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**Extraction of the Michel Parameters from
Muon Decay by a Maximum Likelihood Method**

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EXTRACTION OF THE MICHEL PARAMETERS FROM MUON DECAY BY A MAXIMUM LIKELIHOOD METHOD

by

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ABSTRACT

We have developed a maximum likelihood method for extracting the Michel parameters, which describe the decay spectrum of stopped polarized muons. The method was applied successfully in a Monte Carlo simulation of our apparatus. We also obtained a complete error matrix for the set of Michel parameters. In an experiment of 10^6 muon decays we can reduce the statistical errors by more than an order of magnitude over the currently accepted limits on the Michel parameters.

In our approved LAMPF experiment, we propose to measure the Michel spectrum of positrons from polarized muon decay and to extract the Michel parameters to an unprecedented accuracy. To achieve an order of magnitude improvement over previous results,¹ we must measure $\sim 10^8$ muon decays in our time projection chamber (TPC). However, because so many data present handling and analysis problems, we have developed a more efficient method for extracting the four Michel parameters.

We compare the maximum likelihood method with the standard procedure, in which one separates (bins) the data in momentum and angle, then performs a least squares fit to the spectrum. The standard procedure, however, has inherent inaccuracies. Unless the bin size is considerably smaller than the resolution, an event can move around in its bin, which results in lost information. To make the two-dimensional bins small enough for our experiment would require 10^4 to 10^6 bins, an awkwardly large number for performing a least squares fit.

With the maximum likelihood method no information is lost because there is no binning. All the data are reduced to about 150 numbers, which are then used to extract the Michel parameters.

In this report we describe the method and present a numerical example based upon the apparatus we will use. We also display the complete Michel parameter error matrix, calculated in such a way that N , the number of events, appears as a parameter. Thus, we can find the statistical errors on the Michel parameters for any amount of data simply by substituting N into our numerical results.

Neglecting the radiative corrections, we describe the Michel spectrum, which is a function of the emitted positron's energy and the angle the positron makes relative to the stationary muon's spin direction. The two variables x and μ in Eq. (1) refer to the energy relative to the maximum ($E_0 = 52.831$ MeV) and cosine of the angle. We write the Michel spectrum as

$$P_0(x, \mu) = \sum_{k=1}^5 a_k F_k(x) \mu^{n_k}, \quad (1)$$

where a_k and n_k are as given below.

k	a_k	n_k
1	1	0
2	ρ	0
3	η	0
4	$\rho\xi$	1
5	$\rho\xi\delta$	1

The four Michel parameters are ρ , η , ξ , and δ ; p is the polarization of the stopped muons in the TPC gas.

The $F_k(x)$ are simple functions of x . For our analysis, we simplified the $F_k(x)$ by ignoring the positron's mass, m , in most cases. The functions then become

$$\begin{aligned} F_1(x) &= 3(1-x)x^2, \\ F_2(x) &= 2(\frac{4}{3}x-1)x^2, \\ F_3(x) &= 3x_0(1-x)x, \\ F_4(x) &= (1-x)x^2, \end{aligned}$$

and

$$F_5(x) = 2(\frac{4}{3}x-1)x^2, \quad (2)$$

where $x_0 = mc^2/E_0$.

For a given event the probability that the energy will fall between x and $x + dx$, the angle between μ and $\mu + d\mu$, is shown as

$$P(x, \mu) dx d\mu, \quad (3)$$

where the probability is related to $P_0(x, \mu)$ by

$$P(x, \mu) = X^{-1} A(x, \mu) P_0(x, \mu) \quad (4)$$

where X is the normalization factor and $A(x, \mu)$ is the TPC acceptance function. The acceptance function reflects not only the geometric acceptance but also any cuts one may wish to impose on the TPC track characteristics. The normalization factor X is a natural result of the condition

$$\int_0^1 dx \int_{-1}^1 d\mu P(x, \mu) = 1. \quad (5)$$

Hence,

$$X = \sum_{k=1}^5 a_k g_k, \quad (6)$$

and

$$g_k = \int_0^1 F_k(x) dx \int_{-1}^1 A(x, \mu) \mu^{n_k} d\mu. \quad (7)$$

The five g_k quantities are the acceptance factors.

To measure N muon decays, we use the set of measurements $\{x_i, \mu_i\}$. The probability of seeing that particular set is

$$G(\rho, \eta, \xi, \delta; \{x_i, \mu_i\}) = \prod_{i=1}^N P(x_i, \mu_i). \quad (8)$$

The maximum likelihood method demands that we jiggle the four Michel parameters until G is maximized. Mathematically, we take the derivatives of $\ln G$ with respect to the Michel parameters and set the results to zero to obtain

$$\frac{X}{N} \sum_{i=1}^N \frac{F_k(x_i) \mu_i^{n_k}}{P_0(x_i, \mu_i)} = g_k, \quad k=2,3,4,5. \quad (9)$$

Although Eq. (9) can also be used for $k = 1$, it is not needed because it is a direct result of the other four parameters.

The four simultaneous equations expressed in Eq. (9) are very awkward. The four Michel parameters, buried in X and P_0 , must be iterated in some way. Each iteration means a new calculation must be made of the sum in Eq. (9). The reading of the millions of (x_i, μ_i) pairs, which are on disk or tape, is very time-consuming. Therefore, if sums are required, it would be more efficient to read the data and do the sums only once. For this reason, we need a way of factoring out the Michel parameters from the sums.

We take $P_0(x, \mu)^{-1}$ and make a Taylor series expansion in the Michel parameters about the V-A values to obtain

$$a_k = a_k^* + \Delta a_k, \quad (10)$$

where a_k^* is the V-A result. Equation (10) is also defined for $k = 1$, but by definition $a_1^* = 1$, and $\Delta a_1 = 0$. Therefore,

$$P_0(x, \mu) = P_0^*(x, \mu) + \sum_{k=2}^5 \Delta a_k F_k(x) \mu^{n_k},$$

and

$$x = x^* + \sum_{k=2}^5 \Delta a_k g_k. \quad (11)$$

The expansion is then

$$p_0(x, \mu)^{-1} = \sum_{m=0}^{\infty} (-1)^m \sum_{k_2=0}^m \sum_{k_3=0}^{m-k_2} \sum_{k_4=0}^{m-k_2-k_3} \frac{m!}{k_2! k_3! k_4! k_5!} \left[\prod_{\alpha=2}^5 (\Delta a_{\alpha})^{k_{\alpha}} \right] \frac{\prod_{\alpha=2}^5 (F_{\alpha}(x)_{\mu})^{n_{\alpha}}}{p_0^*(x, \mu)^{m+1}}, \quad (12)$$

where $k_5 = m - (k_2 + k_3 + k_4)$. Equation (12) can be simplified by breaking it up and adding the missing factors from Eq. (9).

$$R_{k_2 k_3 k_4 k_5}^k = \frac{(-1)^m m!}{k_2! k_3! k_4! k_5!} \frac{1}{N} \sum_{i=1}^N \frac{\prod_{\alpha=2}^5 (F_{\alpha}(x_i)_{\mu_i})^{n_{\alpha}}}{p_0^*(x_i, \mu_i)^{m+1}}, \quad (13)$$

$k = 2, 3, 4, 5,$

and

$$Q_{Mk} = \sum_{m=0}^M \sum_{k_2=0}^m \sum_{k_3=0}^{m-k_2} \sum_{k_4=0}^{m-k_2-k_3} R_{k_2 k_3 k_4 k_5}^k \prod_{\alpha=2}^5 (\Delta a_{\alpha})^{k_{\alpha}}. \quad (14)$$

Equation (9) now becomes

$$\left(x^* + \sum_{\ell=2}^5 g_{\ell} \Delta a_{\ell} \right) Q_{Mk} = g_k, \quad k = 2, 3, 4, 5. \quad (15)$$

Note that we have introduced a cutoff M in the infinite sum. It turns out that M need not be very large. Of more importance, however, is the isolation of the coveted sums in the R factors; that is, we have reduced the millions of (x_i, μ_i) pairs to only a few R factors.

To carry out the iteration in a_k , we define a modified Q_{Mk} .

$$Q'_{Mk} = \sum_{m=2}^M \sum_{k_2=0}^m \sum_{k_3=0}^{m-k_2} \sum_{k_4=0}^{m-k_2-k_3} R_{k_2 k_3 k_4 k_5}^k \prod_{\alpha=2}^5 (\Delta a_{\alpha})^{k_{\alpha}}. \quad (16)$$

We also define

$$\begin{aligned} h_{k2} &= R_{1000}^k, \\ h_{k3} &= R_{0100}^k, \\ h_{k4} &= R_{0010}^k, \\ h_{k5} &= R_{0001}^k. \end{aligned} \quad (17)$$

Then Eq. (15) can be written as

$$\sum_{\ell=2}^5 h_{k\ell} \Delta a_{\ell} = \chi^2 + \frac{g_k}{\sum_{\ell=2}^5 g_{\ell} \Delta a_{\ell}} - R_{0000}^k - Q_{Mk}^{\prime}, \quad k = 2, 3, 4, 5. \quad (18)$$

To use Eq. (18), we first calculate the inverse of $h_{k\ell}$ because it is used many times. We then set $Q_{Mk}^{\prime} = 0$ and $\Delta a_{\ell} = 0$ on the right-hand side of Eq. (18) and solve for Δa_{ℓ} on the left-hand side. With $Q_{Mk}^{\prime}$ still zero, we enter the results into the right-hand side and again solve for Δa_{ℓ} on the left-hand side. The iteration continues until Δa_{ℓ} converges. We then enter the $Q_{Mk}^{\prime}$ term with $M = 2$ and continue the iteration until Δa_{ℓ} converges. We then enter $M = 3$, etc., until successive values of M give the same results.

We successfully applied the maximum likelihood method, using the above iteration scheme, in a Monte Carlo run of 5×10^8 events. In fact, we needed to go up only to $M = 2$ to extract the correct Michel parameters (the V-A values in this case) to within one or two standard deviations. Table I shows the results from the Monte Carlo run.

The last column in Table I, taken from a separate Monte Carlo run of 3×10^8 events, shows the type of error that results when the iteration is stopped at $M = 2$ instead of $M = 3$. Of course, we loaded the deck slightly by generating Monte Carlo data with the V-A values. As the statistics increase, we would expect Δa_{ℓ} to decrease and, correspondingly, the cutoff to decrease or stay the same; still, the results confirm that M need not be large.

Table II shows the number of R factors required for a given cutoff.

Before beginning the error analysis, we present a slightly different notation for the Michel parameters (Table III). The notation b_k , which symbolizes the bare Michel parameters, is related to a_k by $a_k = b_k c_k$.

To determine the error matrix, we use $E_{k\ell}$ as the error matrix and $E_{k\ell}^{-1}$ as an element of the inverse of E . The maximum likelihood method gives us a prescription for finding E^{-1} .

$$E_{k\ell}^{-1} = - \frac{\partial^2 \ln G}{\partial b_k \partial b_{\ell}}, \quad k, \ell = 2, 3, 4, 5. \quad (19)$$

TABLE I
MONTE CARLO RESULTS

<u>Michel Parameter</u>	<u>Monte Carlo</u>	<u>V-A</u>	<u>M = 3 Correction</u>
ρ	0.757 ± 0.004	3/4	0.000008
η	0.22 ± 0.11	0	0.0003
ξ	1.005 ± 0.018	1	0.000009
δ	0.757 ± 0.012	3/4	0.000008

TABLE II

NUMBER OF R FACTORS
REQUIRED FOR CUTOFF

M	R Factors
1	20
2	60
3	140
4	280
5	504

TABLE III

MICHEL PARAMETERS

<u>k</u>	<u>b_k</u>	<u>c_k</u>
1	1	1
2	ρ	1
3	η	1
4	ξ	p
5	δ	p ξ

Note that G is defined in Eq. (8). Working out the derivatives, we obtain

$$E_{k\ell}^{-1} = N c_k c_\ell \left[\frac{1}{N} \sum_{i=1}^N \frac{F_k(x_i) F_\ell(x_i) \mu_i^{n_k+n_\ell}}{P_0(x_i, \mu_i)^2} - \frac{g_k g_\ell}{\chi^2} \right]. \quad (20)$$

In principle, we could use the Michel parameters that were extracted by using the maximum likelihood method instead of the V-A values to determine the error matrix. However, because there is so little difference between the two, we chose to use the V-A values.

The error matrix could be determined by a Monte Carlo calculation, which makes the results depend on N. However, it is more efficient to change Eq. (20) into an analytical expression, which, except for the N factor in front, is independent of N. This change is possible because $N^{-1} \sum_{i=1}^N$ is an average. More precisely, if we let $f(x, \mu)$ stand for some reasonably well-behaved function of x and μ , then

$$\frac{1}{N} \sum_{i=1}^N f(x_i, \mu_i) \longrightarrow \int_0^1 dx \int_{-1}^1 d\mu P(x, \mu) f(x, \mu),$$

where N is large. By making the above substitution, we get

$$E_{k\ell}^{-1} = N c_k c_\ell \left[\frac{1}{\chi} \int_0^1 F_k(x) F_\ell(x) dx \int_{-1}^1 \frac{A(x, \mu) \mu^{n_k+n_\ell}}{P_0(x, \mu)} d\mu - \frac{g_k g_\ell}{\chi^2} \right]. \quad (21)$$

Note the complete isolation of N. Now we can write

$$E_{k\ell} = \frac{1}{N} e_{k\ell}, \quad (22)$$

where $e_{k\ell}$, a sort of normalized error matrix, is the inverse of Eq. (21) without the N.

Based upon the configuration of our apparatus, including cuts on TPC tracks, we integrated Eq. (21) numerically and found $e_{k\ell}$. The results are as follows.

$$e_{k\ell} = \begin{Bmatrix} 5.186 \times 10^0 & 1.037 \times 10^2 & 3.781 \times 10^0 & 2.879 \times 10^0 \\ & 3.751 \times 10^3 & 8.546 \times 10^1 & 7.552 \times 10^1 \\ & & 9.709 \times 10^1 & 5.340 \times 10^1 \\ & & & 4.065 \times 10^1 \end{Bmatrix}.$$

The correlation matrix is

$$c_{k\ell} = \begin{Bmatrix} 1.000 & 0.744 & 0.169 & 0.198 \\ & 1.000 & 0.142 & 0.193 \\ & & 1.000 & 0.850 \\ & & & 1.000 \end{Bmatrix}.$$

Here we evaluate the statistical errors of the Michel parameters, $\sqrt{e_{kk}/N}$, for $N = 10^6$ events. Table IV shows the current parameter limits given in Ref. 2. The accuracy of every parameter is improved by at least an order of magnitude. For η , we see a vast improvement. Note, however, that our results refer to the statistical errors only, whereas the numbers quoted by Sachs and Sirlin² include all sources of error from several experiments.

With such a reduction in the errors, we must determine the limits to be set on the coupling constants. The following coupling constants are written in the same manner as shown in Ref. 2.

$$x_i = \left(\frac{|g_i|^2 + |g_i'|^2}{|g_V|^2 + |g_V'|^2} \right)^{1/2}, \quad i = S, P, T, A, \quad (23)$$

and

$$\phi_{VA} = \cos^{-1} \left(\frac{\text{Re}(g_A^{*'} g_V' + g_A' g_V^{*'})}{(|g_V|^2 + |g_V'|^2)^{1/2} (|g_A|^2 + |g_A'|^2)^{1/2}} \right). \quad (24)$$

Using Derenzo's method⁸ to determine the parameter limits, we present in Table V our results and the limits we can set from our experiment.

TABLE IV
MICHEL PARAMETER ERRORS

Parameter	$\sqrt{e_{kk}/N}$	Current Accuracy ²
ρ	0.00023	0.003
η	0.0061	0.21
ξ	0.00099	0.014
ζ	0.00064	0.009

TABLE V
LIMITS ON THE COUPLING CONSTANTS

<u>COUPLING CONSTANT</u>	<u>OUR LIMITS</u>	<u>DERENZO'S LIMITS^a</u>
Axial vector	$0.988 \leq x_A \leq 1.052$	$0.76 \leq x_A \leq 1.20$
Tensor	$x_T \leq 0.027$	$x_T \leq 0.28$
Scalar	$x_S \leq 0.048$	$x_S \leq 0.33$
Pseudoscalar	$x_P \leq 0.048$	$x_P \leq 0.33$
Vector axial vector phase	$0^\circ \leq \Phi_{VA} \leq 2.6^\circ$	$0 \leq \Phi_{VA} \leq 15^\circ$

If we limit the other couplings to zero, we get more symmetrical limits, $0.988 \leq x_A \leq 1.012$, on the axial vector to vector coupling.

The maximum likelihood method seems to be a more efficient way of extracting the Michel parameters from the many data we expect. Just as important, even before the beam is turned on, the method gives us a definitive statement on the expected statistical errors for our apparatus. The results look encouraging. If we can limit the systematics to the same level, a data run of 10^8 events will verify (or possibly even disprove) the V-A values with a sensitivity that is an order of magnitude better than shown in previous experiments.

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