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By

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Particle Diffusion in Tokamaks With Partially Destroyed
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ABSTRACT

A Hamiltonian formalism is developed for the drift orbit trajectories of particles in toroidal systems in the presence of stochastic fields. The equations of motion are integrated numerically to investigate the modification of neoclassical diffusion in a Tokamak due to the onset of stochasticity. Quasilinear diffusion is observed for fields with well developed stochasticity. A significant increase in the diffusion coefficient is observed below the stochastic threshold for electrons, whereas ions are typically not affected until the magnetic field has become quite stochastic.

The effects of asymmetry on particle orbits and transport in magnetized plasmas is a topic of considerable interest especially in the toroidal magnetic fusion program. These effects are of two basic types. First, asymmetric magnetic fields may not have magnetic surfaces. That is, a magnetic field line may wander over a finite volume of space instead of lying on one of a set of nested toroidal surfaces as ideally assumed. Since charged particles, to lowest order in the gyroradius, follow field lines, stochastic field regions have poorer plasma confinement than regions with nested magnetic surfaces. Although the partial destruction of magnetic surfaces is a probable cause for the observed anomalous transport, calculations of the magnitude of the transport coefficients in such a field have thus far been limited to rough estimates.^{1,2,3}

Second, even if nested surfaces exist, the asymmetries in a magnetic field result in a slow drift motion across the magnetic surfaces, which also degrades the confinement.

Magnetic coordinates have proven invaluable in the study of the effects of asymmetries on orbits and transport when exact surfaces exist.⁴ By magnetic coordinates, we mean coordinates in which the magnetic field lines are straight. Since these coordinates assume exact magnetic surfaces, the generalization of the methodology to arbitrary field structures is nontrivial. In this paper, a method of studying drift orbits and transport in stochastic magnetic fields using magnetic coordinates is developed. The basic assumption is that there is a magnetic field configuration with nested surfaces which neighbors the field configuration of interest. The method is applied to the important question of the modification of the tokamak neoclassical transport due to stochasticity.

The magnetic coordinates which are most useful for orbit studies,

are ψ , θ , and χ with

$$\vec{B} = \nabla\psi \times \nabla\theta \quad , \quad (1)$$

$$\vec{B} = \nabla\chi + \beta\nabla\psi \quad . \quad (2)$$

For locally curl-free fields, β is zero. These coordinates are closely related to ψ , θ , and ϕ coordinates which have the periodicities of a torus.⁵ The coordinate ψ labels the flux surface, $\theta = \theta_0 + \phi/q(\psi)$ and $\chi = g(\psi)\phi + I(\psi)\theta$ with $q(\psi)$ the inverse rotational transform, $cq/2$ the poloidal current outside the surface ψ , and $cI/2$ the toroidal current inside the surface.

Unfortunately, magnetic coordinates are well defined in toroidal geometry only when magnetic surfaces exist. Even the presence of small islands greatly complicates these coordinates. However, the simplicity and elegance of the magnetic coordinate representation of drift motion can be preserved in the analysis of motion in stochastic fields if it is assumed that a neighboring magnetic configuration with nested surfaces exists. More precisely, we assume the exact magnetic field $\vec{B}_e$ is given by

$$\vec{B}_e = \vec{B} + \nabla \times (\alpha \vec{B}) \quad , \quad (3)$$

with $\vec{B}$ having exact, nested, magnetic surfaces. The function α , which has units of length, is used to represent the islands and ergodic regions of the field $\vec{B}_e$.

To understand how α introduces magnetic islands and stochasticity, consider the general Fourier representation of a function of position in a

torus.

$$\alpha(\psi, \theta, \phi) = \sum_{n,m} \alpha_{nm}(\psi) \cos(n\phi - m\theta + \phi_{nm}) \quad (4)$$

Topology changes to a magnetic configuration are dominated by $\vec{B}_e \cdot \vec{\nabla} \psi / \vec{B} \cdot \vec{\nabla} \phi$ and

$$\frac{\vec{B}_e \cdot \vec{\nabla} \psi}{\vec{B} \cdot \vec{\nabla} \phi} = - \sum_{n,m} (mg + mI) \alpha_{nm} \sin(n\phi - m\theta + \phi_{nm}) \quad (5)$$

Magnetic islands are produced about each rational surface $q_r = m/n$.

In the normal ordering of a Tokamak, which is $\epsilon \ll 1$ and $q = 1$ with $\epsilon = a/R$, the ratio of the minor and major toroidal radii, the magnetic flux is $\psi = r^2/2$ and the half width of the islands is

$$\left(\frac{\Delta r}{r}\right)_{nm}^2 = \left| \frac{4q^2}{r(dq/dr)} \frac{B_\phi}{B_\theta} \frac{\alpha_{nm}}{r} \right| \quad (6)$$

where $q(r)$ is the safety factor. One finds $\alpha/r \sim 10^{-3}$ gives islands with $\Delta r/r \sim 10\%$. Clearly, α can be very small, as small as a gyroradius, and still give a rich structure to the field. If islands about neighboring rational surfaces overlap, the field structure becomes stochastic.

To develop an equation for the drift velocity of a particle in the field $\vec{B}_e = \vec{B} + \vec{\nabla} \times \alpha \vec{B}$, consider the guiding center velocity in the field $\vec{B}$, which has nested magnetic surfaces. This velocity is⁶

$$\vec{v} = \frac{v_{||}}{B} (\vec{B} + \vec{\nabla} \times \rho_{||} \vec{B}) \quad (7)$$

with $\rho_{||}$, the parallel gyroradius, equal to the parallel velocity of the particle $v_{||}$ divided by the cyclotron frequency, eB/mc . The parallel

gyroradius is taken to be a function of energy E , magnetic moment μ , and position. The energy is given by

$$E = \frac{1}{2} m v_{\parallel}^2 + \mu B + e\Phi , \quad (8)$$

with Φ the electrostatic potential.

The obvious modification of the guiding center velocity which includes the effects of a small perturbation described by α is

$$\vec{v} = \frac{v_{\parallel}}{B} [\vec{B} + \vec{v} \times (\rho_{\parallel} + \alpha)\vec{B}] , \quad (9)$$

with the equation for the energy unchanged. In the limit of zero gyroradius

$$\vec{v} = \frac{v_{\parallel}}{B} \vec{B}_e ; \quad (10)$$

so particles follow the exact field lines. The only error in Eq. (9) is the neglect of direct changes in the field strength produced by α . In the presence of a parallel current, these changes in field strength are proportional to α . Otherwise, they are proportional to α^2 . Changes in the field strength effect the perpendicular drift velocity which is also proportional to the gyroradius. Topology changes, which are proportional to $\alpha^{1/2}$, modify the parallel motion and are clearly dominant for α much smaller than the system size.

If we think of $\vec{B}$ as the magnetic field, the modified drift velocity, Eq. (9), conserves particles, energy, and magnetic moment when used in a drift kinetic equation

$$\vec{v} \cdot \nabla f = C(f) \quad . \quad (11)$$

Using Eq. (9), drift equations in magnetic coordinates can be derived. For simplicity assume $\vec{B}$ is locally curl free. The appropriate magnetic coordinates are $\vec{B} = \nabla\chi = \nabla\psi \times \nabla\theta_0$. The guiding center equations are given by a Hamiltonian

$$H(\rho_c, \chi, \psi, \theta_0) = \frac{1}{2} (\rho_c - \alpha)^2 B^2 + \mu B + \Phi \quad , \quad (12)$$

where we have introduced dimensionless units by normalizing B to its value on axis, B_0 , distances to the minor radius a , and frequencies to the gyro frequency $\omega_0 = eB_0/mc$. The energy is thus given in terms of the basic unit $E_0 = ma^2 \omega_0^2$. A more convenient parameter than the energy is the ratio of gyro radius to system size, given in our units by $\rho/a = (2E)^{1/2}$. The canonical momentum is $\rho_c = \rho_\parallel + \alpha$. Hamilton's equations are

$$\begin{aligned} \frac{d\chi}{dt} &= \frac{\partial H}{\partial \rho_c} \quad , & \frac{d\rho_c}{dt} &= - \frac{\partial H}{\partial \chi} \quad , \\ \frac{d\theta_0}{dt} &= \frac{\partial H}{\partial \psi} \quad , & \frac{d\psi}{dt} &= - \frac{\partial H}{\partial \theta_0} \quad . \end{aligned} \quad (13)$$

In effect, the use of ρ_c as a canonical variable follows the standard Hamiltonian prescription of adding the vector potential to the momentum to obtain the canonical momentum. As can be verified directly, these equations correctly reproduce the lowest order guiding center drift equations. In addition, they automatically possess all the well known desirable properties associated with Hamiltonian systems such as energy and phase space conservation.

Energy conserving pitch angle scattering can also easily be added to Eq. (13) to allow the realistic calculation of particle diffusion. Define $\lambda = v_{\parallel}/v$, then a Lorentz collision operator is obtained⁴ by changing λ after each time step through

$$\lambda' = \lambda(1 - \nu\tau) \pm [(1 - \lambda^2)\nu\tau]^{1/2}, \quad (14)$$

where τ is the length of the time step, and ν the collision frequency with $\nu\tau \ll 1$, and the $\pm$ indicates that the sign is chosen randomly, but with equal probability for each sign.

We have used this formalism to investigate the modification of neoclassical particle diffusion in a Tokamak due to the onset of stochasticity. In a Tokamak, with the usual ordering $\epsilon = a/R \ll 1$, the magnitude of the magnetic field in our dimensionless units is given by

$$B = 1 - \epsilon(2\psi)^{1/2} \cos(\theta), \quad (15)$$

and the toroidal flux is $\psi = r^2/2$, $\theta = \theta_0 + \epsilon\chi/q(\psi)$, and $\phi = \epsilon\chi$.

The local diffusion coefficient is defined by

$$D(E, r) = \frac{1}{2} \frac{\langle (\delta r(t))^2 \rangle}{t}, \quad (16)$$

where angular brackets indicate that an average is performed over an ensemble of particles with random values of θ , ϕ , and pitch, but with energy E and initial minor radius r fixed. The particles are followed for a length of time t sufficiently large so that the motion is truly diffusive. In practice, several different values D_j are averaged for each of several different values

of t_j , and the correct value of D obtained by extrapolating to $t = \infty$.

We will be concerned with low collision frequency, i.e., collision time large compared to transit time. Three qualitatively different regimes exist. Parameterize the magnitude of α in terms of a stochasticity parameter s , with α of the form $\alpha \sim s \cos(n\phi - m\theta)$. Thus $s \approx (1/m)(\delta B/B)$. If $s \ll s_c$ where s_c is the threshold for stochasticity⁸ of the exact field $\vec{B}_e$, the particles diffuse according to the neoclassical banana diffusion rate, approximately given by⁷

$$D_a = [2\varepsilon(\psi)]^{1/2} \sigma q^2 E v / \varepsilon^2 \quad (17)$$

where $\varepsilon(\psi)$ is the local inverse aspect ratio and $\sigma = 0.689$. If $s \lesssim s_c$ a significant fraction of space is filled with magnetic islands, which give rise to enhanced transport much in the way banana orbits contribute to neoclassical diffusion. We will refer to this process as island-banana diffusion. Diffusion in the Taylor Chirikov model due to the combination of islands and a collision term has been investigated previously,^{9,10,11} but the results for $s < s_c$ cannot be directly compared because the collision term used only produced motion within the unperturbed magnetic surface, and because of the quite different particle orbits in the toroidal system.

Finally if $s > s_c$, the diffusion is given by the particles streaming along the exact magnetic field,^{2,3} $D_{QL} = v_{\parallel} D_m$, where D_m , the diffusion constant for the field itself, has been calculated using quasilinear theory,¹² which is equivalent to making a random phase approximation for the different modes in α . Assuming a uniform distribution in pitch we have $\langle |\rho_{\parallel}|^2 \rangle = p/2$ giving

$$D_{QL} = \left(\frac{\rho}{a}\right) \frac{\pi}{8\psi\epsilon} \sum_{m,n} m^2 \alpha_{m,n}^2(\psi) . \quad (18)$$

The Hamiltonian equations (13) have been integrated numerically using a fourth order Kutta-Merson process. After each orbit time step the pitch was changed using the Lorentz collision operator, Eq. (14). The code is vectorized to make full use of the MFE Cray computer, and is sufficiently fast to allow determination of diffusion constants for realistic ion and electron parameters. The time step is controlled by monitoring the fractional change in energy per time step, which is typically 10^{-8} , resulting in an error in energy conservation of one part in 10^5 for an entire determination of D . In the limit of small gyroradius the code simply traces field lines and can be used to produce a Poincaré section displaying the islands produced by the

α_{nm} .

Since the diffusion is measured with respect to the unperturbed flux surfaces, care must be taken to make the time interval long enough so that nondiffusive excursions about the unperturbed flux surfaces caused by nonstochastic field distortion are insignificant compared to actual diffusion. This presents a real problem only below stochastic threshold for very small values of collision frequency, where the actual diffusion is quite small. To eliminate error in these cases, besides integrating typically for hundreds of toroidal circuits, we have employed two procedures. First, instead of simply recording the value of ψ_j at time t_j , we integrate along the orbit to find

$$\psi_j(t_j) = \psi(0) + \frac{1}{(\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{dt}{\delta} [\psi_j(t) - \psi(0)] \exp\left(-\frac{(t-t_j)^2}{\delta^2}\right) , \quad (19)$$

where $\delta \ll t_j$. This procedure averages out the nondiffusive orbit excursions. Second, we have also reevaluated the diffusion using the same time intervals but with no collisions. Below stochastic threshold this value is simply the error in D due to the nondiffusive excursions about the unperturbed flux surfaces. Where practical, t_j was chosen long enough so that this error was not significant. In addition, below stochastic threshold the local diffusion is a sensitive function of the distance from a rational surface, varying typically by a factor of 10. The values of D for $s < s_c$ were obtained through a spatial average, $D^{-1} = 1/\Delta r \int dr D^{-1}(r)$ over the domain of the modes.

We have used parameters typical for present fusion research devices, 1 Kev temperatures, plasma density of $n = 10^{13} \text{ cm}^{-3}$, magnetic field strength of 30 KG, $a = 30 \text{ cm}$, and $\epsilon = 1/4$. The safety factor $q(\psi)$ was taken to be linear in ψ , ranging from 1.0 on axis to 4.0 at the wall. The harmonic perturbing fields α_{nm} were chosen to be of equal amplitude, and well localized about their rational surfaces, i.e.,

$$\alpha(\psi, \theta, \phi) = s \sum_{n,m} \exp[-(\psi - \psi_r)^2/b^2] \cos(n\phi - m\theta + \phi_{nm}) \quad , \quad (20)$$

where $q(\psi_r) = m/n$, ϕ_{nm} are fixed random phases; and b was taken to be 0.04. To simplify the interpretation of results we used 3 modes, with $n = 4$ and $m = 7, 8$, and 9 . This gives a roughly uniform domain of harmonics from $r = 0.5$ to $r = 0.7$, and in this domain we have approximately $|\delta B/B| \sim 5s$. The modes chosen have radial separation of $\delta r = 0.07$ and stochastic threshold⁸ occurs approximately for $s = 2.5 \times 10^{-5}$. Sixty four particles were placed initially at $r = 0.6$, with random values of θ , ϕ , and pitch.

Shown in Fig. 1 are the diffusion coefficients for ions and electrons, as a

function of the stochasticity parameter s , given in terms of the neoclassical value, obtained with $s = 0$, D_{nc} . The neoclassical value was found to agree with the approximate analytic expression Eq. (17) to within a few percent for electrons but was typically a factor of three larger for ions. Also shown are the quasilinear values, approximated by averaging Eq. (18) over the domain considered. It was found impractical to obtain the electron diffusion constant for very small values of s . Nevertheless, it is clear from Fig. 1 that a fairly significant modification of the electron neoclassical diffusion rate exists below the nominal stochastic threshold whereas the ions are relatively unaffected. This is readily understood in terms of island banana diffusion, which is important for electrons, but not for ions, because of the large neoclassical ion diffusion.

We have also investigated the dependence of D on the number of harmonics and on their width, but only the expected differences in behavior are found. In Fig. 2 is shown the Poincaré plot of the magnetic field structure for $s = 10^{-5}$. Although good surfaces still exist, the dynamics is strongly affected by the island structure.

Neutrality of the plasma would of course not permit large differences in electron and ion diffusion to occur, but similar modification of electron heat transport is to be expected. In particular the electron heat flux arises in the same manner as the diffusion, leading to a value approximately 20 times the neoclassical value at the observed fluctuation level of $s = 10^{-5}$. This enhancement is clearly sensitive to mode density and gyro radius. Direct association of the observed mechanism with anomalous electron energy confinement in Tokamaks must await more detailed simulations using observed mode spectra. The spectrum we have chosen is however sufficient to illustrate the diffusion mechanisms operating in the different regimes for both plasma species.

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Figure Caption

Fig. 1. The electron and ion diffusion coefficients, normalized to their neoclassical values, D_{nc} , as a function of the amplitude of the perturbing fields, s . Here $v_e/\omega_e = 1.8 \times 10^{-8}$, $v_i/\omega_i = 10^{-6}$, $a/\rho_e = 12,000$, and $a/\rho_i = 200$. Also shown are the quasilinear values, and the stochastic threshold. The errors shown are due to the nondiffusive orbit excursions made by the particles, and the statistics.

Fig. 2. The Poincaré plot of the exact magnetic field for the value $s = 10^{-5}$. The harmonic content is $m = 7, 8, 9$ and $n = 4$. Shown is a section near the $q = 2$ surface, $r = 0.58$.

Fig. 1

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Fig. 2