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IMPROVED PERFORMANCE OF TANDEM MIRRORS WITH THERMAL BARRIERS

T. B. Kaiser

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ABSTRACT

As originally conceived, tandem mirrors with thermal barriers are limited by MHD instabilities to values of central-cell β that are significantly lower than those attainable in tandems without barriers. Much of the degradation in performance is due to the introduction of a new region of unfavorable field line curvature in the central cell. From a variational calculation that maximizes central cell fusion power subject to constraints on the vacuum field energy and allowable interchange instability drive, I derive an analytic expression for a central cell magnetic field strength profile for which the new instability drive is greatly reduced. As a result, central cell performance is enhanced to a point approaching that of tandem mirrors without barriers. Formulas showing how β_c and fusion power scale with design parameters are also derived.

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I. Introduction

In tandem mirrors with well-pumped thermal barriers,¹ the dominant drive for flute and ballooning interchange instability arises from the unfavorable field line curvature that accompanies the squeezing of field lines as they leave the solenoid and pass through the high-field barrier coils. Moreover, it has been observed numerically that this drive decreases if the transition from the solenoidal field to the barrier field occurs more gradually, i.e., over a larger distance. This in turn allows a higher peak β at the center of the solenoid and a corresponding improvement in the fusion power and Q .

In this note I derive an analytic expression for the optimal shape of $|B(z)| \equiv B(z)$ between the barrier peaks from a variational calculation which maximizes the fusion power in the central cell for a fixed amount of vacuum field energy² and the maximum amount of flute-interchange drive allowed by the pressure-weighted average favorable curvature of the plug and barrier regions. The entire calculation is done within the context of the long-thin approximation and axisymmetric geometry.

II. Variational Calculation

If $\bar{n}_c$ and r_c are the plasma density and radial scale length and L the half-length of the central cell, the fusion power is proportional to $\pi r_c^2 \bar{n}_c^2 \int_0^L dz/B$, which for fixed central-cell temperature is proportional to

$$P = \pi r_c^2 \bar{n}_c^2 \int_0^L \frac{dz}{B} \quad (1)$$

where β_c is the plasma β at $z = 0$.

A convenient measure of the vacuum field energy is

$$\mathcal{E} = \pi r_B^2 B_c \int_0^L \frac{dz}{B} \frac{B^2}{8\pi} \quad , \quad (2)$$

where r_B is the radial scale length of B at $z = 0$, and the flute-interchange drive³ is

$$D = \int_0^L \frac{dz}{B} 2p \frac{rr''}{2\psi} \quad (3)$$

where p is the central-cell pressure, $r(z)$ the trajectory of a field line and ψ the magnetic flux, $\psi = \frac{1}{2} B_c r(0)^2$.

In axisymmetry, $r = \sqrt{2\psi/B}$, so it is convenient to introduce the variable

$$\rho = B^{-1/2} \quad . \quad (4)$$

With (4), Eqs. (1) - (3) can be written

$$P = \pi r_c^2 B_c^5 \beta_c^2 \int_0^L dz \rho^2 \quad (1')$$

$$\mathcal{E} = \frac{1}{8} r_c^2 B_c \int_0^L dz \rho^{-2} \quad (2')$$

$$D = \frac{1}{4\pi} B_c^2 \beta_c \int dz \rho^3 \rho''' \quad (3')$$

I look for the function $\rho(z)$ which maximizes the functional $P[\rho]$ with the constraints $\mathcal{E}[\rho] = \mathcal{E}_0$ and $D[\rho] = D_0$. Thus, I introduce Lagrange multipliers $\tilde{\lambda}_1, \tilde{\lambda}_2$ and maximize

$$J[\rho] = P - \tilde{\lambda}_1 (D - D_0) - \tilde{\lambda}_2 (\mathcal{E} - \mathcal{E}_0)$$

$$= \pi r_c^2 B_c^5 \int_0^L dz (\rho^2 - \lambda_1 \rho^3 - \lambda_2 \rho^{-2}) + \tilde{\lambda}_1 D_0 + \tilde{\lambda}_2 \mathcal{E}_0$$

where for convenience the Lagrange multipliers have been redefined. The appropriate Euler equation is

$$3\lambda_1 (\rho \rho')' - \lambda_2 \rho^{-4} - 1 = 0 \quad (5)$$

or, with the transformation

$$v = \rho^2, \quad (6)$$

$$u = \frac{dv^2}{dz}, \quad (7)$$

$$\frac{du^2}{dv} - \frac{4}{3\lambda_1} \left(1 + \frac{\lambda_2}{v^2}\right) = 0 \quad (8)$$

The appropriate boundary conditions are $B'(0) = B'(L) = 0$, and $B(0) = B_c$, $B(L) = B_{mb} = RB_c$, where R is the vacuum mirror ratio. In terms of (v, u) these conditions become

$$v(0) \equiv v_0 = B_c^{-1}, \quad (9)$$

$$v(L) \equiv v_L = (RB_c)^{-1}, \quad (10)$$

$$u(0) = u(L) = 0 \quad (11)$$

The solution to Eq. (8) is

$$u = \pm \left[\frac{4}{3\lambda_1} \left(v - \frac{\lambda_2}{v} \right) + C \right]^{1/2} \quad (12)$$

where C is an integration constant. Imposition of Eq. (11) results in

$$-C = \frac{4}{3\lambda_1} \left(v_0 - \frac{\lambda_2}{v_0} \right) = \frac{4}{3\lambda_1} \left(v_L - \frac{\lambda_2}{v_L} \right)$$

$$\Rightarrow \lambda_2 = -v_0 v_L \quad (13)$$

$$\Rightarrow \frac{3\lambda_1 C}{4} = -(v_0 + v_L) \quad (14)$$

with which Eq. (12) becomes

$$u = \pm \left[C \frac{(v_0 - v)(v - v_L)}{v(v_0 + v_L)} \right]^{1/2} \quad (15)$$

Using Eq. (7), then,

$$\frac{dv}{dz} = \pm \left[C \frac{(v_0 - v)(v - v_L)}{v(v_0 + v_L)} \right]^{1/2},$$

inspection of which shows that v must decrease monotonically from v_0 to v_L . Therefore,

$$\begin{aligned} \left(\frac{C}{v_0 + v_L} \right)^{1/2} (L - z) &= \int_{v_L}^v dt \left[\frac{t}{(t - v_L)(v_0 - t)} \right]^{1/2} \\ &= 2 \left\{ v_0^{1/2} E \left[\sin^{-1} \sqrt{\frac{v_0(v - v_L)}{v(v_0 - v_L)}} \right], \sqrt{\frac{v_0 - v_L}{v_0}} \right. \\ &\quad \left. - \sqrt{\frac{(v_0 - v)(v - v_L)}{v}} \right\} \end{aligned} \quad (16)$$

where E is an elliptic integral of the second kind. The integration constant C is fixed by applying the boundary conditions (9) and (10):

$$L \left(\frac{C}{v_0 + v_L} \right)^{1/2} = 2 B_c^{-1/2} E \left(\sqrt{\frac{R - 1}{R}} \right) \quad (17)$$

where $E \left(\sqrt{\frac{R - 1}{R}} \right)$ is a complete elliptic integral of the second kind.

Combining Eqs. (9), (10), (16) and (17) gives

$$\begin{aligned} E \left(\sqrt{\frac{R - 1}{R}} \right) \left(1 - \frac{z}{L} \right) &= E \left[\sin^{-1} \sqrt{\frac{R - B/B_c}{R - 1}}, \sqrt{\frac{R - 1}{R}} \right] \\ &\quad - \left[\left(\frac{B}{B_c} - 1 \right) \left(\frac{B_c}{B} - \frac{1}{R} \right) \right]^{1/2} \end{aligned} \quad (18)$$

which implicitly determines B as a function of z.

The Lagrange multipliers λ_1 , λ_2 have been expressed in terms of L, B_c and R by Eqs. (13) and (14):

$$\begin{aligned} \lambda_1 &= -\frac{1}{3} \frac{L^2 B_c^2}{E^2 \left(\sqrt{\frac{R - 1}{R}} \right)} \\ \lambda_2 &= -\frac{1}{R B_c^2} \end{aligned}$$

Therefore, only one of the parameters (L , B_c , R) can be chosen freely, once the constraints $D = D_0$, $\mathcal{E} = \mathcal{E}_0$ have been applied. Consider first the vacuum field energy, Eq. (2'):

$$\begin{aligned}\mathcal{E} &= \frac{1}{8} r_B^2 B_c^2 \int_0^L dz \rho^{-2} = \frac{1}{8} r_c^2 B_c^2 \int_{v_0}^{v_L} \frac{dv}{uv} \\ &= \frac{1}{8} r_B^2 L B_c^2 \frac{F\left(\sqrt{\frac{R-1}{R}}\right)}{E\left(\sqrt{\frac{R-1}{R}}\right)}\end{aligned}\quad (19)$$

where F is a complete elliptic integral of the first kind.

The flute-interchange drive, Eq. (3') is

$$\begin{aligned}D &= \frac{\beta B_c^2}{4\pi} \int_0^L dz \rho^3 \rho'' \\ &= \frac{\beta B_c^2}{4\pi} \left[\rho^3 \rho' \Big|_0^L - \int_0^L dz 3(\rho \rho')^2 \right]\end{aligned}$$

after an integration by parts. The boundary terms vanish because of boundary conditions (11), leaving

$$\begin{aligned}D &= - \frac{3\beta B_c^2}{4\pi} \int_0^L dz (\rho \rho')^2 \\ &= - \frac{3\beta B_c^2}{16\pi} \int_{v_0}^{v_L} dv u \\ &= - \frac{\beta c}{4\pi R L} E\left(\sqrt{\frac{R-1}{R}}\right) \left[(R+1) E\left(\sqrt{\frac{R-1}{R}}\right) - 2F\left(\sqrt{\frac{R-1}{R}}\right) \right]\end{aligned}\quad (20)$$

Application of the constraints $\mathcal{E} = \mathcal{E}_0$ and $D = D_0$ determines two of the parameters (L , B_c , R) in terms of the remaining one. This, in

turn, determines the fusion power, Eq. (1'):

$$\begin{aligned}
 P &= \pi r_c^2 B_c^5 \beta_c^2 \int_0^L dz \rho^2 = \pi r_c^2 B_c^5 \beta_c^2 \int_{v_0}^{v_L} dv \frac{v}{u} \\
 &= \frac{\pi}{3} r_c^2 B_c^4 \beta_c^2 L \left[2 \frac{(R+1)}{R} - \frac{1}{R} \frac{F\left(\sqrt{\frac{R-1}{R}}\right)}{E\left(\sqrt{\frac{R-1}{R}}\right)} \right] \quad (21)
 \end{aligned}$$

If the R-dependence is isolated, Eqs. (19) - (21) can be written

$$\mathcal{E} = \frac{1}{8} r_B^2 L B_c^2 \hat{\mathcal{E}}(R)$$

$$D = - \frac{B_c}{4\pi L} \hat{D}(R)$$

$$P = \frac{\pi}{3} r_c^2 B_c^4 \beta_c^2 L \hat{P}(R)$$

where

$$\hat{\mathcal{E}}(R) = \frac{F\left(\sqrt{\frac{R-1}{R}}\right)}{E\left(\sqrt{\frac{R-1}{R}}\right)}$$

$$\hat{D}(R) = \frac{1}{R} E\left(\sqrt{\frac{R-1}{R}}\right) \left[(R+1) E\left(\sqrt{\frac{R-1}{R}}\right) - 2F\left(\sqrt{\frac{R-1}{R}}\right) \right]$$

$$\hat{P}(R) = 2 \frac{R+1}{R} - \frac{1}{R} \frac{F\left(\sqrt{\frac{R-1}{R}}\right)}{E\left(\sqrt{\frac{R-1}{R}}\right)}$$

The dimensionless functions $\hat{\mathcal{E}}$, $\hat{D}$ and $\hat{P}$ are shown in Fig. 1 for $1 \leq R \leq 50$.

III. Application to MFTF-B

As an example of the benefit to be gained by optimizing the shape of $B(z)$, consider the MFTF-B "base case", for which the design parameters are $B_c = 1.5 \times 10^4$ Gauss, $L = 10^3$ cm, $R = 7.69$, the flute-interchange drive is $D_{BASE} = (2.97 \beta_p - 1.21 \beta_c) \times 10^{-4} \text{ cm}^{-1}$ (β_p = plasma β at center of plug), and the vacuum field energy is $\mathcal{E}_{BASE} = 4.7$ MJ. If a value of $\mathcal{E}_0 = 1.5 \mathcal{E}_{BASE}$ is accepted, then it is straightforward to derive a self-consistent set of design parameters for the optimally-shaped $B(z)$ that is identical to the base-case set, but which yields significantly improved performance parameters, β_c and P :

	FLUTE-INTERCHANGE LIMIT	BALLOONING-INTERCHANGE LIMIT @ $\beta_p = .40$	FUSION POWER PARAMETER (cm^{-3})
BASE CASE	$\beta_c / \beta_p = 0.71$	$\beta_c = .095$	2.4×10^{33}
OPTIMUM $B(z)$	$\beta_c / \beta_p = 1.56$	$\beta_c = .204$	8.5×10^{33}

The critical β for both flute and ballooning interchange is enhanced by a factor of 2.2 and the fusion power by a factor of 3.5. The optimum field shape given by Eq. (18) is shown in Fig. 2 with the base-case field shape shown for comparison.

For a given set of design parameters, L , R and B_c or B_{mb} and a given allowance of flute-interchange drive, D_0 , the functional form for $B(z)$ given by Eq. (18) gives the highest fusion power attainable. It is natural to inquire, then, whether higher values of the performance parameters might be achieved by varying the design parameters. The most

interesting case is the one in which the central cell plasma radius is determined by the plug:

$$r_c^2 \beta_c = r_p^2 \beta_p \quad (22)$$

where r_p and β_p are the plasma radius and magnetic field strength at the plug midplane. The barrier pressure which weights the interchange drive in that region scales with the central cell pressure, i.e., as $\beta_c \beta_c^2$. Thus, the net drive external to the central cell is

$$D_o = D_{po} \beta_p - D_{bo} (\beta_c / \beta_{co})^2 \beta_c, \quad (23)$$

where D_{po} and D_{bo} are positive constants determined by the magnetic field in the plug and barrier, respectively, and β_{co} is a convenient reference field strength, e.g., $\beta_{co} = (\beta_c)_{\text{BASE CASE}}$.

The design parameter space is three-dimensional: among the five quantities $\beta_o, R, L, \beta_c, B_{mb}$, there are two relations, $\beta_o = \beta(R, L, \beta_c)$ and $B_{mb} = R \beta_c$. Selection of parametric coordinates is a matter of convenience. Since designs for given B_{mb} or L are of interest and the dependence of β_c and P on R is complicated, I choose (L, B_{mb}, R) . Then

$$\beta_o = \frac{1}{8} r_p^2 L \left(\frac{B_{mb}}{R} \right)^2 \hat{\beta}(R), \quad (24)$$

$$\beta_c = B_{mb} / R,$$

and the variation of the performance parameters P and β_c over design parameter space is obtained from Eqs. (20) and (23)

$$\frac{\partial \beta_C}{\partial P} = \frac{4\pi D_{bo} L}{D(R) + \frac{4\pi D_{bo}}{2} \left(\frac{B_{mb}}{R}\right)^2 L} \quad (25)$$

and Eq. (21)

$$P = \frac{\pi}{3} B_p (4\pi r_p D_{po} \beta_p)^2 \frac{\left(\frac{B_{mb} L}{R}\right)^3 P(R)}{\left[D(R) + \frac{4\pi D_{bo}}{2} \left(\frac{B_{mb}}{R}\right)^2 L\right]^2} \quad (26)$$

Calculation with Eqs. (24) - (26) shows that at the MFTF-B base case design point, i.e., $L = 10^3$ cm, $B_{mb} = 11.5$ T, $R = 7.69$, $D_{po} = 2.97 \times 10^{-4}$ cm⁻¹, $D_{bo} = 1.21 \times 10^{-4}$ cm⁻¹, the following scalings hold:

$$\left(\frac{\partial \beta_O}{\partial L}\right)_{R, B_{mb}} < 0, \quad \left(\frac{\partial \beta_C}{\partial L}\right)_{R, B_{mb}} < 0, \quad \left(\frac{\partial P}{\partial L}\right)_{R, B_{mb}} < 0 \quad (27)$$

$$\left(\frac{\partial \beta_O}{\partial B_{mb}}\right)_{L, R} > 0, \quad \left(\frac{\partial \beta_C}{\partial B_{mb}}\right)_{L, R} < 0, \quad \left(\frac{\partial P}{\partial B_{mb}}\right)_{L, R} < 0 \quad (28)$$

$$\left(\frac{\partial \beta_O}{\partial R}\right)_{L, B_{mb}} < 0, \quad \left(\frac{\partial \beta_C}{\partial R}\right)_{L, B_{mb}} > 0, \quad \left(\frac{\partial P}{\partial R}\right)_{L, B_{mb}} \begin{cases} > 0, R < 5.75 \\ < 0, R > 5.75 \end{cases} \quad (29)$$

Thus the simplest method of improving performance is the obvious one of increasing the length of the central cell [Ineqs. (27)]. From Eq. (25) it is clear that if $L \gg \frac{D(R)}{D_{bo}} \frac{B_{co}}{4\pi} \left(\frac{R}{B_{mb}}\right)^2$ the central cell contribution to the interchange drive becomes negligible, so that the

limiting value of β_c/β_p is determined by the barrier and plug regions.

The maximum β_c achievable in this way is

$$\lim_{L \rightarrow \infty} \beta_c/\beta_p = \frac{D_{p0}}{D_{b0}} B_{co}^2 \left(\frac{R}{B_{mb}} \right)^2 = 2.45$$

where the numerical value results from using the base case parameters. A plot of β_c/β_p as a function of L is given in Fig. 3, from which it is clear that not a lot is to be gained in β_c by increasing L . Inspection of Eq. (25), however, shows that the fusion power increases without limit as $L \rightarrow \infty$. Of course the same is true of the vacuum field energy, β_0 [Eq. (24)].

Changing the strength of the barrier field has mixed results: near the base case design point an increase in fusion power is accompanied by a decrease in β_c , and vice versa [Ineqs. (28)]. The degradation in β_c arises from the fact that increasing B_{mb} at constant R raises B_c , which increases the interchange drive in the barrier [Eq. (23)], requiring a lower β_c .

As implied by the last of Ineqs. (29), there is an optimum central cell mirror ratio, $R = 5.75$. This is seen in Fig. 4, where the solid curve shows the fusion power in arbitrary units [Eq. (26)] as a function of β_c . The base case value is also shown for comparison. The variation of β_c/β_D with R is shown in Fig. 5, where the solid curve plots the right side of Eq. (25). The base case value is again shown for comparison.

It should be remembered that the preceding analysis considers only interchange instability. Minimization of the effects of interchange is useful as a coil design strategy, but at high β machine performance is, of course, limited by ballooning modes. Since tandem design studies to date have indicated that designs that are good from the point of view of

interchange stability are also good from the point of view of ballooning stability, the strategy is a reasonable one. To illustrate this point, the TANDEM BALLOON code was used to compute the limiting value of β_c for central-cell magnetic field profiles given by Eq. (18) for various values of R , and $\alpha_p = 0.4$. The results, $(\beta_c)_B$, are plotted as solid circles relative to the scale on the right side of Fig. 5. As the figure makes clear, to a very good approximation, $(\beta_c)_B$ scales directly with β_c/α_p . The same scaling is observed for the fusion power, as indicated in Fig. 1, where the data plotted as solid circles were derived from the same TANDEM BALLOON code calculations used to obtain $(\beta_c)_B$. Comparison of Figs. 4 and 5 demonstrates the important point that increasing β_c does not necessarily lead to an increase in the fusion power.

IV. Discussion and Summary

Conventionally designed tandem mirrors with thermal barriers are limited by MHD instabilities to values of central cell R that are significantly lower than those attainable in tandems without barriers. Much of this degradation is due to the introduction of a new region of unfavorable field line curvature in the central cell. From a variational calculation that maximizes central cell performance subject to constraints on the vacuum field energy and allowable interchange instability drive, I have derived an analytic expression for a central cell magnetic field strength profile for which the new instability drive is greatly reduced. As a result, central cell performance is enhanced to a point approaching that of tandem mirrors without barriers. Formulas showing how β_c and fusion power scale with design parameters that are potentially useful for further design improvements were also derived.

A basic assumption in the calculation is the axisymmetry of the magnetic field. If this assumption were relaxed, in order to treat the thermal barrier region, for example, then (still within the long-thin approximation) two functions, $B(z)$ and the flux-surface ellipticity, would be required to completely specify the magnetic field. In the appropriate variational calculation, both of these functions would have to be independently varied in order to optimize the performance parameters. One further complication would be variable plasma pressure. Such a calculation is inherently much more complicated than the one presented here, and would probably require a numerical approach. In cases where the dominant instability drive arises in the barrier region rather than in the solenoid, however, it would be the appropriate one to do.

Notes and References

1. D. I. Baldwin, B. G. Logan and T. F. Fowler, "An Improved Tandem Mirror Fusion Reactor," UCID 18156 (1979).
2. This constraint was suggested to me by L. D. Pearlstein. Practically, it or something similar is necessary for obvious economic reasons; mathematically, it provides a constraint on the class of admissible solutions to the variational problem that is crucial to its being well-posed.
3. T. B. Kaiser, "A Computationally Convenient Criterion for Low-Interchange Stability of Tandem Mirrors," UCID 18181, 1979.

Figure Captions

1. Dimensionless functions $\hat{K}(R)$, $\hat{D}(R)$, $\hat{P}(R)$, defined in the text, as a function of central-cell mirror ratio, R .
2. Optimum magnetic field profile, $B(z)$, given by Eq. (18), with base-case field shown for comparison. Central cell midplane is at $z = 0$, barrier field peak at $z = 1000$.
3. B_c/B_p as a function of central-cell half-length, L , for magnetic field profiles given by Eq. (18) and base-case design point parameters.
4. Central cell fusion power as a function of B_c for magnetic field profiles given by Eq. (18) and base-case design point parameters.
5. Same as Fig. 3 with B_c as the independent variable.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5