce all energy-loss problems for Derenzo.
- Complete tracking information eliminated most external backgrounds.

On the other hand, there were problems which would be difficult to reduce significantly below their level in the Derenzo measurement, so this type of experiment is not a good candidate for an improved measurement of η . An example is the momentum-dependent measuring efficiency which Derenzo found to vary (in a somewhat non-monotonic manner) from 80% to 98% over the observed region of the spectrum.

In discussing Derenzo's experiment, it should be pointed out that a few improvements could be made to his analysis. The most significant change would be to correct for bremsstrahlung in the field of electrons. In hydrogen, where electron and nucleus have the same charge magnitude, this is comparable in size to the nuclear bremsstrahlung effect,⁶ but correction was made for only the latter. It must be admitted that bremsstrahlung in the electron field is somewhat less likely for incident positrons with energies of only a few MeV, and so we must consider the problem in a bit more detail to justify this claim of similar effects. The justification is that most of the spectrum distortion arose from high-energy positrons which emitted a photon early in their trajectory and were then misjudged as having had a lower energy; at these high energies (typically 40 MeV), the cross section for bremsstrahlung in the field of electrons is only a few percent less than that in the

⁶E. Haug, *Z. Naturforsch.* 30 a, 1099 (1975).

proton field. One should, therefore, almost double Derenzo's bremsstrahlung correction and thereby change the measured value of η by about -0.08 . However, as long as one is reviewing the bremsstrahlung analysis, one should restore a minor term which Derenzo dropped from his bremsstrahlung formula,⁷ shifting η by roughly $+0.02$.

Also, a small correction can be made because of the evolution of theory for the value of I_{adj} (the adjusted ionization potential) for hydrogen, which appears in the muon energy-loss equation by which Derenzo calibrates the density of the liquid hydrogen in his bubble chamber. This has evolved from 18.7 eV up to 21.8 eV.⁸ The result is that Derenzo's liquid hydrogen was 2% denser than was calculated with the best value of I_{adj} available at that time. This would change the measured value of η by about -0.02 .

Finally, one can consider the second-order radiative corrections.⁹ These would change Derenzo's value by about -0.01 . Combining all of these effects, one could more accurately quote the Derenzo result as $\eta = -0.21 \pm 0.21$. Since the net change is less than half the quoted uncertainty, the improvements are of only marginal significance. Also, had Derenzo quoted the correlated value for ρ using this corrected value, his answer would have been little different: $\rho = 0.7514 \pm 0.0026$ instead of $\rho = 0.7518 \pm 0.0026$.

3.2 Indirect Measurements

It is possible to determine combinations of α/A and β/A in muon decay by measuring P_{T_1} , the component of the decay positron's transverse polarization in the plane defined by the muon spin and positron momentum. An independent determination of α/A , β/A or a different combination gives one an indirect measurement of $\eta = (\alpha - 2\beta)/A$. This was

⁷H. W. Koch and J. W. Motz, *Rev. Mod. Phys.* **31**, 920 (1959).

⁸M. J. Berger and S. M. Selzer, in *Proceedings of Hawaii Conference on Charge States and Dynamic Screening of Swift Ions*, 1982, pp.57-74.

⁹A. V. Kuznetsov and N. V. Mikheev, *Sov. J. Nucl. Phys.* **31**(1), 136 (1980).

originally established by Kinoshita and Sirlin¹⁰ and later emphasized by Scheck.¹¹

Dropping terms linear in x_0 from our previous formula for $d^2\Gamma^{(0)}(x, \theta, \phi, \psi)/dx d(\cos\theta)$, one obtains

$$P_{T_1} = \frac{6(1-x)\eta + 16(1-\frac{3}{4}x)\frac{\beta}{A}}{6(1-x) + \frac{4}{3}\rho(4x-3) - \xi \cos\theta \left[2 - 2x + \frac{4}{3}\delta(4x-3)\right]}.$$

From this, the advantage of measuring P_{T_1} , rather than η directly, is immediately obvious: there is no suppression by an m_e/M_μ factor. The disadvantage, of course, lies in the difficulty of a polarization measurement. To equate such a polarization-derived value of the spectrum parameterization with spectrum shape measurements at mostly lower energy, the assumptions made are that the derivative-free, local, four-fermion point interaction suffices and that the low-energy spectrum shape is determined only by the Lorentz structure of the interaction.

A measurement of P_{T_1} has, in fact, been done;¹² this yielded a value of η with improved accuracy:

$$\eta = -0.011 \pm 0.085.$$

The value of η was extracted from measurements of P_{T_1} at eight energies, taking advantage of the slightly different energy dependence on η and β/A ; the two parameters are still highly correlated in the fit. The values for α/A and β/A are somewhat less correlated; the values quoted were

$$\alpha/A = 0.015 \pm 0.052 \quad \text{and} \quad \beta/A = 0.002 \pm 0.018.$$

Another indirect approach is to exploit the relationships which exist between the muon decay parameters, using the combinations known to high accuracy to improve the limits on those which are less well known. H. Burkard *et al.* performed a global analysis, using experimental values for ξ'' , ρ , δ , $\xi\delta/\rho$, ξ' , α/A , α'/A , β/A and β'/A . The resulting limits

¹⁰T. Kinoshita and A. Sirlin, Phys. Rev. **108**, 844 (1957).

¹¹F. Scheck, Phys. Rep. **44**, 188 (1978).

¹²H. Burkard *et al.*, Phys. Lett. **160B**, 343 (1985).

on α/A and β/A were

$$\alpha/A = 0.0004 \pm 0.0043 \quad \text{and} \quad \beta/A = 0.0039 \pm 0.0062 ,$$

which in turn yield

$$\eta = -0.007 \pm 0.013 .$$

Burkard et al. do not include detailed information about the exact sources of the limits on particular parameters, and one might wonder whether there might be significant changes due to more recent values, such as the A. Jodidio et al. result¹³ and erratum¹⁴ for $\xi\delta/\rho$. Also, there were certain approximations; the correlation between the fitted value of ρ and η was ignored, giving results not quite consistent with the input data.

The first thing to note is that the input values for ξ'' , α/A , α'/A , β/A and β'/A had little effect on any part of the fit because of their relatively low accuracy. The limits on β/A come mostly from the lower limit on $\xi\delta/\rho$ and the mathematical inequalities between the parameters; those on α/A come from the values for ρ , δ , $\xi\delta/\rho$, ξ' and, again, the mathematical inequalities, with most of the sensitivity being to the lower limit on ξ' . Thus, the fitted values for neither α/A nor β/A should have been much affected by the concerns mentioned above.

¹³B. Balke et al., Phys. Rev. **D34**, 1967 (1986).

¹⁴B. Balke et al., Phys. Rev. **D37**, 237 (1988).

Chapter 4

Experimental Apparatus

4.1 Spectrometer

The axial-focusing spectrometer used in this experiment, named “Comus” after the ancient Roman god of revelry, is similar in concept to the nuclear beta-ray spectrometers traditionally used in nuclide studies. An excellent, general reference on this type of spectrometer is that of Siegbahn.¹

However, since muon decay energies are an order of magnitude higher, many aspects of the spectrometer had to be scaled up drastically. This higher-momentum design allowed some simplifications (such as a thicker “source” and a vacuum window), but also forced the sacrifice of certain refinements. In this last category are iron-free designs, helical collimators and the multitude of calibration sources which exist at energies of a few MeV.

A vertical cross section through the Comus spectrometer is shown in Figure 8. Two features of the spectrometer have been indicated for clarity, even though they do not lie on this vertical plane: the collimator-suspension cables (shown as dotted lines) and the four flux-return yokes of the magnet.

A general remark about the spectrometer design: it used a minimum of coincidence

¹K. Siegbahn, in *Alpha-, Beta-, and Gamma-ray Spectroscopy*, edited by K. Siegbahn (North-Holland, Amsterdam, 1965), Vol. I, pp. 79-202.

counters for accepted particles and no drift chambers or other position-sensitive detectors. This was out of necessity; no more than about 50 mg/cm² of low-Z matter could be allowed before momentum selection in the spectrometer, if adequate corrections were to be possible for the resulting spectrum distortion. The e^+ had to pass through nearly this much material to escape the muon-stopping counter itself, on the average. Even in the detector, it is doubtful that the information gained by use of a tracking device would have justified the resulting scattering, annihilation, efficiency and geometry problems. A related remark is that the central region of Comus was evacuated to about 60 millitorr; interactions with the residual air were negligible.

4.1.1 Magnetic Field

The magnetic field was nearly symmetric around the spectrometer axis and roughly symmetric across the midplane bisecting that axis. It was, however, not a uniform field, the field strength instead following a bell-shaped curve along the axis, as plotted in Figure 9. There were significant radial components of the field at points off-axis and away from the midplane, as implied.

The large end pieces and 312-turn coil of the Comus electromagnet came from the former LBL 25-inch bubble chamber magnet. Four iron, flux-return yokes were added to reduce both the fringe field outside the spectrometer and the power requirements. (1400 amperes at 110 volts were needed for the highest momentum measurements after the yokes were installed.) The reduction in the fringe field made magnetic shielding for the photomultiplier tubes of the various counters more practical; with the yokes in place, none of the tubes had to be shielded against more than 150 gauss.

One problem associated with the fringe field was not completely eliminated—the deflection of the incoming muon beam. In the normal, data-taking configuration of Comus, the beam traveled near the spectrometer and perpendicular to its axis; it was vertically deflected at the highest field settings. The deflection that remained after shielding the beam line was corrected by unbalancing the last quadrupole magnet of the beam line with

a current source, causing it to steer the beam. When one uses a single current source to effect this type of unbalancing act, it is, of course, necessary to adjust the main power supply for the “quad”, so that the dipole component is superimposed on the same quadrupole strength as before.

Naturally, the iron-based design of the magnet resulted in nonlinear variation of the magnetic field with excitation. Nonlinearity first appeared at a tune momentum of about 30 MeV/c, but was not significant there. Magnet hysteresis was handled partly by basing calibrations on field measurements (rather than current) and partly by tuning the spectrometer with a set protocol (from higher to lower field values). Eddy currents, which were induced in the magnet end pieces whenever the field was changed, were allowed to decay for several minutes before data-taking commenced. Thus, the low-momentum portions of the spectrum were measured with stable spectrometer characteristics.

Even near the endpoint, the change in the field *shape* was at a barely noticeable level. The acceptance has been calculated to increase by no more than 0.3%; the momentum calibration, relative to the field at the center of the magnet, shifted upward by 0.06%. These effects would have had no major effect on the measurement of η , even if correction for them had not been made, as the spectrum shape is relatively insensitive to the overall momentum calibration and line shape.

4.1.2 Magnetic-Field Monitoring

The magnetic-field monitoring consisted of one Hall probe, together with three NMR probes whose overlapping ranges spanned the necessary region. The four probes were placed in geometrically similar positions in the magnet, away from edges of the iron yokes, and just outside the vacuum vessel. The placement is discussed further in Appendix B.

For the devices used in this experiment, the Hall probe accuracy was 0.25%, while that of the NMR was 0.001% from 5-40 MeV/c: the magnetic field gradients at the probe locations were substantial, but more accuracy was needed than that practical with a Hall probe. This led to a scheme to eliminate the field gradients locally, as discussed in Ap-

pendix B, so that NMR could be used. The probes themselves were of the LBL/CERN type, except that the preamplifier design was improved to allow operation at the low-field settings.

The Hall probe, on the other hand, provided a convenient monitor, was an aid in tuning the NMR and gave more reliable values near the endpoint. The absolute accuracy was only $\pm 0.25\%$ over the required range, but the long-term reproducibility (the relevant figure of merit, after calibration against the NMR) was $\pm 0.06\%$. This Hall probe was sensitive only to the field component parallel to the spectrometer axis, but was placed where the other components were negligible.

4.1.3 Particle Trajectories

In order to explain the rest of the spectrometer design, particle trajectories in the magnetic field are first discussed. Cylindrical coordinates with the obvious convention are used. From the graphs in Figure 10, it is apparent that particles follow a spiral path with dr/dz almost constant, except near the center of the spectrometer where dr/dz changes sign abruptly. The total rotation of ϕ , the azimuthal angle of the trajectory in the spectrometer, is close to 90° for the accepted particles.

Also note the existence of a reasonably good ring-focus, a position at which particle paths with the same momentum intersect, independent of initial polar angle. The momentum-selection collimators were placed in this region.

However, as is usual for this spectrometer type, there is not a true focus at the detector position; the minimum radius of the detector, 5.08 cm, is just enough to contain almost all of the desired tracks. The radial distribution of these tracks is shown in Figure 11. as found by Monte Carlo in the absence of scattering.

4.1.4 Acceptance Characteristics

Since the η parameter is a characteristic of unpolarized muon decay, it is best measured with either unpolarized muons or a spectrometer which averages out the decay asymmetry.

A priori, one would probably prefer the unpolarized muon approach, simply using an unpolarized beam, or, failing that, stopping the beam in a depolarizing target. The former was not practical for this experiment, since the high stopping density of a surface muon beam was needed, and these beams are intrinsically polarized. The latter was not possible because an active target was needed to provide a target-detector coincidence signal for background rejection; plastic scintillator was the most practical choice, and this is only partially depolarizing.

A spectrometer acceptance that cancels spectrum components proportional to $\cos \theta$, where θ is the angle from the muon spin direction, has therefore been used. This condition will obviously be fulfilled if the acceptance is symmetric under 180° rotations around an axis at $\theta = 90^\circ$. An axial-focusing spectrometer, with its azimuthal symmetry, meets this requirement for muon spins perpendicular to the axis. In fact, because of its higher symmetry, the muons can be allowed to precess around the axis—eliminating the need to control the fringe field of the spectrometer magnet at the muon position and reducing certain potential systematics. However, one must not be quite so casual: real collimators are not precisely centered, real spectrometers are not exactly perpendicular to the beam direction, and the real magnetic field may not lie precisely along the spectrometer axis at the muon's position. These potential systematics are discussed in Appendix A.

Once one has canceled the decay asymmetry, the next design requirement is that the transmission be fairly large; the probability that a muon will yield an event at low-momentum spectrometer tunes is low. This arises from the small momentum bite of the spectrometer (smaller at low-lying data points, being proportional to P_{tune}) and the concentration of the decay weight at high energies. On the other hand, spherical aberration limits the angular acceptance, if a good ring-focus is to exist at the momentum-selection collimators. The compromise between these requirements resulted in an angular acceptance that is roughly triangular, with a peak near 16.0° and extending between about 13.5° and 18.3° . This is plotted in Figure 12, averaged over the accepted momenta. At the peak accepted momentum, the solid angle is $\Omega/4\pi = 0.60\%$. The contour plot of Fig-

ure 13 shows how the angular and momentum acceptances are correlated, as found from calculated tracking results.

Averaged over angles, the line width is $\Delta P_{accept}/P_{tune} = 2.8\%$ (FWHM). The shape is shown in Figure 14; this is also averaged over starting positions on $T1$, but the shape is almost independent of this. A larger target, with resulting position dependence, could have been used in principle, but could have caused problems from any movement of the beam spot during data-taking.

4.1.5 Target-Area Layout

The vacuum vessel in the target area, as elsewhere, was constructed of aluminum. This was desirable to reduce the scattering and showering of charged particles, and the gamma rays from radiative muon decay and bremsstrahlung produced no high-energy fluorescence X rays. Those X rays which were produced had energies below 1.6 KeV and extremely short attenuation lengths, so that few escaped the aluminum itself. Thus, annoying backgrounds in the target-area anti-counters were minimized. A drawing of the target area is shown in Figure 15, to aid in understanding the discussion that follows.

$T1$ — μ^+ -stop Counter

$T1$ was constructed of NE110 plastic scintillator with height $\times$ width of $\frac{1}{2}$ " $\times$ $\frac{3}{4}$ ", and thickness of 0.0838 cm; its long dimension was rotated by about 45° around the vertical so that it presented a roughly half inch square face toward both the beam and the spectrometer acceptance. It served several purposes:

- a target for stopping muons;
- in coincidence with Tb , a count of stopping muons;
- one input to the event trigger;
- a time-of-flight coincidence for background rejection; and

- a way to reject many events scattering out of the target plane (using the large energy deposition typical of such events).

In order to perform these functions, several conflicting requirements had to be met. $T1$ had to be:

- thin and of low- Z material to minimize bremsstrahlung and scattering;
- thick enough to stop the incident muons;
- of a material that would depolarize the muons, reducing problems from spectrometer asymmetries;
- small enough that the spectrometer's acceptance was nearly constant over it;
- large enough to contain most of the beam spot, giving an adequate event rate and reducing edge effects.

In the compromise solution chosen, $T1$ did cause significant spectrum distortions due to intrinsic physical processes, and these will be discussed in Chapter 5. Systematic effects due to muon polarization were not completely eliminated, as stopping muons retain 25% of their original polarization in plastic scintillator,² but this represents a factor of four improvement. The edge effects of $T1$ were reduced by collimating the beam just upstream of the target area.

In future references to this counter, mention will be made of $T1_\mu$ and $T1_e$. These are logic signals in the electronics; $T1_\mu$ is produced by large pulses in $T1$, and $T1_e$ by the relatively small ones, corresponding roughly to stopping muons and escaping positrons, respectively. The separation is not perfect, with about 4% of decay positrons passing through enough scintillator to generate the $T1_\mu$ signal.

²J. Brewer, private communication.

Tb Counter

This counter was 1" square and positioned perpendicular to the muon beam, 1" upstream from the center of $T1$; the thickness was 0.0769 cm. It served three functions:

- Large signals from muons which passed through Tb , generating the logic signal Tb_μ , were used in coincidence with $T1_\mu$ to count muon stops in the target. This eliminates the ambiguity in $T1_\mu$.
- Small signals from the positrons which passed through Tb , generating the logic signal Tb_e , were used in anti-coincidence with $T1_e$ in the event trigger; this eliminated the possibility that a beam e^+ could Bhabha scatter a target electron into the spectrometer acceptance, or itself be scattered into it. While Monte Carlo calculations predict this effect to be negligible, the anti-counter provided additional security. Also rejected were decay positrons from muons stopped in Tb .
- The incident muon momentum was degraded, allowing them to stop at the desired location in $T1$.

A1, A2, A3 Anti-counters

It is possible for an $e^\pm$ to enter the acceptance of the spectrometer in other than a direct line from the target; this could happen by scattering/showering from the walls of the vacuum vessel after a decay in $T1$. The increased path length of this indirect path would seldom be enough to reject the events by the time-of-flight discrepancy, so the $A1$, $A2$ and $A3$ anti-counters were positioned to intersect the path of most possible scatters in the target area. The exception to this is scattering from the thin, flat light guide of $T1$. Even in this case, only a small area near $T1$ is vulnerable, and, being thinly constructed of low- Z material (Lucite), the problem is further minimized. The remaining small correction is included in the target-area Monte Carlo calculation.

The other purpose served by these counters (mostly by $A1$ and $A2$) is to reject the majority of events with a substantial Bhabha scatter in $T1$. This is discussed in more

detail in Section 5.1.3. Note that Bhabha scattering and bremsstrahlung of high-energy particles in *A1* and *A2* can, themselves, contribute low-energy events. These events are, in principle, self-vetoing, but add to the large number of events that must be rejected by *A1* and *A2*. A few scatters from the *A1* bevel pass the applied cuts, but almost all of those from *A2* are vetoed by *A1*.

For an annular counter to be highly efficient, care must be taken to avoid insensitive areas near the inner radius. Multiple light guides could have solved this problem under different circumstances, but space constraints precluded this for *A1* and *A2*. The alternate solution was to divide each annular counter into several thin rings, so that total internal reflection could be achieved, even for light coming from very near the inner radius of each ring. The rule that must be obeyed by each ring follows from simple optics and geometry:

$$n \frac{r_i}{r_o} > 1 ,$$

where n is the index of refraction of the scintillator, while r_i, r_o are the inner and outer radii of the ring, respectively. Subdivision into three rings was adequate for *A1*, while five were required for *A2* because of its smaller inner radius.

4.1.6 Collimators

K1

K1 consists of two aluminum tanks of lead shot, surrounding the vacuum vessel downstream of the target. It shields the spectrometer against particles from the beam line, decaying muons which missed the target, as well as from positrons originating in the target and making large-angle scatters from material near the target area.

K2

The large front collimator, *K2*, provided mostly protection against high-energy particles creating showers in the rear part of the spectrometer, thereby contaminating low-energy

measurements; it consists of 33.3 radiation lengths of lead. An aluminum plate is on the back of the collimator, and aluminum also covers the beveled surface.

K_2 does not set the minimum acceptance angle in the spectrometer as it might appear; this is determined by the downstream collimators. The chance that high-energy particles striking the edges of K_2 would contribute low-energy events was thus greatly reduced.

K_3

This collimator is a lead ring, beveled on its inner radius. It partially determines the high-momentum edge of the acceptance, substantially narrowing the resolution. The need to have more than one exit baffle is typical of axial-focusing spectrometers.

K_4, K_5, C_1, C_2

These collimators are near the ring-focus of the spectrometer and are the primary determining factors of the acceptance. With a non-active collimator slit, especially one which is beyond the strongest magnetic field, there is a danger that high-energy particles will pass through the edges of the collimator without much scattering and be accepted. Since low-energy particles would be scattered more, they would have less chance of being accepted; hence, the measured spectrum would be distorted.

The anti-counter on the inner radius of the slit is C_1 , that on the outer radius is C_2 . The counters were both constructed from $\frac{1}{4}$ " scintillator, placed on the bevels of the K_4 and K_5 lead collimators, respectively. C_1 and C_2 have no reflective wrappings on the surfaces facing the slit, a practice which is desirable to eliminate an inactive covering and possible because all other unwrapped counters in the vicinity are also veto counters. It is topologically necessary, of course, that the light guide for C_1 cross the slit between the counters; for this reason, the Lucite portion of the light guide did not begin until after the crossing. Most events in which the light guide was struck were thus rejected, and they caused little difficulty.

The light guides on both counters cause an additional complication: optical consider-

ations leave a gap on each counter where the guide bends away from the scintillator ring. One result is that some unvetoes trajectories miss the $T2$ trigger counter. These constitute about 1% of the number of good events and have to be considered in the analysis of $\overline{T2} \cdot T3 \cdot T4$ events. The line shape is also somewhat affected.

One other detail about $C1$ and $C2$ is that, while positioned beyond the strongest magnetic fields, the field strength in their vicinity is still enough that a fairly high-momentum particle from the detector or vacuum window is required to reach them. This is important since low-momentum $e^\pm$ are frequently backscattered from the detector shower. Had these counters been placed in a region of very low field near the detector, events would have been vetoed in an energy-dependent (and difficult-to-calculate) manner, with no resultant advantage. Vetoes by $e^\pm$ and gamma rays from the detector can still occur, of course, but the fraction of vetoed events is small and relatively independent of energy.

4.1.7 A, B, C, D Anti-counters

These four counters, each covering 90° of azimuthal angle, together make a cylinder of scintillator around the region through which accepted trajectories travel. Several purposes are served:

- Many high-momentum particles from $T1$ hit the outer radius and back plate of the vacuum vessel, as well as the spectrometer magnet, and some particles from the resulting showers mimic good events. Impacts on these counters by the primary particles vetoed many such events.
- Even if the primary particles hit inactive material (such as the faces of the $K3$ or $K4$ collimators) rather than these counters, they were sometimes struck by a particle in the resulting shower or, more frequently, by an annihilation gamma ray, and still vetoed the event.
- Some events, in which a hard Bhabha scatter occurred in $T1$, were rejected when the spectrometer was tuned to accept the lower-momentum member of the pair. This, of

course, was when such rejection was most needed.

- Some external backgrounds, such as cosmic-ray showers, were also vetoed by these counters. This is in addition to the rejection provided by the time-of-flight coincidence.

4.1.8 Detector

In coincidence with $T1_e \cdot \overline{Tb}_e$, the detector provided the event trigger. It also gave a measure of the energy of the incident particle and, by virtue of its division into three geometrical regions, allowed the study of background events. This will be elaborated below.

Counters $T2$, $T3$, $T4$

These counters were positioned slightly downstream of the spectrometer vacuum window in the order listed:

- $T2$: 4" ϕ , 0.365 cm thick scintillator
- $T3$: 6" ϕ , 0.310 cm thick scintillator
- $T4$: an annular scintillator, with an outer diameter of 8" and an inner diameter of 4"; it is 0.318 cm thick. Because of the hole in its center, this counter was equipped with two separate light guides and phototubes, so that no areas were insensitive. An analog sum was performed on the outputs of the two tubes.

Together, these counters divide the 8" ϕ detector surface into three regions:

1. $T2 \cdot T3$
2. $\overline{T2} \cdot T3 \cdot T4$
3. $\overline{T3} \cdot T4$

The size and location of the counters were chosen so that most good events struck the $T2 \cdot T3$ region; the purpose of the other regions was to study contaminating events for comparison with the calculated corrections.

NaI Detector

The cylindrical NaI(Tl) crystal has a diameter of 8" and a length of 10"; one inch corresponds to 0.98 radiation lengths. It provided an energy measurement of the detected particle, including the annihilation and bremsstrahlung γ 's in the detector shower, when corrected for energy deposition in the upstream counters and other material.

Unfortunately, the same high-Z components which allow efficient conversion of γ -rays tend to backscatter low-energy e^+ . Backscattering, along with energy absorption, also occurred in the window of the NaI detector, consisting of aluminum, MgO reflective powder and a certain amount of rubber and plastic. Thus, although the resolution of the NaI crystal for a low-energy γ -ray is 8.4%, the resolution for low-energy charged particles is worse and has a long tail toward low measured energy. The situation was helped by backscattered particles passing through the $T2$, $T3$ and/or $T4$ counters a second time and depositing additional energy.

Even with only modest energy resolution, however, the NaI was extremely valuable. Particles with higher-than-expected energy were efficiently rejected—very important at low P_{tune} settings. At moderate-to-high P_{tune} settings, most contaminating particles had much lower energy than the desired ones, and the energy resolution was sufficient to reject a large fraction.

4.1.9 M_1, M_2

These counters (not illustrated) were positioned opposite to the spectrometer acceptance and separated from each other, so that positrons originating from $T1$ could cause a coincidence between them. Thus, a diagnostic was provided on the rate of muons stopping in, or near, $T1$, independent of that counter. The solid angle subtended by the coincidence was such that the rate was larger than that of good events in the lower part of the spectrum: the solid angle was 0.0068% of 4π steradians.

4.2 Electronics

The high-speed logic for the experiment consisted mostly of 100 MHz NIM modules and was reasonably simple. The event trigger was chosen to minimize bias to the extent consistent with a reasonable data-taking rate: $T1_e \cdot \overline{Tb_e} \cdot (T2 + T4)$. The discriminator settings for these signals, except for Tb_e , were set conservatively, allowing almost all serious cuts on the data to be done off-line.

A simplified diagram of the trigger electronics is shown in Figure 16; coincidences formed for the various scalers are not shown, nor are various coupling elements to eliminate DC-offset levels and the like. The discriminators used were mostly of LeCroy types 821Z and 621Z, except for an LBL-designed constant-fraction discriminator for the NaI. The LeCroy units were not internally terminated, which eliminated the need to split the signal between the ADC's and discriminators.

The anti-counter electronics were quite simple: each anti-counter had an ADC and a TDC, and was connected to a CAMAC scaler. Most were also monitored in various combinations by visual scalers. The NaI electronics were a straightforward analog sum of the outputs from the four phototubes; a constant-fraction discriminator on this formed the TDC stop, while ADC's recorded the individual outputs, as well as the sum.

4.2.1 $T1$ Electronics

$T1$ presented several non-trivial problems, mostly because the counter had to detect a small positron pulse which followed a large muon pulse, usually by less than a few microseconds. Thus, the counter not only needed a large dynamic range, but had to have very little signal-correlated noise, e.g. afterpulsing and cable reflections. A further complication came from the thinness of the scintillator through which the exiting positrons could be allowed to pass. With an average positron signal involving only 15 photoelectrons, Poisson statistics added significantly to the width of the pulse-height distribution. Combined with variations in path length through the scintillator (arising from muon straggling and the wide range

of accepted track angles with respect to $T1$ —recall that the normal to the $T1$ plane was rotated from the spectrometer axis), the total variation in positron pulse heights for good events was large. This increased the vulnerability of the counter to noise problems.

The problems were addressed in several ways:

- The photomultiplier tube was selected for low noise and afterpulsing; an Amperex XP-2230 was found to be a significant improvement over the RCA 8575 tubes used on the other counters. The output of the tube was clipped to reduce the tail of the muon pulse.
- For a portion of the data, two discriminators triggered on the positron signal, the second level being set 2.8 times as high as the first. Both were recorded in separate TDC's. The first, with the low threshold, would fire on almost any positron exiting $T1$, but was vulnerable to recording an extraneous pulse in the TDC before the desired positron pulse arrived. This same discriminator was used for the event trigger.
- The high-threshold discriminator, on the other hand, missed some positrons, but rarely triggered on afterpulsing, reflections and noise. Thus, a restrictive time-of-flight cut could still be made off-line for those events in which the first $T1$ TDC was stopped prematurely. The combination was much more desirable than either possibility by itself.
- The positron discriminators were placed in the experimental area near the $T1$ counter, minimizing problems with cable reflections and noise pickup; the analog signals to the electronics room were amplified at this same location.

4.2.2 Time-Correlation Electronics

A valuable diagnostic in a muon decay experiment is the time correlation between the arrival of a muon and the detection of a decay positron. For this reason, a nanosecond-resolution clock (of TRIUMF design and construction) was used to measure the time, up to $4.1\ \mu\text{s}$, between the arrival of a muon and the triggering of an event.

When the mean time between muon arrivals is not much greater than the time-out period of the clock, the situation is complicated by muon pile-up in the target, and it is not possible to assign an event to a particular muon with certainty. That was the case in this experiment to some extent, with a mean interval between muons of, typically, 20 μ s. Thus, to reduce distortions in the diagnostic information—such as the exponential decay of the muons with the correct lifetime—electronics were needed to correct for this pile-up. This is standard in the field of muon-spin resonance (μ SR) and will not be discussed here in detail.

The electronics used to label events in which a pile-up problem had occurred, and to regulate start/stop pulses to the clock, consisted of a quad-width NIM module designed and constructed at LBL for use in μ SR experiments. The pulses to the clock were internally gated by fast ECL logic, specifically MECL III. The other logic was from the slower MECL 10K series, except for the microsecond-scale time gates which were implemented with Schottky TTL timing chips. The timing performance of such a specialized design is better than can be obtained with the usual crate or two of standard NIM electronics.

4.3 Data-Acquisition System

Data were taken with a PDP-11/60 computer linked, through Unibus and a Kinetic Systems 3912 crate controller, to a single CAMAC crate. Events were written onto 1600 BPI magnetic tape in blocks of 16, with a readout of the scalers added to the end of each buffer. The recorded data for normal (i.e. not scaler or comment) events were mostly the ADC and TDC information for all counters (excepting $M1$ and $M2$, for which the coincidence was only scaled), plus information on the timing of the muon and positron, relative to previous muons and positrons.

The PDP-11 used the TRIUMF program DA³ to read out the CAMAC crate and to write the data to tape. Each event required 1 ms to acquire, resulting in a computer

³T. Miles and A. Satanove, IEEE Trans. Nucl. Sci. NS-30, 3746 (1983).

deadtime of no more than a few percent for the usual runs. The on-line analysis was performed by MULTI⁴ on a time-available basis. This is a flexible system which allows substantial modification of the analysis task without recompilation.

⁴Fermilab MULTI User's Guide, PN-97.5.

Chapter 5

Distortions of the Spectrum

A detailed understanding of the spectrometer is crucial to correct distortions in the measured spectrum. At the energies studied, e^+ are vulnerable to significant scattering, annihilation and energy loss in the materials traversed, and inactive media degrade energy resolution. In addition, a class of problems arises because the number of e^+ produced in the low-energy regime is so small compared to the number at high energies. Thus, improbable processes by which high-energy particles can yield events at low-momentum tunes cause distortion. More quantitatively, if Comus is tuned to a central momentum of $P_e = 6.15 \text{ MeV}/c$ and there is a probability of 1.5×10^{-8} that an e^+ of higher energy will yield an event at this setting, the contamination is 1%, the same enhancement as if η had been increased by +0.09.

Thus, quite unlikely processes can cause substantial problems at low energies; for tunes below a few MeV/c , many effects are so large as to be unmanageable. For this reason, as well as the decreasing statistical sensitivity of this experiment to η at low energies, the lower limit to the energy range was about 5 MeV.

5.1 Target-area Effects

There are several distorting processes that occur in the μ^+ -stop counter, $T1$, and the surrounding anti-counters. While it is not entirely possible to separate them in a calculation (one being forced to perform Monte Carlo simulations in order to handle the problem properly, since the target is not “thin” in the sense that all effects can be added linearly) it is useful to discuss the processes separately.

5.1.1 Multiple Scattering

In an ideal (but impractical) experiment, one would stop muons at the center of a tiny sphere of scintillator in the place of $T1$. In such a system there would be no angular anisotropy of the e^+ from unpolarized muons. However, the finite size of the muon beam spot requires that the $T1$ scintillator be fairly large in the directions transverse to the beam, while the need to minimize the amount of matter through which the decay e^+ must pass, or from which it might scatter, requires that it be thin. A moderately wide, flat target is then unavoidable.

When e^+ are emitted in directions nearly normal to the plane of $T1$, they experience relatively little scattering; for e^+ initially directed toward the spectrometer acceptance (in this case $\theta_T < 66^\circ$, where θ_T is the angle of the e^+ trajectory from the normal to $T1$) the scattering is typically a few degrees or less and is not of major significance. However, those e^+ which are emitted with $\theta_T \approx 90^\circ$ may pass through a very substantial thickness of scintillator before reaching an edge—up to 2.29 cm. For e^+ with energies of a few MeV, it is almost certain they will scatter out of one of the faces of $T1$ before they travel this far, while most 40 MeV e^+ , for example, will not deviate far from their original direction. This effect is illustrated by Monte Carlo results in Figure 17, which show a substantial anisotropy for 6 MeV positrons. (The asymmetry with respect to $\theta_T = 90^\circ$ derives from the average muon being deposited less than halfway through the scintillator.) Fortunately, the spectrometer does not accept particles at the angles most affected and the distortion

is significant only for the lower-energy data points.

Much of this effect can be eliminated by rejecting events which have an unusually large energy deposition in $T1$. However, the effect of such a cut is complicated, and there is no alternative to Monte Carlo calculation to determine it.

Two scattering problems also occur in the light guide attached to $T1$:

- The light guide is twice the thickness of the $T1$ scintillator, so there is a small overhang which particles leaving $T1$ can strike. However, similar numbers of particles are scattered into, and out of, the spectrometer's angular acceptance, so the net effect is not large.
- The light guide, lying in the plane of $T1$, can scatter particles into the spectrometer acceptance in the same way as $T1$ itself tends to do. The problem is reduced by the anti-counters $A1$ and $A2$, which veto particles whose source is not very close to $T1$. The few remaining events cannot be rejected on the basis of their energy deposition (since they will seldom pass through enough Lucite to produce the required amount of Čerenkov light) and correction is made for them.

5.1.2 Continuous Energy Loss

While a distinction between Bhabha scattering and continuous energy loss for an e^+ is artificial, for calculational purposes one must choose some energy threshold above which to track scattered e^- , and below which to treat the energy loss as continuous. It is also helpful conceptually. Fortunately, the shape of the spectrum in the measured region is insensitive to the value of this threshold, as will be discussed in Section 5.1.3.

There are two parts of the spectrum which are most affected, relatively, by the continuous energy loss:

- The endpoint region: the effect here can be nearly 100%, since the whole spectrum shifts to lower energy and the measured intensities at those energies separated from the endpoint by less than the average energy loss are drastically reduced.

- The low-energy region of the spectrum: this can be understood by considering the Michel spectrum without radiative corrections, $F(x) \equiv d\Gamma^0(x)/dx = -2x^2(3-2x)$.

In this approximation,

$$\frac{1}{F(x)} \frac{dF(x)}{dx} = \frac{6(1-x)}{x(3-2x)}.$$

It is clear that a fixed energy loss causes a fractional change in the spectrum that is divergent at $x = 0$. While the effect is modified by both radiative corrections and an accurate energy-loss treatment, this illustrates the basic trend; the continuous energy loss for an e^+ is, in fact, nearly constant over the energy range of interest.

The results of a more careful analytic calculation are shown in Figure 18.

5.1.3 Bhabha Scattering

The Comus spectrometer did not distinguish between e^+ and e^- , which merely spiral in opposite directions around the spectrometer axis. In principle, energy measurement in the detector could distinguish between them, but our energy resolution was not sufficient for event-by-event determination. Thus, an e^- which was Bhabha scattered out of $T1$ was easily mistaken for an e^+ of equal momentum coming directly from μ^+ decay. The probability for Bhabha scattering in a thickness dx is given by¹

$$d^2\mathcal{O}_{Bhabha} = \frac{2\pi r_e^2 n_e}{\beta_0^2} \frac{dT_1}{T_1^2} \left\{ 1 - \beta_0^2 \left[f_1(y)\epsilon - f_2(y)\epsilon^2 + f_3(y)\epsilon^3 - f_4(y)\epsilon^4 \right] \right\} dx,$$

where we have defined

$$\begin{aligned} f_1(y) &= 2 - y^2, \\ f_2(y) &= 3 - 6y + y^2 - 2y^3, \\ f_3(y) &= 2 - 10y + 16y^2 - 8y^3, \\ f_4(y) &= 1 - 6y + 12y^2 - 8y^3, \\ y &= \frac{1}{T_0+2} \quad \text{and} \quad \epsilon = \frac{T_1}{T_0}. \end{aligned}$$

T_0 is the kinetic energy of the initial e^+ , T_1 is the kinetic energy of the scattered e^- , r_e is the classical electron radius and n_e is the electron density of the material. Both T_0 and T_1

¹H. J. Bhabha, Proc. R. Soc. London, Sect. A **154**, 195 (1936).

are given in units of m_e .

The first point to notice is that $d^2\phi_{Bhabha}$ is divergent as $T_1 \rightarrow 0$. Thus, one might *a priori* expect a very large amount of e^- contamination in low-energy measurements. (Actually, the divergence is prevented by atomic effects, but this has little practical importance here.) The contamination is drastically reduced in the Comus spectrometer, however, by the target-area veto counters which will usually be struck by the coincident e^+ and veto the event. Because Bhabha scattering (ignoring soft photons) is a two-body process, the scattering angle of the e^- is fixed and given by

$$\cos \theta_{scat} = \sqrt{\frac{1 + \frac{2}{T_0}}{1 + \frac{2}{T_1}}}.$$

A little trigonometry gives the more directly useful quantity, θ_{sep} , the angle separating the final e^+ and e^- directions:

$$\cos \theta_{sep} = \frac{1}{\sqrt{\left(1 + \frac{2}{T_1}\right) \left(1 + \frac{2}{T_0 - T_1}\right)}}.$$

For scattered e^- with energies of a few MeV, θ_{scat} tends to be fairly large. For example, an e^- of 6.15 MeV/c momentum will be separated by at least 24.4° from the correlated e^+ . This makes a veto in the A1, A2 counters likely, and the effect is reduced by an order of magnitude, to a manageable level.

Another concern is that the e^+ might be accepted, with the e^- causing vetoes in an energy-dependent way. Such energy dependence would exist if $d^2\phi_{Bhabha}$ significantly varied with T_0 in the region where the cross section is large—for small ϵ . It would also exist if θ_{sep} were a strong function of T_0 . Accurate calculation would then be difficult, complicated by multiple scattering and energy loss in T1, and the absolute calibration and position-dependent light collection efficiency for the veto counters. Fortunately, both $d^2\phi_{Bhabha}$ and θ_{sep} are insensitive to T_0 when ϵ is small. Another way in which energy-dependent veto probability might arise is through deflection of the e^- by the spectrometer's fringe field. This problem is circumvented by the choice of cuts on A1 and A2: they must be

set high enough to avoid vetoes from the extremely low-momenta e^- subject to deflection at the higher spectrometer settings.

The primary effect of the very low-energy e^- is then to reduce the number of good events uniformly. Even so, one might worry that too many events will be lost in this manner, given the nearly divergent cross section for low-energy Bhabha scatters; this does not occur because the *very* low-energy e^- are stopped in *T1* and there is a finite threshold for event rejection in the veto counters. An estimate of the overall Bhabha scattering effect is shown in Figure 19.

5.1.4 External Bremsstrahlung

Bremsstrahlung, like most other energy-loss processes, has a nearly divergent cross section for producing low-energy secondaries—photons, in this case. The divergence is $1/k$, where k is the photon energy. However, the situation is simpler than for Bhabha scattering; there is little chance that the secondary photons will either cause or veto events. The *T1*, *Tb*, *A1*, *A2*, *A3*, *A*, *B*, *C* and *D* scintillators are all fairly thin and carbon ($Z = 6$) is their heaviest component element, so interactions in them are unlikely for photons of high enough energy to cause vetoes.

With regard to vetoing an event, another important thing to consider is the angular separation between the secondary e^+ and the photon. Naturally, being a three-body process, there is a continuous distribution of angles for bremsstrahlung, but the average rms angle between the photon and secondary electron for $Z = 6$ is roughly $0.6 \frac{\ln \gamma_0}{\gamma_0}$ radians.² This rms angle is only a weak function of the energy division between secondaries, with the coefficient varying from about 0.50 to 0.75 as the photon's share of energy varies from 100% to 0%. As an example, if a 40 MeV e^+ emits a photon and is reduced in energy to 5 MeV, the angle separating the secondaries will typically be 1.8 degrees. When the e^+ is accepted, it is unlikely the photon will strike *A1* or *A2*, although it may hit *A*, *B*, *C* or *D*. About 0.05% of all events, otherwise good, are vetoed in this way.

²M. Stearns, Phys.Rev **70**, p.836 (1949).

Still, bremsstrahlung has an important effect on the muon decay spectrum. It is possible for an e^+ to transfer most of its energy to the photon, so that a high-energy e^+ may spuriously appear in the low-energy part of the spectrum. Figure 20 shows the approximate effect on the spectrum shape.

One other comment on bremsstrahlung in the $T1$ counter: carbon and hydrogen are of low enough Z that bremsstrahlung in the field of electrons cannot be ignored in comparison to nuclear bremsstrahlung. Further, the energy dependence of this process is somewhat different from nuclear bremsstrahlung. This will be discussed further in Section C.5.

5.1.5 In-Target Annihilation

The e^+ , as they pass through $T1$, risk annihilation with the e^- in the scintillator. The energy dependence of this process can be seen from the following approximate formula³ for the probability of an e^+ of energy γ_e (in units of m_e) annihilating in a thickness dx of material with electron density n_e :

$$d\phi_{\text{annih}} = 1.60\pi r_e^2 n_e \gamma_e^{-\frac{7}{9}} dx .$$

The effect on the spectrum is fairly small, around 0.1%, for the energies used in this experiment. The explanation is that the target is thin and the process is quite unlikely to produce spurious, low-energy events from high-energy decays.

5.1.6 Internal Bremsstrahlung

Internal bremsstrahlung, i.e. radiative decay, results in very large corrections to the spectrum, but the radiative corrections already discussed include them as a subset, and they are known analytically to adequate accuracy. However, the implicit assumption was made in the previous discussion of radiative corrections that the real photons were not detected, an assumption which is not entirely valid in this experiment: the photons may cause vetoes in

³H. Messel and D. F. Crawford, *Electron-Photon Shower Distribution Function* (Pergamon, Oxford, 1970), p. 17.

the A , B , C , D , $A1$, $A2$ or Tb anti-counters. Section 5.1.4 stated that vetoes from external bremsstrahlung photons were unlikely, but that photon flux was an order of magnitude less than from internal bremsstrahlung in this experiment; the effect cannot be dismissed without study.

To be more quantitative, we use the formula of Fronsda1 and Überall,⁴ in which $y = E_\gamma/E_e(max)$ and y_0 is the minimum value of y considered, and create the following table of branching ratios.

x	Γ_γ/Γ			
	$y_0 = 10^{-3}$	$y_0 = 10^{-2}$	$y_0 = 10^{-1}$	$y_0 = 0.5$
0.105	29.0%	26.2%	22.2%	11.3%
0.205	17.0%	13.5%	9.4%	3.4%
0.305	14.3%	10.4%	6.1%	1.4%
0.405	13.3%	9.1%	4.6%	0.5%
0.505	12.8%	8.3%	3.7%	0.0%
0.605	12.3%	7.7%	3.0%	0.0%
0.705	11.9%	7.1%	2.2%	0.0%
0.805	11.2%	6.3%	1.4%	0.0%
0.905	9.9%	4.8%	0.0%	0.0%

Thus, an average γ -detection efficiency of a few percent could result in substantial spectrum shape changes. For the geometry and data cuts used in this experiment, 0.5% of events are lost from this source, and the shape distortion between $x = 0.10$ and $x = 0.95$ is no more than 0.11%. Looser cuts on the veto counters would reduce the loss of events, but *increase* the distortion. The near-cancelation in the energy dependence of several effects in this experiment (the cross section, the photon-positron angular separation and photon detection efficiency) is fortuitous; a different experiment could have larger (or smaller) distortion.

⁴C. Fronsda1 and H. Überall, Phys. Rev. **113**, 654 (1959).

For finite cuts, however, the spectrum near the endpoint will show more distortion. As the phase space for photons of adequate energy to generate a veto vanishes, the event rate depression also vanishes. Thus, there is a 0.5% correction in the spectrum region used to confirm the spectrometer line shape.

5.2 Momentum-Dependent Detection Efficiency

The particle-detection efficiency in our spectrometer shows a certain amount of momentum dependence, especially at tunes of a few MeV/c. The physical processes responsible will be discussed in the subsections below. Together, these processes make the detector efficiency complicated enough that there is no alternative to a Monte Carlo calculation, which is done with the EGS code. Results are shown in Figure 21 for the positron-detection efficiency. In order to eliminate energy-dependent effects discussed elsewhere, these events were required to not impact any material between $T1$ and the downstream edge of the $C2$ collimator. The figure does include effects due to the last set of collimator-suspension cables and the vacuum window, as well as the detection counters themselves.

5.2.1 Scattering Effects

The optics of Comus are designed so that almost all unscattered tracks will pass within the bounds of the $T2$ scintillator. However, particles must cross the vacuum window and 0.7% of them strike the cables centering the $K4$ collimator. The vacuum window is 0.025 cm thick aluminum, while there are four stainless steel cables: one of 0.16 cm diameter and three of 0.12 cm diameter. Some particles are significantly scattered, especially those which would have passed near the outer radius of $T2$, and either miss $T2$ or only graze a corner. This problem is linked to the 3.5 cm gap which separates the vacuum window from $T2$ and the even larger gap between the cables and $T2$, making scattering angles of only a few degrees significant. The rms scattering angle for relativistic particles varies as P_e^{-1} .

Another scattering effect which reduces detection efficiency is backscattering from the

counter wrappings and scintillator. Although this material is thin and low Z , this probability is not negligible for low-energy particles. The $T2 \cdot T3$ requirement will usually reject the event if backscattering occurs upstream of some plane in $T3$, the position of the plane being determined by the cut on $T3$. The total observed-energy cut may reject events in which the backscatter occurs even later; this is especially true at higher P_{tune} settings, where this cut essentially requires energy deposit in the NaI.

5.2.2 Annihilation

Positrons annihilate in the collimator-suspension cables near the detector, the vacuum window, $T2$, $T3$ and their wrapping material; once annihilation γ 's are formed, charged particles do not usually reappear until the NaI(Tl) crystal is reached. Thus, $T2$ and/or $T3$ may have little, or no, energy deposited in them, and detector inefficiency results. As mentioned in Section 5.1.5, annihilation is energy-dependent. The energy loss by a low-energy e^+ in the early part of the detector can be a significant fraction of its energy, and the energy-loss tail increases the annihilation probability, moving particles closer to the divergence at $E_e = 0$. There is also the energy-dependent effect of backward-traveling secondaries from the shower in the NaI that may trigger $T2$ or $T3$, despite an annihilation.

In-flight annihilations can also occur after $T3$, of course. In the NaI crystal itself, they have little effect; in its window, the effect is to reduce losses in the remainder of the window, increasing the energy deposit in the NaI.

5.2.3 Unintended Vetoes

Since $C1$, $C2$ are relatively close to the detector, there is a small probability that particles in the detection shower will enter the spectrometer and strike them. Very low-energy $e^\pm$ are reflected by the magnetic field, of course, but high-energy ones and γ 's are not. The TDC's on these counters cannot separate this class of vetoes: the extra delay is not enough to be clearly resolved (the counters are about a meter long, being wrapped on the collimators) and it is necessary to cut so tightly on the ADC for these counters that the TDC has

often not been triggered anyway. Typically, these vetoes reduced the detection efficiency by 0.5%.

5.3 Effect of Suspension Cables

Due to the mass of the collimators suspended within Comus (about 255 kilograms) and the need for stability when the spectrometer is moved, several sets of cables are necessary for support and centering. The set furthest downstream has been mentioned for its contribution to detector inefficiency, but the upstream cables are qualitatively different in their effect, since they are struck by a less-filtered distribution of particles. The $e^\pm$ leaving $T1$ risk encountering these cables and interacting, so that an event can be either added or lost.

There are two sets of stainless steel cables which support the $K2$ collimator; each set has four cables, 0.23 cm in diameter. Two of these cables are secured to $K2$ by small brackets which are exposed to the incident particles. The weight of the $K4$ collimator is offset by a third set of cables, two pairs, 0.12 cm in diameter.

These cables are rather opaque, in the sense that events accepted without the cables are seldom accepted when they are hit. At 6.15 MeV/c only 0.11% of the impacting particles are still accepted; at 50 MeV/c the fraction accepted rises to 6%. With the three cable sets blocking 2.5% of the spectrometer aperture, the net distortion of the spectrum is 0.15%.

The larger effect of these cables is to introduce spurious events, either by elastic scattering or by bremsstrahlung and Bhabha scattering. The effect of elastic scatters persists throughout most of the spectrum at a non-negligible level, because of the small angular deflections needed to change the fate of a particle (of order 2 degrees). The bremsstrahlung and Bhabha scattering, on the other hand, affect mostly the low-energy portion of the spectrum. These effects are modified by the presence of veto counters, analogous to the situation with $T1$. A calculation of the net effect of the cable sets as a function of energy is shown in Figure 22.

Included in this figure are "second-order" events in which particles, after scattering

from the cables, shower in the spectrometer back plate or collimators to produce spurious events. These are not totally negligible because scattering allows high-energy particles to reach locations at which they constitute a significant fraction of the incident particles. Explicitly: at low P_{tune} , the magnetic field does not deflect high-energy particles enough to hit the vulnerable parts of the spectrometer back plate; scattering does. The fractional size of the effect increases at low energies; at $P_{tune} = 6.15$ MeV/c the contribution is 0.17%.

5.4 Effect of the $K1$ Collimator

It is possible for $e^\pm$ tracks originating on the back or inner bevel of the $K1$ collimator to be accepted. Thus, in general, one would expect that showering in the $K1$ collimator, or scattering from its inner bevel, would lead to spurious events. Muon decays in the beam collimator, etc. undoubtedly lead to event triggers in this way, but these do not have an in-time signal in $T1$ and are background that can be subtracted in time-of-flight spectra. Further, $A1$ and $A2$ veto nearly all charged particles which might strike $K1$ from the target.

The remaining possibility is that the γ ray from a radiative μ^+ decay in $T1$ will cause showering in $K1$, while the e^+ avoids hitting the various veto counters. Because of the strong angular correlation between the γ and e^+ in radiative decay, this is unlikely: only about 14% of such events escape vetoes from $A1$ or $A2$. This residual fraction is not completely negligible, however, and is not reduced much by energy-deposition cuts, since the particles accepted from this source are not much lower in momentum than the desired ones. The contamination is 0.21% at $P_{tune} = 6.15$ MeV/c, but falls rapidly at higher settings (because of the increasing rate of normal events, more than any reduction in the rate of events from $K1$ scatters).

5.5 Effect of the $K2$ Collimator

Showers directly through this collimator produce only a small number of events. The thickness of $K2$, combined with the shielding effect of $K4$ and the magnetic field, was

adequate to reduce the probability of a spurious event from this source to less than 0.07% at $x = 0.1$. The contamination drops rapidly at higher tunes, since the absolute trigger rate is nearly constant and total-energy cuts in that case are effective in rejecting these low-energy particles.

A more significant source of events is showering by high-energy particles on the edges of $K2$. These events are suppressed by the fact that $K2$ is not a boundary of the acceptance for particles originating at $T1$, but there is a region of phase space in which particles leaving $K2$ can be accepted. On the bevel, this is centered at $0.89 \cdot P_{tune}$ and an angle 16 degrees away from typical incident trajectories. The effect on the spectrum is of order 1%, including a small second-order effect in which events are generated by rescattering from the back plate; this is shown in Figure 23.

5.6 Effect of the $K3$ Collimator

Particles which strike the beveled inner surface of this ring at grazing incidence have a significant chance of scattering into the acceptance. Because small scattering angles (of order 2 degrees) are involved, the particles need penetrate only slightly into the lead, losing a negligible amount of energy. As a result, the probability of these grazing-incidence particles being accepted is relatively energy-independent, as is shown in Figure 24. The exception is in the endpoint region.

On the other hand, particles which penetrate far into the beveled surface and suffer hard, inelastic collisions are usually deflected by a large angle and have little probability of being accepted. This is especially true because of the relatively strong magnetic field which exists in this region: it is not possible for $e^\pm$ with less than about half of the tuned momentum to reach $T2$. These arguments also apply to particles hitting the upstream face of $K3$. Despite their large number, they have no significant effect at low energies and only a small effect at high energies.

5.7 Effect of the K_4 Collimator

The K_4 collimator is somewhat vulnerable to penetration. This is not because its 5.1 cm of lead is unreasonably thin, but because of the large number of particles incident upon it, compared to those in the spectrometer acceptance—a ratio of around 26. This effect is a major source of spurious triggers at spectrometer tunes of 30 MeV/c and greater.

Many of these events can be discarded: they mostly occur at high energies where the detector energy resolution is good, and these spurious triggers tend to involve mostly small energy deposits. Although complete separation is not possible, the effect on most of the spectrum is small when tight energy-deposition cuts are made. The most noticeable spectrum distortion appears in the line shape measurement at the endpoint: the good event rate falls to zero, while these events (which tend to come from particles below the tune momentum) fall off more slowly.

5.8 Effect of the K_5 Collimator

The particles incident on K_5 have momenta similar to the normal events, and most have to penetrate several centimeters of lead to reach the detector. Thus, spurious events at high P_{tune} are eliminated by the energy-deposition cut, while they very seldom occur at low P_{tune} . There is also no large enhancement factor from the spectrum weight of high-energy particles. This collimator does what it should do; the corrections for it are negligible.

5.9 Effect of the C_1 , C_2 Veto Counters

Due to the proximity of C_1 and C_2 to the detector, shower particles from it can hit these counters and veto good events. This reduction of detection efficiency was discussed in Section 5.2.3.

For C_1 , particles striking the beveled surface do so at grazing incidence, so that $e^\pm$ can scatter into the acceptance before depositing enough energy to allow rejection. This is more

likely for low-momentum particles, which tend to scatter at larger angles for a given distance traversed. The fraction of accepted events from this source has been calculated by Monte Carlo to vary from 1.2% at $P_{tune} = 6.15$ MeV/c down to 0.14% at $P_{tune} = 50$ MeV/c.

Also, $e^\pm$ from $T1$ strike the upstream face of $C1$; a small region adjacent to the beveled surface is essentially “dead”, in that the particle trajectories pass through so little scintillator that vetoes are unlikely. Thus, the spectrometer line shape broadens slightly toward low momenta. In addition, the effect has a P_{tune} dependence, since low momentum particles are more likely to be scattered out of the acceptance by this edge. By calculation, these events make up about 0.28% of accepted events at high momenta, falling to 0.12% at $P_{tune} = 6.15$ MeV/c—a net distortion of 0.16% in this range. The combined effect of scattering from the $C1$ bevel and face is shown in Figure 25.

In addition, the geometry of $C1$ is non-ideal, meaning azimuthally asymmetric. The scintillator leaves a gap as it rises away from the $K4$ collimator, then overlaps itself for a short distance, crosses the momentum-selection slit and finally joins the light guide. The low-momentum edge of the line shape is affected by both the gap and by the double thickness of $C1$: the former adds a long tail, the latter reduces the FWHM. The crossing of the slit between $C1$ and $C2$ just reduces the total acceptance.

Similar phenomena occur for $C2$. The $e^\pm$ striking the $C2$ bevel do so at fairly steep angles and have little chance to scatter out without enough energy deposit to cause a veto. However, $e^\pm$ that hit the most downstream part of the bevel will exit from the downstream face, sometimes traversing so little scintillator that no veto results. These tend to hit or not hit $T2$, as they would have done anyway. About 0.45% of the events passing all cuts have gone through this corner, but the principal effect is a slight broadening of the spectrometer line shape toward high momenta. Aside from this, the net spectrum shape distortion is around 0.14%—acceptance being reduced by this much at $P_{tune} = 6.15$ MeV/c, compared to high-momentum tunes, due to the increased scattering. The calculated effect is shown in Figure 26.

The geometry of this counter is also non-ideal. The long strip of scintillator is wrapped

circularly, leaving a small gap on the inside radius just before it overlaps itself. This introduces a long tail onto the high-momentum edge of the spectrometer line shape. It also substantially increases the number of particles which trigger *T3* without hitting *T2*.

5.10 Effect of the Comus Back Plate

The back plate of the Comus vacuum vessel is a 3.8 cm thick plate of aluminum, which corresponds to 10.3 g/cm^2 or 0.43 radiation lengths. Obviously this is not adequate to absorb the showering completely from moderate or high-energy particles that might strike it, although it attenuates the energy of the shower and increases its angular spread. Shielding was never, however, its main purpose.

There are several spectrometer characteristics that make this discussion relevant. First, the magnetic field in Comus deflects particles far above the tune momentum by a few degrees, as one would expect. However, because the field increases in strength with increasing radius, there is a tendency for high-energy particles to be deflected onto paths which are roughly parallel to the spectrometer axis, and at a fairly large radius from it. Many of these paths thread between the thick lead shielding of the *K5* collimator and the *A,B,C,D* anti-counters, striking the relatively unprotected area of the back plate left for the *A,B,C,D* counter light guides to exit the spectrometer. Finally, the material separating this portion of the backplate from *T2* is relatively thin aluminum and provides only minimal additional shielding.

There are, fortunately, other circumstances which mitigate the problem. Many $e^\pm$ are vetoed as intended by *A,B,C,D*; others are vetoed by Čerenkov emission in the light guides of these counters. A few particles will be attenuated by the magnetic shields and support members between the back plate and *T2*. Finally, the distance between the under-shielded part of the back plate and *T2* is large enough that the probability of false events is reduced due to a solid angle factor. These reductions are very significant.

Nonetheless, a problem remains due to the large number of particles which hit this area

and their relatively high energy. The time-of-flight of these particles hardly differs from that of normal events, so TDC cuts do not help. The total-energy cut is extremely helpful at high P_{tune} settings, since the detector energy resolution is good there and events from the back plate tend to be of low energy. In the absence of this cut, the contamination reaches a maximum of 4% near $P_{tune} = 30$ MeV; the cut reduces it to 0.8%. With all cuts applied, the maximum calculated contamination is 1.2% at $P_{tune} = 17$ MeV/c. At low energies, the Monte Carlo calculation needs to be supplemented by the empirical $\overline{T2} \cdot T3 \cdot T4$ data. Figure 27 shows the effect on the spectrum as obtained from the combination of the two.

Chapter 6

Data Analysis and Results

6.1 Monte Carlo Calculations

Many separate Monte Carlos, as opposed to one large one, were used to study various aspects of the spectrometer. This simplified the use of variance-reduction schemes and reduced the overall required CPU time to a practical, though still high, level. When the resulting inaccuracy was negligible, products of two or more small effects from separate Monte Carlos were often ignored.

In general, the philosophy of the Monte Carlos was to record experimentally observable quantities (i.e., energy depositions in the counters) event-by-event, along with quantities useful for variance reduction and comprehension. Events were stored in a compact form and, in most cases, could be kept accessible in computer disk files. Counters in which energy was deposited during an event were indicated in a mask word, and the quantities deposited in those counters were stored as 16-bit words.

Experimental considerations, such as counter calibrations, resolutions and cuts were then imposed on these files. This allowed one to vary cuts, study correlated quantities and adjust detector parameters in the analysis with relative ease. Except as noted, the basic interaction physics in each Monte Carlo was that provided by the EGS4 code system, with modifications as discussed in Appendix C.

In order to include magnetic deflection of the charged particles, field maps were calculated by the POISSON¹ code in the approximation of azimuthal symmetry. Measurements at a field 10% above the endpoint found the four flux-return yokes allowed a deviation from symmetry at the $\pm 0.25\%$ level, which, given the 2.8% line width and 90° rotation of transmitted tracks, does not materially affect the results. Then, exploiting the cylindrical symmetry of the problem, an analytic function was fitted to the field on the spectrometer axis; such a function, together with an adequate number of its derivatives, can reproduce the field away from the axis. Particle trajectories were found according to the prescription of Lindgren and Schneider,² in which the equations of motion, containing a power series expansion of the field at each point, are numerically integrated. Accepted tracks calculated by this approach typically deviated by only 0.01 cm from those found by the more usual Runge-Kutta integration method, and were obtained in a small fraction of the time. Also, the data could be held in a few parameters, as opposed to a full field map.

6.1.1 Incident Beam Studies

Incident muons pass through a thin beam line window and the *Tb* counter before stopping in *T1*, whose normal was turned 42° from the beam. Large angle scatters in the window or *Tb* prevent some particles from striking *T1*—and bias the stopping depths in *T1* toward slightly larger values, since these particles would usually stop more shallowly than average in a semi-infinite region. Also, the quantity of interest, the muon deposition depth below the *T1* surface, is strongly coupled with scattering angles and transverse transport by the *T1* rotation—to a much greater degree than in the usual situation of normal incidence.

For these reasons, the usual values of range and, especially, straggling were modified. A heavily rewritten and extended version of the TRIM85 code (discussed in Appendix D) was used to determine the applicable parameters. The results were checked against, and found to be consistent with, measured range curves obtained by placing varying amounts

¹M. T. Menzel and H. K. Stokes, Report LA-UR-87-115 (1987).

²I. Lindgren and W. Schneider, Nucl. Instrum. Methods **22**, 48 (1963).

of degrader between Tb and $T1$.

This code was also used to study the lead collimator which was just upstream of the Tb counter and which restricted the beam spot width and movement on $T1$. The aperture of this collimator was tapered to allow tighter collimation of the beam than normally possible. Despite the grazing-incidence particles on this bevel, it was found that only a small number of muons were scattered from the collimator onto $T1$ or its light guide; these reduced the average deposition depth by 0.06%. The EGS code was used to check for similar problems for the positrons in the beam, finding none. The explanation is that particles losing a substantial fraction of their energy in the collimator are almost certainly scattered so much that they miss $T1$, while the others present no direct difficulty.

6.1.2 Target Effects Study

The $T1$, Tb , $A1$, $A2$, $A3$, A , B , C and D counters were included in the target-area Monte Carlo. These counters were studied as a group because of the possibility of vetoes from correlated particles, although the A , B , C and D veto counters were also present in other studies. The Monte Carlo used the PRESTA macros as an important addition to EGS4, because of scattering in $T1$.

The number of events required to study the target adequately is very large, partly because of the small, but significant, probability of a high-energy decay positron yielding a low-energy event. Preliminary studies were done to find a distribution of starting events in phase space that substantially reduced the number required (by several orders of magnitude over nature's crude, but effective, approach). The necessary number of high-energy events is finally set by the statistics in the tails of the bremsstrahlung and Bhabha-scattering distributions, while multiple scattering sets the number at low energies. The statistical fluctuations in these distributions could have been lowered only with great difficulty, completely rewriting EGS. For the approach used, 54 million events were started.

In part, the actual variance reduction was accomplished by distributing the event weight very regularly in phase space, eliminating most of the statistical variation due to starting

conditions. The separation between occupied points in phase space was small enough that the spectrometer's angular acceptance and momentum bite, along with physical effects, smoothed the discrete distribution adequately. (Multiple scattering, of course, helps to smooth the discrete distribution in angle, but not by as much as one might naively expect. For example, if particles are started at polar angles spaced by 1° , and scatter by a degree or so on the average, the resulting distribution will still be peaked at the starting angles, if they are more than a few degrees from the axis: the problem is two dimensional and many of the scattering angles add in quadrature, rather than linearly, to the starting angles.)

Events are efficiently used when more are started in phase space regions with an acceptance probability near 50%, and fewer are started in regions where acceptance is either very likely or unlikely. A weighting factor for each event allowed the retention of the correct probability density. In general, events were started so as to minimize the standard deviation of the resulting distribution for the number of events used. However, very large weighting factors were avoided so that, in the real world of limited statistics, the tails of the variance distribution were not too large: the probability distribution for the accepted event weight does not necessarily resemble a Gaussian and grave errors can be made by treating it as such. (As an example, if there is a probability of 0.001 that a particle causing a 1% effect will be accepted, the average effect is 0.001% and the standard deviation is 0.03%. If one draws only a single particle from this distribution, the 99.9% chance that the result will be low by only 0.001% does not compensate well for the occasional 1% spikes, and the probability of the 1% spikes is obviously not at the 30σ level of a Gaussian distribution.) Regions of phase space found to have utterly negligible effects were left unpopulated, although these regions were not very large at the lowest energies studied.

The magnetic field was ignored at this stage. Except for extremely low-momentum particles (less than a few hundred KeV/c), particles upstream of A1 move through the vacuum in straight lines. The extremely low-momentum particles are not important because they are far below the range over which the spectrometer was tuned for spectrum measurements, and the cuts on the target-area counters were set high enough that these particles do not

cause vetoes. It is therefore necessary to run only a single set of target-area Monte Carlos, as opposed to doing one set for each of many P_{tune} settings.

The distribution of particle starting positions within the target was that predicted by the incident beam Monte Carlos, including the effects of collimation and straggling. Events yielding one or more particles downstream of A1 with a polar angle in the accepted range were written to an intermediate file on magnetic tape. These trajectories were traced to their intersection with a plane transverse to the spectrometer axis at the center of T1. The event record then consisted of the direction cosines, intersection point and particle energy, as well as the energy deposited in the counters. Any accompanying charged particles downstream of A1 were recorded at the A1 position, while photons directed at A, B, C or D were allowed to strike them, with any resulting energy deposition recorded.

These files were then inspected by a second program containing maps of two phase-space regions: the one for which a charged particle would strike the A, B, C or D anti-counters, and the spectrometer acceptance. Events in which a charged particle hit A, B, C or D were assumed to be vetoed efficiently (in order to reach these anti-counters, the particles necessarily have high momentum). The program also applied the detection efficiency, as determined by a separate set of Monte Carlos, for the $T2 \cdot T3$ and $\overline{T2} \cdot T3 \cdot T4$ event classes. To use the event statistics most efficiently, the efficiencies were applied as factors to the event weights; the probability that each counter ADC would pass experimental cuts, when the resolution was folded with the energy deposition, was determined; the product gave the overall probability. (The alternative, inefficient approach would be to select from the ADC distributions randomly and compare with the cuts for these specific values.) Four histograms in P_{tune} , separate for the two triggers and for e^+ and e^- , were then incremented with the appropriate weight for each field setting at which the event could be detected.

In this approach, the histograms have been automatically smoothed by the spectrometer line width. However, because this is very narrow at low momenta, and because single events with large weight are found here (from Bhabha scatters or bremsstrahlung), further smoothing was needed. A third-level program thus combined histograms from sepa-

rate Monte Carlo runs, divided these by a function representing the approximate result, smoothed them and then multiplied by the same function. Different functions were used for electron and positron histograms, because of their very different shapes. This avoids most of the error from blindly smoothing a nonlinear function. The statistical variation in the final histogram is better than 0.2%, and was calculated for decay energies down to 6 MeV.

There is one further complication: the spectrum shape is the thing which we are trying to measure; it is not available as pure input to a Monte Carlo. The solution is to parameterize the spectrum as

$$\frac{d\Gamma(x, \eta)}{dx} = \frac{d\Gamma_1(x)}{dx} + \eta \frac{d\Gamma_2(x)}{dx} ,$$

which is possible using Eq. 2.4. Grotch has already set $\rho = 3/4$, so it does not appear as a second parameter. (Deviation of ρ from $3/4$ is studied later in an approximate way, neglecting products of this deviation with the radiative corrections and experimental spectrum distortion.) It is not necessary to run separate Monte Carlos to calculate the experimental modifications to Γ_1 and Γ_2 ; one simply applies different initial event weights to the same events.

6.1.3 Internal Bremsstrahlung Studies

The principal target-area Monte Carlo involved only the decay positrons, not the internal bremsstrahlung photons. This was natural because Grotch has integrated over the photon phase space, assuming that they are not detected, and an acceptable approximation because most were, in fact, not detected. Their effects were studied separately in two specialized programs.

The first program found the probability that correlated photons would veto events and the variation of this with positron energy. Rather than using EGS to make a detailed simulation of each event, the photon-interaction physics were applied in a much faster, simpler way. Tables of the probability that a photon of given energy would cause a veto for a

given path length in a veto counter were calculated for specific cuts; these tables were interpolated to find the probability that the photons accompanying an accepted positron would veto the event. Fortunately, the veto probability was found to be small and insensitive to the anti-counter cuts in the spectrum region used to determine η , as discussed in Section 5.1.6. The correction was finally applied to the calculated spectra.

The radiative-decay photons also produce contaminating $e^\pm$ on the $K1$ collimator and the aluminum vacuum chamber walls in its vicinity. This is studied with EGS4, finding the probability that photons will produce accepted $e^\pm$, without the correlated positron vetoing the event.

6.1.4 Collimator Studies

The collimator surfaces downstream of $A1$ were all studied with the same basic program—but separately, allowing events to be distributed in an efficient manner and making the relevant physics more transparent. It also made interpolation much simpler, as will be discussed below. The surfaces studied were the

- the upstream face and bevel of $K2$,
- the upstream face and outer radius of $K3$,
- the beveled edge of $K3$,
- the upstream face of $C1$,
- the beveled surface of $C1$,
- the upstream face of $C2$,
- the beveled surface of $C2$,
- the upstream face of $K4$,
- the upstream face of $K5$, and

- the spectrometer back plate.

The required CPU time was reduced by factoring effects into several parts. Typically, three functions were found—

- The physical processes giving events from particles incident on a surface usually vary slowly with energy, but are expensive to simulate. (Typically one second of VAX-780 CPU time is needed for each event, this slowness being due to the extensive showers which particles can produce, as well as the cost of transporting charged particles in a magnetic field.) These Monte Carlos were therefore only run at a moderate number of specific spectrometer tunes, spaced in roughly 1 MeV/c steps at low momenta, up to 10 MeV/c steps at high momenta. Events were run and saved, the experimental counter resolutions and cuts being applied later. After the cuts were applied, giving the probability that a particle incident on a surface would yield an event for a given trigger, the Monte Carlo results were (usually) smoothed by hand. These energy-dependent probabilities were then interpolated by three-point, Lagrange interpolation.
- The number of particles incident on a surface can be rapidly varying with P_{tune} , but is relatively cheap to calculate when only direct tracks from $T1$ are considered. Normalized to the number of accepted particles when the central incident momentum is tuned, this value can be cheaply calculated at closely-spaced points as needed; it changes rapidly when the spectrum endpoint excludes impacts as P_{tune} increases, or when the acceptance at the normalization tune approaches the endpoint.
- Finally, the number of particles entering the acceptance at the normalization tune was available from the target-area Monte Carlo. By using these results, some higher-order corrections were included in, at least, an approximate way.

The approximations involved in this sort of separation are usually small; the principal errors occur when particles from the endpoint are incident on a surface, so that the particle

distribution, as well as number, may vary rapidly. In some cases, Monte Carlos were closely spaced in P_{tunc} to reduce this error.

The input file of trajectories encountering each surface of interest was created by tracking particles from the predicted beam spot distribution on $T1$. They were uniformly started within some region of phase space whose boundaries had been determined to contain all trajectories which could reach the surface from $T1$. The intersection with, and momentum vector at, the surface of interest was recorded for those events which did not first strike another surface. The weights and momenta for these events were later scaled at each chosen value of P_{tunc} .

6.1.5 Upstream Support-Cable Studies

The three upstream sets of support cables were also Monte Carloed at only a limited number of spectrometer tunes, for the same reasons as above. To a reasonable approximation, the three cable sets lie on conic surfaces. Simulated particle trajectories were traced through the spectrometer, and their intersections with these surfaces were recorded. For each cone, a file was made of the trajectories that would be successful in the absence of the cable set; another was made of those that would fail after crossing the surface.

Next, an azimuthal angle was chosen from the range in which the trajectory could intersect the given cable set, the event was weighted appropriately and the particle was tracked back to its original impact on the cable. The motivation for this approach was to reduce the size of the input file and the cost of producing it. Particles were transported through the cable (approximated as a cylinder of iron with the measured global density) by EGS4; particles exiting the cable were tracked through the magnetic field until they again encountered material. If at least one particle from the shower reached the vacuum window at the spectrometer exit, the starting conditions of the event, including the random number seeds, were recorded. These were only a miniscule fraction of the events started.

The events could then be reproduced, and the same code used to Monte Carlo the spectrometer downstream of $A1$ was used to simulate the remainder of the event. Energy

depositions in counters and other event-specific information were recorded for later use, as before.

6.1.6 Detection Efficiency Studies

Detection efficiency was studied in a way similar to that used for the collimator impacts; the same basic code was also used. Files of positrons striking the vacuum window and last set of collimator-support cables were created at several spectrometer tunes by tracking particles, saved by the target-area Monte Carlo, through the spectrometer. Separate efficiencies were found for e^+ and e^- , as these differ considerably, but the number of e^- events produced by Monte Carlo was too small; thus, the e^+ input file was used again with opposite charge. The approximation is insignificant.

The result variance was reduced by using the known, average fraction (0.71%) of events hitting the last set of collimator-support cables; this was useful because these events represent a major fraction of the inefficiency at some spectrometer tunes. Energy depositions in the counters were recorded as in the previously-discussed studies, again allowing later application of counter resolutions and cuts.

It was necessary that these studies include the azimuthal irregularities in the $C1$ and $C2$ counters, as these affect the particle distribution on the trigger counters, and, hence, the detection efficiency. Runs were spaced rather closely for tunes in the endpoint region, as the particle distribution changes rapidly.

6.2 Applying Cuts

In general, the measured spectrum shape will depend on the counter pulse-height cuts chosen. In this experiment, the sensitivity is large for some counters: the dead areas of $C1$ and $C2$, for example, are proportional to the applied cuts; and the loss of detection efficiency at a given energy due to annihilation is sensitive to the cut on $T3$. Uncorrected, both of these effects roughly mimic a shift in η . Therefore, it is necessary to apply the experimental

counter resolutions and cuts carefully to the Monte Carlo results. The required information comes from several sources, depending on the counter.

Most TDC spectra were corrected for walk, due to pulse-height variation, using a simple, two-parameter, exponential function of the ADC—with excellent results. *A1* and *A2* are more complicated because the ADC and TDC are also correlated due to the attenuation of light as it reflects (and is delayed) around the circumference of the counter rings. The sum of two exponentials, with four parameters, substantially improves the quality of the correction although this form has no particularly strong basis. *C1* and *C2* have a similar problem in principle, but the light attenuation is much less and the trigger rate in these counters is so low that there is no motivation to make small improvements in the TDC resolution. To improve resolution further and simplify their TDC spectra, the high-rate counters, *T1*, *A1* and *A2*, are calibrated in picoseconds and referenced to the calibrated TDC of a low-rate trigger counter, *T2* or *T4*.

T1, Tb

The *T1* ADC energy calibration was determined by passing beam positrons. The resolution for energy deposition is dominated by photoelectron statistics; the calibration of photoelectrons per deposited MeV was the value for which photoelectron statistics, combined with the Monte Carlo energy-deposition distributions of beam and decay positrons, reproduced the measured distributions.

The ADC was corrected for rate, by comparing the ADC distributions at the same P_{tune} for different beam rates, and for magnetic field effects when $P_{tune} > 37$ MeV/c. It was also corrected in a time-dependent way for the tail and reflections of the muon pulse, which had effects up to 2 μ s afterwards, by using the nanosecond clock reading.

The ADC cuts on this counter were moderately tight, rejecting about 3% of the events, especially from the high energy-loss tail. This was possible because positron energy loss is nearly constant over the momentum range studied, and desirable because most events scattering out of the *T1* plane were thereby rejected, along with some Bhabha scatters and

decays of muons that stopped in the edges of $T1$. Even tighter cuts would have rejected more undesirable events, but would also have increased the vulnerability to inaccuracies in the ADC corrections and calibrations.

The lower ADC cut was made at the point where noise and good events made roughly equal contributions to $P_{tune} = 6.15$ MeV/c runs. Thus, the accepted event sample contained a certain amount of contamination, which was handled by using TDC information. A simple cut was not possible because of the high rates in $T1$; the TDC was not infrequently stopped before the desired pulse by beam positrons, muon pulse reflections, noise, other decay positrons, etc. Its statistical use is, however, extremely important and will be discussed in Section 6.3.

The Tb counter was calibrated by oscilloscope measurements of passing beam positron pulse heights. It was part of the hardware trigger, and no off-line cuts were applied, but the calibration for the discriminator level was needed in the target-area Monte Carlo.

A1,A2,A3

The $A1,A2$ counters were calibrated partly by the band of ~ 80 -KeV lead X rays and partly by passing $e^\pm$. The X rays, produced by particle impacts on $K1$ and $K2$, were visible in high- P_{tune} measurements—due to the high statistics and relatively few hits in the target-area counters in these runs. The $A3$ calibration relied upon the X rays, since $A3$ is struck by very few high-energy $e^\pm$ in triggered events. Position dependence of the light-collection efficiency in all three was determined with passing electrons from a ^{106}Ru source.

The ADC cuts in $A1$, $A2$ and $A3$ were chosen to be above the level of the lead X rays to simplify the calculation of the veto effects; Monte Carlos need not then propagate particles, including photons from internal bremsstrahlung, into the lead collimators to find the resulting fluorescence photons. Further, vetoes from the very low-energy $e^\pm$ which can be deflected by the P_{tune} -dependent magnetic fringe field, are avoided when an energy deposition of at least 90 KeV is required. The cut is still low enough to reject efficiently any energetic $e^\pm$ striking $A1$, $A2$, $A3$.

The TDC spectra of $A1$, $A2$ and $A3$ are also affected by low-energy $e^\pm$. These particles can pass through the apertures of $A1$ and $A2$ initially, only to be reflected from the spectrometer's magnetic field and strike an anti-counter on the return pass. Such an effect is obviously a function of P_{tune} . Delayed vetoes can also result from particle impacts on the collimators. One might, in principle, prefer to reject events with a delayed veto, since many are caused by Bhabha scatters from the target. However, accurate calculation is difficult, and they cannot be separated from uncorrelated impacts on these counters.

Correction was made for $A1$ or $A2$ vetoes, uncorrelated with an event, in a statistical way, subtracting the TDC background in them for each run and adding vetoes that would have occurred if an early firing had not already stopped the TDC. (A small increase in the uncertainty of the data results, of course, and is included in the statistical error.) Correlations between $A1$ and $A2$ were necessarily included in this TDC correction. Also, because of the existence of delayed, correlated vetoes, only early vetoes were used to determine the background level.

A,B,C,D

The A,B,C,D counters were calibrated using positrons from the target with the spectrometer magnet turned off, since, under usual conditions, charged particles pass through these counters at a wide variety of angles and do not give a clear peak. This was checked against the 341 KeV Compton edge (from 511 KeV annihilation gamma rays) which is visible in the ADC spectra for high- P_{tune} measurements. The ADC cuts were made as close to the pedestals as possible—below this Compton edge and far below the level for passing charged particles. This helps to dispose of events in which a collimator has been struck, as well as those with direct hits. While small energy depositions (mostly from annihilation quanta) veto about 0.12% of the good events, the fraction depends only weakly on energy.

C_1, C_2

The C_1, C_2 counters were calibrated by the positrons that traversed them at high- P_{tune} settings. These results were fitted to Monte Carlo calculations incorporating resolution measurements done with a ^{106}Ru source.

The C_1, C_2 ADC's were cut as close to the pedestals as possible, eliminating most particles which grazed the bevel, or struck an edge, of a counter. This reduced the correction for, and therefore the uncertainty of, rejecting flawed events. Also, a small upper limit on the path length through scintillator reduces the probable scattering angles and the resulting energy-dependence of detection efficiency.

About 0.5% of events deposit energy in C_1, C_2 from showers in the detector, and it would be desirable to retain these events, identifying them by their TDC signature. The expected delay is 4 ns in C_1 and 3 ns in C_2 . Unfortunately, they cannot be cleanly separated from other events because the optical length of these counters—bent circularly, but observed from one end—is nearly 6 ns. There is a further complication from the range of pulse heights which are important. One must simply correct for the loss of these events; the fraction is only weakly dependent on P_{tune} .

 $T_2, T_3, T_4, \text{NaI}$

The T_2, T_3 and T_4 counters were calibrated by fitting the measured ADC distributions to Monte Carlo results. The light-collection efficiency varies only slightly over the surfaces of these counters, as determined with a ^{106}Ru source; photoelectron statistics dominate the resolution. There are small gain losses at high spectrometer settings; the correction at the endpoint is 0.7% for T_2 and 2.5% for T_3 .

The ADC cuts on these counters were made low enough to avoid rejecting any significant number of events from the main distributions although, especially for T_3 , some annihilation events were inevitably lost and rely on the calculated detection efficiency for compensation. There are also indirect ADC cuts on these counters via the total-energy deposition cuts.

The NaI calibration was somewhat complicated. A variable attenuator before the ADC

was set to five different values over the range of spectrum measurements, to reduce the dynamic range problem. The attenuator settings were cross-calibrated with 1% corrections, typically, to the nominal values. This done, the energy loss in the NaI window was checked, since this was not known exactly: the manufacturer, Harshaw Chemical Company, only specifies (and knows) the unpacked geometrical thickness of the window components, which is vague for a material like MgO powder. (Naturally, when used as a γ -ray detector, this is not very important.) Thus, the most probable energy losses were compared to Monte Carlo prediction for several low-momentum tunes, and a thickness for the MgO powder was assigned to bring these into agreement. Finally, the magnetic field degrades the phototube gain for $P_{\text{tune}} > 40$ MeV/c; shielding of these tubes is difficult because of the large NaI diameter, and the gain loss reaches 7% at the endpoint.

With calibrations for all counters in the detector, an estimate of the total particle energy was possible. As discussed earlier, this estimate is not highly precise because of backscattering, inactive materials, photoelectron statistics, etc. There is also some effect from the finite range of the ADC's: the Monte Carlos must include the ADC upper limit for T2, T3 and T4, since some backscattered events lead to overflows.

The upper limit on the total energy was chosen to reject very few good events ($\leq 0.05\%$, based on the Monte Carlo results), but is useful because many background events were at much higher energies, especially at low spectrometer tunes. The lower cut was more aggressive, cutting about 0.5% of the good events from each run, since there is no clear separation between contaminating and good events. It succeeds in rejecting as many as half of the undesirable events. This level of cut rejects events fairly far out in the tail of the distribution for good events, and is not very sensitive to small calibration errors. An even tighter cut was not made because the energy-deposition distribution for normal events is beginning to rise rapidly by this point, so that additional gains would have had a much smaller rejection ratio. More importantly, a cut on the total energy can very easily bias the spectrum, through calibration errors or physics approximations.

6.3 Background Subtraction

The $T1$ TDC spectrum is made up of several components: normal “good” events, which occur in a narrow distribution; background events for which the same signal caused both the trigger and TDC stop, and which occur in an almost flat distribution over the combined gate widths of the $T1$ and $T2$ counters; and both good and background events in which the TDC was stopped early by an uncorrelated signal. The uncorrelated TDC stops decrease proportionally to the probability of a TDC stop at a given time, falling to zero by the end of the $T1 \cdot T2$ overlap range.

The lower and upper bounds on this overlap range will be called t_1, t_2 . The background is estimated by first finding the maximum likelihood values of the parameters b and u in the function

$$Q(t) = b + u \int_{t_1}^t S(t') dt' ,$$

where $S(t')$ is the measured TDC spectrum at time t' . The function is fitted to the TDC spectrum for $t \in (t_1, t_2)$, exclusive of the spike corresponding to normal events. $Q(t)$ represents events in which the $T1$ TDC is not stopped by the departure of a positron that hits $T2$. The background in the overlap region is then

$$B(t) = b + u \int_{t_1}^t B(t') dt' ,$$

with $B(t)$ being found from a numerical fit to the data, using the previously determined values for b and u . The fraction of good events in the overlap region is

$$f_{good} = \frac{\int_{t_1}^{t_2} [S(t) - B(t)] dt}{\int_{t_1}^{t_2} S(t) dt} ,$$

which is also the fraction of good events for $t < t_1$, since these early TDC stops are followed, in reality by the overlap region which triggered the event.

This argument is, however, somewhat flawed: events with early TDC stops do not necessarily have the identical characteristics as those without them, i.e. it may not be purely statistical which events have early TDC stops. Several $T1_e$ pulses are often triggered by an arriving muon, and events which closely follow the muon arrival are both more likely

to have early TDC stops and to have been triggered, in part, by a $T1_e$ pulse not related to the $T2$ pulse. Because of this correlation, events for each run were separated into three classes. The first class was composed of events in which only one muon had stopped in $T1$ in the 4-8 μs preceding the event, and in which the event occurred at least 1 μs after the muon arrival; the other two classes were composed of events which were more likely to have involved extraneous $T1_e$ pulses, the one class more so than the other. The background subtraction was done separately for each class, giving somewhat different results from the naive approach. This was especially true for the $\overline{T2} \cdot T3 \cdot T4$ events, which contain a larger fraction of background. The background subtraction for these events used the same approach as for $T2 \cdot T3$, although the discussion above only refers to the latter.

6.4 Magnetic-Field Corrections

The magnetic field shape in the spectrometer changes somewhat as the spectrum endpoint is approached, due to saturation effects in the iron. There is some change in the volume through which particles travel, affecting the tuned momentum by 0.05% at the endpoint. Although not of much significance, this was corrected in the analysis.

However, the field shape is more affected at larger distances from the magnet center, and the largest effect is the shift of the field at the NMR and Hall probes, relative to the fields deflecting accepted particles. This affects the measurement of the tuned momentum by 0.30%. Small shifts also occur in the probe cross calibrations, since they are not in exactly equivalent positions in the magnet. Corrections are made with a combination of measurements and calculation.

6.5 Scaler Corrections

The $T1_\mu$ and Tb signals create hardware event vetoes, hence deadtime. Also, run-to-run calibration is based on knowing the number of muons stopped in $T1$, given by the $T1_\mu \cdot Tb_\mu$ scaler. The involved scalers must be known accurately, and corrections are

necessary because conditions were not identical for all runs. When needed, rates were extracted through scalers dedicated to counting oscillator pulses.

While deadtime due to event readout was incorporated into the gates for most scalers, other corrections had to be applied in the analysis. An imperfection in the data-acquisition system enabled the scalers slightly before events could occur that would be recorded (for most later runs, data acquisition was manually enabled to eliminate this problem); the computer system also occasionally lost data buffers (depending on event rate and, hence, P_{tune}). Because the number of triggered events was recorded as a scaler, and all scalers were recorded with each data buffer, the scalers could be trivially adjusted to correct for these errors.

Pulse pile-up effects were not negligible in some scalers and needed careful correction, especially $T1_\mu$ (because of its relatively long veto into the trigger of 154 ns) and Tb (because of its high rate). The long $T1_\mu$ veto also caused the loss of 6.8% of potential events from each muon due to the time correlation between muon arrivals and events, though this is not of great import in a spectrum shape measurement, except in understanding the spectrometer acceptance.

Each event was assigned a weight of

$$W_{event} = e^{R_{T1_\mu}(\Delta_{T1_\mu} + \delta_1)} e^{R_{Tb}(\Delta_{Tb} + \delta_1)} e^{\Delta_{T1_\mu}/\tau_\mu},$$

where R_{T1_μ} and R_{Tb} are the average, deadtime-corrected $T1_\mu$ and Tb rates for the buffer holding the event; Δ_{T1_μ} and Δ_{Tb} are the logic pulse widths; δ_1 is the scaler deadtime after each pulse; τ_μ is the muon lifetime. The deadtime correction was done by solving a trivial transcendental equation; as an example, for a measured $T1_\mu$ rate of R''_{T1_μ} ,

$$R''_{T1_\mu} = R_{T1_\mu} e^{-R_{T1_\mu}(\Delta_{T1_\mu} + \delta_1)}.$$

Newton's method converges very quickly to the solution for R_{T1_μ} . The uncertainty in the correction as a whole introduces about 0.08% error in any given run.

Another correction must be applied to the $T1_\mu \cdot Tb_\mu$ scaler: the coincidence was between the 154 ns $T1_\mu$ and the 20 ns Tb_μ pulses, leaving a window between the end of Tb_μ and

the end of $T1_\mu$ during which a muon could strike Tb , miss $T1$ and still be counted on the $T1_\mu \cdot Tb_\mu$ scaler. For a typical run, where $R_{T1_\mu} = 50$ KHz and 39% of the muons that hit Tb went on to miss $T1$, the correction is 0.65%. The correction formula is

$$\frac{R_{T1_\mu \cdot Tb_\mu}}{R'_{T1_\mu \cdot Tb_\mu}} = 1 - \frac{R_{Tb_\mu}}{R'_{T1_\mu \cdot Tb_\mu}} + e^{\left(R'_{T1_\mu \cdot Tb_\mu} - R_{Tb_\mu}\right) \Delta T_{b_\mu}} \left[1 - e^{-R_{T1_\mu}(\Delta T_{1_\mu} - \Delta T_{b_\mu} - \delta_2 - \delta_3)}\right],$$

where $R_{T1_\mu \cdot Tb_\mu}$ is the real, and $R'_{T1_\mu \cdot Tb_\mu}$ is the measured rate of $T1_\mu \cdot Tb_\mu$ coincidences corrected only for deadtime; R_{Tb_μ} is the muon rate in Tb (including those missing $T1$); ΔT_{b_μ} is the width of the Tb_μ pulse; δ_2 is the minimum time, between the end of one pulse and the beginning of the next, for the coincidence module to produce two distinct output pulses; and δ_3 is the pulse overlap requirement of the coincidence module. As before, the rates are the average for the buffer holding the event, corrected for pulse pile-up. A complication is that Tb_μ was not recorded on a regular basis. Some information is available— Tb was recorded, but the positron:muon ratio varies enough to leave a 0.19% uncertainty in the number of stopped muons in a run, relative to other runs. This does not cause a major problem, because the statistical uncertainty in the number of events in a given run is much larger than this.

6.6 $\overline{T2} \cdot T3 \cdot T4$ Events

A challenging indicator of the accuracy of the spectrum distortion modeling for Comus is the event class ratio $\overline{T2} \cdot T3 \cdot T4 / T2 \cdot T3$ (where both classes are understood to have had all other trigger criteria, cuts and time-uncorrelated background subtractions applied as well). In an ideal spectrometer, almost no $\overline{T2} \cdot T3 \cdot T4$ events would have occurred; impacts on $T4$ are very unlikely unless particles scatter, annihilate or pass through “imperfections” in the $C1$, $C2$ anti-counters. These events thus provide an independent check of spectrum contamination calculations—but are in no way a measurement of the contamination, since the number of events in the $T2 \cdot T3$ and $\overline{T2} \cdot T3 \cdot T4$ classes have no simple relationship.

The calculated ratio, together with experimental data, is plotted against x_p in Figure 28, where $x_p \equiv P_e / P_e(max)$. No free parameters are used in this figure. The error bars

exhibited for the experimental data are statistical only. The relative flatness of the plot for $x_p \in (0.2, 0.6)$ is not an intrinsic characteristic, but results from the particular cuts chosen. For example, removing the total detected-energy cuts would result in a variation in $\overline{T2} \cdot T3 \cdot T4 / T2 \cdot T3$ of several percent in this range.

In general, the prediction and data appear to be in good agreement, although a few points near the endpoint do not coincide. Regardless, the value of η is obtained from the lower parts of the spectrum, and the endpoint calibration is not sensitive to small inaccuracies. The discrepancy at low momenta is of concern, however. The source cannot be conclusively established, but it probably derives from scattering of $e^\pm$ on the spectrometer backplate and vacuum can. Veto inefficiency in the A, B, C, D anti-counters would permit this. While a high efficiency is expected, the 100% assumed in the Monte Carlo was certainly overly optimistic. Calculation of the inefficiency is not possible because many of the vetoes arise from Čerenkov light in the twisting, adiabatic light guides. The geometry, both for the incident particles which deflect in the magnetic field to intersect these guides and for the directional light passing through the guides, is complex.

Thus, one must subtract additional background from the data, based on the discrepancy and the calculated ratio of $T2 \cdot T3 / \overline{T2} \cdot T3 \cdot T4$ events from scattering in this region. The background is reduced at higher momentum tunes because fewer e^+ reach the vulnerable part of the vacuum vessel and total energy cuts reject more background; the fractional effect is even more strongly reduced. The fitted value of η is changed by -0.082, a 1σ shift. The ratio $\overline{T2} \cdot T3 \cdot T4 / T2 \cdot T3$, with the additional background added to the prediction, is plotted against x_p in Figure 29. This single-parameter fit, using the calculated energy dependence of the contaminating events, has a $\chi^2 / (\text{degree of freedom})$ of 7.2/7 for the lower eight points in the graph. (It is also encouraging to note that the inclusion of this correction improves the $\chi^2 / (\text{degree of freedom})$ for the fit of the data to the $T2 \cdot T3$ spectrum from 11.9/11 to 10.0/11).

6.7 Extracted Value of η

Having finally obtained a theoretical prediction for $d\Gamma(x_p, \eta)/dx_p$ including experimental effects, this was fitted to the data with three parameters: the spectrometer momentum calibration, the acceptance calibration and η . The fit was done iteratively in two parts: the minimum χ^2 fit for acceptance calibration and η were found (with an approximate momentum calibration) for $x_p \in (0.117, 0.756)$, followed by a fit of the momentum calibration to the endpoint data. The theoretical spectrum depends weakly on the acceptance calibration and the fitted values were thus used to correct it. This procedure was repeated a few times until the parameters converged, giving a $\chi^2/(\text{degree of freedom})$ of 10.0/11 and the value

$$\eta = -0.080 \pm 0.088 \text{ (for } \rho \text{ constrained to } 3/4\text{)}.$$

The experimental data, with statistical error bars only, are plotted on the Monte Carlo-calculated theory curve in Figure 30, with $\eta = -0.080$ and ρ constrained to $3/4$. Nearby data points have been combined in the figure for clarity. Calculations did not extend below 6 MeV, and data at lower momenta were not included in the fit. The Monte Carlo curve is not completely smooth; this stems partly from statistical fluctuations and partly from real spectrum distortions. Figure 31 shows the data divided by the theory curve, along with a dashed line showing where data for $\eta = 0$ would be centered.

Unless the statistics in the spectrum mid-range are extremely high, compared to those in the lower part of the spectrum, a significant correlation between the acceptance calibration and η will appear in the fit; for these data, the correlation coefficient is -0.884. The uncertainty in the fitted acceptance normalization is $\pm 0.42\%$.

It was not possible to determine the acceptance more accurately than this from first principles; the actual acceptance was about 4% lower than calculated. (Neither, of course, was it ever intended to use the calculated acceptance in fitting the spectrum. The required accuracy of geometrical measurements would be around 0.01 mm, and several other parameters would need to be known to high accuracy.)

The statistical error on η would decrease with the inclusion of higher momentum points because these help to provide amplitude normalization. On the other hand, the systematic errors increase near the endpoint due to several rapidly varying effects there, and the sensitivity to the ρ parameter becomes excessive. Thus, data above 40 MeV were excluded from the fit.

Instead of fixing ρ to 3/4, the measurement can be specified using the published experimental value of $\rho = 0.7518 \pm 0.0026$:

$$\eta = -0.058 \pm 0.088 \quad (\text{for } \rho \text{ constrained to } 0.7518).$$

For $\Delta\rho = 0.0026$ and the data in the range $x_p \in (0.117, 0.756)$, the corresponding uncertainty in η is 0.032.

6.8 Systematic Error Estimate

Systematic error estimates for η , determined using data in the range $x_p \in (0.117, 0.756)$, are given in Table 6.1. Correlated effects have been directly added in the table entries, while those which should be largely uncorrelated have been added in quadrature.

The column labeled “Shift in η_{fit} ” is the shift an *uncorrected* systematic effect would cause in the determined value for η . The errors assigned to “theory” in the next column are due to neglected higher-order QED terms, uncertainty in atomic effects, etc. in the cross sections for particle interactions. The errors given are those that remain after approximate corrections were applied, if known. Inaccuracies in large-angle scattering, and hence in the detection efficiency, were approximately corrected in the results, rather than inside the EGS4 code. This was also done for Bhabha scattering. Bremsstrahlung in the electron field in $T1$ was largely corrected by modifying EGS4, but an additional correction was applied to the results. Simple, after-the-fact corrections were not possible in thick regions.

The errors assigned to “EGS4” are the uncompensated errors which exist in the modified version of EGS4 that was used, in excess of the errors in the best available theory. In light

materials these errors arise mostly from the multiple scattering treatment; in lead, the inaccuracies in the treatment of annihilation and bremsstrahlung also contribute.

The next column—calculational and experimental uncertainties—includes all other effects. Geometrical approximations, counter calibration errors, Monte Carlo statistics, etc. are included.

The total systematic error in the fitted value of η is estimated to be ± 0.056 , not including the uncertainty in ρ . This is broken down somewhat in Table 6.1; the error estimates are discussed in more detail in Appendix E, and other references are given in Table 6.2. The “miscellaneous” entry includes events caused by conversion of γ 's which start from the target or $K2$, events from shower penetration through $K2/K4$ and errors from NMR probe cross-calibration.

6.9 Conclusions

In the accepted, existing theoretical treatments of the muon decay spectrum and its radiative corrections, the W^+ vector boson has been taken to be infinitely heavy. As discussed in Section 2.4, this yields an excellent approximation to the exact muon decay spectrum, provided that there are no tensor, scalar or pseudoscalar couplings. Since there is no evidence of these couplings in this or other measurements, one may rely upon the radiative corrections that were used—primarily those calculated by Grotch to first order. The higher-order corrections had little effect on the lower and middle portions of the spectrum used in fitting a value of η .

The distortions to the theoretical spectrum, which are inevitable when measuring the low-energy end of the spectrum (especially from stopped muons), have been studied and are understood. The calculation of distortions through variance-reduced and interpolated Monte Carlos has given predictions in close agreement with experimental results, although experimental input was required at low energy. Systematic errors due to other phenomena (such as errors in scalar values and magnetic-field calibrations) made only minor contribu-

tions to the overall error.

Thus, it has been possible to largely correct the instrumental effects in the spectrum measurement and to make the experiment self-normalizing. For ρ constrained to $3/4$, the best-fit value was $\eta = -0.080 \pm 0.088(\text{statistical}) \pm 0.056(\text{systematic})$. This value is in agreement with the V-A prediction of zero, and there is no indication of exotic physics, such as massive mixed neutrinos, in the $P_e \in (6.15, 40)$ MeV/c range of the spectrum. The determination of η is somewhat sensitive to ρ , and values of ρ different from $3/4$ would shift the fitted value by $\Delta\eta = 12.2(\rho - 3/4)$.

Source	Shift in η_{fit}	$\Delta\eta$ (theory)	$\Delta\eta$ (EGS4)	$\Delta\eta$ (exp., calc.)	$\Delta\eta$
μ^+ depth	+0.408			± 0.029	± 0.029
Back plate	+0.133	± 0.006	± 0.004	± 0.027	± 0.028
$T1$	+0.408	± 0.009	± 0.014	± 0.017	± 0.025
Detector ineff.	-0.218		± 0.005	± 0.014	± 0.015
μ^+ spin angle				± 0.014	± 0.014
Cables	-0.057	± 0.003	± 0.005	± 0.009	± 0.011
$K2$	+0.046	± 0.005	± 0.005	± 0.009	± 0.011
$C1$	+0.069		± 0.002	± 0.009	± 0.009
P_{max} calib.				± 0.008	± 0.008
Beam centering				± 0.007	± 0.007
Muon stops				± 0.006	± 0.006
$K3$	+0.014	± 0.002	± 0.002	± 0.004	± 0.005
Line shape				± 0.005	± 0.005
$C2$	-0.008		± 0.002	± 0.002	± 0.003
Asymmetry				± 0.002	± 0.002
Miscellaneous	-0.020	± 0.002		± 0.004	± 0.005
Total		± 0.013	± 0.018	± 0.056	± 0.056

Table 6.1: Error estimates for systematic effects on the fitted value of η (after corrections).

Source	Geometry	References for Effect
μ^+ depth	Fig. 15, Sec. 4.1.5	Sec. 6.1.1, E.2.1
Back plate	Fig. 8	Fig. 27, Sec. 5.10, 6.6, E.2.2
T1	Fig. 15, Sec. 4.1.5	Fig. 17-20, Sec. 5.1, 6.1.2, E.2.3
Detector ineff.	Fig. 8, 11, Sec. 4.1.8	Fig. 21, Sec. 5.2, 6.1.6, E.2.4
μ^+ spin angle		Sec. A.2, E.2.5
Cables	Fig. 8	Fig. 22, Sec. 5.3, 6.1.5, E.2.6
K2	Fig. 8, Sec. 4.1.6	Fig. 23, Sec. 5.5, 6.1.4, E.2.7
C1	Fig. 8, Sec. 4.1.6	Fig. 25, Sec. 5.9, 6.1.4, E.2.8
P_{max} calib.	Sec. 4.1.1	Sec. 6.4, B.4, E.2.9, 4.1.2
Beam centering		Sec. A.1, E.2.10
Muon stops	Fig. 15, 16	Sec. 6.5, E.2.11
K3	Fig. 8, Sec. 4.1.6	Fig. 24, Sec. 5.6, 6.1.4, E.2.12
<u>Line shape</u>	Fig. 13, 14, 30	Sec. E.2.13
C2	Fig. 8, Sec. 4.1.6	Fig. 26, Sec. 5.9, 6.1.4, E.2.14
Asymmetry		Sec. A.3, E.2.15
Miscellaneous	Fig. 8	Sec. 5.4, 5.7, 6.1.4, E.2.16

Table 6.2: References to information on the systematic errors.

Appendix A

Muon-Polarization Effects

Since a stopping muon retains about 25% of its initial polarization in plastic scintillator, one must ensure that one's spectrometer cancels the asymmetries in the muon decay. In other words, one must verify that it is the unpolarized decay spectrum being measured in an experiment, and not something else. While an azimuthally symmetric, axial-focusing spectrometer whose axis and magnetic field at the muon position is perpendicular to the muon spin will do this in principle, one must consider the effects of deviations from the idealized situation.

A.1 Misaligned $\vec{B}$ at Target

In the Comus spectrometer, there is a magnetic field at the target which points along the symmetry axis of the spectrometer, defined as $\hat{z}$. The muons are polarized in a direction perpendicular to this, defined here as $\hat{x}$, so that their spins precess around $\hat{z}$. As long as the spectrometer is azimuthally symmetric around $\hat{z}$ (the possibility that it is not will be discussed later), there is no difficulty. However, suppose that the magnetic field has non-zero B_x or B_y components: the muon spins will precess around $\vec{B}$, not $\hat{z}$. These radial field components could arise from several sources—

- The spectrometer magnet might be slightly asymmetric.

- The incoming beam pipe is surrounded by magnetic shielding near the target which might, in principle, distort the field there.
- The earth's field is certainly not along the spectrometer axis and, at low P_{tune} would cause a significant problem if the flux lines were not adequately shunted away by the nearby iron.

A second system of coordinates $\hat{x}', \hat{y}', \hat{z}'$ and angles θ_B, ϕ_B are now defined. The direction of $\hat{z}'$ is defined to be along $\vec{B}$ at the muon position, separated from $\hat{z}$ by the angle θ_B . The coordinate transformation is given by

$$\hat{x}' = \hat{x} \cos \theta_B \cos \phi_B + \hat{y} \cos \theta_B \sin \phi_B - \hat{z} \sin \theta_B$$

$$\hat{y}' = -\hat{x} \sin \phi_B + \hat{y} \cos \phi_B$$

$$\hat{z}' = \hat{x} \sin \theta_B \cos \phi_B + \hat{y} \sin \theta_B \sin \phi_B + \hat{z} \cos \theta_B .$$

If $\hat{\mu}(\theta)$ is the unit vector along the initial spin direction of the muon

$$\hat{\mu}(0) = \hat{x} = \hat{x}' \cos \theta_B \cos \phi_B - \hat{y}' \sin \phi_B + \hat{z}' \sin \theta_B \cos \phi_B .$$

At later times, $\hat{\mu}$ will precess around $\hat{z}'$ with angular frequency ω :

$$\begin{aligned} \hat{\mu}(t) = & \hat{x}'(\cos \omega t \cos \theta_B \cos \phi_B + \sin \omega t \sin \phi_B) + \\ & \hat{y}'(\sin \omega t \cos \theta_B \cos \phi_B - \cos \omega t \sin \phi_B) + \\ & \hat{z}' \sin \theta_B \cos \phi_B . \end{aligned}$$

In spectrometer coordinates this would be

$$\begin{aligned} \hat{\mu}(t) = & \hat{x} [\cos \omega t + \sin^2 \theta_B \cos^2 \phi_B (1 - \cos \omega t)] + \\ & \hat{y} [(1 - \cos \omega t) \sin^2 \theta_B \cos \phi_B \sin \phi_B + \sin \omega t \cos \theta_B] + \\ & \hat{z} \sin \theta_B [(1 - \cos \omega t) \cos \theta_B \cos \phi_B - \sin \omega t \sin \phi_B] . \end{aligned}$$

Consider now a decay positron whose polar and azimuthal angles in the spectrometer, θ_a and ϕ_a , are such as to allow acceptance. The positron's unit direction vector is then given by

$$\hat{e} = \hat{x} \sin \theta_a \cos \phi_a + \hat{y} \sin \theta_a \sin \phi_a + \hat{z} \cos \theta_a .$$

Defining $\gamma(t)$ as the angle between the muon spin and the positron momentum, and integrating over all ϕ_a with uniform weight:

$$\int_0^{2\pi} \cos \gamma(t) d\phi_a = 2\pi \cos \theta_a \sin \theta_B [\cos \theta_B \cos \phi_B (1 - \cos \omega t) - \sin \omega t \sin \phi_B] .$$

At this point the decay asymmetry must be introduced: it is known with good accuracy to obey the $V - A$ expectation of

$$\alpha(x) = P_\mu \frac{2x - 1}{3 - 2x} ,$$

where P_μ is the muon polarization. (For the range of x involved in this experiment, $|\alpha(x)| < 0.09$.) Further defining τ as the muon lifetime, the probability that a positron at x will be accepted, is given by

$$\begin{aligned} \mathcal{P}(x) &\propto \int_0^\infty \frac{dt}{\tau} \exp\left(-\frac{t}{\tau}\right) \int_{\theta_1}^{\theta_2} \sin \theta_a d\theta_a \int_0^{2\pi} [1 + \alpha(x) \cos \gamma(t)] d\phi_a \\ &\propto A \left[1 + \alpha(x) \langle \cos \theta_a \rangle \sin \theta_B \frac{\tau\omega}{1 + \tau^2\omega^2} (\tau\omega \cos \theta_B \cos \phi_B - \sin \phi_B) \right] , \end{aligned}$$

where $A \equiv 2\pi(\cos \theta_1 - \cos \theta_2)$ and $\langle \cos \theta_a \rangle \equiv \frac{1}{2}(\cos \theta_1 + \cos \theta_2)$. When $\theta_B = 0$ or $\omega = 0$, this reduces to a form in which $\mathcal{P}(x)$ is independent of $\alpha(x)$, as it must. The same would be true if $\cos \theta_2 = -\cos \theta_1$. Otherwise, it is helpful to note that

$$(\tau\omega \cos \theta_B \cos \phi_B - \sin \phi_B) \leq \sqrt{1 + \tau^2\omega^2}$$

in placing an upper limit on the size of the effect.

Now the discussion will be restricted to the case of Comus. Realizing that $\langle \cos \theta_a \rangle \approx 1$ and using the measured field at the target to determine that $\tau\omega = 31.1x$, the approximate upper limit on the fractional distortion of the spectrum for $x > 0.1$ (the approximations not being valid for very small x) is:

$$\frac{\Delta \mathcal{P}(x)}{\mathcal{P}(x)} < \alpha(x) \sin \theta_B .$$

Thus, in this experiment, the muon precession rate is fast enough so as to not cause tune-dependence of the spectrum.

This result can be used to find, for example, the spectrum distortion that could result from the target not being centered in the magnetic field of Comus. From field measurements,

$$\sin \theta_B = (0.0244 \text{ cm}^{-1})r$$

at the target position for small radial displacements, r . Applying this measurement to the above results,

$$\frac{\Delta \mathcal{P}(x)}{\mathcal{P}(x)} < \alpha(x) \frac{0.0244}{\text{cm}} r .$$

A.2 Misaligned Muon Spin

Consider now another possibility—that the muon spin is not quite along the $\hat{x}$ direction as assumed, but is at an angle ϵ from the $\hat{x}$ - $\hat{y}$ plane. This might, for example, arise from misalignment of the spectrometer with respect to the beam line. (Deviations which are not out of that plane make no difference.) It will be assumed that no other asymmetries exist: that the spectrometer acceptance is azimuthally symmetric and the magnetic field at the target is uniform and along $\hat{z}$. In the same notation as Section A.1,

$$\hat{\mu}(t) = \hat{x} \cos \omega t \cos \epsilon + \hat{y} \sin \omega t \cos \epsilon + \hat{z} \sin \epsilon .$$

Then the angle between the muon spin and an accepted positron is given by

$$\cos \gamma(t) \equiv \hat{\mu}(t) \cdot \hat{e} = \sin \theta_a \cos \epsilon (\cos \phi_a \cos \omega t + \sin \phi_a \sin \omega t) + \cos \theta_a \sin \epsilon ,$$

which leads us to the equation

$$\begin{aligned} \mathcal{P}(x) &\propto \int_0^\infty \frac{dt}{\tau} \exp(-\frac{t}{\tau}) \int_{\theta_1}^{\theta_2} \sin \theta_a d\theta_a \int_0^{2\pi} [1 + \alpha(x) \cos \gamma(t)] d\phi_a \\ &\propto A [1 + \alpha(x) \langle \cos \theta_a \rangle \sin \epsilon] . \end{aligned}$$

There is, of course, no dependence upon ω in this equation since the angle between the muon spin and the spectrometer acceptance is constant; any distortion of the spectrum would arise from the energy-dependence of $\alpha(x)$. For the Comus spectrometer, again using $\langle \cos \theta_a \rangle \approx 1$, the upper limit on the fractional error is then obtained as

$$\frac{\Delta \mathcal{P}(x)}{\mathcal{P}(x)} \leq \alpha(x) \sin \epsilon .$$

A.3 Azimuthal Asymmetry

Suppose now that the acceptance of the spectrometer is not uniform around the axis. To study this situation, an extreme case will be considered: the elimination of a portion of the azimuthal acceptance from ϕ_1 to ϕ_2 . This might correspond roughly to the effect of the strip of scintillator in Comus which passes from the C1 anti-counter to its light guide—through the region of the particle trajectories. The previously used notation is retained, and it is assumed that no other asymmetries exist. Then the angle between the muon spin and an accepted positron immediately follows from previous results:

$$\cos \gamma(t) \equiv \hat{\mu}(t) \cdot \hat{e} = \sin \theta_a (\cos \phi_a \cos \omega t + \sin \phi_a \sin \omega t) .$$

Thus,

$$\begin{aligned} \mathcal{P}(x) &\propto \int_0^\infty \frac{dt}{\tau} \exp\left(-\frac{t}{\tau}\right) \int_{\theta_1}^{\theta_2} \sin \theta_a d\theta_a \times \\ &\quad \left\{ \int_{\phi_1}^{\phi_2} [1 + \alpha(x) \cos \gamma(t)] d\phi_a + \int_{\phi_2}^{2\pi} [1 + \alpha(x) \cos \gamma(t)] d\phi_a \right\} \\ &\propto A \left(1 - \frac{\Delta\phi_a}{2\pi}\right) + \frac{\alpha(x)/2}{1 + \omega^2 \tau^2} \left[\left(\theta_2 - \frac{\sin 2\theta_2}{2}\right) - \left(\theta_1 - \frac{\sin 2\theta_1}{2}\right) \right] \times \\ &\quad [(\sin \phi_2 - \sin \phi_1) + \omega \tau (\cos \phi_1 - \cos \phi_2)] . \end{aligned}$$

For the sake of clarity, the trigonometric functions can be expanded and the definition $\Delta\phi_a = \phi_2 - \phi_1$ made. The following equation is valid for small $\Delta\phi_a$ and the relatively small polar angles in Comus:

$$\frac{\mathcal{P}(x)}{A} \propto 1 - \frac{\Delta\phi_a}{2\pi} \left\{ 1 + \frac{\alpha(x)/3}{\cos \phi_1} \frac{1}{1 + \omega^2 \tau^2} \frac{\theta_2^3 - \theta_1^3}{\theta_2^2 - \theta_1^2} \left[(\omega \tau \tan \phi_1 + 1) + \frac{\Delta\phi_a}{2} (\omega \tau - \tan \phi_1) \right] \right\} .$$

Restricted to the case of Comus when $\Delta\phi_a/2\pi \ll 1$ and $x > 0.1$, this can be simplified, much as in Section A.1, to get

$$\frac{\Delta\mathcal{P}(x)}{\mathcal{P}(x)} < 0.0043 \frac{\Delta\phi_a}{2\pi} \frac{\alpha(x)}{x} .$$

The x factor in the denominator arises from the fact that, when Comus is tuned to a high momentum, the muon spins more quickly in the spectrometer field, and there is better time averaging of the decay asymmetry. In other words, an axial magnetic field at the target helps to eliminate the effect of any spectrometer asymmetries, though small asymmetries

such as the aforementioned light guide on $C1$ would not be much of a problem anyway: the fractional spectrum distortion at $x = 0.1$ is only a negligible 0.005% for a muon polarization of 0.25. In fact, any reasonable effort to produce an azimuthally uniform acceptance would avoid problems in this particular experiment.

Appendix B

Field Gradient Compensation

B.1 NMR Requirements

The most precise instrument used to monitor the magnetic field in the Comus spectrometer was a CERN NMR magnetometer.¹ This instrument requires that the magnetic field at the water sample be homogeneous to within about 0.02% per cm, which is far smoother than the field of Comus at any practical location for a probe. It was therefore necessary to smooth the field artificially over a region large enough to contain the NMR sample, without significantly perturbing the field beyond this small region.

B.2 Field-Derivatives Cancellation

With a probe much smaller than the magnet, as was the case for Comus, it will generally suffice to cancel the first derivatives of all significant field components at the center of the NMR sample. This, of course, may not be true when the field has a very high curvature (such as near the corner of an iron yoke), but one would usually not choose such a location for a probe.

Since the field has no time-dependence and the NMR sample carries no current, there

¹K. Borer and G. Fremont, CERN 77-19 (1977).

are four conditions imposed on the field derivatives by Maxwell's equations:

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = 0 .$$

The remaining derivatives to eliminate at the probe center are

$$\frac{\partial B_x}{\partial x} = \frac{\partial B_x}{\partial y} = \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial y} = \frac{\partial B_z}{\partial z} = 0 .$$

Several of these conditions could be achieved through the symmetries of the magnet. With right-handed Cartesian coordinates $\hat{x}, \hat{y}, \hat{z}$, whose origin is not on the magnet's axis, $\hat{x}$ is defined to be tangential to a circle around the symmetry axis, $\hat{y}$ to be radially inward and $\hat{z}$ to be parallel to the symmetry axis. Then, because Comus is axially symmetric with no significant azimuthal field components, $B_x = 0$ at $x = 0$ and, therefore,

$$\frac{\partial B_x}{\partial y} = \frac{\partial B_x}{\partial z} = 0 .$$

One more condition can be satisfied by appropriately choosing the value of z at which the probe is placed. If B_r is the radial field component at a distance r from the symmetry axis, then

$$\frac{\partial B_x}{\partial x} = \frac{B_r}{r} .$$

Thus, this derivative vanishes where $B_r = 0$, which will be true for some z near the spectrometer midplane at any radius—for a spectrometer which is nearly symmetric across this midplane.

Alternatively, one could choose a value of z at which there was an extremum in B_z , thereby eliminating $\partial B_z / \partial z$. Both this and the $B_r = 0$ condition would be fulfilled at a single point if the magnet were completely symmetric across the midplane, but this was not quite true for Comus. Forced to choose which of the two conditions to satisfy, one should note that it is much easier to act upon two derivatives of the same field component for engineering reasons. Thus, z was chosen such that

$$\frac{\partial B_x}{\partial x} = 0$$

and the gradient-compensating coils were used to impose

$$\frac{\partial B_z}{\partial y} = 0 \quad \text{and} \quad \frac{\partial B_z}{\partial z} = 0 .$$

The NMR sample was a cylinder of water about four times as long as its diameter. Since it is difficult to cancel gradients along the long dimension of the sample, the cylinder axis was placed along $\hat{x}$: at large radii in Comus, the field variation is very small in this (azimuthal) direction over the few millimeters of the sample length. At this point, the natural design for the compensating coils is a configuration of long filaments, parallel to $\hat{x}$; one can calculate the fields given by blocks of four filaments at the corners of a rectangle and build the “coils” from several of these blocks. For simplicity, the finite radius of the wires will be neglected, as this has only a small effect on the design.

B.2.1 Canceling $\frac{\partial B_z}{\partial z}$

Recall that the magnetic field a distance r from a current I on an infinite filament is

$$B_\phi(r) = \mu_0 \frac{I}{2\pi r}$$

in MKS units. Consider now the field at $(y, z) = (0, 0)$ from four filaments placed at (a, b) , $(a, -b)$, $(-a, b)$ and $(-a, -b)$, where the current in the first and third filaments is in the $\hat{x}$ direction and in the $-\hat{x}$ direction in the others. From symmetry, B_z and all of its even derivatives with respect to z vanish, while the first non-trivial derivatives are:

$$\frac{\partial B_z}{\partial z} = -\frac{4}{b^2} \frac{\mu_0 I}{\pi} \frac{q}{(1+q^2)^2} \quad \text{and} \quad \frac{\partial^3 B_z}{\partial z^3} = \frac{48}{b^4} \frac{\mu_0 I}{\pi} \frac{q(q^2-1)}{(1+q^2)^4} ,$$

where $q \equiv a/b$. Much of the design approach is immediately obvious from these equations.

- One must choose values of q for each of several sets of filaments which, taken together, minimize $\partial^3 B_z / \partial z^3$ and maximize $\partial B_z / \partial z$. The higher derivatives are not important in Comus.
- One should minimize b , so as to reduce the current required to produce a given field gradient, to the extent that this is consistent with allowing adequate conductor

size. Since the surface area available for passive cooling scales as b^2 , while the heat produced for a given gradient compensation scales as, at least, b^4 , this is clearly desirable.

B.2.2 Canceling $\frac{\partial B_x}{\partial y}$

The procedure for canceling this gradient is similar. Again, the field is found at $(0,0)$ from four filaments placed at $(\pm a, \pm b)$, except that they now all carry a current I in the $-\hat{x}$ direction. B_z and all its even derivatives with respect to y vanish, while

$$\frac{\partial B_z}{\partial y} = \frac{2}{b^2} \frac{\mu_0 I}{\pi} \frac{q^2 - 1}{(1 + q^2)^2} \quad \text{and} \quad \frac{\partial^3 B_z}{\partial y^3} = \frac{12}{b^4} \frac{\mu_0 I}{\pi} \frac{1 - 6q^2 + q^4}{(1 + q^2)^4}.$$

Much as before, the higher derivatives can be ignored and several values of q can be chosen which, taken together, minimize $\partial^3 B_z / \partial z^3$ and maximize $\partial B_z / \partial z$. Again, one should try to minimize b .

B.3 Other Considerations

When the absolute size of the gradients to be compensated is large, substantial current densities are required in the compensating coils. As an example, when Comus was tuned to the muon decay endpoint, $\partial B_z / \partial y = 220$ gauss/cm. About 60 amps were required to cancel this gradient in the coils built for this experiment. While the absolute size of the current stems from the use of only four sets of conductors (which maximized the conductor filling factor and simplified the conductor positioning), the current density would be high for any design and implies substantial heating. It is for this reason that printed circuit techniques² were rejected in favor of a machined coil form and #14 copper wire. Even then, forced air cooling was required.

²The printed circuit compensating coils discussed by K. Borer and G. Fremont for use with the CERN NMR probes were rated at 20 gauss/cm; these probes are substantially thinner than the LBL-designed probes and compensation therefore requires less current.

B.4 Accuracy

In measuring a field with very large gradients, it may seem that the positioning requirements for the probe and its compensating coils will be far too stringent to be met; in a field with a gradient of a couple of percent per centimeter, it would seem that the accuracy of NMR is degraded with a mispositioning of order 10^{-3} cm. However, if the probe is fixed into position, one can still obtain readings that are proportional to the field, and proportionality is all that is needed in this experiment.

Similarly, it is not necessary that the compensation coils be machined and positioned to high accuracy; provided that the current in the coils is proportional to the field being measured, a misalignment of the coils produces no error in the *linearity* of the measurement, even though it may cause a net field at the electronic center of the probe sample. In practice, this is accomplished by adjusting the shape of the NMR resonance line to a minimum width for each measurement. This uses the fact that the field shape is nearly constant. It was found that the linearity, judged from the comparison of probes in the region in which their ranges overlapped, and the reproducibility, judged against the magnet shunt voltage, Hall probe, and the other NMR probes, were excellent. The intrinsic precision of the probes, 10^{-5} , was not degraded in the range below 40 MeV at which data were used directly to find η .

Near the endpoint, the accuracy is reduced, but the high precision of the NMR probes was not needed there. The non-linearity of the measurements is probably the result of the changing field shape and the torque that the compensating coils experience in the large magnetic field.

Appendix C

Monte Carlo Physics — EGS

C.1 Introduction to EGS

The EGS4 Code System¹ is a general purpose program for the simulation of electromagnetic showers. It extends the capabilities of, and corrects errors in, EGS3.² This latter code was used in the earlier part of the analysis of this experiment, until EGS4 was released and could be implemented. The total dynamic kinetic energy range is claimed to be from a few tens of KeV to a few thousand GeV for charged particles and from 1 keV to a few thousand GeV for photons. This covers the energy range with which a muon decay experiment is likely to concern itself. The relevant physics are largely contained within this code system, with the user being mostly required to initialize the particle characteristics and trajectories, to handle of the geometry for particle propagation and to extract information from the particle showers. For high-quality results, the user must also carefully choose various parameters (such as thresholds for discrete particle treatment), balancing CPU time requirements against accuracy.

Given the historical development of EGS for medical physics and high-energy calorimetry, it is not surprising that time is not handled in the program at all. This means, barring

¹W. R. Nelson, H. Hirayama and D. W. O. Rogers, SLAC 265 (1985).

²R. L. Ford and W. R. Nelson, SLAC 210 (1978).

a major rewrite of the code, that one must be very careful that the cuts in a real experiment do not change the calculated results. Positronium lifetimes (of order 1 ns) in the relevant materials in this experiment are so short as not to present a serious problem, but time-of-flight delays must sometimes be considered in applying cuts to TDC's.

The various physics functions are contained in EGS as piecewise-linear fits which deviate by less than 1% from the theory; this is not usually a limiting factor in the accuracy. The physics which this code implements has been carefully examined to determine its accuracy and the necessary changes. A general purpose code will not fit every situation, and EGS is no exception.

C.2 Continuous Energy Loss

It appears that EGS was designed for use with thick media, where shower statistics dominate the spread in energy deposition. When a charged particle is moved through a step, EGS simply uses the average value for the restricted energy loss. Of course, if an interaction is hard enough to result in a particle which will be tracked in its own right, this large energy loss is taken care of accurately; what is not properly accounted for is the distribution of "continuous" energy losses around the average restricted energy loss—as represented by the Landau³ or Blunke-Leisegang⁴ distributions, for example. In principle, a solution is to set the threshold for discrete particle treatment much lower than the lowest average energy loss expected in a region, but this is extremely costly in terms of CPU time.

Many of the regions in the spectrometer are thin, and relatively soft interactions contribute to the width of the energy-loss distribution. However, in many applications related to this experiment, it is necessary that this energy-loss straggling be included; many of the problems of interest are nonlinear in energy and are dependent upon the higher moments of the energy-loss distribution. One obvious example is the placing of an ADC cut upon

³L. Landau, *J. Phys. USSR* **8**, 201 (1944).

⁴I. Blunck and S. Leisegang, *Z. Physik* **128**, 500 (1950).

a counter, though there are also many examples among the physical processes which the particles undergo.

In such a situation, one would like to propagate a particle over a fairly long step and choose the energy loss from a known distribution depending on the incident particle's characteristics, the material and step size. This is easily implemented: EGS provides the average energy loss and one need only randomly choose a value from the energy-loss distribution whose average is equal to this. The particular one chosen was the Blunke-Leisegang distribution, which is somewhat broader and more accurate than the Landau distribution for thin lamina, becoming very similar for thick lamina. It is composed from the sum of four Gaussian curves whose relative weights, widths and centroids were chosen from empirical data. In order to choose a properly weighted variable from this distribution, one need only choose which of the four Gaussians to use, based on their relative weights, and then randomly sample from a Gaussian of this width and centroid.

There are, in general, problems of internal consistency with this approach, appearing in some dependence of the energy-straggling width on the step sizes taken to travel a given distance. This arises from the fact that the tails of distributions like that of Landau or Blunke-Leisegang do not join smoothly with the discrete energy-loss distribution; the tails are not of the correct shape and include energy losses past the threshold for discrete production. The distributions are accurate only in the region around the average energy-loss peak. Nonetheless, straggling can be incorporated in this way with reasonable accuracy, provided that the step size is small enough that the tail of the continuous loss distribution does not extend significantly past the discrete treatment threshold and that a trajectory through a region is not heavily subdivided. Also, errors are usually partially corrected by folding the energy-loss distribution with a resolution curve to reproduce the measured width of the ADC of a counter.

Another potential problem is the choice of the threshold for discrete production. If this threshold is too low, the calculation becomes hopelessly time-consuming, while, if too high, the real particles may escape from a thin lamina (such as T_1) and cause effects (such

as vetoes) which will not be reproduced in the Monte Carlo. The solution lies in realizing that the main situation in which one is really concerned about these low energies is the vetoing of events in A1,A2 by low-energy Bhabha scatters from T1. Since the probability of a Bhabha scattering is almost independent of the initial e^+ energy, one does not have to know accurately the probability that these Bhabha scatters will result in a veto. Even though this may be somewhat inaccurate in an absolute sense, there will be little distortion of the spectrum *shape*, and one is interested in nothing else in this experiment.

The accuracy in EGS of the continuous energy loss for charged particles is taken to be 1.3% for this experiment. An uncertainty of 0.7% arises from the adjusted ionization potential ($I_{adj} = 78 \pm 7$ eV for carbon),⁵ and another 0.5% arises from the parameterization of the Sternheimer treatment of the density effect. Curve fits to the energy loss introduce another 1% error. The threshold for discrete treatment of photons is low enough in the applications for this experiment that radiative losses can be ignored in estimating the error in the continuous energy loss.

C.3 Path-Length Restriction

While EGS limits the length of a step to the extent that multiple scattering will usually not cause an unreasonable deviation from the straight-line path, this restriction is sometimes inadequate because of the way in which scattering is handled (i.e., a new direction is chosen *after* each step through the medium, even for substantial scattering angles). As a result, there are situations in which a tighter restriction upon the path length must be imposed:

- If the medium under consideration is a scintillator, the correct criteria for limiting the path length may be the amount of energy deposited during a step. Consider, for example, a charged particle entering a veto counter. In the real world, it is possible that this particle will scatter out of the counter before a significant amount of energy is deposited. However, if the possibility of scattering is not even considered

⁵International Commission on Radiation Units and Measurements Report 37, 1984.

in the Monte Carlo simulation until after the first path length has been traversed, the scattering may not be relevant any more—enough energy may have already been deposited in the calculation to produce an event “veto”.

- If a charged particle is travelling close to, and nearly parallel to, a material boundary, the calculated results can depend strongly upon the path-length limit. The reason, of course, is that a tiny scattering may allow the particle to escape from the region in a much shorter distance than would be needed if a straight path were followed. One place where this problem arises is in the scattering of grazing incidence particles from collimators. Another is in the muon-stopping target, T1, which is very thin compared to its height and width; some of the e^+ will be produced with initial trajectories nearly in the target plane, but can occasionally scatter into the spectrometer acceptance. EGS will naively move particles a distance of a centimeter or more before considering the possibility of a scatter, while the real particles will often scatter out of the target well before that distance and, thus, be scattered at smaller angles than found by EGS. For these reasons, EGS may give erroneous values for both the energy deposition in the target and the likelihood of a particle scattering into the spectrometer acceptance, unless one stringently limits the allowable path length per step.

A package of macros referred to as PRESTA⁶ is available to improve the particle transport in EGS4. In addition to including a boundary crossing and lateral transport algorithm, so that the above problems are more accurately handled, it handles the multiple scattering path-length correction in a way that is consistent with the multiple scattering treatment. This means that results are nearly independent of step size—which is not always the case with uncorrected EGS4 at low energy.

Most of the programs used to simulate the Comus spectrometer incorporated PRESTA. This macro package was generalized to allow region-by-region adjustment of the algorithm, increasing the speed and/or accuracy in most applications.

⁶Alex F. Bielajew and David W.O.Rogers, Report PIRS 042, 1986.

C.4 Magnetic Fields

EGS does not contain a built-in capability to track particles through magnetic fields and always moves particles along piecewise linear paths. Multiple scattering in the material regions of Comus largely obscures the effects of the magnetic fields, except in light materials. The only low-Z materials in the high-field regions of Comus are veto counters, and these will certainly veto events in which a particle travels a significant distance through the counters.

The vacuum regions do not allow this same facile simplification, unless the step size were to be decreased from near infinity to near zero, in which case accuracy would not be maintained due to round-off error. Thus, propagation in these regions was handled by adding subroutines which receive the characteristics of charged particles as they enter a vacuum region and transport them through the field until they encounter matter again. The new direction cosines and position are given to EGS and propagation continues as before.

These magnetic field effects are, in fact, vital in determining the effect of showers which come from particle impacts upon collimators, support wires and other matter suspended within the stronger parts of the magnetic field in Comus. It is sometimes possible for particles from these showers to strike anti-counters and/or the trigger counters. Whether or not this happens, especially for the lower-energy charged particles in the shower, is largely determined by the magnetic field. Similar comments pertain to the backscattering of particles from the detector into the anti-counters in the spectrometer.

C.5 Bremsstrahlung

EGS treats bremsstrahlung and pair production in the field of the electrons by multiplying the cross section of the nuclear process by a correction factor: the different energy dependences are not considered. Further, the correction factor is valid only at extremely high energies. This does not matter in the case of pair production, which does not have a major

impact in this experiment; bremsstrahlung, though, is important.

While the relative inaccuracy is small for heavy elements (nuclear bremsstrahlung being larger by a factor of $O(Z)$), the distinction between the two processes can be important for materials containing only light elements. Plastic scintillator, in which carbon ($Z = 6$) is the heaviest component, is a potential problem. This is especially true for low-energy primaries, for which the electron to nuclear ratio is much less than the high-energy limit. Thus, EGS4 has been modified to treat bremsstrahlung in the field of electrons more accurately, using a table of cross sections found from numerical integrations of a formula derived by E. Haug⁷, which is too lengthy to justify its inclusion here. Also incorporated were the suggestions of the ICRU⁸ with regard to the treatment of the shielding problem and the formula's applicability to e^+ above energies of 5 MeV, since Haug's result is, strictly speaking, applicable only to the scattering of two free e^- .

In this way EGS4 was given the correct stopping power due to bremsstrahlung in the electron field, but this was still done with a multiplicative correction factor to avoid extensive changes in the code. This is to say that the differential cross section was still not correct. For NE 110, hard bremsstrahlung was overemphasized by about 4.5%, while very soft bremsstrahlung was too small by as much as 17%.

Another thing to consider is that bremsstrahlung processes are not given the correct angular distributions for the final particle directions in EGS. In fact the lepton is not assumed to change direction when emitting bremsstrahlung, and the photon is always taken to be emitted at the same polar angle with respect to the lepton. The justification for the former, as given by the authors of EGS, is that multiple scattering dominates over the deflection from bremsstrahlung. In materials thicker than about 0.0025 radiation lengths, this is true. Fortunately, the absolute values of the angles involved are small enough that there is no significant effect in this experiment even when this condition is not met. The use of the average for the photon angle also presents no problem: the angles involved are

⁷E. Haug, *Z. Naturforsch* **30a**, 1099 (1975).

⁸International Commission on Radiation Units and Measurements, Report 37 (1984).

small and it is unlikely that the photons will be detected in the veto counters. Even when they are detected, it is unlikely that including the narrow distribution of photon angles would affect the outcome: almost none hit the A1,A2 counters in any case, and most of those hitting the A,B,C,D counters would also do so for small changes in angles.

The overall accuracy of the bremsstrahlung cross section in NE110, differential in photon energy, is taken to be 8% for positron energies of 5-53 MeV. The inaccuracy of the screening functions used causes 0.5% error and approximations in the treatment of compounds introduces a 0.2% error. The uncertainties in the theory for the stopping power due to the nuclear process are considered to be 3-5%⁹ for $e^\pm$ below 50 MeV, and 3% above 50 MeV. Uncertainties in the differential cross-section are twice this. The error from hard bremsstrahlung in the field of electrons is known, and its effect in the target can be roughly corrected.

The accuracy in heavy materials, such as lead, is comparable—although the sources of error are different. The largest error comes from applying an overall constant to make a Coulomb correction for incident particle energies below 50 MeV, while this should be a function of the energy of the outgoing photon. There is also an error of up to 1% from interpolation of this correction factor between Z values, while hard bremsstrahlung in the field of electrons is in error by only 0.4%.

The other major error arises from the difference in screening corrections for e^+ from those applied for e^- ; EGS uses the latter screening functions for both processes. In NE 110 the differences are negligible above 2 MeV, but in lead the corrections appear to be important up to hundreds of MeV. They are expected^{10,11} to reduce the positron bremsstrahlung cross-section by 8% at 53 MeV, 18% at 14 MeV and 29% at 6 MeV from that of electrons.

⁹International Commission on Radiation Units and Measurements Report 37 (1984).

¹⁰I. J. Feng, R. H. Pratt and H. K. Tseng, Phys. Rev. A24, 1358 (1981)

¹¹International Commission on Radiation Units and Measurements, Report 37, p. 49 (1984)

C.6 Bhabha Scattering

The separate treatment of Bhabha scattering and bremsstrahlung in the field of an electron is fundamentally artificial, since Bhabha scattering is always accompanied by real photon emission (Bloch-Nordsieck argument) and electrons recoil from the passing of the virtual photon in bremsstrahlung. However, they are treated separately both in the literature and in EGS.

The radiative corrections can, in principle, have a large fractional effect on the cross section for Bhabha scattering. This is especially true for large energy transfers, since the energy carried off by real photons drives the cross section to zero.¹² Unfortunately, available calculations of the radiative corrections are not directly applicable to this experiment, having been done in the soft photon approximation,^{13,14,15} for virtual photons only,¹⁶ for ultra-relativistic energies,¹⁷ for large energy transfers¹⁸ or for specific experiments.

The Bhabha scatters of most importance to this experiment are those in which positrons of 20-53 MeV lose 10-50% of their energy to relatively low-energy electrons. The soft photon calculations predict reductions of a few percent, but have a photon energy cutoff in a logarithmic term and so do not give a definite prediction; plausible cutoffs yield reductions of 1-10%. The virtual photon calculations of Furlan and Peressutti predict a reduction of about 1% in the probability of Bhabha scatters for energy transfers of around 20%. The ultra-relativistic formula predicts a reduction of 5-6%, and, in the large energy transfer formula, the cross section is lowered by 10%. Thus, a reasonable estimate for the cross-section reduction is $(5 \pm 4)\%$.

¹²A. C. Hearn, P. K. Kuo and D. R. Yennie, *Phys. Rev.* **137**, 1950 (1969).

¹³A. I. Akhiezer and R. Polovin, *Doklady Akad. Nauk USSR*, **90**, 55 (1953).

¹⁴M. Redhead, *Proc. Roy. Soc. (London) A* **220**, 219 (1953).

¹⁵R. Polovin, *JETP*, **31**, 449 (1956), *Sov. Phys. JETP*, **4**, 385 (1957).

¹⁶G. Furlan and G. Peressutti, *Nuovo Cimento* **19**, 830 (1961).

¹⁷A. I. Akhiezer and V. B. Berestetskii, in *Quantum Electrodynamics* (Interscience Publishers, New York, 1965), p. 730.

¹⁸A. C. Hearn, P. K. Kuo and D. R. Yennie, *Phys. Rev.* **137**, 1950 (1969).

C.7 Annihilation

EGS considers only two-photon annihilation, without radiative corrections, and ignores the one- and three-photon processes. Single photon annihilation, with $\sigma_{1\gamma} \propto (\alpha Z)^4$, is important in heavy materials, and about 12% of the annihilations in lead occur through this process. However, above about 1 MeV, bremsstrahlung dominates annihilation for γ -ray production in lead, so that shower development might not be strongly affected at energies of a few MeV. In light materials the effect is totally negligible.

Three photon annihilation is best considered along with radiative corrections to the two-photon process, since the calculated divergences in these α^3 terms cancel and leave an unambiguous result. A calculation, including the effects of hard photons, has been done¹⁹ in the high-energy approximation; in this approximation, the error in the two-photon cross section is 26% at 6 MeV, decreasing to 4% by 53 MeV. The errors in the corrections are presumably similar in relative size. Applying their corrections to the usual two-photon cross section, $\sigma_{2\gamma}^*$, the total annihilation cross section can be closely approximated by

$$\sigma_{2\gamma} + \sigma_{3\gamma} = \sigma_{2\gamma}^* (1 + \epsilon), \quad \epsilon = (0.6\%) \cdot \gamma^{0.23}$$

for 5-60 MeV positrons. Thus, the errors in the EGS annihilation formula are around 1.1-1.7% at the relevant energies.

C.8 Multiple Scattering

EGS4, as implemented with the PRESTA macros, handles multiple scattering in a self-consistent way, using the theory of Molière²⁰ corrected by a factor $(\theta/\sin\theta)^{\frac{1}{2}}$. (An apparent typo, which caused a small deviation from this, has been repaired in the EGS4 code.)

Bethe²¹ states that the sum of the three functions used by Molière approximates this theory to within 1%, and the interpolation of these functions in EGS is accurate to within

¹⁹S. I. Eidelman and E. A. Kuraev, Nucl. Phys. B143, 353 (1978).

²⁰G. Molière, Z. Naturforsch. 3a, 78 (1948).

²¹H. A. Bethe, Phys. Rev. 80, 1256 (1953).

0.5%. Based on Bethe's quantitative comparison with the exact theory of Goudsmit and Saunderson,²² it does not appear that multiple scattering uncertainties at small angles should be important when the step sizes in EGS are not large.

One shortcoming in the EGS treatment that must be noted is this: scattering from atomic electrons below the threshold for discrete treatment is included as a multiplicative correction to Molière scattering, approximating the electrons as, basically, protons. This is obviously incorrect for scattering at angles of more than 90°. For a discrete treatment limit of 89 KeV kinetic energy, such as was typically used in running EGS, the maximum pseudo-elastic single scattering angle of 6 MeV positrons from electrons should actually be 2.8°. For 50 MeV positrons, the angle becomes 0.35°. The approximation can be significant in light materials, such as NE110, where single scatters at large angles are overstated by 17%. Fortunately, backscattering has fairly low probability in light materials. The primary effect in this experiment was to reduce the calculated detection efficiency slightly; this was corrected by hand.

Calculations of backscattering from active media (such as the T3 counter) are biased by another effect which is inherent in a condensed history Monte Carlo such as EGS. Particles are propagated a finite distance before "continuous" effects like multiple scattering are considered. This means that energy deposition by a backscattered particle in a scintillator will be overstated, since it usually results from a single nuclear scatter that occurred before the end of the step. Reducing step sizes near the surface, as is done by PRESTA, reduces the bias (but does not eliminate it, since the step sizes are not reduced below some finite limit). In principle, it would be nearly possible to eliminate this problem by treating single scatters of more than some cutoff angle as a discrete process to be included in the reaction cross section.

In their evaluation of the accuracy of the treatment of backscattering by PRESTA, Biela-jew and Rogers²³ found that the fraction of energy reflected from a semi-infinite slab showed

²²S. A. Goudsmit and J. L. Saunderson, *Phys. Rev.* **57**, 24 and **58**, 36 (1940).

²³Alex F. Bielajew and David W.O. Rogers, Report PIRS 042, pp.29-30, 1986.

a significant dependence on the Monte Carlo step size limit. This effect did not seem to appear in the version of the code used in this experiment at nearly the same level, and it is possible that the dependence is overstated. Step sizes are, nonetheless, restricted to reduce the problem.

On the other hand, very short step sizes cannot be used indiscriminately as this violates the requirement that the scattering be *multiple* enough for the theory to apply. This is a potential problem for grazing incidence scattering of low-momentum particles in high-Z materials. In this case, scatters of one or two degrees can be important. Short step sizes can also compromise accuracy with computational round-off error problems, especially in complicated, non-rectilinear geometries.

C.9 Showers in Thick Material

The accuracy of EGS in thick materials is difficult to estimate and undoubtedly varies considerably, depending on the details of the problem. Problems which sample only the tails of distributions will tend to be much less accurate than others. For example, the calculated penetration by monochromatic electrons through a region whose thickness nearly equals their range will be very sensitive to small errors in the energy loss distribution, while, if the region were half as thick, errors in the energy loss distribution might be nearly irrelevant. Most calculations for this experiment are not highly sensitive to the inaccuracies in EGS, as incident particles are continuous in energy and angle and tend to smear the distributions. In general, the Monte Carlos of thick regions are taken to have around 10% accuracy in the physics, with additional errors for statistics, geometry, counter calibrations, etc.

Appendix D

Monte Carlo Physics — TRIM

The stopping distribution of heavy particles for one special case—that of a fixed energy beam normally incident on a homogeneous, semi-infinite medium—is fairly well understood. However, analytic calculations are not generally applicable to real experiments; measurements find both shorter average ranges and asymmetric stopping distributions about the average because of large angle scattering. More complex geometries cause further complications.

Realistic stopping distributions can be obtained through Monte Carlo calculation. The TRIM85¹ code implements this for low-energy ions of hydrogen through uranium incident on a wide variety of materials. It must be noted, however, that the source code provided does not contain all of the features attributed to it; alternative versions were used to obtain some of the results shown in the book by Ziegler *et al.*

This code has been rewritten extensively, so as to apply it to the problem of a μ^+ stopping in a complex geometry. Geometrical, input and output routines are separated from the routines involved with physics, emulating the approach used in the EGS package. The physics in TRIM85 is improved as needed, including the insertion of intrinsic straggling and an iterative algorithm to find the energy loss during a step. The TRIM85 dE/dx values

¹J. F. Ziegler, J. P. Biersack and U. Littmark, *The Stopping and Range of Ions in Solids* (Pergamon Press, New York, 1985) Vol. 1.

for low-energy protons are joined to those available for muons at higher energies. The revised code does an excellent job of reproducing the available experimental results—for example, agreement with the helium bubble chamber measurements of Derrick *et al.*² is at the 0.3% level for surface muons.

²M. Derrick *et al.*, Phys. Rev. A2, 7 (1970).

Appendix E

Systematic Effects

E.1 General Comments

Systematic errors in the determination of η are sensitive to the portion of the spectrum used in the fit, and, in an experiment where the amount of data gathered at any particular point on the spectrum is somewhat arbitrary, the errors are affected by the distribution of the statistics. For example, many systematics would have had dramatically more effect on the fitted value of η if data close to the endpoint had been used in the fit; effects which have a fairly uniform impact on the rest of the spectrum, such as errors in the calculated spectrometer line shape, may suddenly vanish at the endpoint and cause significant shape distortion.

Therefore, the systematic error estimates quoted for this experiment were usually determined by applying an effect to the spectrum, refitting the data to this modified spectrum and noting the shift in the fitted value of η . When an effect caused run-to-run variation at the same nominal spectrometer tune, this was inappropriate and a Monte Carlo approach was taken: for a number of separate fits to the spectrum, the effect was applied randomly to the data points, and the standard deviation of the resulting distribution of fitted η values was taken as the error.

E.2 Error Estimates

E.2.1 Muon Depth

The muon depth below the $T1$ surface was primarily calculated from the beam line momentum, the material upstream of $T1$ and the energy loss rate in NE110 scintillator. The beam line setting was uncertain to $\pm 0.5\%$, giving a muon range error of $\pm 1.75\%$, with another 1.5% arising from uncertainty in the adjusted ionization potential for NE110. Combining the errors in quadrature, and taking into account that only the last 42.2 mg/cm^2 of the 130.7 mg/cm^2 muon range was in $T1$, the depth in $T1$ is in error by

$$(130.7/42.2) \cdot \sqrt{1.75^2 + 1.5^2}\% = 7.1\% .$$

Also, the amount of material upstream of $T1$ is uncertain by $\pm 0.7 \text{ mg/cm}^2$, making a $(0.7/42.2) \cdot 100\% = 1.7\%$ effect on the depth in $T1$. The total uncertainty, then, is 7.2% , and the resultant error in η is ± 0.029 , since the $T1$ effect shifts the fitted value of η by $+0.408$.

E.2.2 Back Plate

The calculation of that portion of the back plate contamination not corrected empirically has moderate uncertainties from bremsstrahlung and Bhabha scattering in the theory and from the scattering treatment in EGS4. These are taken to be less for the empirically corrected spectrum, since the contamination is *measured* at a similar location in this case. Use of the measured $\overline{T2} \cdot T3 \cdot T4 / T2 \cdot T3$ ratio to correct the Monte Carlo calculation of scattering in the spectrometer back plate, discussed in Section 6.6, caused the fitted value of η to shift by -0.082 . The error is dominated by the limited statistics available for that fit: $0.082 \times (\pm 30\%) = \pm 0.025$.

E.2.3 Target Area

For convenience, the target-area effects are divided into small and large energy losses. Uncorrected, small energy losses cause a $+0.250$ shift in the fitted value of η , while hard

collisions cause a shift of +0.212.

T1: Continuous Energy Loss Since the accuracy of the energy loss known for carbon at these energies is at the 0.9% level (Section C.2), the uncertainty in η from this is ± 0.002 . With EGS4 there is also an inaccuracy of about 3% due to the soft radiative energy loss, and a 1% parameterization error, giving ± 0.008 in η .

Then, reserving the muon depth uncertainty to Section E.2.1, one calculational uncertainty is the exact beam spot distribution and, hence, the number of particles affected by the edges of T1 and the light guide that slightly overhangs the scintillator. Also, the 42° rotation of T1 means that particles entering one side of the spectrometer acceptance deposit more energy in T1 than those entering the other. Combined with a left-right asymmetry in the acceptance of about 5%, the average energy loss is shifted. Each of the above should affect the average energy loss by about a percent, giving an uncertainty in η of ± 0.004 .

T1: Hard Interactions Bhabha scattering (Section 5.1.3 and Figure 19) and external bremsstrahlung (Section 5.1.4 and Figure 20) make large contributions to the effect that T1 has on the spectrum shape. The theoretical uncertainty (Sections C.6 and C.5) in the effect on η is about 4.5% or ± 0.009 . In EGS4, there is a further uncertainty from the multiple scattering treatment, giving an error in η of ± 0.012 .

The uncertainty in the calculation is estimated to be 8%, arising from the suppression of Bhabha scattering by the veto counters, counter calibrations and geometrical uncertainties. The resulting uncertainty in η is ± 0.017 .

E.2.4 Detector Inefficiency

Uncorrected, this effect shifts the fitted value of η by -0.218. After the results have been approximately corrected for the shortcomings of EGS4 in annihilation (Section C.7) and backscattering (Section C.8) probabilities, the scattering treatment in EGS4 leaves an uncertainty in η of about ± 0.005 .

Uncertainties in the calculation of all of the effects discussed in Section 5.2 (scattering in the vacuum window and fourth set of collimator-suspension cables, backscattering from the trigger counters, annihilation, accidental vetoes), as well as in the total energy cut discussed on page 82, contribute in roughly equal measure to the ± 0.014 error in the fitted value of η . The largest cause of the momentum-dependence of the detection efficiency—annihilation—actually contributes somewhat less error than the other processes. Annihilation is mostly sensitive to the calibration of the $T2$ and $T3$ counters, and these are known to 2% accuracy. The other inefficiencies were calculated to an accuracy of only about 10%.

E.2.5 Muon Spin Direction

As discussed in Section A.2, spectrum distortion can result if the muon spin is not perpendicular to the spectrometer axis. In this experiment, it is estimated that a $\pm 0.5^\circ$ misalignment of the spectrometer with respect to the muon beam line could have existed. For a muon polarization of 25%, this $\pm 0.5^\circ$ misalignment would give an error in η of ± 0.014 .

E.2.6 Scattering from Upstream Cables

The scattering from the three upstream sets of collimator-support cables was discussed in Section 5.3, with the calculated effects shown in Figure 22. The inclusion of this correction changes the determined value of η by -0.057 .

Low-energy events tend to derive from bremsstrahlung and Bhabha scattering, so the relevant theory is known to about 5% for these events, corresponding to a ± 0.003 error in η . The high-energy events tend to come from scattering, and the EGS4 physics behind this part of the correction should be accurate at the 4% level for $Z \approx 26$, corresponding to a ± 0.005 error in η . The accuracy also suffers from the approximation of the stranded cable as a cylinder, since scattering near the surface of the cables represents a disproportionately large fraction of the events. Between this and Monte Carlo statistics, the uncertainty in η is estimated to be at the ± 0.009 level.

E.2.7 K_2

The theory accuracy for showering in thick lead is around 10%. EGS4 brings another 10% error from its handling of bremsstrahlung and annihilation in lead, and of multiple scattering in the aluminum shell. This gives ± 0.005 uncertainty in η from each. The calculational accuracy in η , due to approximations in the Monte Carlo and statistics, is ± 0.009 .

E.2.8 C_1

The EGS4 multiple scattering treatment is expected to give rather inaccurate answers for the scattering from the upstream face of C_1 ; the error in this -0.010 effect on η is ± 0.002 . The calibration accuracy on C_1 is about 6%, giving an uncertainty of ± 0.004 in η from this. Geometry and Monte Carlo statistics give a further uncertainty of ± 0.008 , for a total of ± 0.009 .

E.2.9 Momentum Calibration

The most accurate spectrometer calibration relies on the spectrum endpoint. The saturation effects that appear in the magnet for $P_{tune} > 40$ MeV/c cause a 0.30% calculated nonlinearity of the magnetic monitors with respect to P_{tune} at the endpoint, as discussed in Section 6.4; since the accuracy of the calculated saturation effects is at the 10% level, an uncertainty of 0.03% remains after the calculated correction.

More importantly, the NMR and Hall probes used for magnetic monitoring (described in Sections 4.1.2 and B.4) began to disagree near the endpoint, leaving the absolute probe calibration in doubt by 0.1%. Adding these two errors in quadrature, the endpoint calibration accuracy is at the 0.11% level. The resulting inaccuracy in η is $\pm 0.008\%$.

E.2.10 Beam Centering

While the spectrometer solid angle is nearly constant over the T_1 counter, the magnetic field varies somewhat in direction and can cause polarized muons to rotate about an axis

not parallel to the spectrometer axis. The change in decay symmetry over the spectrum can then cause distortion, as discussed in Section A.1.

The collimation of the beam and placement accuracy of $T1$ within the magnetic field allows a displacement error of 0.25 cm for the beam spot center with respect to the magnetic field axis. Applying the results of Section A.1 for 25% muon polarization to the spectrum, one finds an error in η of ± 0.007 .

E.2.11 Muon Stops

The number of muons stopping in the $T1$ counter, used for normalizing each run, comes from the $T1_\mu \cdot Tb_\mu$ scaler. As discussed in Section 6.5, the uncertainty in this is about 0.19% after corrections have been made. Adding a Gaussian-distributed error with a standard deviation of 0.19% to each run randomly and refitting the data, the resulting uncertainty in η is found to be ± 0.006 .

E.2.12 K^3

The majority of the events from this collimator made grazing impacts on the beveled inner radius, scattering in very short distances. The multiple scattering theory is beginning to break down at the lower energies studied because of the very short step sizes, while, at the same time, the finite step size in EGS4 is probably too long for highly accurate results. Also, the calculation is sensitive to the microscopic surface condition of the bevel.

E.2.13 Spectrometer Line Shape

Based on the quality of fit of the data at the spectrum endpoint, either of the extreme tails of the spectrometer line shape could hold about 1% more or less of the total accepted weight than calculated. This is consistent with the accuracy of the geometry of the $C1$ and $C2$ veto counters used in calculations. The resulting error in η is ± 0.005 .

E.2.14 $C2$

The EGS4 multiple scattering treatment is expected to give rather inaccurate answers for the scattering from the bevel of $C2$; the error in this -0.008 effect on η is ± 0.002 . The calibration accuracy on $C1$ is 12%, giving an uncertainty of ± 0.001 in η from this. Geometry and Monte Carlo statistics give a further uncertainty of ± 0.002 , for a calculational error of ± 0.002 .

E.2.15 Azimuthal Asymmetry

The spectrometer asymmetry, resulting from the double layer of scintillator on part of $C1$, $C1$'s traversal of the acceptance to connect to its light guide, the collimator-suspension cables, possible misalignment of collimators, and the ADC cuts in the $T1$ counter (which introduce some asymmetry because of $T1$'s rotation) is of the order of 10%. Using the results of Section A.3 to modify the spectrum for a 10% spectrometer asymmetry and a muon polarization of 25%, the effect on η turns out to be a negligible ± 0.001 .

E.2.16 NMR Probe Cross-calibration

Since three separate NMR probes with overlapping ranges are used to span the spectrum, the relative calibration of the probes had to be measured to transfer the endpoint calibration. The highest to middle-range probes were cross-calibrated to 0.0030%, while the middle to lowest-range probes were cross-calibrated to 0.0015%. Choosing probe cross-calibrations randomly from Gaussians with these standard deviations, the resulting error in η was found to be ± 0.002 .

Figure 1. Positron energy spectrum from unpolarized muon decay, without radiative corrections. The spectrum is for $\rho = 3/4$, $\eta = 0$ and is normalized to 1.

Figure 2. Relative sensitivity of the decay spectrum to η , as a function of x ; radiative corrections are included and eliminate the divergence at $x = 0$.

Figure 3. Time required to acquire data for a statistical accuracy on η of ± 0.088 at various spectrometer tunes, assuming an average muon stopping rate of 40K/sec on the target. This does not include the time for calibrations, including the amplitude calibration.

Figure 4. Positron energy spectrum from unpolarized muon decay, including radiative corrections. The spectrum is for $\rho = 3/4$, $\eta = 0$ and is normalized to 1.

Figure 5. Relative effect of first-order radiative corrections on the unpolarized decay spectrum.

Figure 6. Relative accuracy of Grotch's approximate decay spectrum for use above $E_s > 3m_e$, relative to his exact first-order expression.

Figure 7. Relative effect of radiative corrections, higher than first order, on the decay spectrum near the endpoint.

Figure 8. Vertical section through the Comus spectrometer. The dotted lines represent the collimator-support cables, projected onto this plane.

Figure 9. Magnetic field on the axis of the Comus spectrometer for $z = 1$.

Figure 10. Typical accepted trajectories calculated for the Comus spectrometer. Note the approximate ring focus in the radial distribution.

Figure 11. Monte Carlo of the radial distribution of impacts at the detector, in the absence of azimuthal irregularities in $C1$ and $C2$ or other non-ideal effects. The radius of the T2 trigger counter is 5.08 cm.

Figure 12. Calculated angular acceptance, averaged over momenta, of the Comus spectrometer at T_1 .

Figure 13. Contours at 10%, 50% and 90% of the peak acceptance, showing the correlations between momentum and polar angle for acceptance at $T1$.

Figure 14. Calculated line shape of Comus spectrometer at $T1$. Note the break in the slope of the high momentum edge from the use of two separate collimators ($K3$ and $C2$).

Figure 15. Horizontal section through the target area.

Figure 17. Monte Carlo results for the anisotropy of the e^+ distribution exiting the T1 target counter. The angle is measured from the normal to the T1 surface; the positrons were initially distributed isotropically with 6 MeV energy.

Figure 18. Relative effect on the spectrum of continuous energy losses in T1.

Figure 19. Relative effect on the spectrum of Bhabha scattering in Tl, including the suppression by veto counters.

Figure 20. Relative effect on the spectrum of external bremsstrahlung in $T1$.

Figure 21. Detection inefficiency of events for a $T2 \cdot T3$ trigger, for particles leaving $T1$ and not impacting further material until past $C2$.

Figure 22. Relative spectrum effect due to the three upstream sets of collimator-suspension cables, as calculated by Monte Carlo. The reduced effect above $x_p = 0.75$ is due to the absence of particles incident on the cables from above the spectrum endpoint, while the rise near $x_p = 1$ is due to the drop in the number of normal events near the endpoint.

Figure 23. Relative spectrum effect of scatters from the *K2* collimator, as calculated by Monte Carlo.

Figure 24. Relative spectrum effect of scatters from the bevel of the K3 collimator, as calculated by Monte Carlo. The drop for $x_p > 0.9$ is due to the absence of incident particles from above the endpoint momentum.

Figure 25. Relative spectrum effect of scatters from C1, as calculated by Monte Carlo.

Figure 26. Relative spectrum effect of scatters from C2, as calculated by Monte Carlo. The dip near $x_p = 1$ arises from the number of particles on C2 falling off more rapidly than the number accepted, since C2 forms part of the high side of the momentum selection slit.

Figure 27. Relative spectrum effect of scatters for the spectrometer back plate, based on Monte Carlo calculation and a fit to the measured $T_2 \cdot T_3 / \overline{T_2} \cdot T_3 \cdot T_4$ ratio.

Figure 28. Comparison of the measured and calculated $T2 \cdot T3 / \overline{T2} \cdot T3 \cdot T4$ ratio. The calculation is from first principles; the measured points are shown with only the statistical errors of the data.

Figure 29. Comparison of the measured $T2 \cdot T3/\sqrt{T2} \cdot T3 \cdot T4$ ratio to a calculation with one free parameter to include scattering from the spectrometer back plate. The measured points are shown with only the statistical errors of the data.

Figure 30. Data points imposed on the corrected theory spectrum for $\rho = 3/4$, and the best-fit values of η and amplitude normalization. Error bars represent only the statistical errors of the data.

Figure 31. Data points divided by the corrected theory spectrum for $\rho = 3/4$, and the best-fit values of η and amplitude normalization. Error bars represent only the statistical errors of the data.