PERSPECTIVES IN THE THEORY OF NUCLEAR
COLLECTIVE MOTION

Abraham Klein
Department of Physics
University of Pennsylvania
Philadelphia, Pennsylvania 19104
U.S.A.

MASTER

UPR-0152T

3/13/80

DISCLAIMER

This book was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

There is no objection from the patent
point of view to the publication or
dissemination of the document(s)
noted in this letter.

BROOKHAVEN PATENT GROUP

5/1/81 By *AK*

INVITED PAPER AT INTERNATIONAL CONFERENCE ON BAND STRUCTURE
AND NUCLEAR DYNAMICS held at New Orleans, Louisiana,
February 27 - March 1, 1980. To be published by North Holland
Publishing Company.

PERSPECTIVES IN THE THEORY OF NUCLEAR
COLLECTIVE MOTION

1

Abraham Klein

Department of Physics
University of Pennsylvania
Philadelphia, Pennsylvania 19104
U.S.A.

This report discusses three different subjects. The first is the development of a generalized version of the VMI (variable moment of inertia) model which ties it to the original form of IBM (interacting boson model) and provides a possibility of fitting vibrational spectra with generalized vibrational formulas. The second is a suggestion for fitting band crossing calculations of the phenomenological type more completely than has hitherto been done into the framework of the VMI method. The third, which we consider the most important and far reaching, is the description of a complete mathematical method for the microscopic derivation of the IBM from a conventional shell model Hamiltonian. In addition to elements already foreseen by previous authors, we propose a solution for the most important problem outstanding not only within the framework of the IBM, but also in all previous work on boson expansion. We refer to the problem of actually selecting, in a general fashion, the most collective excitations. We introduce a criterion that the subspace constructed from these excitations should possess an average energy which is lower than the rest of the shell model space and explain how this criterion can be implemented in practice.

I. INTRODUCTION

It is a privilege to speak in the same session as the discoverers or co-discoverers of the two most important contributions to nuclear collective phenomenology since the pioneering efforts of Bohr and Mottelson. Since for much of the past decade I have been off playing dilettante in other areas of physics, I also view the present paper as my effort at rehabilitation as a serious nuclear structure theorist.

In choosing my subject matter, I was partly guided by what has, in its organization, been the unifying conception for the conference. Though the program is unusually heavily loaded with theory, we have purposely excluded the more highbrow and ambitious theories of nuclear collective motion, in particular where these have so far not had an extensive interface with experiment. I believe that these theories are currently undergoing a vigorous and promising development, as evidenced by some recent works, still in preprint form, for

example those of Marumori¹⁾ and Zelevinsky²⁾. In his round table remarks, David Rowe will have an opportunity, if he so chooses, to chastize us for our neglect and to discuss his own approach with Rosensteel.^{3,4)}

Let us then briefly outline the subjects to which we shall address our attention. (References will all be given in the body of the text.)

In Sec. II, we address two possibly related questions, an extension of Variable Moment of Inertia (VMI) concepts to the Interacting Boson Model (IBM) and the physically unexplained fact that the VMI (certainly aimed in its original formulation as an extension or generalization of the geometrical model of deformed nuclei) continues to provide as good a description as can be found phenomenologically, for the given number of parameters, of transition and even vibrational nuclei. Restricting ourselves to the first version of IBM, which emphasized symmetries, we show that a phenomenology can be constructed which incorporates VMI-like ideas into IBM. At the same time we thereby achieve a generalized version of VMI which allows us to fit vibrational nuclei with generalized vibrational formulas which are alternatives to VMI fits and often superior to strict VMI formulas. We give a mathematical explanation for the success of the VMI formalism and at the same time raise physical questions concerning its validity for such nuclei.

In Sec. III, an interlude, we criticize existing, purely phenomenological band crossing calculations (as opposed to core-quasiparticle coupling methods), asserting that, as carried out up to now, they do not incorporate the full power and flexibility of VMI ideas. We illustrate with the aid of a two-band calculation how this may be done.

In Sec. IV we turn from phenomenology to microscopic theory. Here we outline what we believe is a complete and implementable method for deriving the IBM (or an IBM) from a conventional shell model Hamiltonian. This method incorporates the previous idea that one maps from the shell model space to the boson space only a convenient piece of the shell model space which is believed to include the collective part. In fact for a single j shell the mapping includes only that part. In the multi-shell problem, it is inconvenient to try to identify only the collective subspace before the mapping (this is what the writer has been trying to do for almost two decades - it is possible but not easy). Therefore we map a larger space - all the angular momentum $J = 0$ and $J = 2$ pairs - onto the boson space. We utilize a method which is an "obvious" extension of previous single j work.

The major new element in our work, absent from previous work, is the introduction of a criterion, which can be implemented without having to know the exact eigenstates, for selecting the collective subspace (of "s" and "d" bosons, for example) from the full boson space. This involves the elevation to a complete dynamical method of the cogent physical criterion that the average energy of a state in the collective subspace be lower than that of any other part of the space.

We describe finally how this idea is currently being tested on a multi-level pairing model prior to its being attempted for a realistic shell-model Hamiltonian.

II. AN EXTENSION OF THE VARIABLE MOMENT OF INERTIA CONCEPT. POSSIBLE RELATION WITH THE INTERACTING BOSON MODEL.

A. Theory of the Yrast Band.

One can easily arrive at the impression that everything worthwhile (and perhaps more) has been said about the Variable Moment of Inertia Model (VMI) and the numerous alternatives and generalizations inspired by it.⁵⁻⁸⁾ Particularly for deformed nuclei and below the region of back bending, the original two-parameter model yields dramatically good fits to the energies of the Yrast band. What is in some ways even more remarkable is that the formalism continues to give good fits to the spectra into the transition region and beyond, including as definitely vibrational nuclei as we can point to, yielding on the average better fits than the formulas derived from the conventional vibrational picture.⁹⁾ Though a full and clear explanation has been provided for the possibility of mathematical extension of the original domain of definition of the formalism into the region of parameter space necessary for the extended applications⁸⁾, a physical understanding of this success remains elusive.

As far as I am aware, only Marshalek has derived a VMI formula beginning with a vibrational framework.¹⁰⁾ In his work he emphasizes the possibility that a so-called vibrational nucleus can rotate once it acquires angular momentum in an excited state. This is consistent with the idea that an anharmonic vibrator may look more and more like a rotator as the angular momentum increases. It may also imply that a "phase transition" occurs at a finite angular momentum and thus, in this picture, there is no continuity with the usual model of a mildly perturbed anharmonic vibrator at low angular momentum. We shall in fact show that indeed there is no such continuity in the VMI formalism but, at the same time give an alternative explanation, purely mathematical, of the strange (but limited) success of the VMI in the vibrational regime. At the same time we may note that Marshalek's work is based on the cranking formalism, which is unjustified for weakly deformed nuclei.

The main purpose of this section, however, is to present a new extension of the VMI concept, which contains the possibility of describing vibrational nuclei by a formula which reduces to the anharmonic vibrational description near the ground state.

We begin with the briefest possible summary of the usual VMI formalism. Here the excitation energy, $E(J)$, of the state with angular momentum J is written as

$$E(J) = \frac{J(J+1)}{2\theta(J)} + U(0) \quad , \quad (2.1)$$

where, for a prescribed form of the "potential energy" $U(0)$, the value of $\theta(J)$, the variable moment of inertia, is determined from a variational condition

$$\left. \frac{\partial E(J)}{\partial \theta} \right|_J = 0 \quad . \quad (2.2)$$

In the standard model $U(0)$ takes the harmonic form,

$$U(0) = \frac{1}{2}C(0-\theta_0)^2 \quad . \quad (2.3)$$

Though the decomposition of $E(J)$ in (2.1) as a sum has physical

significance, as we shall see below, there remains even within this framework considerable freedom in the choice of the function $U(\theta)$. This point has been discussed fully in previous work^{11,6)} and will not concern us further in this work.

Another generalization¹²⁾ which appears to be phenomenologically useful (though of uncertain theoretical significance) is to replace 0 in the first term of (2.1) by an arbitrary power θ^n , at the same time retaining (2.3). By this means a third (continuous) parameter n , enters the picture; the main point which we wish to extract from this suggestion is that this parameter (with a value less than unity) improves the fit for transitional and vibrational nuclei.¹³⁾ Up to now, to our knowledge, no one has questioned the use of the factor $J(J+1)$ in the first term of (2.1), though this J dependence, for small J , has theoretical justification only in the regime of deformed nuclei - Marshalek's discussion¹⁰⁾ apart. Perhaps this acceptance is due, at least in part, to the remarkable fact already mentioned that the VMI fits vibrational spectra better than the usual vibrational formula.⁹⁾

Another remarkable fact is that no one has so far attempted to adjoin VMI Model considerations to the Interacting Boson Model (IBM)¹⁴⁾, the second important new development in low energy phenomenology of the past decade.

We now show that just as the VMI can be viewed as a generalization of the standard geometrical model, in which the concept of fixed average shapes is replaced, in the phenomenology, by the idea of shapes and correlations dependent on the angular momentum (and ultimately other quantum numbers), so too the IBM can be generalized in a like sense. This is not so surprising if one remembers that the various "dynamical" symmetries unearthed by the IBM can also be identified in the geometrical model. This observation is, however, unnecessary to carry through the appropriate generalization. What is necessary is that we reexamine the theoretical ideas leading to Eqs. (2.1) and (2.2)¹⁵⁻¹⁸⁾ and realize that they permit additional flexibility beyond what has hitherto been exploited.

We illustrate the basic nature of the argument by considering the Yrast band. We expose our assumptions:

(i) We are given a set of model state vectors, $|J, X_i(J)\rangle$, referring to the state of angular momentum J (suppressing all other quantum numbers in the notation). These depend on a set of parameters $X_i(J)$, $i=1, \dots$ which measure deformation, pairing, and the mixing with quasi-particle excitations. In general the $X_i(J)$ represent dynamical degrees of freedom which are here being treated quasi-statically (and therefore approximately). If the J dependence of the $X_i(J)$ is neglected, the state vectors reduce to those of some prescribed collective model which agrees with the lowest energy data. [Thus, this model need not be rotational. Secondly, we do not have to describe how to obtain this set of model states. This is the job of the microscopic theory. It is even possible to generalize our assumption by supposing that the model state of given J depends on the parameters for other states, i.e., the states may be functionals of the parameter functions $X_i(J)$]

(ii) Let $K(J, X_i(0))$ be the excitation energy of the ground state band of the prescribed collective model, i.e.

$$K(J, X_i(0)) \equiv \langle J, X_i(0) | H | J, X_i(0) \rangle \\ - \langle 0, X_i(0) | H | 0, X_i(0) \rangle, \quad (2.4)$$

where H is the Hamiltonian. The models we shall consider are those for which

$$K(J, X_i(0)) \cong aJ + bJ^2. \quad (2.5)$$

In fact, this includes, for suitable values of a and b , all the limiting dynamical symmetries of the IBM at low J and interpolates between them.

(iii) To derive (2.1), for instance, the excitation energy $E(J, X_i(J))$ is decomposed into a sum of two terms

$$E(J, X_i(J)) \equiv K(J, X_i(J)) + V(X_i(J)), \quad (2.6)$$

where $K(J, X_i(J))$ generalizes (2.4), namely

$$K(J, X_i(J)) \equiv \langle J, X_i(J) | H | J, X_i(J) \rangle \\ - \langle 0, X_i(J) | H | 0, X_i(J) \rangle, \quad (2.7)$$

and

$$V(X_i(J)) = \langle 0, X_i(J) | H | 0, X_i(J) \rangle \\ - \langle 0, X_i(0) | H | 0, X_i(0) \rangle. \quad (2.8)$$

We next assume that, as a generalization of (2.5), we may write

$$K(J, X_i(J)) \equiv (aJ + bJ^2)/\theta(J), \quad (2.9)$$

where $\theta(J)$, satisfying $\theta(0) = 1$, is the tracking or scaling variable which replaces, for our formulation, the variable moment of inertia. In all previous treatments of VMI (for the Yrast band alone) the left hand side of (2.9) has been identified with the first term of (2.1). Therefore the present proposal will contain VMI as a limiting case.

Actually, we believe that physical considerations dictate a more general assumption than (2.9), which we shall elaborate hereafter (see Eq. 2.15). But first we shall complete the story associated with (2.9).

(iv) The remaining assumptions parallel those made in previous accounts.¹⁵⁻¹⁸ Thus the values $X_i(0) \equiv X_{i0}$ are determined by the requirement that the potential energy (2.8) have a minimum at $J=0$, namely

$$(\partial V / \partial X_{i0}) = 0, \quad \det |\partial^2 V / \partial X_{i0} \partial X_{j0}| > 0. \quad (2.10)$$

Furthermore, the $X_i(J)$ are supposed slowly varying functions of J and also slowly varying functions of $\theta(J)$, so that we may write

$$\delta X_i(J) \equiv X_i(J) - X_{i0} \equiv \kappa_i [\theta(J) - 1]. \quad (2.11)$$

From these assumptions, we derive the form equivalent to that given in Eq. (2.3), namely

$$V(X_i(J)) - V(X_i(0)) \equiv U(0) \cong \frac{1}{2} \gamma (\theta - 1)^2, \quad (2.12)$$

$$\gamma = \sum_{ij} \kappa_i \kappa_j \partial^2 V / \partial X_{i0} \partial X_{j0}. \quad (2.13)$$

Altogether assumptions (i) - (iv) imply that the excitation energy $E(J)$, (2.6) can be written in the form

$$E(J) = \left[(aJ + bJ^2)/\phi \right] + \frac{1}{2} \gamma (0-1)^2 \\ = \left[\frac{J + \kappa J(J-2)}{\phi} \right] + \frac{1}{2} \gamma (\phi - \phi_0)^2, \quad (2.14)$$

the second form involving rescaling and redefinition of constants. This formula represents a generalization of VMI containing IBM as a limiting case. We emphasize that this generalization has been achieved by insisting on a form interpretable as an analytic continuation (in J) of the polynomial valid at small J given by (2.5).

Let us consider briefly the possible physical significance to be ascribed to the first term of (2.14). (Of course this will be modified by the presence of the potential energy term). It describes an anharmonic vibrator with a constant ratio of anharmonic to harmonic term as a function of angular momentum, only the scale of the spectrum changing with angular momentum. A more flexible description would also allow the relative anharmonicity to vary with angular momentum. Such an increased flexibility can be incorporated into the formalism by means of separate scaling variables for each term of (2.5). An appropriate generalization of (2.14) is

$$E(J) = \frac{J}{\phi_1} + \frac{J(J-2)}{\phi_2} + \frac{1}{2} \sum_{i,j=1}^2 \kappa_{ij} (\phi_i - \phi_{i0}) (\phi_j - \phi_{j0}). \quad (2.15)$$

This formula contains five parameters: ϕ_{10} , ϕ_{20} , κ_{11} , κ_{22} , and $\kappa_{12} = \kappa_{21}$.

We judge this to be too many parameters for the amount of data available for the Yrast bands of non-rotational nuclei, though it might be of interest to investigate Eq. (2.15) in general terms to see if it predicts the phenomenon of forking of bands.¹⁸⁾ Our present purpose in introducing (2.15) is in order to have a common parent for special cases other than (2.14) which will additionally prove useful when we confront the data below.

B. Multiband Generalization.

Multiband generalizations of the VMI have been studied by a number of authors^{5,18-20)}, but in every case the generalization has been restricted to some form of rotational model applicable to strongly deformed nuclei. Again we note that this is an unnecessary restriction. We shall illustrate with the SU(5) or vibrational limit, though the mode of reasoning will clearly apply to any regime. We quote formulas only, since the "derivation" involves reasoning which is an obvious extension of that applied to the rotational models.

We start with the formula for the excitation energy valid within the SU(5) or quadrupole anharmonic vibrator limit of the IBM²¹⁾,

$$E(n,v,J) = \epsilon n + \frac{1}{2} \alpha n(n-1) \\ + \beta (n-v)(n+v+3) + \gamma [J(J+1) - 6n], \quad (2.16)$$

where n is the phonon number (number of "d" bosons), v the seniority, and ϵ , α , β , γ are parameters. In general we require ϵ to be an order of magnitude larger than the other parameters for a good vibrational spectrum. VMI-type generalizations of various kinds can be considered. We can utilize anywhere from one overall tracking

parameter to four (one for each term of (2.16)). The "potential energy" that one adds to (2.16) then has a corresponding degree of complexity. Thus, in the general case, where we would use four scaling functions, $\theta_i(n, v, J)$, $i=1, \dots, 4$, we add to the modified form of (2.16), the term

$$U(\theta_i) = \frac{1}{2} \sum C_{ij} (\theta_i - 1) (\theta_j - 1) ,$$

$$C_{ij} = C_{ji}, \det |C_{ij}| > 0 . \quad (2.17)$$

Simplifications are achieved by imposing constraints among the θ_i . It should be clear, then, how this method applies in principle, to any collective model.

C. Comparison with Experiment:

We turn then to selective comparison of our ideas with experiment. We consider first a set of four nuclei with very similar spectra, namely ^{150}Sm , ^{152}Gd , ^{154}Dy , and ^{156}Er with $R_4 = (E(4)/E(2))$ taking the values 2.316, 2.194, 2.233, and 2.315 respectively. Selected results of calculations which have been carried out for these nuclei are presented in Tables 1-4. The second column of each table contains the experimental results taken from a recent compilation²²⁾. The third and fourth columns contain the phenomenological predictions of the anharmonic vibrator model, with parameters defined by the equation

$$E(J) = \phi_0^{-1} [J + xJ(J-2) + yJ(J-2)(J-4)] . \quad (2.18)$$

The approximation AVM1 refers to the approximation $y=0$, whereas in AVM2, all three parameters have been utilized. The last row headed $\langle |AE| \rangle$ is in every instance the average numerical deviation from experiment of the levels not utilized in the fit; as indicated, we have determined our parameters by requiring exact fits to the first few energies.

Aside from the interest of the results of AVM for the purposes of comparison with the other calculations, they definitely show the limitations in accuracy of the $SU(5)$ symmetry in pure form for these nuclei though judging from AVM2, ^{154}Dy and ^{156}Er appear to be rather good vibrational nuclei.

Table 1. Yrast band - ^{150}Sm

J	Exp (Mev)	AVM1 $\phi_0 = 5.9882$ $x = 0.07894$	AVM2 $y =$ 4.909×10^{-3}	$x=0.60$ $\kappa = 8.722 \times 10^{-4}$ $\phi_0 = -7.6782$	VAVM $C = 6.5693 \times 10^{-5}$ $\theta_0 = 66.356$
2	0.33395	-	-	-	-
4	0.77335	-	-	-	-
6	1.27885	1.31820	-	1.27913	-
8	1.8371	1.9685	1.8111	1.8363	1.8301
10	2.432	2.724	2.331	2.436	2.417
12	3.048	3.585	2.798	3.073	3.033
14	(3.646)	4.552	3.175	3.743	3.675
16	(4.306)	5.624	3.421	4.443	4.339
$\langle AE \rangle$		0.537	0.347	0.053	0.020

Table 2. Yrast band -- ^{152}Gd

J	Exp (Mev)	AVM1 $\phi_0 = 5.8099$ $x = .048491$	AVM2 $y =$ 6.815×10^{-4}	$x = 0.40$ $\kappa = .001673$ $\phi_0 = -3.1469$	VAVM $C = 1.4027 \times 10^{-4}$ $\theta_0 = 117.88$
2	0.34424	-	-	-	-
4	0.75524	-	-	-	-
6	1.2274	1.2330	-	1.2281	-
8	1.7470	1.7776	1.7551	1.7495	1.7529
10	2.300	2.389	2.333	2.311	2.325
12	2.884	3.067	2.954	2.908	2.936
14	3.499	3.812	3.615	3.537	3.583
16	4.143	4.623	4.308	4.194	4.262
$\langle \Delta E \rangle$		0.184	0.078	0.021	0.057

Table 3. Yrast band ^{154}Dy

J	Exp (Mev)	AVM1 $\phi_0 = 5.9773$ $x = 0.05813$	AVM2 $y =$ 1.631×10^{-3}	$x = 0.40$ $\kappa = 1.7879 \times 10^{-3}$ $\phi_0 = -2.2855$	VAVM $C = 9.2614 \times 10^{-5}$ $\theta_0 = 98.886$
2	0.3346	-	-	-	-
4	0.7470	-	-	-	-
6	1.2241	1.2372	-	1.2239	-
8	1.7479	1.8052	1.7528	1.7510	1.7524
10	2.3052	2.4510	2.3200	2.3199	2.3225
12	2.8945	3.1746	2.9126	2.9251	2.9275
14	3.5110	3.9760	3.5175	3.5627	3.5628
$\langle \Delta E \rangle$		0.192	0.011	0.020	0.027

Table 4. Yrast band ^{156}Er

J	Exp (Mev)	AVM1 $\phi_0 = 5.8072$ $x = 0.07869$	AVM2 $y =$ 2.154×10^{-3}	$x = 0.15$ $\kappa = .03444$ $\theta_0 = 5.0382$	VAVM $C = 2.5933 \times 10^{-4}$ $\theta_0 = 71.067$
2	0.3444	-	-	-	-
4	0.7972	-	-	-	-
6	1.3406	1.3584	-	1.3343	-
8	1.9589	2.0280	1.9568	1.9412	1.9561
10	2.633	2.806	2.628	2.608	2.630
12	3.315	3.692	3.336	3.328	3.354
$\langle \Delta E \rangle$		0.159	0.009	0.015	0.015

The last column of each table gives results obtained from the formulas

$$E(J) = aJ + 0^{-1}J(J-2) + \frac{1}{2}C(\theta - \theta_0)^2 \quad (2.19)$$

This represents a limiting form of Eq. (2.15) in which $\phi_1 = a^{-1}$ = constant is chosen to fit $E(2)$ and therefore $E(2) = 2a^{-1}$ in every case. Consequently it has not been listed separately in the tables. In this model only the anharmonicity scales and the scaling variable $\theta(J)$ is determined for each J by the usual minimization conditions

$$\left(\partial E(J)/\partial \theta\right)_J = 0 \quad (2.20)$$

Since the last two terms vanish for $J = 2$, the constants C and θ_0 are determined by fitting $E(4)$ and $E(6)$. Equation (2.19) is put forward here as a three parameter formula alternative to (2.14) which reduces to a simple polynomial in J as $J \rightarrow 0$ and agrees with experiment. To understand this assertion, we must first complete the description of the remaining calculations.

The remaining column headed by a value of x refers to calculations based on Eq. (2.14) which we have chosen to call generalized VMI. The value $x = (1/3)$ is the VMI, $x = 1/2$ corresponds to a suggestion of Marshalek¹⁰⁾ and $x \ll 1$ is necessary for a vibrational picture. There is, however, a minimum value of x for which this formalism makes sense. It is, in fact, the value listed under ΔVMI , for this value is achieved in the limit $\kappa \rightarrow \infty$, $\phi \rightarrow \phi_0$, and $\frac{1}{2}\kappa(\phi - \phi_0)^2 \rightarrow 0$. We carried out calculations for a range of values of x , starting with the minimum value and increasing x until we achieved the (roughly) optimal result which is displayed. These show: (i) The VMI proper is closely optimal for two of the nuclei, Gd and Dy, but for Sm we must go beyond even Marshalek's value and for Er we find a vibrational value, $x = 0.15$. (ii) Three of the best fits occur for negative values of the moment of inertia, thus beyond the original domain of definition of VMI, but not beyond the extended domain.⁸⁾ (iii) Most noteworthy, there is nothing to choose numerically between a suitably generalized VMI and the theory marked VAVM. Now the latter definitely represents a vibrational picture; this is ascertained in the first instance from the ratio $(a\theta_0)^{-1} = (\frac{1}{2}E(2)\theta_0)^{-1} = 0.089, 0.049, 0.060, 0.082$ for the four nuclei under study. Can one say anything as clear cut about the generalized VMI solutions? What do they mean and why do they work?

For one thing, they do not fall within the framework of the picture of small J behavior predicated on Eq. (2.5), i.e. containing IBM limiting behavior. The easiest way to see this is to consider the case $\phi_0 = 0$. In this case the generalized VMI, utilizing the variational condition, leads to a simple closed formula

$$E(j, \phi_0 = 0) = \frac{3}{2} \kappa_0^{1/3} \{J + x_0 J(J-2)\}^{2/3} \quad (2.21)$$

which is certainly not analytic at $J=0$. To this formula, the potential energy contributes $(1/3)$ of the total. For negative ϕ it contributes successively a greater percentage of the energy. This, plus the non-analytic form (2.21), should, in the writer's opinion raise serious questions concerning the use of the generalized VMI for small or negative moments of inertia, especially since we are now able to supply an alternative associated with more conventional physics.

Needless to say suspicion is not proof, but in any event if a formula like (2.21) indeed represents true physics, what that physics is has yet to be clarified. Equation (2.21) does, at the least serve one very useful purpose, i.e., of clarifying mathematically why a numerically reasonable solution can be found within the frame-

work of the generalized VMI. Thus instead of continuing to negative ϕ_0 with the original formula, let us instead try the reasonable continuation

$$E(J) = \{ J + xJ(J-2) \}^{2/3} / 4 + \frac{1}{2} \kappa (\phi - \phi_0)^2, \quad \phi_0 > 0, \quad (2.22)$$

applicable to the domain $x > x_0$, where x_0 is the value determined by (2.21), and therefore $\phi_0 \rightarrow (3/2 \kappa_0)^{1/3} - 1$ for $x \rightarrow x_0$ from above. However by repetition of the previous argument, when $\phi_0 \rightarrow 0$, we will obtain the formula

$$E(J, \phi_0=0) = \frac{3}{2} \kappa_1^{1/3} \{ J + x_1 J(J-2) \}^{2/3} \quad (2.23)$$

These considerations suggest that for negative ϕ_0 , we can represent the generalized VMI, at least approximately by an interpolating formula of the structure

$$E(J) \cong \alpha \{ J + xJ(J-2) \}^{(2/3)^n} \quad n \geq 1 \quad (2.24)$$

treating n as a third continuous parameter besides α and x . The fractional power gives fairly precisely for suitable n the increasing relative compression of the spectrum for large J , as required by experiment. We shall return to this point in connection with the study of the Pd isotopes to which we turn below.

Our minimal conclusion so far is that we have uncovered a new, physically motivated, description for transition nuclei within the generalized scaling framework described here. It is distinct from the VMI solution, which is contained within the same general framework, and is at least as accurate as the latter. We have also raised some questions concerning the physical meaning of the extended VMI solution, though none of our arguments can be claimed to be definitive.

Some additional insight plus further illustration of some of our previous remarks can be achieved by considering the three Pd isotopes ^{102}Pd , ^{100}Pd , ^{98}Pd with $R_4 = 2.293, 2.218, 1.786$ respectively. The data presented in tables 5-7 was kindly provided by Trudy Goldhaber. For ^{102}Pd there is minor emendation of previously published data²³⁾ and for ^{98}Pd it is in agreement with published data.²⁴⁾ For the case of ^{100}Pd there is nothing to add to previous comments, since the results are qualitatively similar to those presented in previous tables. In the cases of both ^{100}Pd and ^{98}Pd we encounter new features.

Let us next consider ^{100}Pd . First, we observe the unusual phenomena that AVM2 gives a (slightly) poorer fit than AVMI. Secondly none of the generalized VMI formulas gives a good fit over the usual range of values of x . The best fit is provided by VAVM, but it is clearly of poorer quality than that for the nuclei considered previously. We have convinced ourselves that it is possible to match the quality of this fit by using Eq. (2.24) with $n = 1.6$, but this point will not be pursued further here in view of the remarks leading to (2.24). Probably the most reasonable overall conclusion (see below) is that ^{100}Pd is simply not very collective.

In ^{98}Pd , we have the only nucleus considered in this discussion with $R_4 < 2$. Here we see that AVMI is ridiculous and that AVM2 is

Table 5. Yrast band - ^{102}Pd

J	Exp (Mev)	AVM1 $\phi_0 = 3.5939$ $x = 0.07318$	AVM2 $y =$ -3.512×10^{-3}	$x = 0.45$ $r = 0.007708$ $\phi_0 = -1.6126$	VAVM $C = 3.8260 \times 10^{-4}$ $\theta_0 = 45.083$
2	0.5565	-	-	-	-
4	1.2759	-	-	-	-
6	2.1113	2.1582	-	2.1115	-
8	3.0131	3.2034	3.0158	3.0367	3.0304
10	3.9927	4.4115	3.9425	4.0362	4.0152
12	5.0549	5.7825	4.8445	5.1001	5.0544
14	6.1386	7.3164	5.6749	6.2213	6.1404
$\langle \Delta E \rangle$		0.512	0.182	0.039	0.011

Table 6. Yrast band - ^{100}Pd

J	Exp (Mev)	AVM1 $\phi_0 = 3.0062$ $x = 0.03202$	AVM2 $y =$ -3.927×10^{-3}	$x = 1.0$ $\kappa = 9.7518 \times 10^{-4}$ $\phi_0 = -21.200$	VAVM $C = 6.6018 \times 10^{-6}$ $\theta_0 = 44.843$
2	0.6653	-	-	-	-
4	1.4158	-	-	-	-
6	2.1888	2.2515	-	2.2398	-
8	2.9874	3.1724	2.9216	3.1291	2.9798
10	3.8688	4.1785	3.5515	4.0747	3.7854
12	4.7607	5.2698	4.0158	5.0701	4.6030
14	5.7061	6.4463	4.2518	6.1104	5.4312
$\langle \Delta E \rangle$		0.361	0.646	0.222	0.131

Table 7. Yrast band - ^{98}Pd

J	Exp (Mev)	AVM1 $\phi_0 = 2.3186$ $x = -0.05341$	AVM2 $y =$ 3.7097×10^{-3}	$x = 0.10$ $\kappa = 0.04563$ $\phi_0 = -0.07264$	VAVM Eq. (2.26) $\xi = 0.01$ $D = 0.1235$ $\psi_0 = 1.2185$
2	0.8626	-	-	-	-
4	1.5409	-	-	-	-
6	2.1117	2.0349	-	2.2338	2.2160
8	2.7722	2.3446	2.6518	2.9547	2.9224
10	3.6438	2.4700	3.2380	3.7057	3.6751
12	4.4450	2.4111	3.9471	4.4865	4.4826
14	5.6977	2.1679	4.8559	5.2957	5.3500
$\langle \Delta E \rangle$		1.448	0.436	0.162	0.134

unimpressive. We observe that VMI ($x = (1/3)$) will give an improved fit but that the best generalized VMI fit has shifted to the vibrational regime with $x = 0.10$. Furthermore VAVM as defined by (2.19) will not work at all (it yields an unstable potential energy). But it is well to remember that Eq. (2.19) is only one special limit of (2.15). An even simpler limit is associated with what may be termed

the variable vibrator model (VVM) in the simple form

$$E(J) = \frac{J}{\psi} + \frac{1}{2} D (\psi - \psi_0)^2, \quad (\partial E(J) / \partial \psi)_J = 0. \quad (2.25)$$

It is not difficult to show that Eqs. (2.25) always lead, for $D > 0$, as required, to a value $R_4 < 2$, so that the limiting formalism VVM was not interesting for any previous case.

To provide a three-parameter formalism we modify the first of Eqs. (2.25) to

$$E(J) = \frac{J}{\psi} + \frac{1}{2} D (\psi - \psi_0)^2 + \xi J(J-2). \quad (2.26)$$

and report a close to optimum fit in the last column of table 7 with $\xi = 0.01$. A suspicion lingers here that VMI ($x=0.10$) and VAVM, Eq. (2.26) do not represent distinguishable physics in the present example.

The relative inability of VMI to describe ^{100}Pd has been interpreted by Trudy Goldhaber²⁵⁾ as indicating that this nucleus is semi-magic because of special quartet binding. We have no argument with this extremely reasonable shell-model picture. Within the purely phenomenological framework being pursued here, however, there is not much to choose between them in the extent to which ^{100}Pd and ^{98}Pd conform to the class of models under study. Neither appears to be very collective in the sense of this study as already remarked above for ^{100}Pd .

III. BAND CROSSING CALCULATIONS AND VMI CONCEPTS.

For strongly deformed nuclei, the original VMI formalism has proved to be truly remarkable in its accuracy up to the region where one encounters backbending. The interpretation of the backbending in terms of the crossing of two rotational bands as the mechanism in most observed cases has now gained universal acceptance. In more detail the interpretation of the second band (superband) as a rotation-aligned band has been shored up by calculations based on the core-particle coupling model²⁶⁻²⁸⁾. The most convincing work of this type is that contained in a series of papers by Reinecke and Ruder.²⁹⁻³¹⁾

To the extent that one accepts the just mentioned work, the problem about to be discussed is not an open one. Nevertheless a number of authors have asked the following question. Suppose one does not assume the K-value for the superband given by the core-particle coupling models, but adopts only the general band coupling formalism³²⁾, can one by a fit to the data determine the favored K-value or values? Calculations reported to date on this problem, some of them quite elaborate^{33,34)}, have not been decisive in deciding this question. In this section we shall suggest that these calculations have not incorporated the VMI philosophy in a sufficiently single-minded manner.

The one-band VMI fails precisely because at least one of the parameters $X_i(J)$ introduced in the general argument of the previous section, that describing band-mixing, becomes rapidly varying and must be treated dynamically. In the following we describe a model in which this is done in a standard way, but at the same time all the other features of VMI are retained.

We consider a two band model with model state vectors $|J, \delta, X_i(J, \delta)\rangle$, $\delta=1,2$. We assume that the diagonal matrix elements of the Hamiltonian are given by VMI-type formulas

$$H_{\delta\delta} = \langle J, \delta, X_i(J, \delta) | H | J, \delta, X_i(J, \delta) \rangle = \frac{J(J+1) - K_{\delta}^2}{2} + \frac{1}{2} C_{\delta} (\theta_{\delta} - \theta_{\delta 0})^2. \quad (3.1)$$

Here K_{δ} is the K value of the band, and we have used a vector notation for the parameters $X_i(J, \delta)$, which describe the "hidden degrees of freedom". It must be admitted at the outset that the use of this formula for the excited (superband) is more problematical than for the ground band.

For the off-diagonal element we assume the form

$$H_{12} = \langle J, 1, X(J, 1) | H | J, 2, X(J, 2) \rangle = h f(J) (1 + B_1 (\theta_1 - \theta_{10}) + B_2 (\theta_2 - \theta_{20})), \quad (3.2)$$

where h , B_1 and B_2 are constants and $f(J)$ takes the conventional values^{3,2)}

$$f(J) = \{J(J+1)\}^{\frac{1}{2}}, \quad K = 1 \rightarrow K = 1, \quad (3.3a)$$

$$= J(J+1), \quad K = 0 \rightarrow K = 0, \quad (3.3b)$$

$$= \{J(J+1)(J+2)(J-1)\}^{\frac{1}{2}}, \quad K = 0 \rightarrow K = 2. \quad (3.3c)$$

Some discussion of (3.2) is required. In the absence of specific model assumptions, the form (3.2) is chosen in accord with VMI philosophy and will receive justification a posteriori if the contributions $|B_{\delta}(\theta_{\delta} - \theta_{\delta 0})|$ remain small compared to unity. As an aside, it is often remarked that VMI is a theory only of the energy, and that there is no recommendation for improvement of other observables. The type of assumption we have made for H_{12} , Eq. (3.2) illustrates that such an assertion is a calumny. From the point of view of theory we can regard H_{12} as a typical scalar observable, and (3.2) can be viewed as a possible VMI theory of this observable. Corresponding assumptions can be developed for any observable if one is not afraid to allow additional parameters to enter the discussion. In so far as some of the observables of greatest interest, such as the BE(2) can be identified with some of the $X_i(J)$, the necessity for additional parameters gives information about which physical parameters contribute to the restoring force constant C in the VMI formalism.

From the secular equation

$$\begin{vmatrix} H_{11} - W & H_{12} \\ H_{12} & H_{22} - W \end{vmatrix} = 0, \quad (3.4)$$

we obtain the usual solutions for each J,

$$W_{1,2} = \frac{1}{2}(H_{11} + H_{22}) \mp \{(H_{11} - H_{22})^2 + 4 H_{12}^2\}^{\frac{1}{2}}, \quad (3.5)$$

but it remains for us to apply variational considerations to determine the values of $\theta_1(J)$, $\theta_2(J)$. For this we need two independent equations, which we can obtain as follows: From the conditions ($i = 1, 2$)

$$\delta W_i = 0 = \frac{\partial W_i}{\partial \theta_1} \delta \theta_1 + \frac{\partial W_i}{\partial \theta_2} \delta \theta_2, \quad (3.6)$$

we conclude that

$$\det \left| \frac{\partial W_i}{\partial \theta_j} \right| = 0. \quad (3.7)$$

This yields after simplification a first condition

$$(H_{22} - H_{11}) Y_1 Y_2 + H_{12} h f(J) (B_2 Y_1 - B_1 Y_2) = 0, \quad (3.8)$$

$$Y_i = C_i (\theta_i - \theta_{i0}) - \frac{\{J(J+1) - K_i^2\}}{0_i^2}. \quad (3.9)$$

Equation (3.8) may be viewed as determining θ_2 as a function of θ_1 . The second required condition is obtained by computing $(d\theta_2/d\theta_1)$ from (3.8) and equating it to the same quantity computed from either of Eqs. (3.6) (which are now compatible). The resulting equation may be written

$$\begin{aligned} Y_1^2 Y_2^2 + \frac{1}{2} (H_{22} - H_{11}) \{ Y_2^2 (dY_1/d\theta_1) - Y_1^2 (dY_2/d\theta_2) \} \\ - H_{12} h f(J) \{ B_2 Y_2 (dY_1/d\theta_1) + B_1 Y_1 (dY_2/d\theta_2) \} \\ + \{ h f(J) \}^2 (Y_1 B_2 - Y_2 B_1)^2 = 0. \end{aligned} \quad (3.10)$$

Eqs. (3.8) - (3.10) replace the usual VMI variational condition.

We have not yet programmed the full calculation described above. We have, however, performed a "back of the envelope" calculation (with a hand-programmable calculator) to try to judge if there is any promise in the above formalism. For this purpose we have chosen to look at the beautiful data for ^{164}Er , where the ground band can be followed above the crossing and the superband below the crossing.³⁶⁾ The energy values of interest to us are listed in Table 8. We have simplified the calculations by choosing $B_2 = 0$ and selecting a fixed (ad hoc) formula for the superband, namely

$$H_{22}(J) = A + \{J(J+1) - K^2\} \{ \theta_{20}^{-1} - \alpha(24-J) \}, \quad (3.11)$$

which describes an almost rigid rotator with small correction term (constant α) to help account for the empirical fact that the effective moment of inertia of the superband decreases with angular momentum. The starting values in our fitting procedure

were obtained by fitting (3.11) to the S-band energies at $J=22$ and 24 and by arbitrarily requiring $H_{22}(J=16) = 3.29$ Mev, which overshoots the experimental energy by 30 keV. The small correction parameter α was then kept fixed throughout the rest of the calculation.

Table 8
Experimental Energies, ^{164}Er

J	G-band (Mev)	S-band (Mev)
2	0.09139	-
4	0.29947	-
6	0.61440	-
8	1.02461	-
10	1.5179	-
12	2.0828	2.5193
14	2.7026	2.8748
16	3.4112	3.2631
18	4.1212	3.7686
20	4.8684	4.3457
22	5.6515	5.0001
24	-	5.7291

There remain then the six parameters: C_1 , θ_{10} ; h , B_1 ; Λ and θ_{20} 15
Utilizing the variational condition

$$\left. \frac{\partial W_1}{\partial \theta_1} \right|_J = 0 \quad (3.12)$$

to determine the variational moment of inertia $\theta_1(J)$, the six parameters were fitted simultaneously to the G-band energies for $J=2, 4, 14$ and to the S-band energies for $J=16, 22$, and 24 , i.e., in every case to energies of the Yrast band. All remaining experimental energies were then calculated.

This method was applied to two models, namely $K=1$ and $K=0$ for the S-band. The anchoring of the Yrast band in three crucial pairs of energies is so restrictive that we find that both models fit the remaining members of the Yrast band to a high degree of accuracy, namely $\langle |AE| \rangle = 1.6$ keV for $K=1$ and 1.5 keV for $K=0$. It is, however, a different story for the unanchored Yrare band, where we find 112.9 keV average deviation per level for $K=1$ and 340.3 keV per level for $K=0$. This difference appears to us to make this subject worth pursuing further.

Another point of interest is the variation with angular momentum of H_{12} arising from the term $B_1 (\theta_1 - \theta_{10})$. For $K=1$, for instance, we found $B_1 = -0.004536$ and $(\theta_1 - \theta_{10})_{\max} = 35.526$ at $J=24$. The product is 0.1621 which is a 16% effect for $J=24$, whereas a corresponding calculation yields about half as much for the neighborhood of the crossing and a negligible effect near the ground state. This would appear to justify the expansion used to treat H_{12} .

IV. MICROSCOPIC FOUNDATIONS OF THE INTERACTING BOSON MODEL (IBM) AND OTHER BOSON EXPANSIONS.

A. Preliminary Observations

We now turn to the other important development in nuclear phenomenology of the past decade, the IBM. We do not wish to become embroiled in any controversy concerning either of two questions which have been raised concerning this theory: (i) Is it new? (ii) Is it improved? Certainly we shall hear a great deal about this subject during the course of this conference. Nevertheless we cannot refrain from inscribing here our view of where the IBM fits into the general scheme of things, what has been accomplished, and what remains to be done.

In order to achieve some perspective, let us recall that when the ideas of the theory of superconductivity were introduced into nuclear physics, it was unnecessary to modify our phenomenological notions concerning the principle collective excitations of even nuclei, i.e., the Bohr-Mottelson theory, or as many seem to prefer to call it these days, the geometrical model. Why? The reason is that these motions can be viewed as excitations of a substrate (the core). Before nuclear superconductivity this substrate was viewed as the assembly of particles moving independently in a self-consistent deformed well. The great technical advance was that all we had to do was change particles to quasi-particles in the substrate and change nothing about the phenomenological description of the excitations on the substrate. Of course the core-particle coupling model now became the core-quasiparticle coupling model.

In at least one further respect, borrowing from the number non-conserving methods of the theory of superconductivity accorded in a very natural way with the existing ideas. In the Bohr-Mottelson

picture, the most important excitations are described in terms of quadrupole bosons. Any number of bosons can, in principle, be added to the substrate, though in practice, the theory was capable in its original forms of describing only low energy excitations. This is completely in accord with the usual methods of superconductivity where fermion particle number is not strictly conserved.

The older phenomenology has nothing to say about the microscopic structure of the constituent bosons. Do they represent particle hole correlations or particle-particle correlations? One answer to this question is provided by the entire tradition or paradigm of nuclear theory which goes back to the introduction of the random phase approximation into the subject³⁶⁾ (see ³⁷⁾ for a review) or the first paper on boson expansions in nuclear theory³⁸⁾ and culminating for this conference in the work of Tamura and his collaborators.³⁹⁻⁴¹⁾ The answer given by this tradition is that the collective excitation is described as a two quasi-particle correlation (as opposed to a quasi-particle - quasi-hole correlation).

It is our contention (or rather our current belief) that there is no fundamental inconsistency between this viewpoint and the viewpoint underlying the IBM, to which we now turn, though there may be conceptual and technical advantages of one or the other approach. After describing the differences between IBM and the previous view, we shall attempt to justify the assertion which initiated this paragraph.

The motivation for the IBM may reside in part in the observation that in improved treatments of nuclear superconductivity it is recognized more explicitly that the basic unit is the correlated pair of particles (not quasi-particles) coupled to angular momentum zero. [We shall continue to talk only about pairs of particles, though if we are beyond the half-shell closure, it becomes convenient to replace particles by holes.] The traditional collective excitation is then interpreted as a pair of particles coupled to $J=2$. In the IBM, these pairs become a pair of fundamental entities, the s and d bosons. The total number of bosons is restricted to be half the number of fermions.

In attempting to assay what the IBM has added to nuclear physics we must remember that there have been two distinct versions of the theory. In the first, no distinction was made between neutrons and protons. This theory could be viewed as a number-conserving generalization of the existing phenomenological theory. The authors of this version must also be credited with emphasizing within this framework the role of certain limiting symmetries and of suggesting that a relatively simple phenomenological Hamiltonian requiring at most the interaction of pairs of bosons would suffice to give a sufficiently good account of nuclear properties. Though the jury is still out on the last suggestion as well as the true significance or necessity of the boson number as a good quantum number, at least one of the limiting symmetries, the $O(6)$, even though shown to be related to the γ -unstable model⁴²⁾ had been largely neglected previously as a choice for the interpretation of experiment.

In the second version⁴³⁾ neutron and proton bosons were accorded separate recognition, thus definitely going beyond previous phenomenology. Agreement with data has certainly been enhanced, but the connection with the limiting symmetries is much less transparent. There is continued insistence on keeping the collective Hamiltonian simple.

The IBM is still a very young theory, even from the phenomenological point of view. We must certainly continue to investigate the potentiality for improving the Hamiltonian, coupling to other degrees of freedom, etc. However these details eventually come out, the IBM will remain important for a number of reasons: (i) A number-conserving theory is a priori a sounder theory than a number non-conserving one and for sufficiently small total number, the differences should be detectable experimentally. (ii) It can be understood more directly than previous phenomenological theories in terms of structures which occur in a fundamental way in shell-model calculations. It is in this sense it has received some shoring up from special shell model calculations^{44,45}). (iii) Finally - and this is what will interest us mostly - it has refocussed attention on the problem of deriving a collective boson description from a shell model description.

Now let us all admit candidly that up to now this problem has not been fully solved within the IBM framework. Our present purpose is to outline a method for achieving this end. In the course of describing this method we shall be led to the conclusion that the extensive work of Tamura and associates is also deficient in at least one element of theory, and that an essential part of our proposal is appropriate to any method of deriving a collective boson Hamiltonian and therefore to that work.

Before descending into the trenches, we first return to the question raised earlier on how we should view the existing work on boson expansions which treat the bosons as quasi-particle pairs. It is then our opinion, that technical details aside, i.e., admitting that they may require some technical improvements, there is no reason why they cannot constitute viable alternatives to the (to be developed) methods which conserve particle number. It is our experience⁴⁶) with related problems (but a different method - the equations of motion method) that even for systems with moderate particle number, a relatively modest improvement over the usual BCS method such as that due to Nogami,⁴⁷) incorporated within the quasi-particle formalism, will bring most of the benefits of number conservation. Another point of conflict, whether we can really stop at fourth order (simple boson-boson interaction), or have to go to sixth order can only be settled ultimately in the trenches.

B. A Multi-Level Boson Correspondence for the IBM.

The general problem is to diagonalize a shell model Hamiltonian approximately by means of a transcription to a boson space. There are at least four essential problems to be solved or decisions to be made before this program can be brought to fruition. Of these the first three discussed below were addressed in some form in the work of Otsuka et al^{48,49}). (i) The choice between particles and quasi-particles. Previously, though particles had been utilized in model calculations involving bosons with $J=0$ ^{50,51}), they had not been utilized in more realistic studies of collective motion involving $J \neq 0$ excitations, except in some preliminary studies on the single j shell by the writer and his coworkers⁵²) within the framework of the equations of motion method. (ii) The size of the original shell model space to be mapped on to the boson space. In previous work this had often been the entire shell model space. Reduction of the size of the space had taken place after the mapping. Another departure from previous work is the suggestion that one work backwards. Start with the boson space of desired structure (s and d bosons) and work backwards to define a fermion space of suitable structure which

will map onto this boson space without extraneous elements in either space. (iii) There are alternate formal methods for carrying through the correspondence from the fermion space to the boson space known as the Belyaev-Zelevinsky method³⁸⁾, the Marumori method^{53,54)} and the Holstein-Primakoff method^{55,56)}. All are equivalent if carried through exactly⁵⁷⁾. Workers choosing quasi-particle methods have all utilized one of the first two methods. Following through on their concept under (ii) the IBM workers^{48,49)} have found it natural to use the Holstein-Primakoff method. Though we cannot go into such details here, this allows them to treat s-boson effects without expansion. There is still an expansion in d-boson effects, however. We shall return to this point in our concluding remarks. (iv) the observations under (i)-(iii) completely characterize the work of Otsuka et al and indeed we shall adopt their suggestions in seeking to extend their work in the direction of greater realism, by allowing more than one single particle level. As illustration only we shall use two levels with the property that the equation

$$\vec{j}_1 + \vec{j}_2 = \vec{2} \quad (4.1)$$

possesses a solution.

To see the essential new problem we note that we shall want to study the correspondence

Fermion Space, Pair Operators		Boson Space, Boson Operators
J=0 $A^{(0)\dagger}(11), A^{(0)\dagger}(22)$	→	$b_1^\dagger, b_2^\dagger$
J=2 $A^{(2)\dagger}(11), A^{(2)\dagger}(22), A^{(2)\dagger}(12)$	→	$b_{11}^\dagger, b_{22}^\dagger, b_{12}^\dagger$ (4.2)

The notation compresses all quantum numbers associated with j_1 into the numerical subscript and similarly for j_2 . In the fermion space there are two monopole pairs and three quadrupole pairs, two pure j_1 or j_2 and according to (4.1), one mixed operator. Each of these five operators is mapped onto the corresponding boson. Below we shall describe how to carry out this mapping. But now we see that in the boson space we have more degrees of freedom than we wish to deal with. How do we choose the s boson and the d boson? We address (and solve) this vital problem in subsection C.

For the remainder of this subsection we study the problem framed by Eq. (4.2). The first thought is to use the mapping given by Otsuka et al and take a direct product of two such mappings. This is too naive because we want to treat all quadrupole operators on an equal footing and there are no mixed quadrupole operators in the direct product without further coupling. A second thought is whether we must couple all operators to a resultant angular momentum in the fermion space before carrying out the mapping, or put less naively, what actual basis is "best" to use in the fermion space. As it turns out the decision is determined by the dynamical method for picking the physical s and d bosons. Therefore we shall describe our (simple) proposal and justify its utility later.

Following Otsuka et al, we design the correspondence looking backwards from the boson space, i.e., we write down the boson basis desired and then infer the required structure of the fermion space. In the boson space we choose a direct product of five subspaces, the $b_1^\dagger$ and $b_2^\dagger$ (kinematical) s-boson subspaces and three SU(5)

"seniority" boson bases for b_{11}^+ , b_{22}^+ , and b_{12}^+ . Each of these subspaces is characterized by states of definite angular momentum, among other quantum numbers, but we do not choose to couple these together, as it would turn out to be an unnecessary complication.

Next, in the fermion space, construct from the operators displayed in (4.2), states of maximum seniority with angular momentum coupled together for each kind of quadrupole pair. To construct these states most simply we must replace the given quadrupole operators by operators of pure seniority defined as follows: Consider a single j and the seniority description of the monopole operators. The operator

$$P_P = \frac{p S_0 - \frac{1}{2}p(\frac{1}{2}p+1) + S_+S_-}{p S_0 - \frac{1}{2}p(\frac{1}{2}p+1)} , \quad (4.3)$$

where

$$\begin{aligned} S &= \frac{1}{2} (\Omega - v) , & \Omega &= j + \frac{1}{2} \\ S_0 &= \frac{1}{2} (n - \Omega) , \\ S_+S_- &= \hat{S}^2 - S_0^2 + S_0 , \end{aligned} \quad (4.4)$$

annihilates the state $|j^{v+p}, n\rangle$ and is unity when acting on the state $|j^v, v\rangle$. Here v is the seniority and n the number of particles in the given shell. Thus the operator

$$D^{(2)+}(11) \equiv P(11) A^{(2)+}(11) , \quad (4.5)$$

where

$$P(11) = P_2(1)P_4(1) , \quad (4.6)$$

is a product of two operators (4.3), definitely increases the seniority by two as it definitely increases the number of particles by two. $D^{(2)}(22)$ is similarly defined, but for the mixed quadrupole, we make the replacement

$$D^{(2)+}(12) = P(12) A^{(2)+}(12) , \quad (4.7)$$

$$P(12) = P_1(1) P_1(2) . \quad (4.8)$$

For the considerations which follow we need a succinct labeling of states. We shall use the same labeling for both the fermion and boson spaces. Let

$$\begin{aligned} n_1 &= \text{no. of } s \text{ fermion pairs in } j_1 \\ &= \text{no. of } b_1 \text{ bosons,} \\ n_2 &= \text{no. of } s \text{ fermion pairs in } j_2 \\ &= \text{no. of } b_2 \text{ bosons,} \\ &\vdots \\ n_{ij} &= \text{no. of } d \text{ fermion pairs described by } A^{(2)+}(ij) \\ &= \text{no. of } b_{ij} \text{ bosons.} \end{aligned} \quad (4.9)$$

Thus if n = number of fermion pairs = number of bosons, we have

$$n = n_1 + n_2 + n_{11} + n_{22} + n_{12} , \quad (4.10)$$

$$N_1 \equiv 2n_1 + 2n_{11} + n_{12} , \quad (4.11)$$

$$N_2 = 2n_2 + 2n_{22} + n_{12} \quad , \quad (4.12)$$

$$v_1 = 2n_{11} + n_{12} \quad , \quad (4.13)$$

$$v_2 = 2n_{22} + n_{12} \quad , \quad (4.14)$$

where N_i = number of fermions in level i and v_i is the associated seniority. A state with quantum numbers (4.10-14) will be labelled

$$|n_1, n_2, n_{11} J_{11} \alpha_{11}, n_{22} J_{22} \alpha_{22}, n_{12} J_{12} \alpha_{12}\rangle_{F \text{ or } B} \quad , \quad (4.15)$$

where J_{ij} refers to the angular momentum quantum numbers and α_{ij} to the remaining quantum numbers, and the subscript F or B, as required, distinguishes fermion from boson space.

Having defined the spaces involved, the technique of obtaining the fermion-boson mapping is essentially the same as that described by Otsuka et al. It is out of place to list formulas here, and even more out of place to describe the elementary but tiresome bookkeeping and angular momentum algebra which is used to derive them. What is necessary is to describe why such a relatively trivial extension of the single j correspondence suffices to do the multi-shell problem.

C. How Can One Determine the "Physical" s and d Bosons?

We turn then to the heart of the matter. This problem is central not only to any eventual success in trying to derive the IBM but also appears in modified form in the older work where, it has also been treated inadequately, or at least questionably in all previous work.

Let us therefore review briefly the history of this subject. Underlying our discussion let us imagine the typical microscopic Hamiltonian utilized, consisting of the sum of a shell model term plus a pairing interaction plus one or more quadrupole-quadrupole interactions. In the vibrational regime, when pairing dominates over quadrupole effects, everyone agrees that the quasi-particle random phase approximation (RPA) is a reasonable first approximation to the collective quadrupole boson and indeed little is lost by ignoring ground state correlations and therefore using the Tamm-Dancoff (T.D.) collective boson. As quadrupole correlations increase i.e., as we attempt to describe transition nuclei the RPA breaks down. At this juncture, the only recourse appears to be to choose the collective T.D. boson or even some schematic approximation to it.

Now what is wrong with this method? It is certainly ad hoc to return for strong quadrupole coupling, faute de mieux, to a method which is justified for weak coupling. Since in the transition to collective variables we throw away all or almost all of the Hamiltonian other than that which depends on the collective variables, without too much hope moreover of honestly checking our approximation in a systematic way, it behooves us to choose that collective subspace in an optimal way at the outset rather than utilize an ad hoc procedure.

Therein lies the heart of our criticism of previous approaches. Either there has been no optimization of a space of choices or else it has been an inadvertent one. We translate the previous verbiage to mean that we must seek a variational criterion. But the

usual variational principle deals with only one state of the system at a time, and along with whatever technical problems we may encounter is not conceptually adequate for the problem at hand.

The essence of nuclear collectivity is that we wish to bring down an entire collective subspace of states - that of the s and d bosons in the current phrasing of the problem - to an average energy which is lower than that of the remaining states of the given nucleus. It is not that difficult, it turns out, to put the criterion just mentioned on a quantitative basis. Since the average energy of a set of states is proportional to the trace of the Hamiltonian over the subspace considered, a necessary condition for determining this subspace is the condition

$$\delta(\text{Tr } H) = 0, \quad (4.16)$$

which can be read that the average energy is stationary.

This variational principle is certainly a consequence of the usual Rayleigh-Ritz variational principle. The real question is whether it can be associated with a full dynamical scheme which can be carried out in practice to some reasonable approximation. In fact, the relation of Eq. (4.16) with a full dynamical scheme for collective motion has been known to the writer for over a decade⁵⁸⁾, but it has recently^{59,60)} been moving into an even larger place in his scheme of things⁶¹⁾. See also⁶¹⁾. In the more recent work, it has been shown how such a variational statement combined with suitable kinematical constraints, arising mainly from commutation relations, provides a complete scheme of quantum mechanics and that in approximate form it leads to the kind of approach to collective motion which has been the writer's stock-in-trade. In fact, the variational approach was also used in a number of model calculations (see 52) for instance) without, however, the full awareness of its role in the structure of the theory.

We now describe how the variational condition (4.16) may be used in a new and essential way to solve the problem of finding the s and d bosons. WHAT WILL EMERGE IS THAT THE VARIATIONAL PRINCIPLE DIVIDES THE PROBLEM INTO THE DESIRED TWO PARTS, IN THAT IT FIRST DETERMINES THE COLLECTIVE HAMILTONIAN. AFTERWARDS THE LATTER MAY BE DIAGONALIZED CONVENTIONALLY. In fact it may be more informative to show in detail how this method is being applied to a test model in order to verify its applicability. From this description, it will become clear - and in any event we shall afterwards explain - why the mapping described in subsection B suffices to carry through the method in the general case.

To test the method we choose a multi-level pairing model (number of levels left indefinite)

$$H = \sum_i h_i N_i - G \sum_{ii'} A_i^\dagger A_{i'}, \quad (4.17)$$

where the operators are defined in terms of the shell model operators by the equations

$$N_i = \sum_m a_{im}^\dagger a_{im} \quad (4.18)$$

$$A_i^\dagger = \sum_{m>0} (-)^{j_i-m} a_{i,m}^\dagger a_{i,-m}^\dagger = (A_i)^\dagger, \quad (4.19)$$

involving a sum over magnetic quantum numbers. The index i

comprises all other single particle quantum numbers, h_i is a single particle energy, and G is the pair coupling constant. The operators $N_i, A_i, A_i^\dagger$ satisfy the algebra $SU(2) \times SU(2) \times \dots$

$$[A_i, A_i^\dagger] = \delta_{ii}, [A_i, N_i] = 0, \quad (4.20)$$

$$[A_i, N_i] = 2\delta_{ii}, A_i^2 = 0, \quad (4.21)$$

where $2N_i = 2j_i + 1$.

Now the first task is to map a basis in the shell model space onto a boson basis. For the model under consideration this is a well-studied problem⁵¹⁾. One maps the fermion states for each level,

$$|p_i\rangle_F = N_p (A_i^\dagger)^p |0\rangle_F, \quad (4.22)$$

where N_p is a normalization constant onto the boson basis

$$|p_i\rangle_B = (p!)^{-1/2} (B_i^\dagger)^p |0\rangle_B. \quad (4.23)$$

Here $B_i, B_i^\dagger$ satisfy the canonical commutation relations

$$[B_i, B_i^\dagger] = 1. \quad (4.24)$$

The total mapping is then the direct product of the mappings for each individual level.

The second task is to choose a version of the fermion pair-boson operator correspondence. Of the three possibilities mentioned in our preliminary remarks, we choose the Holstein-Primakoff version because it is the most highly summed form of correspondence and the analogue of what we wish to do in the real problem. The well-known^{50,51)} formulas are

$$(N_i)_B = 2B_i^\dagger B_i, \quad (4.25)$$

$$(A_i^\dagger)_B = (\Omega_i)^{1/2} B_i^\dagger \left[1 - \frac{B_i^\dagger B_i}{\Omega_i} \right]^{1/2}. \quad (4.26)$$

If we substitute (4.25,26) into (4.17) we obtain the corresponding Hamiltonian in the boson space,

$$H_B = 2 \sum_i h_i B_i^\dagger B_i - G \sum_{ii'} (\Omega_i \Omega_{i'})^{1/2} B_i^\dagger \left[1 - \frac{B_i^\dagger B_i}{\Omega_i} \right]^{1/2} \left[1 - \frac{B_{i'}^\dagger B_{i'}}{\Omega_{i'}} \right]^{1/2} B_{i'}, \quad (4.27)$$

So far all we have done is to map the multi-level pairing problem onto a boson space. As long as we remember that only boson basis states are permitted for which the square roots in (4.27) yield real numbers, we have not changed (or indeed simplified) the original problem. The next step is the crucial one. We introduce a linear, unitary (ultimately orthogonal) transformation from the original, kinematical, bosons B_i to a new set of boson, S_λ , to be chosen according to the dynamical criterion we have introduced in (4.16), and expressed by the standard transformation equations,

$$B_i^\dagger = \sum_\lambda \alpha_{i\lambda} S_\lambda^\dagger, \quad (4.28)$$

$$S_{\lambda}^{\dagger} = \sum_i \alpha_{i\lambda}^* B_i^{\dagger}, \quad (4.29)$$

$$\sum_{\lambda} \alpha_{i\lambda}^* \alpha_{i'\lambda} = \delta_{ii'}, \quad (4.30)$$

$$\sum_i \alpha_{i\lambda}^* \alpha_{i\mu} = \delta_{\lambda\mu}. \quad (4.31)$$

A typical square root factor in (4.27), which can be handled exactly within the context of the direct product basis constructed from the $B_i^{\dagger}$, becomes

$$\left[1 - \sum_{\lambda\mu} \frac{\alpha_{i\mu} \alpha_{i\lambda}^* S_{\lambda}^{\dagger} S_{\mu}}{\Omega_i} \right]^{\frac{1}{2}}, \quad (4.32)$$

and can be dealt with in the calculations which follow (which will utilize the direct product basis constructed from the $S_{\lambda}^{\dagger}$) only by partial expansion.

To illustrate our viewpoint we consider an approximation which has some formal analogy with the real physical problem involving the s and d bosons.

We assume that the low-lying states can be constructed largely from the basis states of two bosons, $S \equiv S_1^{\dagger}$ and $T \equiv S_2^{\dagger}$. Let $\alpha_{i1} \equiv \alpha_i$, $\alpha_{i2} \equiv \beta_i$. We truncate the sum in (4.28), and (4.29) becomes unapplicable, similarly (4.30). The orthonormalization conditions (4.31) become

$$\sum_i \alpha_i^* \alpha_i = \sum_i \beta_i^* \beta_i = 1, \quad (4.33)$$

$$\sum_i \alpha_i^* \beta_i = \sum_i \beta_i^* \alpha_i = 0. \quad (4.34)$$

Since (4.27) conserves the total number of bosons, this translates into the conservation of

$$N = N_S + N_T \equiv S^{\dagger} S + T^{\dagger} T. \quad (4.35)$$

Again, it will be difficult to deal with the radical (4.32) unless we can assume that $N_T \ll N$ and we shall therefore proceed with this assumption. Thus, with the help of (4.35) we write (4.32) as

$$(4.32) = \left[1 - \frac{\alpha_i^* \alpha_i N}{\Omega_i} + \frac{T^{\dagger} T}{\Omega_i} (\alpha_i \alpha_i^* - \beta_i \beta_i^*) - \frac{S^{\dagger} T}{\Omega_i} \alpha_i \beta_i^* - \frac{T^{\dagger} S}{\Omega_i} \beta_i \alpha_i^* \right]^{\frac{1}{2}}. \quad (4.36)$$

Retaining only the first two terms under the radical and expanding the remainder to first order only - to illustrate procedure - we derive an approximate boson Hamiltonian of the form

$$H_B = H_d + H_{od}, \quad (4.37)$$

where H_d is diagonal in the direct product basis constructed from $S^{\dagger}$ and $T^{\dagger}$ and is given by the expression

$$H_d = e_1 S^\dagger S + e_2 T^\dagger T + v_{12,12} S^\dagger T^\dagger T S + v_{22,22} T^\dagger T^\dagger T T \quad (4.38)$$

where, with the definition

$$r_i \equiv \Omega_i^{-1/2} \left[1 - \frac{(N-1) \alpha_i \alpha_i^*}{\Omega_i} \right]^{1/2} \quad (4.39)$$

the coefficients in (4.38) are given by the expressions

$$e_1 = 2 \sum_i h_i |\alpha_i|^2 - G \left| \sum_i r_i \alpha_i \right|^2 \quad (4.40)$$

$$e_2 = 2 \sum_i h_i |\beta_i|^2 - G \left| \sum_i r_i \beta_i \right|^2 \quad (4.41)$$

$$\begin{aligned} v_{12,12} = & -\frac{1}{4} G \sum_{ii'} r_i \alpha_i r_{i'} \alpha_{i'}^* \{ r_i^{-2} (\alpha_i \alpha_i^* - \beta_i \beta_i^*) + (i \leftrightarrow i') \} \\ & - \frac{1}{2} G \sum_{ii'} r_i \beta_i r_{i'} \alpha_{i'}^* \{ r_i^{-2} \alpha_i \beta_i^* + (i \leftrightarrow i') \} \\ & + \text{C. C.} \quad (4.42) \end{aligned}$$

$$v_{22,22} = -\frac{1}{2} G \sum_{ii'} r_i \beta_i r_{i'} \beta_{i'}^* \{ r_i^{-2} (\alpha_i \alpha_i^* - \beta_i \beta_i^*) + (i \leftrightarrow i') \} \quad (4.43)$$

Corresponding expressions for H_{od} will not be quoted here because they do not enter into the present discussion, i.e., they do not enter into the calculation of $\text{Tr } H_B$ if we use the designated basis.

We are prepared to apply (4.16), but before we can do so, we must render a decision based on physical considerations. Thus the underlying idea here is that the ground state is dominated by the basis vector

$$\frac{(S^\dagger)^N}{(N!)^{1/2}} |0\rangle \quad (4.44)$$

and that the low-lying states are mainly $T^\dagger$ excitations. Generally speaking, it makes mathematical but not physical sense to compute the trace over the entire space of fixed N ,

$$\frac{(S^\dagger)^{N-n}}{\{(N-n)!\}^{1/2}} \frac{(T^\dagger)^n}{(n!)^{1/2}} |0\rangle \quad (4.45)$$

This is because the parameters of the Hamiltonian will dictate that some new excitation will enter (in energy) before the very high excitations associated with $T^\dagger$. The method is, however, completely flexible since subspaces can be enlarged or pared, and once a choice has been made, it is often trivial to calculate the quantity

$$\begin{aligned} F(\underline{\alpha}, \underline{\beta}, N) & \equiv N^{-1} \text{Tr } H_B \\ & = N^{-1} \text{Tr } H_d \quad (4.46) \end{aligned}$$

which now depends on the vectors $\underline{\alpha}$, $\underline{\beta}$ and the value of N , in addition to the parameters of the original Hamiltonian.

Finally then, we require that (4.46) be stationary with respect to the choice of the vectors $\underline{\alpha}$, $\underline{\beta}$, subject to the constraints (4.33)

and (4.34). We demand

$$0 = \delta K = \delta F - \epsilon_1 \delta \sum_i \alpha_i \alpha_i^* - \epsilon_2 \delta \sum_i \beta_i \beta_i^* - \epsilon_{12} \delta \sum_i \alpha_i \beta_i^* - \epsilon_{21} \delta \sum_i \alpha_i^* \beta_i. \quad (4.47)$$

A well-known consistency requirement for the Lagrange multipliers is that

$$\epsilon_{21} = \epsilon_{12}^* \quad (4.48)$$

With the definitions

$$\partial F / \partial \alpha_i^* \equiv X_i \quad (4.49)$$

$$\partial F / \partial \beta_i^* \equiv Y_i, \quad (4.50)$$

we obtain the non-linear equations

$$X_i - \epsilon_1 \alpha_i - \epsilon_{12}^* \beta_i = 0, \quad (4.51)$$

$$Y_i - \epsilon_2 \beta_i - \epsilon_{12} \alpha_i = 0, \quad (4.52)$$

which, together with (4.33) and (4.34) constitute enough equations to determine α , β and the Lagrange multipliers. It is consistent here to choose all quantities to be real.

With the help of the constraints applied to (4.51) and (4.52), we can derive the equations

$$\epsilon_1 = \sum_i \alpha_i^* X_i, \quad (4.53)$$

$$\epsilon_2 = \sum_i \beta_i^* Y_i, \quad (4.54)$$

$$\epsilon_{12}^* = \sum_i \beta_i^* X_i, \quad (4.55)$$

$$\epsilon_{12} = \sum_i \alpha_i^* Y_i. \quad (4.56)$$

the equality of the last two equations providing a consistency check.

As of this writing although our program is working, the numerical analysis associated with the equations just described is incomplete and therefore will not be reported. Our purpose in giving this account is to show that it yields a well-defined calculational procedure of a standard non-linear algebraic type, and, except for the (non-essential) algebraic complexity of the realistic case, provides a prototype for the latter, as we discuss below.

The equations we have described contain several well-known limiting cases: (i) If we keep only the S^z boson, our equations characterize a number-conserving BCS approximation, carried out within the boson framework. For a single j level, it becomes exact. (ii) If the full complement of bosons is retained, but only single excitations are allowed, and the equations are linearized, we obtain an approximation equivalent to the T. D. approximation.

D. Concluding Observations.

We should now be able to understand why the mapping described in subsection B is adequate for the purposes for which it is intended. Thus the bosons b_i and b_{ij} indicated under (4.2) will play the same role as the kinematical bosons B_i of the pairing discussion. Special linear combinations will be sought, separately for the $J=0$ and $J=2$ bosons which will render $\text{Tr } H$ stationary in a suitable subspace and this together with constraints will determine the "physical bosons".

In searching the literature for some previous realization that there was some need for a criterion to characterize the most collective modes, we have become aware of only one previous attempt in this direction. In B. Sorensen's last paper on boson expansions⁶², he invokes a variational criterion which we can view as an approximate alternative to our proposal: He replaces the trace criterion by the requirement of minimization of the first excitation energy of the nucleus. The equations for the determination of the collective excitation are the T. D. approximation to an effective Hamiltonian wherein one quadrupole coupling parameter is varied to obtain the minimized energy.

Let us recall once more that the correspondence defined under (4.2) involves seniority bases. The ensuing boson expansion appears to have as parameter the ratio of the average occupation number of d bosons to some measure of the total degeneracy. Though we now believe that these series converge rapidly enough for transitional nuclei so that the method will be viable for them, it is still not clear, though one remains optimistic, why the same method should work for strongly deformed nuclei. Therefore it remains of interest to search for correspondences alternative to the seniority bases favored thus far, correspondences which start from the strong coupling limit.

We remark finally that the method espoused here for the determination of the physical s and d bosons applies as well, with some technical modification, to the quasi-particle approach to boson expansions (Sorensen's work was within the quasi-particle framework) and represents therefore a missing element in current work.

E. Acknowledgement

The work described in this section is being carried out in collaboration with Dr. M. Vallieres of Drexel University. Work supported by U.S. Department of Energy under contract No. EY-76-C-02-3071.

References

1. T Marumori, A. Hayashi, T. Tomoda, A. Kuriyama, and T. Maskawa, University of Tokyo Institute for Nuclear Study Report INS-Rep.-347 (1979).
2. V. G. Zelevinsky, NORDITA preprint ISSN-0106-2646 (1979).
3. G. Rosensteel and D. J. Rowe, Phys. Rev. Letters 38 (1977) 10.
4. D. J. Rowe and G. Rosensteel, Proc. Int. Colloquium on Group Theoretical Methods in Physics, ed. P. Kramer and A. Rieckers (Springer-Verlag, 1978).
5. M. A. J. Mariscotti, G. Scharff-Goldhaber and B. Buck, Phys. Rev. 178 (1969) 1864.

6. G. Scharff-Goldhaber, C. B. Dover and A. L. Goodman, in Ann. Rev. Nucl. Sci. (Annual Reviews, Inc., Palo Alto, 1976) Vol. 26, p. 239.
7. S. M. Harris, Phys. Rev. 138 (1965) B509.
8. G. Scharff-Goldhaber and A. Goldhaber, Phys. Rev. Lett. 24 (1970) 1349.
9. T. K. Das, R. M. Dreizler, and A. Klein, Phys. Rev. C2 (1970) 632.
10. E. R. Marshalek, Phys. Rev. C4 (1971) 1710.
11. M. I. Stockmann and V. G. Zelevinsky, Phys. Letters 41B (1972) 19.
12. A. N. Mantri and P. C. Sood, Phys. Rev. C7 (1973) 1294.
13. J. L. Wood and R. W. Fink, Nucl. Phys. A224 (1974) 589.
14. Interacting Bosons in Nuclear Physics, ed. F. Iachello (Plenum Press, New York, 1979).
15. J. E. Draper, Physics Letters 32 (1970) B581.
16. T. K. Das, R. M. Dreizler, and A. Klein, Phys. Letters 34 (1971) B235.
17. A. B. Volkov, Phys. Letters 35 (1971) B299.
18. B. C. Smith and A. B. Volkov, Phys. Letters 47 (1973) B193.
19. T. K. Das, R. M. Dreizler, and A. Klein, Phys. Rev. Letters 25 (1970) 1626.
20. S. M. Abecasis, Nucl. Phys. A180 (1972) 485.
21. A. Arima and F. Iachello, Ann. of Phys. (N.Y.) 99 (1976) 253.
22. M. Sakai and Y. Gono, U. of Tokyo Inst. for Nucl. Study report INS-J-160 (1979).
23. G. Scharff-Goldhaber, M. Mckeown, A. H. Lumpkin and W. F. Diel, Jr., Phys. Letters 44B (1973) 416.
24. W. F. Diel Jr. and G. Scharff-Goldhaber, Bull. Am. Phys. Soc. 23 (1978) 555.
25. G. Scharff-Goldhaber, J. Phys. G: Nucl. Phys. 5 (1979) L207.
26. F. S. Stephens and R. S. Simon, Nucl. Phys. A183 (1972) 257.
27. C. Flaum and D. Cline, Phys. Rev. C14 (1976) 1224.
28. F. S. Stephens in Elementary Modes of Excitation in Nuclei, Course LXIX, Proc. International School of Physics, "Enrico Fermi", ed. A. Bohr and R. A. Broglia, (North-Holland, Amsterdam, 1977) p. 172.
29. M. Reinecke and H. Ruder, Zeit. f. Physik A279 (1976) 93.
30. M. Reinecke and H. Ruder, Zeit. f. Physik A282 (1977) 407.
31. M. Reinecke and H. Ruder, Zeit. f. Physik A290 (1979) 395.
32. A. Bohr and B. R. Mottelson, Nuclear Structure (W. A. Benjamin, Reading, 1975) Vol. II, Chap. 4.
33. R. A. Broglia, A. Molinari, G. Pollaro and T. Regge, Phys. Lett. 57B (1975) 113.
34. Y. El Masri, J. Vervier, and A. Faessler, Nucl. Phys. A279, (1977) 223.

35. N. R. Johnson, D. Cline, S. W. Yates, F. S. Stephens, L. L. Reidinger, and R. M. Konning, Phys. Rev. Letters 40 (1978) 151.
36. M. Baranger, Phys. Rev. 120 (1960) 957.
37. D. J. Rowe, Nuclear Collective Motion (Methuen, London, 1970), Chaps. 13, 14.
38. S. T. Beliaev and V. G. Zelevinsky, Nucl. Phys. 39 (1962) 582.
39. T. Kishimoto and T. Tamura, Nucl. Phys. A192 (1972) 264.
40. T. Kishimoto and T. Tamura, Nucl. Phys. A270 (1976) 317.
41. T. Tamura, K. Weeks, T. Kishimoto, Phys. Rev. C20 (1979) 307.
42. J. Meyer-ter-Vehn, in ref. 14, Chap. 5.
43. F. Iachello in ref. 14, Chap. 1.
44. K. T. Hecht, J. B. McGrory and J. P. Draayer, Nucl. Phys. A197 (1972) 369.
45. J. B. McGrory in ref. 14, Chap. 3.
46. C. T. Li, P. K. Chattopadhyay, A. Klein, and M. J. Vassanji, Phys. Rev. C19 (1979) 2002.
47. Y. Nogami, Phys. Rev. 134 (1964) B313.
48. T. Otsuka, A. Arima, and F. Iachello, Nucl. Phys. A309 (1978) 1.
49. T. Otsuka, Unpublished thesis, Dept. of Physics, U. of Tokyo (1978).
50. S. C. Pang, A. Klein, and R. M. Dreizler, Ann. of Phys. (N.Y.) 49 (1968) 477.
51. M. Kleber, Phys. Letters 30B (1969) 588.
52. M. Vallieres, A. Klein, and R. M. Dreizler, Phys. Rev. C7 (1973) 2188.
53. T. Marumori, M. Yamamura, and A. Tokunaga, Prog. Theoret. Phys. (Kyoto) 31 (1964) 1009.
54. S. Y. Li, R. M. Dreizler and A. Klein, Phys. Rev. C4 (1971) 1571.
55. T. Holstein and H. Primakoff, Phys. Rev. 58 (1950) 1098.
56. E. R. Marshalek, Nucl. Phys. A224 (1974) 221.
57. E. R. Marshalek, Nucl. Phys. A161 (1971) 401.
58. G. Do Dang, G. J. Dreiss, R. M. Dreizler, A. Klein and Chi-Shiang Wu, Nucl. Phys. A114 (1968) 501.
59. A. Klein and C. T. Li, Physica 96A (1979) 243.
60. A. Klein, C. T. Li and M. Vassanji, U. of Pennsylvania report UPR-0138T (1979).
61. D. Janssen, Yad. Fiz. 28 (1978) 935. (Trans. Sov. J. Nucl. Phys. 28 (1978) 479.)
62. B. Sorensen, Nucl. Phys. A217 (1973) 505.