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THE USE OF ENDOCHRONIC PLASTICITY
FOR MULTI-DIMENSIONAL SMALL
AND LARGE STRAIN PROBLEMS

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ABSTRACT

The endochronic plasticity theory was proposed in its general form by K. C. Valanis. An intrinsic time measure, which is a property of the material, is used in the theory. The explicit forms of the constitutive equation resemble closely those of the classical theory of linear viscoelasticity. Excellent agreement between the predicted and experimental results is obtained for some metallic and non-metallic materials for one dimensional cases.

No reference on the use of endochronic plasticity consistent with the general theory proposed by Valanis is available in the open literature. In this report, we derive the explicit constitutive equations that are consistent with the general theory for one-dimensional (simple tension or compression), two-dimensional plane strain or stress and three-dimensional axisymmetric problems.

It is found that the one-dimensional form used by Valanis is only an approximation of the real simple tension test. Hence, the endochronic parameters obtained by his method should not be directly used for two or three dimensional problems.

I. INTRODUCTION

The endochronic theory of material behavior was proposed by K. C. Valanis [1], and uses the principle that the history of deformation is defined in terms of a "time scale" which is not the real time, but is in itself a property of the material. No use of the classical yield surface concept is required in this theory. It was used by Valanis [2] to predict the mechanical response of aluminum and copper under conditions of complex strain histories. One constitutive equation described with remarkable accuracy and ease of calculation many phenomena, such as cross-hardening, loading and unloading loops, cyclic hardening as well as the effect of pre-shearing on tension behavior. Z. P. Bazant and his co-workers further developed the theory to describe the liquefaction of sand [3] and the inelastic behavior and failure of concrete [4]. The use of a two-dimensional endochronic constitutive relation in dynamic transient analysis of shells was first considered by H. C. Lin [5].

Basically, the endochronic theory uses an "intrinsic time" in place of real time in the viscoelastic constitutive equations. In other words, the time convolution integrals present in viscoelasticity are replaced by the "intrinsic time" convolution integrals. Obviously, the accuracy of an endochronic model in describing the behavior of a real material is dependent on the form of the "relaxation" function, G , and on the definition of intrinsic time. Important material phenomena may be lost when either is not defined properly. In such a case, the applicability of an endochronic model must be limited to a restricted class of problems.

The endochronic plasticity of the simplest form predicts weak unstable behavior for a material when it is subjected to external force. In other words, the material creeps when it receives work generated by external agency such as force. An extensive study of the one dimensional endochronic material

under dynamic condition and its comparison with classical elastoplastic predictions are given in [6]. It was found that the endochronic plasticity theory gives reasonably good results compared to conventional theory particularly for strains that are not very small. The slightly unstable (creep) behavior may even be a positive factor in some engineering applications.

However, it is noted that the constitutive equations for one-dimensional simple tension and compression tests in [1] is inconsistent with the general formulation presented in the same reference, and the limited reference [5] on its use in two dimensional case contains an "error" in the definition of the intrinsic measure. The purpose of this report is to derive the particular constitutive equations for problems that have engineering interest such as plane strain, plane stress, and axisymmetric problems.

Numerical experimentations are performed and the results are compared with other theories when available.

II. GENERAL THEORY OF ENDOCHRONIC PLASTICITY

The general endochronic constitutive equations can be derived through thermodynamic considerations of irreversible process by the use of hidden or internal variables. We summarize the resulting equations from [1] without thermal effect and time effect, for an isotropic material:

$$s_{ij} = \int_0^z G(z-z') \frac{\partial e_{ij}}{\partial z'} dz' \quad (1)$$

$$\frac{1}{3} \sigma_{kk} = \int_0^z B(z-z') \frac{\partial \epsilon_{kk}}{\partial z'} dz' \quad (2)$$

where

$$dz = \frac{d\xi}{f(\xi)} \quad (3)$$

$$(d\xi)^2 = \frac{(\beta_2 - \beta_1)}{3} d\epsilon_{ii} d\epsilon_{jj} + \frac{\beta_1}{2} d\epsilon_{ij} d\epsilon_{ij} ; d\xi \geq 0 \quad (4)$$

Here $s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$ is the deviatoric stress tensor, $e_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_{kk} \delta_{ij}$ is the deviatoric strain tensor, z is the intrinsic time, f is a function of ξ ; G , B , β_1 , and β_2 are material parameters which may depend on z and $\frac{1}{3} \sigma_{kk}$ is the hydrostatic pressure. Index notations are used in writing these equations and repeated indices represent summation. Since small strains are assumed in [1], no distinctions between various stresses and strain measures are necessary. The choices for the functions G and B , and the material parameters β_1 and β_2 are not completely arbitrary. They are subjected to the requirements necessary for thermodynamic stability.

The following assumptions were used in [1] for the simplest endochronic constitutive equation for an isotropic material:

$$G(z) = 2\mu_0 e^{-\alpha z} ; \quad \mu_0, \alpha = \text{constants}$$

$$B(z) = B_0 = \text{constant} \quad (5)$$

$$f(\xi) = 1 + \beta \xi ; \quad \beta = \text{constant}$$

Substituting eq. (5) into eqs (1) to (4), we obtain the following:

$$s_{ij} = 2\mu_0 \int_0^z e^{-\alpha(z-z')} \frac{\partial e_{ij}}{\partial z'} dz'$$

$$\sigma_{kk} = 3B_0 \varepsilon_{kk}$$

$$\alpha dz = \frac{\alpha d\xi}{(1+\beta\xi)} = \frac{\alpha_1 (\beta d\xi)}{1 + \beta\xi} ; \quad \alpha_1 = \frac{\alpha}{\beta} \quad (6)$$

$$(\beta d\xi)^2 = K_1 d\varepsilon_{ij} d\varepsilon_{ij} + K_2 d\varepsilon_{ij} d\varepsilon_{ij} ; \quad \xi = \int d\xi ; \quad \beta d\xi \geq 0$$

where K_1 and K_2 may depend on ε_{ij} .

The differential equation corresponding to the above equations can be obtained by differentiating the first of eq (6) with respect to z and then combining the resulting equation with the second of eq (6). It is given by

$$d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl} - (\alpha dz) (\sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}) \quad (7)$$

where

$$C_{ijkl} = 2\mu_0 \delta_{il} \delta_{jk} + (B_0 - \frac{2}{3}\mu_0) \delta_{ij} \delta_{kl} \quad (8)$$

and δ_{ij} is the Kronecker delta.

Equation (8) indicates that C_{ijkl} is an isotropic fourth order tensor. Its physical meaning can be brought to light by assuming the existence of a virgin natural state in which $\sigma_{ij}^0 = \epsilon_{ij}^0 = \beta\xi^0 = 0$. If this state is disturbed such that infinitesimal stress, strain, and intrinsic time, $d\sigma_{ij}$, $d\epsilon_{ij}$ and $\beta d\xi$ are generated and the new intrinsic time and stress are $\beta\xi = \beta\xi^0 + \beta d\xi = \beta d\xi$ and $\sigma_{ij} = \sigma_{ij}^0 + d\sigma_{ij} = d\sigma_{ij}$, respectively. Then eq (7) becomes the following, after rearranging terms

$$\left(1 + \frac{\alpha_1 \beta d\xi}{1 + \beta d\xi}\right) d\sigma_{ij} = C_{ijkl} d\epsilon_{kl} + \frac{\alpha_1 \beta d\xi}{3(1 + \beta d\xi)} (d\sigma_{kk}) \delta_{ij} \quad (9)$$

Since the disturbances are infinitesimal, we can neglect the first order terms in the presence of unity and neglect second order terms in the presence of first order terms. By doing so, we have

$$d\sigma_{ij} \rightarrow C_{ijkl} d\epsilon_{kl} \quad \text{as } d\epsilon_{kl} \rightarrow 0 \text{ and } d\sigma_{ij} \rightarrow 0 \quad (10)$$

provided α_1 , α and β are bounded. Equations (10) and (8) now can be interpreted to mean that C_{ijkl} are the elastic constants of the material when it is subjected to infinitesimal disturbance from its natural virgin state.

Hence the simple endochronic constitutive equations can be summarized in matrix notations as

$$\begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{Bmatrix} = \frac{E_0}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{Bmatrix} d\epsilon_{11} \\ d\epsilon_{22} \\ d\epsilon_{33} \\ d\epsilon_{12} \\ d\epsilon_{23} \\ d\epsilon_{31} \end{Bmatrix}$$

$$- \frac{\alpha dz}{3} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \quad (11)$$

where E_0 and ν are the elastic modulus and Poisson's ratio of the material,

$$\alpha dz = \frac{\alpha_1 \beta d\xi}{1 + \beta \xi} \quad (12)$$

and

$$\begin{aligned} (\beta d\xi)^2 &= K_1 (d\epsilon_{11} + d\epsilon_{22} + d\epsilon_{33})^2 \\ &+ K_2 [(d\epsilon_{11})^2 + (d\epsilon_{22})^2 + (d\epsilon_{33})^2 + 2(d\epsilon_{12})^2 + 2(d\epsilon_{23})^2 + (d\epsilon_{31})^2] \end{aligned} \quad (13)$$

Equations (11) to (13) define a simple endochronic material provided the material parameters α_1 , K_1 and K_2 are known.

III. GENERAL ENDOCHRONIC PLASTICITY FOR LARGE STRAINS

It is possible to extend an existing constitutive law for small displacement problems to large strain and large displacement/rotation problems in many ways. The choice of one method over another is usually based on the efficiency of that method. In other words, the choice depends heavily upon the simplicity of the resulting constitutive equations. The mathematical requirements for a constitutive law are that they should be frame indifferent, etc.; the physical requirements are that no stress increment is generated when the material undergoes rigid body motion, etc. For large strain problems, the measure of distortion or strain must be zero when the material is subjected to rigid body motion lest the constitutive equations become too complicated. Four candidates are qualified for this purpose; they are corotational strain [7], Eulerian strain, Lagrangian strain and rate of deformation tensors. The rate of deformation is chosen as the measure of distortion because; 1) it is linear, hence superposition is possible, 2) its vanishing indicates rigid body motion, and 3) the vanishing of the trace of the rate of deformation tensor indicates incompressibility. The Eulerian stress tensor is the best candidate for the measure of force once the rate of deformation is taken as the measure of distortion, since the rate of work done by the stress is given by the product of the Eulerian stress tensor and the rate of deformation tensor. Other types of stress such as the Kirchhoff stress tensor of first or second kind can also be used as a measure of force in formulating the constitutive equations provided the transformations between these stress tensors and the Eulerian stress tensor are known. However, use of the rate of deformation and the Eulerian stress tensors result in the simplest constitutive equations.

A particularly efficient formulation is further obtained by using the corotational coordinates methods (described in [7]) in which all variables are

referred to a coordinate system that rotates with the material. An extension of classical elastoplasticity for small displacement problems to large displacement, large strain problems are achieved using identical forms of the constitutive equations. We shall assume that the same form of endochronic constitutive equations can be applied for large strain problems in the corotational coordinates when the corotational rate of deformation and Eulerian stress are used in their proper places.

We summarize some results from [7] without derivation in the following:

$$\begin{aligned}\widehat{v}_{ij} &= \frac{1}{2} \left[\widehat{\epsilon}_{ik} (\delta_{kj} + \widehat{\epsilon}_{kj})^{-1} + \widehat{\epsilon}_{jk} (\delta_{ki} + \widehat{\epsilon}_{ki})^{-1} \right] \\ \widehat{\epsilon}_{ij} &= \frac{\partial \widehat{\epsilon}_{ij}}{\partial t} \\ \widehat{\epsilon}_{ij} &= \widehat{\epsilon}_{ji} = \frac{1}{2} \left(\frac{\partial \widehat{u}_i^{\text{def}}}{\partial X_j} + \frac{\partial \widehat{u}_j^{\text{def}}}{\partial X_i} \right)\end{aligned}\tag{14}$$

where $\widehat{\epsilon}_{ij}$ is the corotational strain tensor, $\widehat{v}_{ij}$ is the corotational rate of deformation tensor, $\widehat{u}_i^{\text{def}}$ is the corotational deformation displacement, X_i is the Lagrangian coordinate (or the initial corotational coordinate) and the superscript -1 on a quantity indicates the inverse of that quantity.

The endochronic constitutive equations for large strains can now be written as

$$\begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{33} \\ \hat{\sigma}_{12} \\ \hat{\sigma}_{23} \\ \hat{\sigma}_{31} \end{Bmatrix} = \frac{E_0}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} \\ \hat{v}_{22} \\ \hat{v}_{33} \\ \hat{v}_{12} \\ \hat{v}_{23} \\ \hat{v}_{31} \end{Bmatrix}$$

$$- \frac{\alpha(dz)}{3(dt)} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{33} \\ \hat{\sigma}_{12} \\ \hat{\sigma}_{23} \\ \hat{\sigma}_{31} \end{Bmatrix} \quad (15)$$

where

$$\alpha dz = \frac{\alpha_1 \beta d\xi}{1+\beta\xi} \quad ; \quad \frac{\alpha dz}{dt} = \frac{\alpha_1}{1+\beta\xi} \left(\frac{\beta d\xi}{dt} \right) \quad (16)$$

and

$$\left(\frac{\beta d\xi}{dt} \right)^2 = \{ K_1 (\hat{v}_{11} + \hat{v}_{22} + \hat{v}_{33})^2 + K_2 [(\hat{v}_{11})^2 + (\hat{v}_{22})^2 + (\hat{v}_{33})^2 + 2(\hat{v}_{12})^2 + 2(\hat{v}_{23})^2 + 2(\hat{v}_{31})^2] \} \quad (17)$$

The physical meaning of $\hat{v}_{ij}$ will be studied later. It is sufficient to note, at this stage, that $\hat{v}_{ij} = \dot{\epsilon}_{ij}$ when the displacements are small. The

above equation, eq (15) is derived assuming a linear relationship between the rate of the hydrostatic stress $\dot{\sigma}_{kk}$ and the rate of volume change V_{kk} . However, this assumption can be easily modified if the material properties suggest a nonlinear relationship. For some metals subjected to very high pressure, for example, Mie-Grüneisen equation of state [8] relating the hydrostatic pressure $p = \frac{1}{3} \sigma_{kk}$ and the density change (or volume change) can be used.

It is not necessary to reformulate the constitutive equations for nonlinear hydrostatic response. The linear hydrostatic response is always computed by eqs (15) to (17), hence the correct hydrostatic response is the sum of the nonlinear portion of the Mie-Grüneisen hydrostatic response and this linear response. By realization of this fact, a simple form of endochronic constitutive equations can be written that is applicable to all magnitudes of strains.

IV. UNIAXIAL STRESS STATE AND SHEARING STATE

(A) Uniaxial stress:

The uniaxial stress condition is realized by requiring that all but one stress component vanish, say σ_{11} or $d\sigma_{11}$. Furthermore, we assume symmetry exists about 1-axis, i.e. $V_{33} = V_{22} = V$, etc. Since the shear components are not coupled with the axial components, the vanishing of shear stresses in eq (16) results in the vanishing of shear strains. Hence, eq (15) can be rewritten as

$$\begin{bmatrix} \hat{\sigma}_{11} \\ 0 \end{bmatrix} = \frac{E_0}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \hat{V}_{11} \\ \hat{V} \end{Bmatrix} - \frac{\alpha dz}{3dt} \sigma_{11} \begin{Bmatrix} +2 \\ -1 \end{Bmatrix}$$

Solving the second of the above equation for the transverse corotational rate of deformation $\hat{V}$ and substituting this result to the first equation, we obtain

$$d\hat{\sigma}_{11} = E_0 \hat{V}_{11} dt - \frac{2}{3} (1+\nu) (\alpha dz) \hat{\sigma}_{11} \quad (18)$$

where

$$\alpha dz = \frac{\alpha_1 \beta d\xi}{1+\beta\xi}$$

$$(\beta d\xi)^2 = \{K_1 (\hat{V}_{11} + 2\hat{V})^2 + K_2 [(\hat{V}_{11})^2 + 2(\hat{V})^2]\} (dt)^2 \quad (19)$$

and

$$\hat{V}(dt) = -\nu \hat{V}_{11}(dt) - \frac{1}{3} \frac{(1+\nu)(1-2\nu)}{E_0} (\alpha dz) \hat{\sigma}_{11} \quad (20)$$

The relationship between the rate of deformation and corotational strain in the $\hat{1}$ -axis direction in the absence of shear strain is given by eq (14)

$$\hat{V}_{11} = \frac{\hat{\epsilon}_{11}}{1+\hat{\epsilon}_{11}} ; \hat{V}_{11} dt = \frac{d\hat{\epsilon}_{11}}{1+\hat{\epsilon}_{11}} \quad (21)$$

In the absence of rigid body motion, the deformation displacements are identical to the displacements. If we further assume that the deformation is homogeneous,

then the last equation of eq (14) can be interpreted as

$$\hat{\epsilon}_{11} = \frac{\Delta \ell}{\ell^o} \quad (22)$$

where ℓ^o is the original length and $\Delta \ell$ is the increment in length. Hence we can say that the corotational strain in a uniaxial stress case is nothing but the conventional engineering strain $\hat{\epsilon}_{11}^{eng}$, i.e. $\hat{\epsilon}_{11} = \hat{\epsilon}_{11}^{eng}$. Since the corotational coordinates are identical to the Lagrangian coordinates when the rigid body translation and rotation are absent, we can eliminate the superscript hat ($\hat{}$) if so desired such as in the case of simple tension test.

Integrating eq (21) with respect to time, we have

$$\hat{D}_{11} = \ln(1 + \hat{\epsilon}_{11}^{eng}) ; \quad \hat{D}_{11} = \int \hat{V}_{11} dt \quad (23)$$

From this, we see that $\hat{V}_{11}$ is the rate of natural or logarithmic strain $\hat{D}_{11}$ in the 1-axis direction. Also noted is the singular behavior of $\hat{V}_{11}$ when $\hat{\epsilon}_{11} = -1$ in eq (21). The physical meaning of this can be seen from eq (22). In other words, eqs (21) and (22) state that a length can be extended to infinitely long, but it can not be compressed to zero length. Hence, if we assume that a material has the property that its true (Eulerian) stress-natural strain constitutive curves are identical in tension and compression, then it will be harder to compress a bar made of this material than to extend it by the same amount of length (engineering strain). The forms of the endochronic constitutive equations and the use of Eulerian stress and rate of deformation as variables in these equations imply the assumption that the true stress-natural strain curves are identical in tension and compression.

Equations (18) to (21) define the endochronic constitutive law for a uniaxial stress condition completely. In the case of small strain, we have $\hat{V}_{11} dt \rightarrow d\hat{\epsilon}_{11}$ by use of eq (21). In this case, these equations reduce to, for uniaxial stress (simple tension or compression)

$$\begin{aligned}
 d\hat{\sigma}_{11} &= E_0 d\hat{\epsilon}_{11} - \frac{2}{3} (1+\nu) (\alpha dz) \hat{\sigma}_{11} \\
 \alpha dz &= \alpha_1 \beta d\xi / (1+\beta\xi) \\
 (\beta d\xi)^2 &= K_1 (d\hat{\epsilon}_{11} + 2d\hat{\epsilon})^2 + K_2 [(d\hat{\epsilon}_{11})^2 + 2(d\hat{\epsilon})^2] \\
 d\hat{\epsilon} &= -\nu d\hat{\epsilon}_{11} - \frac{1}{3} \frac{(1+\nu)(1-2\nu)}{E_0} (\alpha dz) \hat{\sigma}_{11}
 \end{aligned} \tag{24}$$

The constitutive equations used in [1], however have the following form:*

$$\begin{aligned}
 d\hat{\sigma}_{11} &= E_0 d\hat{\epsilon}_{11} - (\bar{\alpha} dz) \hat{\sigma}_{11} \\
 \bar{\alpha} dz &= \bar{\alpha}_1 \bar{\beta} d\xi / (1+\bar{\beta}\xi) \quad ; \quad \bar{\alpha}_1 = \bar{\beta} / \bar{\alpha} \\
 (\bar{\beta} d\xi)^2 &= \bar{\beta}_1^2 (d\hat{\epsilon}_{11})^2
 \end{aligned} \tag{25}$$

Equation (24) indicates that the transverse strain depends on the axial strain, the elastic modulus, the intrinsic time and the current stress, while this dependence is absent in eq (25). Also, the intrinsic time $\beta d\xi$ in eq (24) is a function of two constants K_1 and K_2 and transverse strain while one constant is assumed in eq (25). Hence, we can say eq (25) is at best an approximation to eq (24). To simplify eq (24) to the form of eq (25), we relax the requirement indicated by the last of eq (24) and assume

$$d\hat{\epsilon} = -\theta d\hat{\epsilon}_{11} \quad ; \quad \nu \leq \theta \leq \frac{1}{2} \tag{26}$$

where θ can be treated as a general Poisson's ratio. The numerical range of θ is limited by the range indicated above from consideration of elasticity and incompressibility of material. However, this should not be interpreted as the physical limits of range for θ . Obviously, θ is not a constant but is a function of E_0 , ν , $d\hat{\epsilon}_{11}$ and $\hat{\sigma}_{11}$; however, we assume θ is a constant. By doing so, we find that the intrinsic time $\beta d\xi$ is given by, from eq (24)

* Equation (25) can be derived from eq (24) by assuming incompressibility i.e. $\nu = 1/2$. However, this assumption introduces indetermination in hydrostatic pressure.

$$(\beta d\xi)^2 = [k_1(1-2\theta)^2 + K_2(1+2\theta^2)] (d\epsilon_{11})^2 \quad (27)$$

A comparison among eqs. (24), (25) and (27) yields the following relation for the endochronic parameters:

$$\begin{aligned} (\bar{\beta}_1)^2 &= (1-2\theta)^2 K_1 + (1+2\theta^2) K_2 \\ \bar{\alpha} &= \frac{2(1+\nu)}{3} \alpha \quad ; \quad \bar{\alpha}_1 = \frac{2(1+\nu)}{3} \alpha_1 \end{aligned} \quad (28)$$

Equation (25) may be integrated explicitly if the axial strain history is simple. By assuming a monotonically increasing axial strain, we find the following relation between the endochronic parameters and other material properties [1]

$$\begin{aligned} \alpha_1 &= \frac{3}{2(1+\nu)} \left[\left(\frac{E_o}{E_p} \right) - 1 \right] \\ (1-2\theta)^2 K_1 + (1+2\theta^2) K_2 &= \left(\frac{E_p}{\sigma_o} \right)^2 \end{aligned} \quad (29)$$

where E_p is the asymptotic slope of the stress-strain curve when ϵ_{11} is large and σ_o is the intersection of this asymptotic straight line with the $\epsilon_{11} = 0$ axis.

Equation (29) indicates that the endochronic material parameters can not be uniquely determined from these conditions even for the approximate formulation. By studying the very definition of the intrinsic time measure $\beta d\xi$ given by eq (4), we note that it is really a scaled measure of the strain energy stored in a material at least for the case of very small strains. Therefore, it appears reasonable to assume that the relationship between K_1 and K_2 is similar to that of the Lamé's constants of linear elasticity theory.

That is

$$K_2 = \frac{1-2\nu}{\nu} K_1 \quad (30)$$

$$K_1 = \left(\frac{E_p}{\sigma_0} \right)^2 / [(1-2\nu)^2 + \frac{(1-2\nu)}{\nu} (1+2\nu)^2] \quad (31)$$

Equations (29), (30) and (31) now defines uniquely the approximate values of α_1 , K_1 and K_2 once the value of θ is assumed.

It is noted that the true values of α_1 , K_1 and K_2 can only be determined by assuming their values and then by comparing the predictions given by eq (24) with experimental results. The previous approximation only gives us some guidance on how to assume their values. It is also noted that the value of α_1 and $\bar{\alpha}_1$ are different by a factor of $\frac{2}{3}(1+\nu)$. This difference will not cause any trouble in one dimensional problems since it can be incorporated into $\bar{\alpha}_1$. However, in multi-dimensional problems, the correct values of α_1 , K_1 and K_2 must be used.

(B) Shearing

The stress-strain relations in a shearing situation are more complicated than the uniaxial stress case. The shearing condition can be treated as a one dimensional problem only when the shear is very small as we shall see in latter section. Consider a rectangular block whose sides are of the length L and is subjected to a homogenous strain state caused by the displacement LU on the top surface as shown in Fig. 1, where U is not necessarily small. The displacement in the X_1 coordinates can be written as

$$\begin{aligned} U_1 &= UX_2 \\ U_2 &= 0 \end{aligned} \quad (32)$$

The motion of the block subjected to this displacement contains rigid body rotation and its amount can be computed from [9] as shown in Fig. 1, where the magnitude of the rotation $\hat{\theta}$ is given by

$$\hat{\theta} = \tan^{-1}\left(\frac{-U}{2}\right) \quad (33)$$

Hence the deformation displacement in the corotational coordinates $\hat{x}_i$ are found from [7] as

$$\begin{aligned} \hat{u}_1^{\text{def}} &= (\cos \hat{\theta}) UX_2 + (\cos \hat{\theta} - 1) X_1 + (\sin \hat{\theta}) X_2 \\ \hat{u}_2^{\text{def}} &= (-\sin \hat{\theta}) UX_2 - (\sin \hat{\theta}) X_1 + (\cos \hat{\theta} - 1) X_2 \\ \hat{u}_3^{\text{def}} &= u_3 \end{aligned} \quad (34)$$

The corotational strain components can be obtained by substituting eq (34) into eq (14). They are

$$\begin{aligned} \hat{\epsilon}_{11} &= \cos \hat{\theta} - 1 \\ \hat{\epsilon}_{22} &= (-\sin \hat{\theta}) U + (\cos \hat{\theta} - 1) = (-\sin \hat{\theta}) U + \hat{\epsilon}_{11} \\ \hat{\epsilon}_{12} &= \frac{1}{2} (\cos \hat{\theta}) U \\ \hat{\epsilon}_{23} &= \frac{1}{2} \frac{\partial u_3}{\partial X_2} \\ \hat{\epsilon}_{13} &= \frac{1}{2} \frac{\partial u_3}{\partial X_1} \\ \hat{\epsilon}_{33} &= \frac{\partial u_3}{\partial X_3} \end{aligned} \quad (35)$$

We now assume that the displacement $u_3 = \text{constant}$ so that $\hat{\epsilon}_{13} = \hat{\epsilon}_{23} = \hat{\epsilon}_{33} = 0$ to simplify the computation for rate of deformations. The strain increments for eq (35) are

$$\begin{aligned} d\hat{\epsilon}_{11} &= -(\sin \hat{\theta}) (d\hat{\theta}) \\ d\hat{\epsilon}_{22} &= -(\sin \hat{\theta}) (dU) - (\cos \hat{\theta}) (d\hat{\theta}) U - (\sin \hat{\theta}) (d\hat{\theta}) \\ d\hat{\epsilon}_{12} &= \frac{1}{2} [-(\sin \hat{\theta}) (d\hat{\theta}) U + (\cos \hat{\theta}) (dU)] \\ d\hat{\epsilon}_{13} &= d\hat{\epsilon}_{23} = d\hat{\epsilon}_{33} = 0 \end{aligned} \quad (36)$$

By substituting eqs (35) and (36) into the first of eq (14), we obtain the rate of deformation as

$$\begin{aligned}
 \hat{V}_{11}dt &= [(1+\hat{\epsilon}_{22})d\hat{\epsilon}_{11} - \hat{\epsilon}_{12}d\hat{\epsilon}_{12}]/\Delta \\
 \hat{V}_{22}dt &= [(1+\hat{\epsilon}_{11})d\hat{\epsilon}_{22} - \hat{\epsilon}_{12}d\hat{\epsilon}_{12}]/\Delta \\
 \hat{V}_{12}dt &= \{[(1+\hat{\epsilon}_{11}) + (1+\hat{\epsilon}_{22})]d\hat{\epsilon}_{12} - (d\hat{\epsilon}_{11}+d\hat{\epsilon}_{22})d\hat{\epsilon}_{12}\}/2\Delta \\
 \hat{V}_{13}dt &= \hat{V}_{23}dt = \hat{V}_{33}dt = 0 \\
 \Delta &= (1+\hat{\epsilon}_{11})(1+\hat{\epsilon}_{22}) - \hat{\epsilon}_{12}^2
 \end{aligned} \tag{37}$$

The singular points of the rate of deformation given by eq (37) can be found by substituting eq (35) into the last of eq (37). By doing so, we find that the only singular situation occurs when $\cos\hat{\theta} = 0$ or $\hat{\theta} = \pm \pi/2$. Since θ and $\hat{\theta}$ are related by $\tan\theta = 2 \tan\hat{\theta}$ (see Fig. 1), the singular points are also given by $\theta = \pm \pi/2$. Physically, these singular points mean that a vertical side of the block can not be pushed to rotate 90 degrees in either direction. The existence of singular points in rate of deformation for both uniaxial and shear states confirms the effectiveness of using the rate of deformation as a measure of distortion.

The stress components can now be found by using eq(37) and eq (15).

They are given by

$$\begin{aligned} \begin{Bmatrix} d\hat{\sigma}_{11} \\ d\hat{\sigma}_{22} \end{Bmatrix} &= \frac{E_o}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} dt \\ \hat{v}_{22} dt \end{Bmatrix} - \frac{(\alpha dz)}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \end{Bmatrix} \\ d\hat{\sigma}_{33} &= -\frac{(\alpha dz)}{3} [2\hat{\sigma}_{33} - \hat{\sigma}_{11} - \hat{\sigma}_{22}] \\ d\hat{\sigma}_{12} &= \frac{E_o}{(1+\nu)} \hat{v}_{12} dt - (\alpha dz) \hat{\sigma}_{12} \\ \hat{\sigma}_{13} = \hat{\sigma}_{23} &= (1+\beta\xi)^{-\alpha_1} \\ (\alpha dz) &= \frac{\alpha_1 \beta d\xi}{1+\beta\xi} \\ (\beta d\xi)^2 &= [K_1 (\hat{v}_{11} + \hat{v}_{22})^2 + K_2 (\hat{v}_{11}^2 + \hat{v}_{22}^2 + 2\hat{v}_{12}^2)] (dt)^2 \end{aligned} \tag{38}$$

It is interesting to note that, for shear components, the vanishing of rate of deformation in one component does not necessarily warrant the vanishing of the corresponding stress component (though the reverse is true) because of the coupling effect of the intrinsic time.

By using eqs (33), (36) and (37), we may express the intrinsic time ($\beta d\xi$) in terms of the very variable U . Hence, it is possible to plot any stress component in corotational coordinates or in spatial coordinates as a function of UL once the values of the endochronic parameters, α_1 , K_1 and K_2 are known. The correct values of these variables for large strain problems are then obtained by comparing the computed results with the experimental results.

As can be seen from the previous derivation, the "simple" displacement field given by eq (32) and constant displacement in the transverse direction result in the generation of all stress components. Most of these stress components are of second order effect, if the displacement U is small, i.e.,

$U \ll 1$, $dU \ll 1$. Under this condition, we have the following approximate formulas

$$\hat{\theta} = U/2; \quad \cos \hat{\theta} = 1; \quad \sin \hat{\theta} = U/2 \quad (39)$$

Substituting this equation into eqs (35) and (36), we have the following results for corotational strains and strain increments:

$$\begin{aligned} \hat{\epsilon}_{11} &= \hat{\epsilon}_{22} = \hat{\epsilon}_{33} = \hat{\epsilon}_{13} = \hat{\epsilon}_{23} = 0 \\ \hat{\epsilon}_{12} &= U/2 \\ d\hat{\epsilon}_{11} &= d\hat{\epsilon}_{22} = d\hat{\epsilon}_{33} = d\hat{\epsilon}_{13} = d\hat{\epsilon}_{23} = 0 \\ d\hat{\epsilon}_{12} &= dU/2 \end{aligned} \quad (40)$$

The rate of deformation is now obtained by combining eq (40) with eq (37)

$$\begin{aligned} \hat{v}_{11} &= \hat{v}_{22} = \hat{v}_{33} = \hat{v}_{13} = \hat{v}_{23} = 0 \\ \hat{v}_{12} dt &= dU/2 \end{aligned} \quad (41)$$

By noting the symmetry between the axial stress components in eq (15) in the absence of $\hat{v}_{11}$, $\hat{v}_{22}$ and $\hat{v}_{33}$, we find that the axial stresses must vanish, i.e., $\hat{\sigma}_{11} = \hat{\sigma}_{22} = \hat{\sigma}_{33} = 0$. Since the parameter α_1 is positive, we conclude that the shear stress in the transverse direction given by the fourth of eq (38) is unity for small strain and zero for large strain. For practical problems, hence, we can treat them as zero, i.e. $\hat{\sigma}_{13} = \hat{\sigma}_{23} = 0$.

The endochronic constitutive equation for the only nonvanishing component can now be stated as, by using eqs (38) and (41)

$$\begin{aligned} d\hat{\sigma}_{12} &= \frac{E_0}{2(1+\nu)} dU - (\alpha dz) \hat{\sigma}_{12} \\ \alpha dz &= \frac{\alpha_1 \beta d\xi}{(1+\beta\xi)} \\ (\beta d\xi)^2 &= \frac{K_2}{2} (dU)^2 \quad \text{or} \quad \beta d\xi = \sqrt{\frac{K_2}{2}} |dU| \end{aligned} \quad (42)$$

This equation is identical in form to eq (25), hence explicit solutions are available [1] if we assume $dU > 0$. The solution for eq (42) subjected to this condition is

$$\hat{\sigma}_{12} = \frac{G_o (1 + \sqrt{\frac{K_2}{2}} U)}{\sqrt{\frac{K_2}{2}}^n} \left\{ 1 - (1 + \sqrt{\frac{K_2}{2}} U)^{-n} \right\}; \quad n = 1 + \alpha_1 \quad (43)$$

where the shear modulus G_o is given by

$$G_o = \frac{E_o}{2(1+\nu)} \quad (44)$$

The parameters α_1 and K_2 can be found by studying the asymptotic behavior of eq (44) at large strain. They are related by the following relations

$$n = \frac{G_o}{G_p} ; \quad \alpha_1 = \frac{G_o}{G_p} - 1 \quad (45)$$

$$\sqrt{\frac{K_2}{2}} = \frac{G_p}{\sigma_{12}^o} \quad \text{or} \quad K_2 = 2 \left(\frac{G_p}{\sigma_{12}^o} \right)^2$$

This constitutive equation and the definition of various symbols are shown in Fig.2.

From the above derivation, we see that two of the three endochronic material parameters, α_1 and K_2 can be obtained from the simple shear test at least for small strain problems. The third parameter K_1 should then be determined from simple tension test or by using some appropriate assumptions such as eq (30). Again as in the uniaxial stress case, the constant $\sqrt{K_2/2}$ can be absorbed into another constant for one dimensional problem. However, the correct value must be used for multidimensional problems.

V. PLANE STRESS STATE

In this section we shall derive the endochronic constitutive equations that are applicable to axisymmetric thin shells. We assume the plane stress situation exists in the direction normal to the shell surface, i.e.

$\hat{\sigma}_{33} = \hat{\sigma}_{13} = \hat{\sigma}_{23} = 0$. We further assume that $\hat{\sigma}_{12}$ is also zero. Under these assumptions, the only non-vanishing corotational strains are $d\hat{\epsilon}_{11}$, $d\hat{\epsilon}_{22}$ and $d\hat{\epsilon}_{33}$. The nonvanishing rate of deformation corresponding to this strain state can be found by using eq (14) to be

$$\begin{aligned}\hat{V}_{11}dt &= d\hat{\epsilon}_{11}/(1+\hat{\epsilon}_{11}) \\ \hat{V}_{22}dt &= d\hat{\epsilon}_{22}/(1+\hat{\epsilon}_{22}) \\ \hat{V}_{33}dt &= d\hat{\epsilon}_{33}\end{aligned}\tag{46}$$

It should be noted that the vanishing of stress component $\hat{\sigma}_{33}$ almost guarantees the nonvanishing of $\hat{V}_{33}$ or $\hat{\epsilon}_{33}$. However, the explicit dependence of the constitutive equations on $\hat{V}_{33}$ can be eliminated by requiring the additional condition $\hat{\sigma}_{33} = 0$ to hold.

Equation (46) again indicates the existence of singular points of the rate of deformation when $\hat{\epsilon}_{11}$ or $\hat{\epsilon}_{22}$ equals -1. This means that the length of a material can not be reduced to zero, as we have studied in uniaxial stress case.

Substituting eq (46) and the condition of plane stress into the general constitutive equation, eq. (15), we have

$$\begin{Bmatrix} d\hat{\sigma}_{11} \\ d\hat{\sigma}_{22} \\ 0 \end{Bmatrix} = \frac{E_o}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \hat{v}_{11}dt \\ \hat{v}_{22}dt \\ \hat{v}_{33}dt \end{Bmatrix} - \frac{(\alpha dz)}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ 0 \end{Bmatrix} \quad (47)$$

The last of the above equation can be solved for $\hat{v}_{33}dt$. Substituting the resulting $\hat{v}_{33}dt$ into the first two of eq (47), we find the constitutive equations to be

$$\begin{Bmatrix} d\hat{\sigma}_{11} \\ d\hat{\sigma}_{22} \end{Bmatrix} = \frac{E_o}{(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \hat{v}_{11}dt \\ \hat{v}_{22}dt \end{Bmatrix} - \frac{1}{3} \frac{\beta d\xi}{(1+\beta\xi)} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \frac{\nu}{(1-\nu)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \end{Bmatrix}$$

where

$$(\beta d\xi)^2 = \left\{ K_1 (\hat{v}_{11} + \hat{v}_{22} + \hat{v}_{33})^2 + K_2 [(\hat{v}_{11})^2 + (\hat{v}_{22})^2 + (\hat{v}_{33})^2] \right\} (dt)^2 \quad (48)$$

and

$$\hat{v}_{33}dt = - \left[\frac{\nu}{(1-\nu)} (\hat{v}_{11} + \hat{v}_{22})dt + \frac{(1+\nu)(1-2\nu)}{3E_o(1-\nu)} \frac{\alpha_1 \beta d\xi}{(1+\beta\xi)} (\hat{\sigma}_{11} + \hat{\sigma}_{22}) \right]$$

Equation (48) is the endochronic constitutive equation for plane stress in the 3-axis direction. The first of these equations can not be directly solved for the stress increment once the rate of deformation is known, not as in the case of uniaxial stress, because the intrinsic time $\beta d\xi$ is not only a function of rate of deformation but also the stress components.

To the best of the author's knowledge, other usages of endochronic constitutive laws for two dimensional problems are in [5] and [10] only. However, the intrinsic time in these references is equivalent to the following form

$$(\beta d\xi)^2 = (d\epsilon_1 + d\epsilon_2)^2 \quad (49)$$

This form of definition of intrinsic time is claimed to be an approximation of the general definition of intrinsic time, eq (13) without proof in [5]. However, eq (49) is not only not an approximation of eq (13), but is incorrect. Mathematically, the intrinsic time is required to be positive definitive (i.e. it vanishes if and only if all the strain increments are zero) by the assumption that all deformations are thermodynamically irreversible in the endochronic theory. Equation (49) allows vanishing values of $\beta d\xi$ for strain paths that are generated when $d\epsilon_1 = -d\epsilon_2$. This means that the deformations are reversible when $d\epsilon_1 = -d\epsilon_2$. Though eq (49) violates the endochronic theory it does not violate thermodynamics. In other words, the use of eq (49) indicates the the material response is purely elastic when $d\epsilon_1 = -d\epsilon_2$ or under simple shearing condition. However, experimental evidences show that the irreversible (or plastic) strains are caused by shearing. For example, a material that yields under a stress of σ^0 in tension will yield under $\frac{1}{2}\sigma^0$ (Tresica yield condition) or $\frac{1}{\sqrt{3}}\sigma^0$ (Von Mises' yield condition) in shear. But the use of eq (49) will predict the corresponding yield stress in shear to be infinite. Furthermore, eq (49) lacks the participation of the strain increment $d\epsilon_{33}$ which is usually present under the assumption of plane stress in 3-axis direction.

VI. PLANE STRAIN STATE

The plane strain in the 3-axis direction is defined as $d\epsilon_{33} = d\epsilon_{13} = d\epsilon_{23} = 0$. We also assume that the shear components in the directions parallel to the shell surface also vanish as in the case of plane stress. Then the nonvanishing components of the rate of deformation can be written as

$$\begin{aligned}\hat{v}_{11}dt &= d\hat{\epsilon}_{11}/(1+\hat{\epsilon}_{11}) \\ \hat{v}_{22}dt &= d\hat{\epsilon}_{22}/(1+\hat{\epsilon}_{22})\end{aligned}\quad (50)$$

and the endochronic constitutive equation eq (15) becomes

$$\begin{Bmatrix} d\hat{\sigma}_{11} \\ d\hat{\sigma}_{22} \end{Bmatrix} = \frac{E_0}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \hat{v}_{11}dt \\ \hat{v}_{22}dt \end{Bmatrix} - \frac{\alpha_1(\beta d\xi)}{1+\beta\xi} \left(\begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \end{Bmatrix} - \frac{E_0 \hat{D}}{3(1-2\nu)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right) \quad (51)$$

where $\hat{D} = \int \hat{v}_{11}dt + \int \hat{v}_{22}dt$

and the intrinsic time is obtained from the general expression, eq (17), by setting

$$\hat{v}_{33} = 0$$

$$(\beta d\xi)^2 = \{K_1(\hat{v}_{11} + \hat{v}_{22})^2 + K_2[(\hat{v}_{11})^2 + (\hat{v}_{22})^2]\}(dt)^2 \quad (52)$$

As can be seen from eqs (48), (51) and (52), the use of endochronic theory in the case of plane strain is much simpler than in plane stress, since the intrinsic time is a function of the rate of deformation only.

The other use of endochronic theory for plane strain case is in [10]. There again, the intrinsic time definition of the form eq (49) is used. By the same line of reasoning as in the plane stress state, we conclude that the use of eq (49) is incorrect for the case of plane strain also.

VII. AXISYMMETRICAL STRESS STATE

The constitutive equation for axisymmetric stress state is sought for applications in problems where the geometry, loading and response have axisymmetry with respect to an axis, say 2-axis. This condition can be visualized by using a cylindrical coordinate system in which the axes r , z and θ are represented by 1, 2 and 3 axes. Due to symmetry with respect to 2-axis, we must have

$$\hat{\sigma}_{13} = \hat{\sigma}_{23} = \hat{\epsilon}_{13} = \hat{\epsilon}_{23} = 0 \quad (53)$$

The rate of deformation components are computed from eq (14) as

$$\begin{aligned} \hat{v}_{11} dt &= [(1+\hat{\epsilon}_{22})d\hat{\epsilon}_{11} - \hat{\epsilon}_{12}d\hat{\epsilon}_{12}]/\Delta \\ \hat{v}_{22} dt &= [(1+\hat{\epsilon}_{11})d\hat{\epsilon}_{22} - \hat{\epsilon}_{12}d\hat{\epsilon}_{12}]/\Delta \\ \hat{v}_{33} dt &= d\hat{\epsilon}_{33}/(1+\hat{\epsilon}_{33}) \\ \hat{v}_{12} dt &= [(2+\hat{\epsilon}_{11}+\hat{\epsilon}_{22})d\hat{\epsilon}_{12} - \hat{\epsilon}_{12}(d\hat{\epsilon}_{11}+d\hat{\epsilon}_{22})]/2\Delta \\ \hat{v}_{13} &= \hat{v}_{23} = 0 \end{aligned} \quad (54)$$

where

$$\Delta = (1+\hat{\epsilon}_{11})(1+\hat{\epsilon}_{22}) - \hat{\epsilon}_{12}^2$$

The endochronic constitutive equation for this axisymmetric case can be obtained by using eqs (54), (15) and (17) as

$$\begin{Bmatrix} d\hat{\sigma}_{11} \\ d\hat{\sigma}_{22} \\ d\hat{\sigma}_{33} \end{Bmatrix} = \frac{E_0}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} dt \\ \hat{v}_{22} dt \\ \hat{v}_{33} dt \end{Bmatrix} - \frac{\alpha_1(\beta d\xi)}{3(1+\beta\xi)} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{33} \end{Bmatrix} \quad (55)$$

$$d\hat{\sigma}_{12} = \frac{E_o}{(1+\nu)} \hat{v}_{12} dt - \frac{\alpha_1 (\beta d\xi)}{1+\beta\xi} \hat{\sigma}_{12} \quad (56)$$

where

$$(\beta d\xi)^2 = \{K_1 (\hat{v}_{11} + \hat{v}_{22} + \hat{v}_{33})^2 + K_2 [(\hat{v}_{11})^2 + (\hat{v}_{22})^2 + (\hat{v}_{33})^2 + 2(\hat{v}_{12})^2]\} (dt)^2 \quad (57)$$

The physical meaning of the singular behavior of the rate of deformation when $\Delta = 0$ can be studied for each individual situation.

VIII. NUMERICAL APPLICATIONS

The various endochronic constitutive equations derived previously relate the corotational true stress to the rate of deformation. Hence these forms of constitutive equations are particularly useful in situations where the rates of deformation or corotational strains are given such as in the displacement-formulation of finite element method. In this section, we assume that the rates of deformation at one instance, say t , are known, the stresses and rates of deformation at time before t are also known and we are looking for the stress increments by using the endochronic constitutive laws.

The numerical integration is much easier for cases where the intrinsic time does not depend on the stress but only on the rate of deformation such as in plane strain and axisymmetric stress states. For these cases, the intrinsic time is completely determined once the rate of deformation is known. By writing the constitutive equations in difference form and using the following notations

$$\begin{aligned}\hat{\sigma}_{ij}(t) &= \hat{\sigma}_{ij}(t-\Delta t) + \Delta \hat{\sigma}_{ij} = \hat{\sigma}_{ij}^0 + \Delta \hat{\sigma}_{ij} \\ \beta \xi(t) &= \beta \xi(t-\Delta t) + \beta \Delta \xi = \beta \xi^0 + \beta \Delta \xi, \text{ etc.} \\ d\hat{\sigma}_{ij} &= \Delta \hat{\sigma}_{ij} ; \quad \beta d\xi = \beta \Delta \xi\end{aligned}\tag{58}$$

the constitutive equations in difference form are obtained as:

For plane strain state,

$$\begin{pmatrix} \Delta \hat{\sigma}_{11} \\ \Delta \hat{\sigma}_{22} \end{pmatrix} = C \left[\frac{E_0}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{pmatrix} \hat{v}_{11} dt \\ \hat{v}_{22} dt \end{pmatrix} - \Delta X \begin{pmatrix} \hat{\sigma}_{11}^0 \\ \hat{\sigma}_{22}^0 \end{pmatrix} - \frac{E_0 \hat{D}}{3(1-2\nu)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \tag{59}$$

where

$$C = (1+\Delta X)^{-1} ; \quad \Delta X = \frac{\alpha_1 \beta \Delta \xi}{1+\beta \xi^0 + \beta \Delta \xi}$$

and

$$\begin{aligned}\hat{D} &= \hat{D}^0 + \hat{v}_{11} dt + \hat{v}_{22} dt \\ \beta \Delta \xi &= \{k_1 (\hat{v}_{11} + \hat{v}_{22})^2 + k_2 [(\hat{v}_{11})^2 + (\hat{v}_{22})^2]\}^{1/2} (\Delta t); \quad \beta \Delta \xi \geq 0\end{aligned}\tag{60}$$

For axisymmetric stress state,

$$\begin{Bmatrix} \Delta \hat{\sigma}_{11} \\ \Delta \hat{\sigma}_{22} \\ \Delta \hat{\sigma}_{33} \end{Bmatrix} = [C_p] \left[\frac{E_0}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} dt \\ \hat{v}_{22} dt \\ \hat{v}_{33} dt \end{Bmatrix} - \frac{\Delta X}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11}^0 \\ \hat{\sigma}_{22}^0 \\ \hat{\sigma}_{33}^0 \end{Bmatrix} \right] \quad (61)$$

$$d\hat{\sigma}_{12} = C \left[\frac{E_0}{1+\nu} \hat{v}_{11} dt - (\Delta X) \hat{\sigma}_{11}^0 \right]$$

where the $[C_p]$ is a 3x3 matrix given by

$$[C_p] = \begin{bmatrix} 1+2\Delta x & -\Delta x & -\Delta x \\ -\Delta x & 1+2\Delta x & -\Delta x \\ -\Delta x & -\Delta x & 1+2\Delta x \end{bmatrix}^{-1} ; \quad \Delta X = \frac{\alpha_1 \beta \Delta \xi}{1 + \beta \xi^0 + \beta \Delta \xi} \quad (62)$$

and

$$\beta \Delta \xi = \sqrt{K_1 (\hat{v}_{11} + \hat{v}_{22} + \hat{v}_{33})^2 + K_2 [(\hat{v}_{11})^2 + (\hat{v}_{22})^2 + (\hat{v}_{33})^2 + 2(\hat{v}_{13})^2]} (\Delta t) ; \quad \beta \Delta \xi \geq 0$$

For these cases, the stress increments $\Delta \hat{\sigma}_{ij}$ can be determined uniquely from eqs (59) to (62) regardless of the magnitude of the increment $\Delta \xi$. However, when the increment is very small, i.e. $\Delta \xi \rightarrow 0$, and we can neglect it in comparison to unity, we have $[C_p] \rightarrow [I]$ where $[I]$ is a unit matrix and $C \rightarrow 1$. Under this condition, we see that the difference constitutive equations are obtainable by simply replacing $\hat{\sigma}_{ij}$ by $\hat{\sigma}_{ij}^0$ in the differential constitutive equation.

The difference constitutive equations for situations where the intrinsic time depends not only on the rate of deformation but also on the stress itself such as in the uniaxial stress state and plane stress state are much more complicated. We study the uniaxial stress state first.

The difference constitutive equations for uniaxial stress state can be obtained, by using eq (24) or eqs (18) to (20), as

$$\Delta \hat{\sigma}_{11} = \frac{E_o \hat{v}_{11} dt - \frac{2(1+\nu)\alpha_1 \beta \Delta \xi}{3(1+\beta \xi^o + \beta \Delta \xi)} \hat{\sigma}_{11}^o}{1 + \frac{2(1+\nu)\alpha_1 \beta \Delta \xi}{3(1+\beta \xi^o + \beta \Delta \xi)}} \quad (63)$$

$$\hat{v} dt = -\hat{v}_{11} dt - \frac{(1+\nu)\alpha_1 \beta \Delta \xi}{3(1+\beta \xi^o + \beta \Delta \xi)} \hat{\sigma}_{11}^o + \frac{E_o \hat{v}_{11} dt - \frac{2(1+\nu)\alpha_1 \beta \Delta \xi}{3(1+\beta \xi^o + \beta \Delta \xi)} \hat{\sigma}_{11}^o}{1 + \frac{2(1+\nu)\alpha_1 \beta \Delta \xi}{3(1+\beta \xi^o + \beta \Delta \xi)}} \quad (64)$$

$$(\beta \Delta \xi)^2 = \{ K_1 (\hat{v}_{11} + 2\hat{v})^2 + K_2 [(\hat{v}_{11})^2 + 2\hat{v}^2] \} (dt)^2 \quad (65)$$

Equation (63) yields a unique stress increment for a rate of deformation provided the value of $(\beta \Delta \xi)$ is known. The governing equation for $(\beta \Delta \xi)$ is obtained by substituting eq (64) into eq (65). The resulting form is a fourth order algebraic equation in terms of $(\beta \Delta \xi)$ [see Appendix for an example]. Since the coefficients of this equation are functions of known quantities, α_1 , E_o , ν , $\beta \xi^o$ and $\hat{\sigma}_{11}^o$, the roots $(\beta \Delta \xi)$'s are readily available. There are four roots for $(\beta \Delta \xi)$'s to this equation. The correct value for $\beta \Delta \xi$ is to be determined by requiring that $\beta \Delta \xi$ must be real and non-negative. Considering the

number of roots for $(\beta\Delta\xi)$, these two conditions may not be sufficient to determine the correct value (e.g. more than one root may be real and positive). Because of the complexity involved in numerically solving this equation and the difficulty in choosing the right root it is not advisable to use eqs (64) and (65) to define $\beta\Delta\xi$.

To circumvent these difficulties, we assume that the increment Δt or $\beta\Delta\xi$ is small such that the following is approximately true.

$$\begin{aligned}\hat{\sigma}_{11} &\doteq \hat{\sigma}_{11}^0 \\ 1 + \beta\xi &\doteq 1 + \beta\xi^0\end{aligned}\tag{66}$$

In other words, the difference step used in integrating the constitutive equation should be sufficiently small such that $\Delta\hat{\sigma}_{11}$ and $\beta\Delta\xi$ are negligible when compared to $\hat{\sigma}_{11}$ and $\beta\xi$. Using this assumption, we can write the uniaxial constitutive equation approximately as

$$\Delta\hat{\sigma}_{11} = E_0 \hat{V}_{11} dt - \frac{2(1+\nu)\alpha_1 \beta\Delta\xi}{3(1+\beta\xi^0)}\tag{67}$$

and the intrinsic time, $\beta\Delta\xi$, is defined by

$$(\beta\Delta\xi)^2 = \{ K_1 (\hat{V}_{11} + 2\hat{V})^2 + K_2 [(\hat{V}_{11})^2 + 2(\hat{V})^2] \} (dt)^2\tag{68}$$

where

$$\hat{V}dt = A + B (\beta\Delta\xi) \quad ; \quad A = -\nu\hat{V}_{11}dt; \quad B = \frac{-(1+\nu)(1-2\nu)\alpha_1\hat{\sigma}_{11}^0}{3E_0(1+\beta\xi^0)}\tag{69}$$

The final algebraic equation for intrinsic time is obtained by substituting eq (69) into eq (68) as

$$a(\beta\Delta\xi)^2 + b(\beta\Delta\xi) + c = 0$$

where

$$a \equiv (4K_1 + 2K_2)BB-1$$

$$b \equiv 2(4K_1 + 2K_2)AB + 4K_1 B\hat{V}_{11}dt$$

$$c \equiv (K_1 + K_2)(\hat{V}_{11}dt)^2 + 4K_1 A\hat{V}_{11}dt + (4K_1 + 2K_2)AA$$

(70)

The situation in plane stress is about the same. The difference equation for the constitutive equation results in a fourth order algebraic equation in terms of the intrinsic time. Again assuming the increments $\beta\Delta\xi$, Δt etc. are so small that the stress and intrinsic time increments are negligible compared to the stress and intrinsic time themselves, i.e. eq (66) is true, we have the following approximate plane stress endochronic constitutive equations:

$$\begin{Bmatrix} d\hat{\sigma}_{11} \\ d\hat{\sigma}_{22} \end{Bmatrix} = \frac{E_0}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \hat{V}_{11}dt \\ \hat{V}_{22}dt \end{Bmatrix} - \frac{\alpha_1(\beta\Delta\xi)}{3(1-\nu)(1+\beta\xi^0)} \begin{bmatrix} 2-\nu & 2\nu-1 \\ 2\nu-1 & 2-\nu \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11}^0 \\ \hat{\sigma}_{22}^0 \end{Bmatrix} \quad (71)$$

$$\hat{V}_{33}dt = A+B(\beta\Delta\xi) \quad ; \quad A \equiv \frac{-\nu}{1-\nu} (\hat{V}_{11} + \hat{V}_{22})dt \quad ; \quad B \equiv \frac{-\alpha_1(1+\nu)(1-2\nu)(\hat{\sigma}_{11}^0 + \hat{\sigma}_{22}^0)}{3E_0(1-\nu)(1+\beta\xi^0)}$$

where the intrinsic time increment is defined by

$$a(\beta\Delta\xi)^2 + b(\beta\Delta\xi) + c = 0$$

and

$$a \equiv BB(K_1 + K_2) - 1$$

(72)

$$b \equiv 2B[A(K_1 + K_2) - K_1(\hat{V}_{11} + \hat{V}_{22})dt]$$

$$c \equiv K_1[(\hat{V}_{11} + \hat{V}_{22})dt]^2 + K_2[(\hat{V}_{11})^2 + (\hat{V}_{22})^2](dt)^2 + 2AK_1(\hat{V}_{11} + \hat{V}_{22})dt + AA(K_1 + K_2)$$

The solution for either eq (70) or eq (72) is given by

$$\beta\Delta\xi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}} \quad ; \beta\Delta\xi \geq 0 \quad (73)$$

In order to uniquely determine the value for $\beta\Delta\xi$, we require that the solution given by eq (73) have one positive and one negative root. This is equivalent to requiring that $ac \leq 0$. This condition further restricts the range of value of K_1 and K_2 . The positive definitiveness property of the intrinsic time requires that K_1 and K_2 should not be both zero, but either one of it can be zero. However, numerical experimentation by using eqs (71) and (72) subjected to a constantly increasing strain i.e. $d\hat{\epsilon}_{11} = d\hat{\epsilon}_{22} = \text{constant}$ show that the solutions indicated by eq (73) are real and opposite in sign for the case $K_1=0, K_2=K$. When the condition $K_1=K; K_2=0$ is used, two real and positive roots are obtained. The correct value for $\beta\Delta\xi$ can not be uniquely determined in this case. This is caused by using the approximate second order algebraic equation. When this happens, either another set of values for K_1 and K_2 should be used or the correct fourth order algebraic equation for $\beta\Delta\xi$ should be utilized. The derivation and final forms for the fourth order equation for plane stress is shown in the Appendix.

IX. EXAMPLES AND DISCUSSIONS

The one dimensional endochronic constitutive equation derived in the previous section suggests that the intrinsic time measure is a function of not only the strain increment but also the stress. Valanis' definition of intrinsic time, eq (25), is incorrect for uniaxial stress, though it is acceptable for simple shear stress as shown by eq (42). Since the constitutive equations for both cases have the same form, as can be seen from eqs (25) and (42), it is expected that the equation used by Valanis [1] should give reasonably good results for the case of uniaxial stress for the axial component. To obtain some confidence in the endochronic plasticity theory, we present the comparison of one dimensional results obtained by using eq (25) and by using the elastoplastic theory. Figure 3 shows the axial response of a copper specimen in a uniaxial stress test where the solid line is the elastoplastic response and the chained line is the endochronic plastic response. The responses are computed by integrating corresponding constitutive equations from the assumed engineering strain history. The rate of deformation is used as the independent variable in both constitutive equations. This is true in all the following numerical examples unless otherwise stated. The strain history used in Fig. 3 is obtained from [11]. The comparison between these two theories seems reasonably good for relatively large strains. Figures 4 and 5 show the enlarged portion of Fig. 3 in regions of various strains. From these, we can say that the gross behavior of the endochronic plastic material is very close to that of the elastoplastic material except at small strains. It is noted in these figures that the hardening of a material after an unloading process is larger for the endochronic material i.e. the endochronic constitutive curve lies above that of the elastoplastic theory. It is also noted that no hysteresis loop is observed during the cycle of loading-unloading-reloading. The lack of hysteresis for this strain history is studied in [7]. It is shown analytically in [7] and numerically in [2] that the hysteresis loop does exist if an additional term is introduced into the first of eq (5). In this case, due to the introduction of additional

material parameters, these can not be uniquely determined from the simple tension test even for the approximate endochronic theory. Figure 6 shows the responses of the previous example as shown in Fig 5 and the responses obtained by using a two term endochronic theory in the form as used in [2]. Clearly, the over hardening effect (compared to elastoplastic theory) and the lack of hysteresis loop (compared to experimental results) can be improved by using more terms in the general relaxation function as given by eq (5).

Figures 7 to 10 show the responses of endochronic and elastoplastic theories under cyclic strain history. It is found that no hysteresis loop is formed in the case of elastoplasticity if the strain never exceeds the initial yield strain while hysteresis loop is always formed in the case of endochronic plasticity as shown in Fig 7. The behaviors under large strains for these two theories compare very well qualitatively as shown in Figs 9 and 10.

The above results are obtained by using the approximate equations developed by Valanis as given by eq (25). Though this approximation yields reasonably good results in the uniaxial stress state, it does not yield any information on the behavior in the transverse direction. The elastoplastic theory, on the other hand, states that the transverse strain can be obtained either from elastic Poisson's effect or from plastic incompressibility depending on whether the deformation is elastic or plastic. It is found that physically reasonable deformations in the transverse direction can be predicted by the endochronic theory if its correct forms, eqs (18) - (20), are used. However, the endochronic material parameters can not be determined uniquely from the simple tension test. Parameter studies must be performed to determine the necessary material properties.

Consider a rod made of aluminum whose material properties are: initial

Young's modulus $E_o = 6.895 \times 10^{10}$ Pa (10×10^6 psi), plastic modulus $E_p = 1.379 \times 10^9$ Pa (2×10^5 psi) and initial yield stress $\sigma_o = 1.576 \times 10^8$ Pa (2.286×10^4 psi). For the approximate endochronic theory in the form given by Valanis, the endochronic parameters can be determined uniquely from E_o , E_p and σ_o . However, for the correct form of endochronic theory, the Poisson's ratio is needed.

Figure 11 shows the response of this endochronic rod with a Poisson's ratio of 0.3 subjected to constantly increasing strain with vanishing transverse stress, i.e. under simple tension test condition. Figure 12 shows the response when $\nu = 1/3$. These results are obtained by substituting $\theta = \nu$, $\theta = (\nu + 0.5)/2$ and $\theta = 0.5$, respectively, into eqs (30) and (31). The original bi-linear elasto-plastic curves are also shown in these figures for comparison. The asymptotic plastic moduli and initial yield stresses obtained by using various values of ν and θ , and the errors compared to the initially assumed values are summarized in Table 1. As can be seen from these results it is possible to find the correct endochronic material parameters by comparing analytical predictions with experimental data. Furthermore, the use of $\theta = 1/2$ seems to give reasonably good result.

The transverse strain in a uniaxial stress situation can be examined by using the correct endochronic constitutive equations. The negative ratio of the transverse and the axial rate of deformation, $-\nu^r$, is obtained by first dividing eq (20) by $\hat{V}_{11} dt$ and then combining the resulting equation with eq (18). It is given by

$$\nu^r = -\hat{V}/\hat{V}_{11} = 1/2 + (\nu - 1/2)E/E_o \quad ; \quad E = d\hat{\sigma}_{11}/(\hat{V}_{11} dt) \quad (74)$$

where E can be regarded as the instantaneous Young's modulus. The value of E depends on the stress state and the strain history of the material. For a material that is subjected to monotonically increasing loading, the value of E is bounded by E_o and E_p and E must asymptotically go to E_o when the strain

is very small and E_p when the strain is very large. Hence, from eq (74), the "effective Poisson's ratio", v^r , must lie between v and $[1/2 + (v-1/2)E_p/E_o]$ for monotonically increasing loading situation. In other words, the effective Poisson's ratio starts from the elastic Poisson's ratio and asymptotically increases to a fixed value of $[1/2 + (v-1/2)E_p/E_o]$. It is interesting that this value is not a constant for all materials. Its value depends on the ratio of plastic and elastic modulus and the elastic Poisson's ratio, and it is one-half only if the material does not have work hardening, i.e. $E_p = 0$.

It should be noted that for linearly elastic material, the definition of Poisson's ratio is very clear since the ratio of transverse and axial strains is the same as that of the strain rates (small strains are usually assumed for linearly elastic analysis). However, this is not true in the current study. We have already defined an effective Poisson's ratio v^r previously. To study the transverse effect in detail for a uniaxial stress condition we further define an apparent Poisson's ratio as the ratio of the total transverse and axial natural strain with a negative sign, i.e.

$$v^a = -(\int \tilde{v} dt) / (\int \tilde{v}_{11} dt) \quad (75)$$

It is only possible to integrate eq (75) numerically for an endochronic material. Figure 13 shows the axial stress-strain curve of an endochronic bar ($E_o = 6.895 \times 10^{10}$ Pa, $E_p = 1.379 \times 10^9$ Pa, $\sigma_o = 1.576 \times 10^8$ Pa, $v = 0.3$ and $\theta = 0.5$) subjected to monotonically increasing axial strain. Figure 14 shows the effective Poisson's ratio, v^r as a function of the axial strain, and Fig 15 shows the apparent Poisson's ratio. The theoretical lower and upper bounds for v^r as given by eq (74) in this case are 0.3 and 0.496, respectively. As can be seen from these figures, both the effective and apparent Poisson's ratios are within these bounds. They start from the elastic Poisson's ratio and increase monotonically with different rates. The rate of the effective

Poisson's ratio approaching a constant value is greater than that of the apparent one. All this qualitatively agrees with experimentally observed phenomena.

The effective and apparent Poisson's ratios for a one-dimensional bi-linearly elastoplastic material can be easily shown to be the same, and they are given by

$$\begin{aligned} \nu^r = \nu^a &= 1/2 + (\nu - 1/2) [E_p/E_0 + (1 - E_p/E_0)\epsilon_0/\epsilon]; & \text{for } \epsilon \geq \epsilon_0 \\ \nu^r = \nu^a &= \nu & \text{for } \nu \leq \epsilon_0 \end{aligned} \quad (76)$$

for monotonically increasing loading where ϵ is the axial natural strain and ϵ_0 is the initial yield natural strain. It is observed, from eqs (74) and (76), that the effective Poisson's ratio for both the endochronic plasticity and the bi-linear elastoplasticity has the same value whether the strain is large or small. Hence it is reasonable to say that the endochronic plasticity is equivalent to nonlinear elastoplasticity for monotonically increasing loading cases.

The bounds on the effective Poisson's ratio for other loading histories will be different from those for the monotonically increasing loading since the value of the instantaneous Young's modulus E as given by eq (74) will not necessarily be bounded by E_0 and E_p . Figures 16, 17 and 18 show the axial stress-strain curve, the effective and the apparent Poisson's ratios for the same material which has been stressed to a certain value and then unloaded to zero stress and reloaded again. It should be noted that no hysteresis loop is expected for this loading history since the stress never changes sign as indicated in Fig 16. It is interesting to note, from Fig. 17, that the effective Poisson's ratio has values very close to 0.1. By using eq (74), we find that the instantaneous Young's modulus corresponding to $\nu^r = 0.1$ is $E = 2E_0$. In other words, the material has a sound speed that is almost twice the initial sound

speed somewhere during the unloading-reloading process. This is confirmed by Fig 16 where the unloading slope is obviously greater than the initial slope in the axial stress-strain curve. In Fig 18, we note a kink that is formed in the apparent Poisson's ratio and an area is formed by this kink and the curve that would be formed for monotonically increasing load. The same type of behavior is observed for a bi-linear elastoplastic material. However, for elastoplastic material, the value of apparent Poisson's ratio is bounded by one-half and no area is formed. This difference is caused by the different unloading-reloading process in these two plasticity theories. Figures 19 to 21 show the responses for the same problem where the axial stress has changed sign during the unloading process. An hysteresis loop is formed as expected in this case as shown in Fig 19. Due to the formation of the hysteresis loop in the axial stress-strain curve, other closed loops are also observed in the Poisson's ratio-strain curves as shown in Figs 20 and 21.

For cyclic loadings, the use of apparent Poisson's ratio to study the transverse effect is not very effective, since artificial singularities may be generated when the axial strains vanish. Hence the transverse strain-axial strain curves are studied instead of the apparent Poisson's ratio. Figures 22 to 24 show the results obtained when the material is subjected to one complete cycle of straining. The residual axial stress and permanent transverse natural strain after completion of a cycle can be measured from Figs 22 and 24 and they are $2.920 \times 10^8 \text{ Pa}$ ($4.235 \times 10^4 \text{ psi}$) and 8.77×10^{-4} , respectively. The stress and strain for the corresponding bi-linearly elastoplastic material can be computed to be $2.806 \times 10^8 \text{ Pa}$ ($4.070 \times 10^4 \text{ psi}$) and 8.14×10^{-4} . The relative error between these plasticity theories is 3.4% for the residual stress and 7.5% for the permanent transverse strain. These discrepancies are caused by the difference in the theories (i.e. all strains are irreversible in endochronic plasticity while portions of the strains are reversible for elastoplasticity), and the use of

approximate endochronic material parameters. It is comfortable to note that the permanent strain is larger in the endochronic theory since the axial stress-strain curve for the corresponding bi-linear elastoplasticity theory is an upper bound of that for the endochronic theory. Figures 25 to 27 and Figs 28 to 30 represent the response of the endochronic material under many cycles of straining where for the former the strain is cycled between zero and certain value without the strain ever being negative while for the latter the strain is cycled between positive and negative values. It is noted that for all these cases the effective Poisson's ratio is bounded between 0.5 and 0.1 for the endochronic plasticity while the bounds for an elastoplastic material are 0.5 and 0.3. Also noted is that the permanent transverse strains grow under cyclic loadings as indicated in Figs 27 and 30.

All the above results are one-dimensional and are obtained by using either eq (24) or eq (25). These results indicate that the endochronic plasticity yields reasonably good results compared to the corresponding bi-linear elastoplastic theory for monotonically increasing loading and cyclic loading particularly when the strains are not small. For certain strains (i.e. strains that lie around the corner of the corresponding bi-linear stress-strain curve) and unloading, the results of endochronic plasticity may be quite different from those of the elastoplastic theory. This difference is caused mostly by deviation of the endochronic stress-strain curve from that of the corresponding bi-linear elastoplasticity and the high "elastic" modulus during unloading process of the endochronic theory. The difference between these theories for small strains is caused by the absence of linear elastic response of the endochronic theory.

In the following, we study the endochronic plasticity for two-dimensional problems. It should be noted that, when the rate of deformation (i.e. rate of natural strain) and true stress are used as the independent and dependent

variables in the one-dimensional constitutive equation, no difference is observed in the final stress-strain curve. In other words, the material has the same behavior under tension and compression when the rate of deformation and true stress are used to measure the behaviors. This can not be expected in the multi-dimensional problems due to the coupling effect of the intrinsic time.

Consider a small rectangular plate whose sides are parallel to the 1 and 2 axis, respectively. Let the thickness of the plate be so small such that plane stress condition exists across the thickness of the plate, and let the sides of the plate be subjected to the deformation as shown in Fig. 31. The endochronic material parameters used for this plate are the same as those used for the previous one-dimensional problem, i.e. $\nu=0.3$, $\theta=0.5$ etc., and eq (48) is used to compute the plane stress endochronic response.

Figures 32 and 33 show the stress-strain curves in the 1 and 2 directions,* respectively when the loading is $d\hat{\epsilon}_{11}=d\hat{\epsilon}_{22}=d\epsilon$ where $d\epsilon$ is a positive constant. Figures 34 and 35 are the results when $d\hat{\epsilon}_{11}=d\hat{\epsilon}_{22}=-d\epsilon$. As expected all these four sets of stress-strain curves are identical except in signs due to the symmetry of loading. Figures 36 and 37 show the response obtained when $d\hat{\epsilon}_{11}=-d\hat{\epsilon}_{22}=d\epsilon$. It should be noted that great differences are observed between the responses in 1 and 2 directions once the strains pass the corner of the stress-strain curve. As can be seen also that the material is stronger in the compression than in the tension as indicated by the response in 2 and 1 direction. This is caused by the coupling of the intrinsic time for multi-dimensional problems and the use of rate of deformation as strain measure, since the one-dimensional results studied previously and the multi-dimensional response shown in Figs 32 to 35 all indicate the same behavior in tension and compression. The asymptotic plastic modulus E_p and the initial yield stress σ_0 can be measured from Figs 36 and 37, respectively. They are: $E_p=4.788 \times 10^8 \text{ Pa} (6.945 \times 10^4 \text{ psi})$, $\sigma_0=8.942 \times 10^7 \text{ Pa} (1.297 \times 10^4 \text{ psi})$ for tension in 1 direction; and $E_p=1.293 \times 10^9 \text{ Pa} (1.875 \times 10^5 \text{ psi})$, $\sigma_0=8.315 \times 10^7 \text{ Pa} (1.206 \times 10^4 \text{ psi})$ for compression in 2 direction.

*The directions of $\hat{1}$, $\hat{2}$ and $\hat{3}$ axes are the same as those of 1, 2 and 3 axes for the case of a plate subjected to in-plane deformation.

It should be noted that this loading history is equivalent to simple shear state along the line that is 45 degrees to the 1 or 2 axis at least for small strains. Hence the shear response can be computed simply as one-half of the sum of the responses in 1 and 2 direction. By doing so, we find the yield stress in shear is $\sigma_{12}^0 \doteq 0.55 \sigma_0$. The yield stress in shear for conventional plasticity depends on the yield criterion used: $\sigma_{12}^0 \doteq 0.58 \sigma_0$ for Von Mises's yield criterion and $\sigma_{12}^0 = 0.50 \sigma_0$ for Tresca yield criterion. Hence, we can say that the endochronic plasticity for the plane stress gives reasonably good results.

It is expected that the plane stress constitutive equation will yield the uniaxial results if the correct strain histories in 1 and 2 directions were known. From our previous experience, we know that the apparent Poisson's ratio increases from elastic Poisson's ratio of 0.3 to 0.5 for monotonically increasing axial strain history. Figures 38 and 39 are the responses obtained when we use $d\hat{\epsilon}_{11} = d\epsilon$ and $d\hat{\epsilon}_{22} = -0.5d\epsilon$; and Figs 40 and 41 are the responses when $d\hat{\epsilon}_{11} = d\epsilon$ and $d\hat{\epsilon}_{22} = -v d\epsilon$. The maximum transverse stress $\hat{\sigma}_{22}$ is about 7.14% of the maximum axial stress $\hat{\sigma}_{11}$ for the first case ($d\hat{\epsilon}_{22}/d\hat{\epsilon}_{11} = -0.5$) and is about 15.65% for the second case ($d\hat{\epsilon}_{22}/d\hat{\epsilon}_{11} = -v$). The uniaxial stress state is obtained when this stress ratio is zero. Figures 42 and 43 are the responses obtained when $d\hat{\epsilon}_{11} = -d\epsilon$ and $d\hat{\epsilon}_{22} = +0.5d\epsilon$, and Figs 44 and 45 are the results when $d\hat{\epsilon}_{11} = -d\epsilon$ and $d\hat{\epsilon}_{22} = +v d\epsilon$. A very interesting phenomenon is observed when the strain histories are reversed in signs. That is when the input (strain history) has changed signs, the change in output (stress-strain curve) does not simply change in sign. This effect explains the experimentally observed different behaviors in tension and compression for materials that are subjected to many cycles of straining as reported by Wardsroth [11]. Valanis [2] claims that small difference is observable in tension and compression when he uses the endochronic theory for cyclic loadings. However, considering the dimensions of the "one-dimensional" specimen, the author feels that the different behavior is generated by the multi-dimensional effect and due to the fact that the same amount of engineering strain does not mean the same amount of rate

of deformation in tension and compression.

The results for plane stress constitutive equations under various reloadings and cyclic straining are qualitatively similar to those obtained for one-dimensional constitutive equation. Same conclusions are true for the plane strain cases. Hence they are not presented here.

The above results are all for static responses only. The usage of endochronic plasticity for dynamic problems are studied in detail for a one-dimensional spring-mass system in [6]. For multi-dimensional problems, we study the response of circular plates subjected to explosive loadings. The plates are made of either aluminum or steel with the following material properties: for aluminum--elastic modulus $E_0 = 7.165 \times 10^{10}$ Pa (1.039×10^7 psi), initial yield stress $\sigma_0 = 2.923 \times 10^8$ Pa (4.240×10^4 psi), plastic modulus $E_p = 3.754 \times 10^8$ Pa (5.444×10^4 psi), ultimate stress $\sigma_u = 3.096 \times 10^8$ Pa (4.490×10^4 psi), density $\rho_0 = 2.680 \times 10^3$ Kg/m³ (5.201 slug/ft³), and elastic Poisson's ratio $\nu = 0.3$; and for steel-- $E_0 = 1.960 \times 10^{11}$ Pa (2.842×10^7 psi), $\sigma_0 = 5.585 \times 10^8$ Pa (8.100×10^4 psi), $E_p = 3.618 \times 10^9$ Pa (5.248×10^5 psi), $\sigma_u = 6.205 \times 10^8$ Pa (9.000×10^4 psi), $\rho_0 = 7.789 \times 10^3$ Kg/m³ (15.112 slug/ft³) and $\nu = 0.3$. All the plates have a diameter of 1.524×10^{-1} m (6 in) and the center portions with a diameter of 5.080×10^{-2} m (2 in) are covered with explosive sheets that are detonated instantaneously. The plates are clamped at the edges between heavy steel rings by means of twelve bolts tightened to a specified value. The details of the experimental set-up are available in [12]. The dynamic responses of the plate are computed using the corotational finite element method [7,13]. The material properties are modeled using either the tri-linear elastoplastic theory or the endochronic theory. The endochronic material parameters are computed from its corresponding bi-linear constitutive curve that has a constant slope of either E_0 or E_p and the value of $\theta = 0.5$ in eqs (30) and (31). Since the one-dimensional elastoplastic and endochronic

stress-strain curves are not exactly the same, the results obtained are expected to be different. Figure 46 is the displacement history at the center of an aluminum plate with a thickness of 1.588×10^{-3} m (1/16 in) subjected to a total impulse of 1.67 N-sec. Figure 47 is the central displacement of an aluminum plate with a thickness of 3.175×10^{-3} m (1/8 in) and subjected to a total impulse of 1.96 N-sec and Fig 48 is the result for a steel plate of 1.588×10^{-3} m (1/16 in) subjected to a total impulse of 2.20 N-sec. The solid and dashed lines in these figures represent the tri-linear elastoplastic and endochronic responses, respectively. It is interesting to note that the magnitudes and arrival times of the peak central displacement predicted by these theories are very close, though the residual vibrations are noticeably larger in the elastoplastic results. The experimental and a finite difference result as shown in [12] also show the existence of residual vibrations with amplitudes closer to that predicated by elastoplastic analysis. This difference in residual vibration should not be used to judge against the use of endochronic theory, since slippage of the plate at its edge is always observed in [12]. From these results, however, we can say that the use of endochronic theory gives comparable results at least for the peak displacement, hence it can be used in safety analysis where only peak responses are of paramount importance.

The impulse in the plate problems is not treated as uniform initial velocity over the central portion of the plate in this study. A better distribution of velocity is obtained by treating the impulse as uniformly distributed pressure applied over a very short time and by using the work-equivalence principle consistent with the finite element formulation. This velocity distribution greatly improves the results compared to those obtained by using uniform velocity. It is observed in [6] that the use of endochronic theory resulted in saving of computation time. This is the consequence of using the incorrect endochronic constitutive equations in the forms as proposed by

[2] and [5]. It is found, in this study, that the endochronic theory enjoys a better numerical efficiency only when the intrinsic time is a function of known quantities such as in the case of plane strain, axi-symmetric stress and general stress states. In other words, the assumption of vanishing stress in some direction results in the dependence of intrinsic time on the unknown quantities such as stress components in other directions. Hence, for uniaxial and plane stress states the endochronic constitutive equations can only be solved either by an iterative technique or by using very small time steps as in the case of elastoplastic analysis. It is found that a much smaller time step is needed for the endochronic theory than for the elastoplastic theory in the case of the aluminum plate with thickness of 3.175×10^{-3} m (1/8 in) i.e. Fig 47. In this case, more total computation time is used for the endochronic theory even though its efficiency is still better during each time step.

X. CONCLUSION

Extensive numerical studies have been performed for both one- and multi-dimensional static and dynamic problems using endochronic and elastoplastic constitutive laws. Most conclusions observed in [6] are found to be correct even though some incorrect definitions for intrinsic time are used there. Those conclusions and the new ones observed in this study are summarized as follows:

- (1) Permanent plastic strains are larger and occur at lower stress levels in the endochronic model than in the elastoplastic model when they are connected by eqs (30) and (31).
- (2) Hysteresis loops occur for an endochronic model when the system is subjected to initial displacement or velocity conditions.
- (3) An endochronic model creeps when subjected to an external force. However, hysteresis loops start forming immediately after removal of the external force.
- (4) Better agreement with the elastoplastic solution can be obtained for a particular range of interest by adjusting values of K_1 and K_2 rather than using eqs (30) and (31), which give the best fit to the bi-linear elastoplastic material only as $\epsilon \rightarrow 0$ and $\epsilon \rightarrow \infty$ and which fits worst around the yield point.
- (5) Physically reasonable qualitative solutions are obtained for the endochronic model for all the problems studied.
- (6) An endochronic model may be unstable in the sense it may "creep" with respect to the intrinsic time. Hence, it may physically creep when subjected to suddenly applied constant loading in dynamic problems.

- (7) The computational efficiency of an endochronic model compared to that of the elastoplastic model depends on the problem. For problems where no condition on the stress is imposed, the endochronic intrinsic time is a function of known quantities, hence the endochronic constitutive equation can be solved in exact form. In these cases much larger time or load step can be used compared to the elastoplastic theory. When this is not true such as in the cases of uniaxial or plane stress state, the reverse may be true.
- (8) The "creep" phenomenon of an endochronic model can be used as a safety factor in design problems.
- (9) The endochronic models yield compatible results at least in peak displacements compared to the results obtained by the corresponding elastoplastic analysis. Hence this theory can be used in problems where peak deformation is of concern such as transient safety analysis.
- (10) The endochronic model has a damping effect. Hence residual vibrations are damped out very fast in this model while no damping is expected for an elastoplastic model. This character can be used to obtain static solution from a dynamic computer program.

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APPENDIX

DETERMINATION OF $\beta\Delta\xi$ FOR PLANE STRESS CONDITION

The endochronic constitutive equations for plane stress in difference form are given by

$$\begin{Bmatrix} \Delta\hat{\sigma}_{11} \\ \Delta\hat{\sigma}_{22} \end{Bmatrix} = \frac{E_o}{(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} dt \\ \hat{v}_{22} dt \end{Bmatrix} - \frac{\beta\Delta\xi}{3(1+\beta\xi^o + \beta\Delta\xi)} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (A-1)$$

$$+ \frac{\nu}{(1-\nu)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11}^o + \Delta\hat{\sigma}_{11} \\ \hat{\sigma}_{22}^o + \Delta\hat{\sigma}_{22} \end{Bmatrix}$$

$$(\beta\Delta\xi)^2 = \{K_1 (\hat{v}_{11} + \hat{v}_{22} + \hat{v}_{33})^2 + K_2 [(\hat{v}_{11})^2 + (\hat{v}_{22})^2 + (\hat{v}_{33})^2]\} (dt)^2 \quad (A-2)$$

and

$$\hat{v}_{33} dt = - \left[\frac{\nu}{(1-\nu)} (\hat{v}_{11} + \hat{v}_{22}) dt + \frac{(1+\nu)(1-2\nu)\alpha_1(\beta\Delta\xi)}{3E_o(1-\nu)(1+\beta\xi^o + \beta\Delta\xi)} (\hat{\sigma}_{11}^o + \Delta\hat{\sigma}_{11} + \hat{\sigma}_{22}^o + \Delta\hat{\sigma}_{22}) \right] \quad (A-3)$$

where $\beta\xi^o$ and $\hat{\sigma}_{ij}^o$ are the intrinsic time and stress at previous increment, which are known.

The stress increment $\Delta\hat{\sigma}_{11}$ and $\Delta\hat{\sigma}_{12}$ can be found in terms of known quantities by rearranging terms in eq (A-1) as

$$\begin{Bmatrix} \Delta\hat{\sigma}_{11} \\ \Delta\hat{\sigma}_{22} \end{Bmatrix} = [c] \left[\frac{E_o}{(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \hat{v}_{11} dt \\ \hat{v}_{22} dt \end{Bmatrix} - \frac{\beta\Delta\xi}{3(1+\beta\xi^o + \beta\Delta\xi)} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \right. \\ \left. + \frac{\nu}{(1-\nu)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}_{11}^o \\ \hat{\sigma}_{22}^o \end{Bmatrix} \right] \quad (A-4)$$

where [c] is a 2x2 matrix and is given by

$$[c] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\beta \Delta \xi}{3(1+\beta \xi^0 + \beta \Delta \xi)} \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \frac{\nu}{(1-\nu)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \quad (A-5)$$

Substituting eqs (A-4) and (A-5) into eq (A-3), we find the rate of deformation in 3-axis direction

$$\hat{v}_{33} dt = - [A+B \frac{X}{1+CX}] \quad (A-6)$$

where

$$\begin{aligned} A &\equiv \frac{\nu}{(1-\nu)} (\hat{v}_{11} + \hat{v}_{22}) dt \\ B &\equiv \frac{(1+\nu)(1-2\nu)}{3E_0(1-\nu)} \left[\frac{E_0}{(1-\nu)} (\hat{v}_{11} + \hat{v}_{22}) dt + \hat{\sigma}_{11}^0 + \hat{\sigma}_{22}^0 \right] \\ C &\equiv \frac{(1+\nu)}{(1-\nu)} \end{aligned} \quad (A-7)$$

$$X \equiv \frac{\alpha_1 \beta \Delta \xi}{1+\beta \xi^0 + \beta \Delta \xi} ; \quad (\beta \Delta \xi) = \frac{(1+\beta \xi^0)X}{\alpha_1 - X}$$

The equation governing the intrinsic time ($\beta \Delta \xi$) can be obtained by substituting eqs (A-5) and (A-6) into eq (A-2). Omitting the lengthy algebra, we write the governing equation as

$$(\beta \Delta \xi)^2 = (K_1 E + K_2 H) + (K_1 F + K_2 I) \frac{X}{1+CX} + (K_1 + K_2) G \frac{X^2}{(1+CX)^2} \quad (A-8)$$

where

$$\begin{aligned} E &\equiv [(\hat{v}_{11} + \hat{v}_{22}) dt]^2 - 2[(\hat{v}_{11} + \hat{v}_{22}) dt] A + A^2 \\ F &\equiv 2B[A - (\hat{v}_{11} + \hat{v}_{22}) dt] \\ G &\equiv B^2 \\ H &\equiv (\hat{v}_{11} dt)^2 + (\hat{v}_{22} dt)^2 + A^2 \\ I &\equiv 2AB \end{aligned} \quad (A-9)$$

a cubic equation. Let a cubic equation be defined as

$$\omega^3 + \hat{p}\omega^2 + \hat{q}\omega + \hat{r} = 0 \quad (\text{A-17})$$

This equation can be reduced to the form

$$\lambda^3 + \hat{a}\lambda + \hat{b} = 0 \quad (\text{A-18})$$

where

$$\begin{aligned} \omega &= \lambda - \hat{p}/3 \\ \hat{a} &= \frac{1}{3}(3\hat{q} - \hat{p}^2) \\ \hat{b} &= \frac{1}{27}(2\hat{p}^3 - 9\hat{p}\hat{q} + 27\hat{r}) \end{aligned} \quad (\text{A-19})$$

Let

$$A = \left(-\frac{\hat{b}}{2} + \sqrt{\frac{\hat{b}^2}{4} + \frac{\hat{a}^3}{27}} \right)^{1/3}, \quad B = \left(-\frac{\hat{b}}{2} - \sqrt{\frac{\hat{b}^2}{4} + \frac{\hat{a}^3}{27}} \right)^{1/3} \quad (\text{A-20})$$

then the solution for eq (A-18) is

$$\lambda = A+B, -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{3} \quad (\text{A-21})$$

The solution for ω is then given by the first of eq (A-19).

The fourth order algebraic equation in terms of X is obtained, by substituting the last of eq (A-6) into eq (A-7) and rearranging terms, as

$$\bar{p}X^4 + \bar{q}X^3 + \bar{r}X^2 + \bar{s}X + \bar{t} = 0 \quad (A-10)$$

where

$$\begin{aligned} \bar{p} &\equiv aC^2 + bC + f - gC^2 \\ \bar{q} &\equiv 2aC(1+\alpha_1C) + b(1-2\alpha_1C) - 2f\alpha_1 - 2gC \\ \bar{r} &\equiv a(1-2\alpha_1C)^2 + b\alpha_1(\alpha_1C-2) + f\alpha_1^2 - g \\ \bar{s} &\equiv 2a\alpha_1(\alpha_1C-1) + b\alpha_1^2 \\ \bar{t} &\equiv a\alpha_1 \\ a &\equiv K_1E + K_2H \\ b &\equiv K_1F + K_2I \\ f &\equiv (K_1+K_2)G \\ g &\equiv (1+\beta\xi^0)^2 \end{aligned} \quad (A-11)$$

Equation (A-10) can be also written as

$$X^4 + pX^3 + qX^2 + rX + s = 0 \quad (A-12)$$

where $p = \bar{q}/\bar{p}$, $q = \bar{r}/\bar{p}$, $r = \bar{s}/\bar{p}$; $s = \bar{t}/\bar{p}$. This equation can be reduced to the form

$$y^4 + \bar{a}y^2 + \bar{b}y + \bar{c} = 0 \quad (A-13)$$

by substituting for y the value

$$X = y - p/4 \quad (A-14)$$

To solve eq (A-12), first solve the cubic equation

$$z^3 + \frac{\bar{a}}{2} z^2 + \left(\frac{\bar{a}^2 - 4\bar{c}}{16}\right) z - \frac{\bar{b}^2}{64} = 0 \quad (A-15)$$

The solution of this equation is indicated later.

If the roots of eq (A-15) are 1, m and n, then the roots of eq (A-13) are:

$$\begin{aligned} y &= -\sqrt{1-\sqrt{m}-\sqrt{n}}, -\sqrt{1+\sqrt{m}+\sqrt{n}}, \sqrt{1-\sqrt{m}+\sqrt{n}}, \sqrt{1+\sqrt{m}-\sqrt{n}}; \text{ if } \bar{b} > 0 \\ \text{and } y &= \sqrt{1+\sqrt{m}+\sqrt{n}}, \sqrt{1-\sqrt{m}-\sqrt{n}}, -\sqrt{1+\sqrt{m}-\sqrt{n}}, \sqrt{1-\sqrt{m}+\sqrt{n}}, \text{ if } \bar{b} < 0 \end{aligned} \quad (A-16)$$

The solution for X is then given by eq (A-14).

The solution of a cubic equation is needed for solving a fourth order equation. In the case of $\bar{p} = 0$, the governing equation (A-9) itself is also

Table 1

Summary of Errors Compared to the Associated Bi-linear Stress-Strain Curve
by Using Various Endochronic Material Parameters

ν	θ	α_1	K_1	K_2	σ'_o, P_a	E'_p, P_a	$(E'_p - E_p)/E_p, \%$	$(\sigma'_o - \sigma_o)/\sigma_o, \%$
0.3	0.3	56.539	44.171	58.894	1.462×10^8	1.290×10^9	-6.47	-7.22
0.3	0.4	56.539	42.535	56.713	1.486×10^8	1.270×10^9	-7.92	-5.70
0.3	0.5	56.539	38.281	51.042	1.566×10^8	1.280×10^9	-7.20	-0.62
0.3	0.3	55.125	57.422	57.422	1.494×10^8	1.370×10^9	-0.67	-5.19
0.3	0.416	55.125	55.682	55.682	1.526×10^8	1.320×10^9	-4.29	-3.16
0.3	0.5	55.125	51.042	51.042	1.606×10^8	1.290×10^9	-6.47	+1.92

ν = Poisson ratio;

$$E_o = 6.895 \times 10^{10} P_a ; E_p = 1.379 \times 10^9 P_a ; \sigma_o = 1.576 \times 10^8 P_a$$

E_p and σ_o are the material parameters for the associated bi-linear stress-strain curve.

E'_p and σ'_o are the material parameters measured from the endochronic stress-strain curve

Figure 1

Shearing state of a rectangular block

Figure 2

Endochronic constitutive curve for small strain in
simple shear state

Figure 3

Elastoplastic (solid line) and endochronic (chain-dashed line) uniaxial stress-strain curves for a copper specimen subjected to certain strain history. The strains are measured in percentage.

Figure 4

Elastoplastic (solid line) and endochronic (chain-dashed line) uniaxial stress-strain curves for a copper specimen subjected to a certain strain history. The strains are measured in percentage.

Figure 5

Elastoplastic (solid line) and endochronic (chain-dashed line) uniaxial stress-strain curves for a copper specimen subjected to certain strain history. The strains are measured in percentage.

Figure 6

Elastoplastic and endochronic uniaxial stress-strain curves for a copper specimen subjected to certain strain history. The endochronic curves are obtained by having the "relaxation function" $G(z)$ in eq. (5) the following form:

$$G(Z) = E_1 + E_2 e^{-\alpha Z}$$

The maximum and minimum value of E_2/E_1 consistent with the formulation in Ref. (2) are ∞ and 78.88 for this material, respectively.

ELASTOPLASTIC

ENDOCHRONIC-REF(2)

ONE TERM $E_2/E_1 = \infty$

TWO TERMS

$E_2/E_1 = 153.60$

$E_2/E_1 = 78.88$

Figure 7

Elastoplastic (solid line) and endochronic (chain-dashed line) uniaxial stress-strain curves for a copper specimen subjected to certain cyclic straining. The strains are measured in percentage. The limits of the cyclic strain are -0.025% and $+0.025\%$, respectively. These limits are smaller than the initial yield strain of the elastoplastic model, which is 0.031% .

Figure 8

Elastoplastic (solid line) and endochronic (chain-dashed line) uniaxial stress strain curves for a copper specimen subjected to certain cyclic straining. The strains are measured in percentage. The limits of the cyclic strain are -0.05% and $+0.05\%$, respectively. These limits are larger than the initial yield strain of the elastoplastic model, which is 0.031% .

Figure 9

Elastoplastic (solid line) and endochronic (chain-dashed line) uniaxial stress-strain curves for a copper specimen subjected to certain cyclic straining. The strains are measured in percentage. The limits of the cyclic strain are -0.6% and $+0.6\%$, respectively. These limits are greater than the initial yield strain of the elastoplastic model, which is 0.031% .

Figure 10

Elastoplastic (solid line) and endochronic (chain-dashed line) uniaxial stress-strain curves for a copper specimen subjected to 10 cycles of straining. The strains are measured in percentage. The limits of the cyclic strain are -0.6% and $+0.6\%$, respectively. These limits are greater than the initial yield strain of the elastoplastic model, which is 0.031% .

Figure 11

Elastoplastic (solid bi-linear line) and endochronic uniaxial stress-strain curves for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to monotonically increasing strain. The endochronic curves are obtained by using eq (24) and various values for θ in eq (31): $\theta=0.3$ —dashed line, $\theta=0.4$ —solid line and $\theta=0.5$ —chain-dashed line. The horizontal axis is the natural strain and the vertical axis is the true stress in psi.

Figure 14

Effective Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to monotonically increasing strain. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the effective Poisson's ratio.

Figure 13

Endochronic uniaxial stress-strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to monotonically increasing strain. The value of θ used is 0.5. The horizontal axis is the natural strain and the vertical axis is the true stress in psi.

Figure 14

Effective Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to monotonically increasing strain. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the effective Poisson's ratio.

Figure 15

Apparent Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to monotonically increasing strain. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the apparent Poisson's ratio.

Figure 16

Endochronic uniaxial stress-strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to a loading-unloading-reloading strain history. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the true stress in psi.

Figure 17

Effective Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to a loading-unloading-reloading strain history. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the effective Poisson's ratio.

Figure 18

Apparent Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to a loading-unloading-reloading strain history. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the apparent Poisson's ratio.

Figure 19

Endochronic uniaxial stress-strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to a loading-unloading-reloading strain history. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the true stress in psi.

Figure 20

Effective Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to a loading-unloading-reloading strain history. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the effective Poisson's ratio.

Figure 21

Apparent Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to a loading-unloading-reloading strain history. The value of θ used is 0.5. The horizontal axis is the axial natural strain and the vertical axis is the apparent Poisson's ratio.

Figure 22

Endochronic uniaxial stress-strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to one cycle of straining. The limits of the cyclic strain are 0 and 0.05, respectively. The horizontal axis is the axial natural strain and the vertical axis is the true stress in psi. ($\theta=0.5$)

Figure 23

Effective Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to one cycle of straining. The limits of the cyclic strain are 0 and 0.05, respectively. The horizontal axis is the axial natural strain and the vertical axis is the effective Poisson's ratio. ($\theta=0.5$)

Figure 24

Transverse-axial strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to one cycle of straining. The limits of the cyclic axial strain are 0 and 0.05, respectively. The horizontal axis is the axial natural strain and the vertical axis is the transverse natural strain. ($\theta=0.5$)

Figure 27

Transverse-axial strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to many cycles of straining. The limits of the cyclic strain are 0 and 0.02, respectively. The horizontal axis is the axial natural strain and the vertical axis is the transverse natural strain. ($\theta=0.5$)

Figure 28

Endochronic uniaxial stress-strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to many cycles of straining. The limits of the cyclic strain are -0.05 and $+0.05$, respectively. The horizontal axis is the axial natural strain and the vertical axis is the true stress in psi. ($\theta=0.5$)

Figure 25

Endochronic uniaxial stress-strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to many cycles of straining. The limits of the cyclic strain are 0 and 0.02, respectively. The horizontal axis is the axial natural strain and the vertical axis is the true stress in psi. ($\theta=0.5$)

Figure 26

Effective Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to many cycles of straining. The limits of the cyclic strain are 0 and 0.02, respectively. The horizontal axis is the axial natural strain and the vertical axis is the effective Poisson's ratio. ($\theta=0.5$)

Figure 29

Effective Poisson's ratio for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to many cycles of straining. The limits of the cyclic strain are -0.05 and $+0.05$, respectively. The horizontal axis is the axial natural strain and the vertical axis is the effective Poisson's ratio. ($\theta=0.05$)

Figure 30

Transverse-axial strain curve for an aluminum specimen with an initial Poisson's ratio of 0.3 subjected to many cycles of straining. The limits of the cyclic strain are -0.05 and $+0.05$, respectively. The horizontal axis is the axial natural strain and the vertical axis is the transverse natural strain. ($\theta=0.5$)

Figure 31

In-plane straining of a plate that is in a state of plane stress, i.e. $\hat{\sigma}_{13} = \hat{\sigma}_{23} = \hat{\sigma}_{33} = 0$

Figure 32

Endochronic stress-strain curve in 1 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to monotonically increasing in-plane strain history of $d\hat{\epsilon}_{11} = d\hat{\epsilon}_{22} = d\epsilon > 0$. The horizontal axis is the integral of the rate of deformation in 1 direction and the vertical axis is the true stress in psi in 1 direction.

Figure 33

Endochronic stress-strain curve in 2-direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to monotonically increasing in-plane strain history of $d\epsilon_{11} = d\epsilon_{22} = d\epsilon > 0$. The horizontal axis is the integral of rate of deformation in 2-direction and the vertical axis is the true stress in psi in 2-direction.

Figure 34

Endochronic stress-strain curve in 1 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to a monotonically decreasing in-plane strain history of $d\hat{\epsilon}_{11} = d\hat{\epsilon}_{22} = -dc < 0$. The horizontal axis is the negative integral of rate of deformation in 1 direction and the vertical axis is the negative true stress in psi in 1 direction.

Figure 35

Endochronic stress-strain curve in 2 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to a monotonically decreasing in-plane strain history of $d\bar{\epsilon}_{11} = d\bar{\epsilon}_{22} = -d\epsilon < 0$. The horizontal axis is the negative integral of the rate of deformation in 2 direction and the vertical axis is the negative true stress in psi in 2 direction.

Figure 36

Endochronic stress-strain curve in 1 direction for an aluminum plate ($\nu=0.3, \theta=0.5$) subjected to an in-plane strain history of $d\hat{\epsilon}_{11} = -d\hat{\epsilon}_{22} = d\epsilon > 0$. The horizontal axis is the integral of rate of deformation in 1 direction and the vertical axis is the true stress in psi in 1 direction.

Figure 37

Endochronic stress-strain curve in 2 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\hat{\epsilon}_{11} = -d\hat{\epsilon}_{22} = d\epsilon > 0$. The horizontal axis is the negative integral of rate of deformation in 2 direction and the vertical axis is the negative true stress in psi in 2 direction.

Figure 38

Endochronic stress-strain curve in 1 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\hat{\epsilon}_{11} = -d\hat{\epsilon}_{22}/2 = d\epsilon > 0$.

The horizontal axis is the integral of rate of deformation in 1 direction and the vertical axis is the true stress in psi in 1 direction.

Figure 39

Endochronic stress-strain curve in 2 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\hat{\epsilon}_{11} = -d\hat{\epsilon}_{22}/2 = d\epsilon > 0$. The horizontal axis is the negative integral of rate of deformation in 2 direction and the vertical axis is the negative true stress in psi in 2 direction.

Figure 40

Endochronic stress-strain curve in 1 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\hat{\epsilon}_{11} = -d\hat{\epsilon}_{22}/\nu = d\epsilon > 0$.

The horizontal axis is the integral of rate of deformation in 1 direction and the vertical axis is the true stress in psi in 1 direction.

Figure 41

Endochronic stress-strain curve in 2 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\hat{\epsilon}_{11} = -d\hat{\epsilon}_{22}/\nu = d\epsilon > 0$. The horizontal axis is the negative integral of rate of deformation in 2 direction and the vertical axis is the true stress in psi in 2 direction.

Figure 42

Endochronic stress-strain curve in 1 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\epsilon_{11} = -d\epsilon_{22}/2 = -d\epsilon < 0$. The horizontal axis is the negative integral of rate of deformation in 1 direction and the vertical axis is the negative true stress in psi in 1 direction.

Figure 43

Endochronic stress-strain curve in 2 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\epsilon_{11} = -d\epsilon_{22}/2 = -d\epsilon < 0$. The horizontal axis is the integral of rate of deformation in 2 direction and the vertical axis is the true stress in psi in 2 direction.

Figure 44

Endochronic stress-strain curve in 1 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\epsilon_{11} = -d\epsilon_{22}/\nu = -d\epsilon < 0$. The horizontal axis is the negative integral of rate of deformation in 1 direction and the vertical axis is the negative true stress in psi in 1 direction.

Figure 45

Endochronic stress-strain curve in 2 direction for an aluminum plate ($\nu=0.3$, $\theta=0.5$) subjected to an in-plane strain history of $d\epsilon_{11} = -d\epsilon_{22}/\nu = -d\epsilon < 0$. The horizontal axis is the integral of rate of deformation in 2 direction and the vertical axis is the negative true stress in psi in 2 direction.

Figure 46

Central displacement history of a clamped aluminum circular plate subjected to an impulsive loading. The radius and thickness of the plate are 3 inch and 1/16 inch, respectively. A total impulse of 1.67 N-sec is applied uniformly over the central circular region of radius 1 inch. The solid line is the elasto-plastic response and the dashed line is the endochronic response.

P8

Figure 47

Central displacement history of a clamped aluminum circular plate subjected to an impulsive loading. The radius and thickness of the plate are 3 inch and 1/8 inch, respectively. A total impulse of 1.96 N-sec is applied uniformly over the central circular region of radius 1 inch. The solid line is the elastoplastic response and the dashed line is the endochronic response.

Figure 48

Central displacement history of a clamped steel circular plate subjected to an impulsive loading. The radius and thickness of the plate are 3 inch and 1/16 inch, respectively. A total impulse of 2.2 N-sec is applied uniformly over the central circular region of radius 1 inch. The solid line is the elastoplastic response and the dashed line is the endochronic response.