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Determination of the pipe stemming load

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CONTENTS

Abstract	1
1. Introduction	1
2. The Derivation of the Basic Equations	4
3. The Solution in the Unstemmed Region	7
4. An Approximation	8
5. The Solution in the Lower Portion of the Stemmed Region When Slipping Occurs on the Pipe-Stemming Interface	9
6. The Solution in the Upper Portion of the Stemmed Region Where Slip Does Not Occur on the Pipe-Stemming Interface	10
7. Guide For Programming	13
Acknowledgments	20
References	20
Appendix I—The Solution without the Approximation	21
I-1. The Solution in the Lower Portion of the Stemmed Region When Slipping Occurs on the Pipe-Stemming Interface	21
I-2. The Solution in the Upper Portion of the Stemmed Region Where Slip Does Not Occur on the Pipe-Stemming Interface	23
Appendix II—The Effect of Temperature on the Determination of the Pipe-Stemming Load	27

DETERMINATION OF THE PIPE STEMMING LOAD

ABSTRACT

A mechanical model for the emplacement pipe system is developed. The model is then employed to determine the force applied to the surface collar of the emplacement pipe, the pipe-stemming load, and the stress along the emplacement pipe as a function of stemming height. These results are presented as integrals and a method for their numerical integration is given.

1. INTRODUCTION

The pipe-stemming load is the portion of the weight of the stemming material carried by the emplacement pipe in a vertical underground test system. The emplacement pipe supports the experimental device from above. The hole containing the emplacement pipe and experimental device is backfilled with stemming material to prevent the escape of material during and after the test. Hamilton¹ did the initial analytical work to determine the pipe stemming load. He suggested that the stemming load, i.e., the weight of the stemming material used to backfill the emplacement hole, was partially carried by the pipe support system and partially carried by the wall of the emplacement hole. The portion of a total increment in stemming weight carried by the pipe support system was argued by Hamilton to be $R_p/(R_p + R_h)$, where R_p and R_h are the pipe and hole radii, respectively. The ratio $R_p/(R_p + R_h)$ is called the *wetted perimeter* fraction. The stemming tests of Cramer² on a 100-ft-length of pipe apparently confirmed the *wetted perimeter* concept. However, Norris³ observed that the collar stemming load data from a number of emplacement systems contradicted Cramer's results. The increase in the collar stemming load in the actual emplacement system was considerably less than that predicted by the *wetted perimeter* concept. Norris surmised that the discrepancy between the actual emplacement pipe collar data on one hand and Hamilton's *wetted perimeter* concept and Cramer's experimental data on the other hand was caused by Cramer and Hamilton not accounting for the "load release due to pipe movement." That is, the downward motion of the support pipe through the stemming material transferred some of the total load carried by the pipe to the emplacement hole wall through the stemming material. Norris observed that the short pipe used by Cramer had an extremely high stiffness compared to the usual systems of long emplacement pipes. Norris states that a prime mechanism for reducing the collar stemming load appears to be displacement of the pipe. A second mechanism is the support given by stemming material under and around the sides of the device-diagnostic package.

Norris presented a more refined model for the pipe-stemming load. He assumed that the pipe stemming load was reduced by "a concentrated spring at the (device-diagnostic) canister with reactive force proportional to the displacement associated with the concentrated canister stemming load." He assumed that the weight of the stemming material carried by the pipe was reduced by the concentrated canister spring and by a distributed spring, called K_{soil} , in the annulus stemming. K_{soil} provided a restoring force proportional to the displacement associated with the wetted perimeter-stemming load. Norris' work led to the development of Stem Code, a computer program to determine the pipe stemming load. An early version of this program is described in Ref. 4.

At that time Stem Code and its rationale was a major advance on the pioneering work by Hamilton. However, the predictions of Stem Code for the pipe stemming load did not agree with the measured data. Modifications of the code were made and new parameters were introduced. Blake⁵ reviewed the situation in 1973. He observed that the spring constant K_{soil} introduced in the Stem Code could be adjusted to make the

predictions of the Stem Code agree with the experimental data. In some cases there was still no agreement between Stem Code and the measured pipe collar loads; the Portulaca event was cited as an example. Blake also noted that the Stem Code predictions were relatively insensitive to many code parameters, and the parameter K_{soil} had no basis in experimental fact.

The present work resulted from a meeting on July 13, 1977 of A. Blake, R. Corallo, D. Maiden, F. Morrison and S. C. Cowin. Two changes in the analysis for the pipe-stemming load calculation were discussed. The first was removal of the assumption that the vertical stress in the stemming was constant, and the second was adjustment of the zero-strain reference state in the stemming material to be consistent with the motion of the support pipe. On July 15, 1977, Cowin wrote a note to the participants in which the governing differential equations were derived, but not solved. It was implied that, since the governing differential equations were linear, the problem might be solved easily. However, the solution of the problem was not simple, as this report shows.

This analysis is an extension of those done by Hamilton¹ and Norris.³ The differences lie in the details of the assumptions. In particular, the idea of Norris³ concerning load release caused by pipe movement is pursued to better understanding of the interaction between the emplacement pipe and the stemming material. Our analysis suggests that load release caused by pipe movement is in excess of that suggested by Norris. The mechanism for this load release is the reversal of the direction of the friction force between the stemming material and the pipe. This occurs because the low stiffness of the long pipe permits large relative deflections of the distal end of the pipe. The contrast in the situation experienced by the near and far end of a highly stemmed long pipe is illustrated in Fig. 1. To demonstrate that this occurs in practice, note that the deflection δ of the steel pipe 1,500 ft long with a cross-sectional area of 16 in.² is 3.5 in. caused by 2/3 of its wetted perimeter load.

FIG. 1. The interaction between the stemming material and the pipe; (a) the stemming material acts down on the pipe near the free surface of the emplacement hole; (b) the stemming material acts up on the pipe near the bottom of the emplacement hole.

$$\delta = \frac{PL}{AE} = \frac{(94,000 \text{ lb}) (1500 \text{ ft} \times 12 \text{ in./ft})}{(16 \text{ in.}^2) (30,000,000 \text{ psi})} = 3.5 \text{ in.}$$

where

$$P = \frac{R_p}{R_h - R_p} (R_h^2 - R_p^2) \gamma \times 1000 \text{ ft} = 94,000 \text{ lb.}$$

On the other hand, the maximum deflection of the stemming material before slip occurs between the pipe and the stemming material is ~1 in. Actually, it is considerably less than 1 in. and the greatest leeway is taken to raise the deflection to 1 in. Thus, it is reasonable to expect that for a long pipe stemmed for a large portion of its length, the effective direction of the pipe-stemming material friction force can reverse itself.

The consequences of this reversal of friction force are significant. For example, the portion of a total increment in stemming weight carried by the pipe support system need not be in the ratio $R_p/(R_p + R_h)$, but can be variable and can be a negative number. The *wetted perimeter* concept holds only for short pipes, not for the long pipes of an emplacement system. It is reasonable to expect that the pipe collar load could vanish. Reports have indicated that this has been observed. For example, strong backs supporting the load have, on some events, been removed after the stemming operation was complete.

The interaction mechanism between the stemming material and the support pipe is treated in detail in this work. Norris³ realized that the downward deflection of the pipe induced a shearing strain in the surrounding stemming material annulus. The zero-strain reference state for the stemming material was taken as the unstemmed pipe deflection at the point in question. Here the zero-strain reference state in the stemming material at a given downhole location is the pipe deflection when the support pipe has been stemmed to that particular downhole location.

The modeled interfacial interaction between the stemming material and the pipe is now described. It is assumed that when the stemming material is emplaced at a specified location, the stemming material exerts a downward force on the pipe. This downward force is assumed to be the maximum that can occur without slipping. As the stemming level rises above this location, the pipe deflects in the direction of the downward force exerted by the stemming material. As the stemming level rises farther, the pipe continues to deflect. The force on the pipe caused by the stemming material decreases and vanishes as the strained stemming material takes the weight of the stemming material previously carried by the pipe. As stemming continues above the specified location, the pipe deflection increases and the force of the stemming material on the pipe reverses its direction and acts upward on the support pipe, opposing its motion. The final stage of the interfacial interaction between the pipe and stemming material occurs when the static friction force is overcome and the pipe slides down through the stationary stemming. The considerations of these mechanisms set this work apart from its predecessors.

The principal results of this report are now described. The input parameters that determine the results and are assumed to be known are the following.

E = Young's modulus for the pipe.

A = Cross-sectional area of the pipe.

w = Weight per unit length of the pipe.

R_p = Radius of the pipe.

R_h = Radius of the emplacement hole.

μ_p = Coefficient of friction between the stemming and the pipe.

μ_h = Coefficient of friction between the stemming and the hole wall.

K = Ratio of lateral to vertical pressure in the stemming material (Janssen's coefficient).

G = Shear modulus of the stemming material.

L = Length of the pipe.

W = Weight of the device-diagnostic canister and other load applied at the end of the pipe.

$u_R(s)$ = History of release displacement at the pipe collar given as a function of s , the downhole distance to the free surface of the stemming material.

The solution is determined by two functions $y(s)$ and $F(s)$, where s is the downhole distance to the free surface of the stemming. The distance from the free surface of the stemming to the point where the pipe begins to slip relative to the stemming material is y . F is the force at the pipe support collar and the focus of this analysis. To determine $F(s)$, $y(s)$ must be determined first. $F(s)$ is given by Eq. (46) and $y(s)$ by Eq. (50). The solution of these equations is not difficult with the present computational facilities.

The derivation of the basic equations is summarized in Sec. 2. These equations are solved in the unstemmed region in Sec. 3, and an approximation to the equations is given in Sec. 4. The solution in the lower portion of the stemmed region where slipping occurs on the pipe-stemming interface is presented in Sec. 5. In Sec. 6 the solution in the upper portion of the stemmed region where slip does not occur on the pipe-stemming interface is developed. Appendix I presents the solution in the case where the approximation of Sec. 4 is not introduced and Appendix II deals with the inclusion of thermal effects.

2. THE DERIVATION OF THE BASIC EQUATIONS

The problem is divided into two parts for this analysis. Consider the partially stemmed hole shown in Fig. 2. The portion of the pipe that is unstemmed lies between 0 and the level of the free surface of the stemming material s , $0 \leq x \leq s$. The stemmed portion of the pipe is between s and L , i.e., all stations x such that $s \leq x \leq L$. The vertical displacement of the pipe at the station x due to stemming up to the station s is denoted

FIG. 2. A schematic of a partially stemmed hole.

by $u(x,s)$. Similarly, we let $T(x,s)$ denote the axial force in the pipe at a station x due to stemming up to the station s . If the hole is unstemmed, then the displacement is denoted by $u(x,L)$ and the axial force by $T(x,L)$. The axial force $T(x,s)$ and the vertical displacement are related by Hooke's law

$$T(x,s) = EA \frac{du}{dx}(x,s), \quad (1)$$

where E denotes Young's modulus for the pipe and A is the cross-sectional area of the pipe.

Equilibrium of the pipe requires that $T(x,s)$ satisfy the equation

$$\frac{dT(x,s)}{dx} + w = 0, \text{ if } 0 \leq x \leq s, \quad (2a)$$

or

$$\frac{dT(x,s)}{dx} + 2\pi R_p \tau(x,s) + w = 0, \text{ if } s \leq x \leq L, \quad (2b)$$

where $\tau(x,s)$ is the shear stress exerted on the pipe by the stemming material, w is the weight per unit length of the pipe, and R_p is the radius of the pipe. The derivation of Eq. (2b) from a free-body diagram of a pipe section is shown in Fig. 3. Equation (2a) follows from Eq. (2b) when $\tau(x,s)$ vanishes. A similar free-body diagram of an annular-shaped slice of stemming material in Fig. 4 shows that

$$\frac{d\sigma}{dx}(x,s) + \frac{\sigma(x,s)}{\ell_1} + \frac{2R_p}{R_h^2 - R_p^2} \tau(x,s) - \gamma = 0, \text{ if } s \leq x \leq L, \quad (3a)$$

and

$$\sigma(x,s) = 0, \text{ if } 0 \leq x \leq s, \quad (3b)$$

FIG. 3. A free-body diagram of the pipe. The sum of the forces for this free-body diagram is:

$$T + dT/dx dx - T + 2\pi R_p \tau dx + w dx = 0,$$

or

$$dT/dx + 2\pi R_p \tau + w = 0,$$

where P is the axial force in the pipe, R_p is the radius of the pipe, w is the weight per unit length of the pipe, and τ is the shear stress the stemming material places on the pipe.

FIG. 4. A free-body diagram of an annular-shaped volume of stemming material. The sum of the forces for this free-body diagram is:

$$\left(\sigma + \frac{d\sigma}{dx} dx \right) \pi (R_h^2 - R_p^2) - \sigma \pi (R_h^2 - R_p^2) + 2\pi R_p \tau dx + \mu_h R_h 2\pi K \sigma dx - \gamma \pi (R_h^2 - R_p^2) dx = 0, \quad \text{or} \quad \frac{d\sigma}{dx} + \frac{2\mu_h R_h K}{R_h^2 - R_p^2} \sigma + \frac{2R_p}{R_h^2 - R_p^2} \tau - \gamma = 0,$$

where σ is the vertical compressive stress in the stemming, γ is the weight density of the stemming material, R_h is the radius of the hole, K is the ratio of horizontal pressure to vertical pressure in the stemming material, and μ_h is the coefficient of friction between the stemming and the casing wall.

where

$$\ell_1 = \frac{R_h^2 - R_p^2}{2\mu_h R_h K} \quad (4)$$

and where $\sigma(x,s)$ is the vertical stress in the stemming, R_h is the radius of the hole, γ is the weight density of the stemming, K is the ratio of horizontal to vertical pressure in the stemming material, and μ_h is the coefficient of friction between the stemming material and the hole wall.

The assumption about the nature of the shear stress that acts between the stemming material and the pipe is now given in detail. As the stemming material falls into place in the hole, we assume that a shear stress τ of maximum magnitude develops and it acts up on the stemming material and down on the pipe, i.e., in the direction of positive τ . The maximum magnitude of the shear stress is $\mu_p K \sigma$, where K is the ratio of the

horizontal to vertical pressure and μ_p is the coefficient of friction between the pipe and the stemming material. These assumptions are the same as those used to obtain Janssen's formulas.⁶⁻⁸ The situation changes as the hole fills, because the deflection of the pipe at previously filled stations is increasing and the pipe tends to move down relative to the stemming material, reducing the shear stress τ . As the downward deflection of the pipe is increased, the shear stress τ changes sign and then increases to a maximum negative magnitude $-\mu_p K \sigma$. Further movement of the pipe occurs by slipping through the stemming material. It is assumed that the shear stress τ changes from $+\mu_p K \sigma$ to $-\mu_p K \sigma$ smoothly and linearly because of the elasticity of the stemming material in shear. With these assumptions the shear stress τ is given by

$$\tau(x, s) = \mu_p K \sigma - \frac{G}{R_h - R_p} [u(x, s) - u(x, x)],$$

if $\tau \geq -\mu_p K \sigma$ and $s \leq x \leq L$; (5)

otherwise there is a slip displacement u_{slip} at the pipe surface given by

$$u_{\text{slip}} = u(x, s) - u(x, x) - \frac{(R_h - R_p)}{G} 2\mu_p K \sigma(x, s), \text{ if } s \leq x \leq L. \quad (6)$$

In Eqs. (5) and (6), G is the shear modulus of the stemming material and $u(x, x)$ is the deflection of the pipe at a station x when the free surface of stemming is at station x . The displacement $u(x, x)$ is subtracted from $u(x, s)$ to obtain a zero-strain initial reference strain state for the stemming material.

The system of equations describing the behavior of the pipe and stemming material are Eqs. (1), (2b), (3), (5), and (6). These equations apply to the stemmed portion of the pipe. Equations (1) and (2a) apply to the unstemmed portion.

3. THE SOLUTION IN THE UNSTEMMED REGION

The vertical displacement $u(x, s)$ and the axial pipe force $T(x, s)$ in the unstemmed region of the pipe can be determined from Eqs. (1) and (2a), subject to two boundary conditions. We let $F(s)$ denote the axial force at the top of the pipe and u_R denote the release displacement, i.e., the adjusted displacement of the pipe at the top of the hole obtained by releasing the pipe after the stemming has been initiated,

$$\underbrace{u(0, s) = u_R(s)}_1, \quad \underbrace{T(0, s) = F(s)}_2. \quad (7)$$

Equation (2a) can be integrated directly using the boundary condition Eq. (7)₂,

$$T(x, s) = F(s) - wx, \quad 0 \leq x \leq s. \quad (8)$$

A differential equation is obtained from Eqs. (1) and (2a) by the elimination of $T(x, s)$,

$$EA u''(x, s) + w = 0. \quad (9)$$

This differential equation may be integrated using the boundary conditions in Eq. (7) and using Eq. (2a),

$$u(x, s) = \frac{F(s)}{EA} x - \frac{w}{2EA} x^2 + u_R(s), \quad \text{if } 0 \leq x \leq s. \quad (10)$$

If the entire length of the pipe is unstemmed, then the tension $T(L, L)$ in the pipe at the bottom must equal the weight of the device-diagnostic package W ; thus from Eq. (8)

$$T(L, L) = W = F(L) - wL. \quad (11)$$

The deflection $u(x, L)$ in the case when the entire length of pipe is unstemmed is then given by the following expression

$$u(x, L) = \frac{Wx}{AE} + \frac{wx}{AE} \left(L - \frac{x}{2} \right) + u_R(L), \quad (12)$$

where $F(L)$ from Eq. (11) has been substituted into Eq. (10) evaluated at $s = L$. It is reasonable to assume that $u_R(L) = 0$ and that will be done in a later section.

4. AN APPROXIMATION

The solution to the complete system of equations is complicated, but not impossible. A solution is described in Appendix I. In this section it is shown that a reasonable approximation in the basic system of equations will reduce the complexity of the calculations. This approximation is based on the observation that the first term in Eq. (3) is small compared to the other three terms, thus Eq. (3) can be approximated by

$$\frac{\sigma(x, s)}{\ell_1} + \frac{2R_p}{R_h^2 - R_p^2} \tau(x, s) - \gamma = 0, \text{ if } s \leq x \leq L. \quad (13)$$

A dimensional analysis of Eq. (3) shows that the first term is proportional to ℓ_1/ℓ_0 , where ℓ_0 is a length used to render the x coordinate dimensionless. For a typical stemming operation $R_h = 36$ in., $R_p = 5$ in., and $\mu_h K \cong 1/4$; thus from Eq. (4) ℓ_1 is about 6 ft. It is reasonable to select ℓ_0 much greater than 6 ft, thus ℓ_1/ℓ_0 is small and the derivative of $\sigma(x, s)$ with respect to x can be neglected. *This does not mean that $\sigma(x, s)$ is a constant*; only that its first derivative is smaller than the other terms of Eq. (3) in the problem under consideration.

Equations (13) and (5) may be solved for $\tau(x, s)$ and $\sigma(x, s)$ in terms of displacement $u(x, s)$,

$$\tau(x, s) = \frac{\gamma \mu_p K (R_h^2 - R_p^2) \ell_1}{R_h^2 - R_p^2 + 2R_p \mu_p K \ell_1} - \frac{G(R_h + R_p)}{R_h^2 - R_p^2 + 2R_p \ell_1} [u(x, s) - u(x, x)], \quad (14)$$

and

$$\sigma(x, s) = \frac{(R_h^2 - R_p^2) \gamma \ell_1}{R_h^2 - R_p^2 + 2R_p \mu_p K \ell_1} + \frac{2R_p \ell_1 G}{(R_h - R_p)(R_h^2 - R_p^2 + 2R_p \ell_1)} [u(x, s) - u(x, x)]. \quad (15)$$

5. THE SOLUTION IN THE LOWER PORTION OF THE STEMMED REGION WHEN SLIPPING OCCURS ON THE PIPE-STEMMING INTERFACE

The determination of the pipe displacement $u(x, s)$, the axial pipe force $T(x, s)$ and the vertical stemming material stress $\sigma(x, s)$ is much more complicated in the stemmed region than is the determination of $u(x, s)$ and $T(x, s)$ in the unstemmed region. The problem is considered in two parts. First we consider the case when $\tau(x, s) = -\mu_p K \sigma(x, s)$ in the lower portion of the stemmed region and $\tau(x, s) > -\mu_p K \sigma(x, s)$ in the upper portion of the stemmed region. In this section the problem of slipping along the lower portion of the pipe stemming interface is considered. In the next section this result is used to obtain a solution to the problem in the no-slipping region above the slipping region.

Let $s + y, y > 0$ denote the downhole distance at which slip first occurs, $0 \leq s + y \leq x \leq L$. Slip is assumed to occur along the pipe for x greater than $s + y$ and less than $L, s + y \leq x \leq L$. Since $\tau(x, s) = -\mu_p K \sigma(x, s)$, it follows from Eq. (13) that

$$\sigma(x, s) = \sigma_0, \tau(x, s) = -\mu_p K \sigma_0, \quad (16)$$

where

$$\sigma_0 \equiv \frac{\gamma \ell_1 \ell_2}{\ell_2 - \ell_1}, \quad \ell_2 = \frac{R_h^2 - R_p^2}{2\mu_p R_p K}. \quad (17)$$

The slip displacement is given by Eq. (6),

$$u_{\text{slip}} = u(x, s) - u(x, x) - \frac{R_h - R_p}{G} 2\mu_p K \sigma_0. \quad (18)$$

The equilibrium condition Eq. (2b) can be rewritten in the form

$$\frac{dT(x, s)}{dx} + w_e = 0, \quad (19)$$

where

$$w_e = w - 2\pi R_p \mu_p K \sigma_0 \quad (20)$$

is the effective weight per unit length of the pipe. w_e is less than w in the region $s + y \leq x \leq L$ because the friction force is acting upward in this region, reducing the effective weight. The solution to Eq. (19) subject to the boundary condition

$$T(x, s) \Big|_{x=L} = W \quad (21)$$

is:

$$T(x, s) = W + w_e (L - x) \text{ for } s + y \leq x \leq L. \quad (22)$$

Note that the pipe tension in the stemmed region of the pipe that has slipped is less than the tension in the same region with no slipping.

6. THE SOLUTION IN THE UPPER PORTION OF THE STEMMED REGION WHERE SLIP DOES NOT OCCUR ON THE PIPE-STEMMING INTERFACE

The system of differential Eqs. (1), (2b), (5) and (13) can be reduced to the solution of a single second-order differential equation with constant coefficients for $u(x,s)$. To obtain this differential equation, first substitute Eqs. (1) and (5) into Eq. (2b) and solve for $\sigma(x,s)$,

$$\sigma(x,s) = \frac{EA\ell_2}{A_s\ell_3^2} [u(x,s) - u(x,x)] - \frac{\ell_2}{A_s} [EAu''(x,s) + w] , \quad (23)$$

where the following notations have been introduced:

$$\ell_3 = + \left(\frac{EA(R_h - R_p)}{2\pi R_p G} \right)^{1/2} , \quad A_s = \pi(R_h^2 - R_p^2) . \quad (24)$$

The differential equation for $u(x,s)$ is obtained by substituting $\sigma(x,s)$ given by Eq. (23) and $\tau(x,s)$ given by Eq. (5) into Eq. (13),

$$u''(x,s) - \lambda^2 u(x,s) = -\frac{w}{EA} - \frac{\gamma A_s}{EA} \frac{\ell_1}{\ell_1 + \ell_2} - \lambda^2 u(x,x) , \quad (25)$$

where

$$\lambda^2 = \frac{\ell_2}{(\ell_1 + \ell_2)\ell_3^2} \quad (26)$$

and λ has the dimension of reciprocal length. If we take a typical emplacement, assuming

$$R_h = 36 \text{ in.}, R_p = 5 \text{ in.}, A = 16 \text{ in.}^2, K\mu_p \cong K\mu_n \cong \frac{1}{4}, G = 2000 \text{ psi, and } E = 30 \times 10^6 \text{ psi,} \quad (27)$$

then from Eqs. (4) and (24)

$$\ell_1 = 5.88 \text{ ft, } \ell_2 = 42.36 \text{ ft, } \ell_3 = 40.55 \text{ ft and } \lambda = \pm (43.3 \text{ ft})^{-1} . \quad (28)$$

The value of $u(x,x)$ occurring in Eq. (25) can be obtained from Eq. (10), thus Eq. (25) can be rewritten as

$$u''(x,s) - \lambda^2 u(x,s) = -\frac{1}{EA} \left[w + \gamma A_s \frac{\ell_1}{\ell_1 + \ell_2} + \lambda^2 x F(x) - \lambda^2 \frac{wx^2}{2} + EA\lambda^2 u_R(x) \right] . \quad (29)$$

This differential equation must satisfy the boundary conditions

$$u(s,s) = \frac{sF(s)}{EA} - \frac{ws^2}{2EA} + u_R(s)$$

and

$$u'(s,s) = \frac{F(s)}{EA} - \frac{w}{EA} s , \quad (30)$$

which are the values of $u(x,s)$ and $u'(x,s)$ given by Eq. (10) evaluated at $x = s$. These conditions are consequences of the continuity of displacement and force at the interface $x = s$.

The solution to Eq. (29), subject to the boundary conditions Eq. (30), is

$$EAu(x, s) = EAu_R(s) \cosh \lambda(x-s) + \frac{\rho_1^2 \rho_2}{\rho_2} \gamma A_s [1 - \cosh \lambda(x-s)] \\ - \frac{wx^2}{2} + F(s) \left[\frac{1}{\lambda} \sinh \lambda(x-s) + s \cosh \lambda(x-s) \right] \\ - \int_s^x [\xi F(\xi) + EAu_R(\xi)] \lambda \sinh \lambda(x-\xi) d\xi. \quad (31)$$

This result is difficult to obtain and is checked by verifying that Eq. (31) satisfies Eq. (29) and the boundary conditions Eq. (30). It is helpful to keep in mind the following formula for the derivative of an integral. If

$$G(x) = \int_{p(x)}^{q(x)} f(x, \xi) d\xi, \quad (32)$$

then

$$G'(x) = \int_{p(x)}^{q(x)} \frac{df(x, \xi)}{dx} d\xi + f[q(x), x] q'(x) - f[p(x), x] p'(x). \quad (33)$$

A derivation of this result is given in Ref. 9. The term $\lambda \sinh \lambda(x-\xi)$ is the Green's function for the differential equation labeled as Eq. (29).

The axial tension $T(x,s)$ in the pipe can be obtained by substituting Eq. (31) into Eq. (1)

$$T(x, s) = \lambda EAu_R(s) \sinh \lambda(x-s) - \frac{\rho_1^2 \rho_2}{\rho_2} \gamma A_s \lambda \sinh \lambda(x-s) - wx + F(s) [\cosh \lambda(x-s) \\ + \lambda s \sinh \lambda(x-s)] - \int_s^x [\xi F(\xi) + EAu_R(\xi)] \lambda^2 \cosh \lambda(x-\xi) d\xi \text{ for } s \leq x \leq s+y. \quad (34)$$

The condition to determine the function $F(s)$ is that $T(x,s)$, given by Eq. (34) and evaluated at $x = s+y$, must equal $T(x,s)$ given by Eq. (22) and also evaluated at $x = s+y$,

$$W + w_e(L-s-y) = \lambda EAu_R(s) \sinh \lambda y - \frac{\rho_1^2 \rho_2}{\rho_2} \gamma A_s \lambda \sinh \lambda y - w(s+y) + F(s) [\cosh \lambda y + \lambda s \sinh \lambda y] \\ - \int_s^{s+y} [\xi F(\xi) + EAu_R(\xi)] \lambda^2 \cosh \lambda(s+y-\xi) d\xi. \quad (35)$$

This is an integral equation for $F(s)$ given the history $u_R(s)$. If there is no yielding at the pipe stemming interface, $y = L - s$, then Eq. (35) can be written

$$W + wL = \lambda EA u_R(s) \sinh \lambda(L - s) - \frac{\ell_3^2 \ell_1}{\ell_2} \gamma A_s \lambda \sinh \lambda(L - s) + F(s) [\cosh \lambda(L - s) + \lambda s \sinh \lambda(L - s)] - \int_s^L [\xi F(\xi) + EA u_R(\xi)] \lambda^2 \cosh \lambda(L - \xi) d\xi. \quad (36)$$

To obtain an expression for $F(s)$, Eq. (36) is differentiated with respect to s ,

$$F'(s) = \frac{-\frac{\ell_3^2 \ell_1}{\ell_2} \gamma A_s \lambda^2 - \lambda EA \frac{du_R}{ds} \tanh \lambda(L - s)}{1 + \lambda s \tanh \lambda(L - s)}. \quad (37)$$

This differential equation can be integrated to obtain

$$F(s) = W + wL + \int_L^s \frac{\left(\frac{\ell_3^2 \ell_1}{\ell_2} \gamma A_s \lambda^2 + \lambda EA \frac{du_R}{d\xi} \tanh \lambda(L - \xi) \right) d\xi}{1 + \lambda \xi \tanh \lambda(L - \xi)}, \quad (38)$$

where the boundary condition $F(L) = W + wL$ is used. This boundary condition is obtained from Eq. (36) by setting $s = L$.

It is more difficult to solve for $F(s)$ when there is some yielding along the pipe-stemming interface. In this case Eq. (35) for $F(s)$ still applies, but in order to solve it one must know y , which is a function of s , $y = y(s)$. The function $y(s)$ is determined by solving for the depth $s + y$ at which yielding first occurs as a function of s . At yield, $\sigma(x, s)$ and $\tau(x, s)$ are given by Eq. (16); from Eqs. (16) and (5) we obtain the condition

$$2\mu_p K \epsilon_0 = \frac{G}{R_h - R_p} [u(y + s, s) - u(y + s, y + s)], \quad (39)$$

which, using Eq. (17) can be rewritten as

$$EA [u(s + y, s) - u(s + y, s + y)] = \frac{2A_s \ell_3^2 \gamma \ell_1}{\ell_2 - \ell_1}. \quad (40)$$

When $u(s + y, s)$ from Eq. (31) and $u(s + y, s + y)$ from Eq. (10) are substituted into Eq. (40), an implicit equation for $y(s)$ is obtained:

$$EA u_R(s) \cosh \lambda y + \frac{\ell_3^2 \ell_1 \gamma A_s}{\ell_2} \left(\frac{\ell_1 + \ell_2}{\ell_1 - \ell_2} - \cosh \lambda y \right) + F(s) \left[\frac{1}{\lambda} \sinh \lambda y + s \cosh \lambda y \right] - (y + s) F(y + s) - EA u_R(y + s) - \int_s^{s+y} [\xi F(\xi) + EA u_R(\xi)] \lambda \sinh \lambda(y + s - \xi) d\xi = 0. \quad (41)$$

The analytical development of the solution stops here. Equations (41) and (35) must be solved simultaneously for the function $y(s)$, which gives the distance from the free surface of the stemming to the point where the stemming begins to slip relative to the pipe, and the function $F(s)$, which is the force necessary to support the pipe. Once $F(s)$ and $y(s)$ have been determined, the function $T(x,s)$ representing the axial force along the pipe is completely determined by Eq. (34).

7. GUIDE FOR PROGRAMMING

In this section a step-by-step guide is outlined for a computer program for determining the pipe stemming load. The first draft of the computer program should omit Steps 7 and 8, because that is where the axial force along the length of the entire pipe is calculated. In Steps 3 and 6 the axial force in the pipe at the earth's surface is calculated, and since this force can be compared with the field data, it is the principal force of interest in the first stage of program development.

Step 1: Introduce the variables.

SYMBOL (typical value)	MEANING
INPUT PARAMETERS	
E (30×10^6 psi)	Young's modulus for the pipe
A (16 in.^2)	Cross-sectional area of the pipe
w (52 lb/ft)	Weight per unit length of the pipe
R_p (5 in.)	Radius of the pipe
R_h (36 in.)	Radius of the emplacement hole
μ_p (0.42)	Coefficient of friction between the stemming and the pipe
μ_h (0.42)	Coefficient of friction between the stemming and the hole wall
K (0.6)	Ratio of lateral to vertical pressure in the stemming material (Janssen's coefficient)
G (2000 psi)	Shear modulus of the stemming material
L	Length of the pipe
W	Weight of the device-diagnostic canister and other load applied at the end of the pipe
$u_R(s)$	History of release displacement at the pipe collar given as a function of s , the downhole distance to the free surface of the stemming material
γ (95 lb/R^3)	Weight per unit volume of the stemming material
INDEPENDENT VARIABLES	
x	Downhole coordinate (see Fig. 2)
s	Downhole distance to free surface of the stemming material (see Fig. 2)
OUTPUT INFORMATION	
ℓ_1, ℓ_2, ℓ_3	Problem parameters of dimension length
λ	Problem parameter of dimension inverse length

A_s	Area of annular region between pipe and hole wall
w_e	Reduced effective weight per unit length of the pipe
σ_0	Stress in the stemming material
$y(s)$	Distance from the free surface of the stemming material to the point where the stemming material begins to slip relative to the pipe (see Fig. 2)
$F(s)$	Force necessary to support the pipe at the earth's surface when there is no slipping between the stemming material and the pipe
s_0	Level of the stemming surface when slipping between the stemming material and the pipe begins at the bottom of the pipe, $x = L$
$F_0(s)$	Force necessary to support the pipe at the earth's surface when there is slipping between the stemming material and the pipe
$T(x,s)$	Axial force in the pipe as a function of position along the pipe x , and as a function of the distance to the free surface of the stemming s , when there is no slipping between the stemming material and the pipe; that is to say s_0 is not in the interval 0 to L , or if it is, then $0 \leq s_0 \leq s \leq L$.
$T_0(x,s)$	Axial force in the pipe as a function of position along the pipe x , and as a function of the distance to the free surface of the stemming s , when there is slipping between the stemming material and the pipe, $0 \leq s \leq s_0 \leq L$.

Step 2: Preliminary calculations.

Step 2a: Using the input data, compute the value of the following quantities

$$\ell_1 = \frac{R_h^2 - R_p^2}{2\mu_h R_h K},$$

$$\ell_2 = \frac{R_h^2 - R_p^2}{2\mu_p R_p K},$$

$$\ell_3 = + \left(\frac{EA (R_h^2 - R_p^2)}{2\pi R_p G} \right)^{1/2},$$

$$\lambda = \left(\frac{\ell_2}{(\ell_1 + \ell_2)\ell_3^2} \right)^{1/2},$$

$$A_s = \pi(R_h^2 - R_p^2),$$

$$q_0 = \frac{\gamma \ell_1 \ell_2}{\ell_2 - \ell_1},$$

$$w_e = w - 2\pi R_p \mu_p K \sigma_0.$$

(42)

Step 2b: Compute du_R/ds from $u_R(s)$. This calculation causes concern because it is suspected that the function $u_R(s)$ should contain sharp changes from one value to another separated by long periods where it is constant. It is feared that the shape changes in $u_R(s)$ will make du_R/ds too high (perhaps infinite), and these shape changes in $u_R(s)$ should be smoothed in a reasonable fashion so that the derivative stays bounded. For example if $u_R(s)$ has the form

then $\frac{du_R(s)}{ds}$ has the form

But if we smooth $u_R(s)$, spreading the shape change, to

then $\frac{du_R(s)}{ds}$ has the form

Step 3: Determination of the function $F(s)$.

The function $F(s)$ is given by Eq. (38), which is written in the form

$$F(s) = W + wL + I(s) , \quad (43)$$

where

$$I(s) = \int_L^s \frac{\left(\frac{\ell_3^2 \ell_1}{\ell_2} \gamma A_s \lambda^2 + \lambda EA \frac{du_R}{d\xi} \tanh \lambda(L - \xi) \right) d\xi}{1 + \lambda \xi \tanh \lambda(L - \xi)} . \quad (44)$$

This integral can be evaluated by any computational method for integrals (e.g. Simpson's rule). The evaluation should commence at $s = L$ and proceed downward in the absolute value of s to $s = 0$. The function $F(s)$ should then be stored.

Prelude to Steps 4 and 5. When there is yielding at the stemming-pipe interface the system of integral Eqs. (35) and (41) must be solved for the functions $y(z)$ and $F(s)$ before the axial force $T(x,s)$ in the pipe can be determined. Equations (35) and (41) can be written in the simpler forms

$$C_1 + C_2 z = F(s) \cosh \lambda(z - s) + [Q(s) - C_3] \sinh \lambda(z - s) - \int_s^z Q(\xi) \lambda \cosh \lambda(z - \xi) d\xi , \quad (45)$$

and

$$C_4 = F(s) \sinh \lambda(z - s) + [Q(s) - C_3] \cosh \lambda(z - s) - Q(z) - \int_s^z Q(\xi) \lambda \sinh \lambda(z - \xi) d\xi , \quad (46)$$

where z is the distance from the top of the pipe to the point where yielding at the pipe stemming interface first occurs,

$$z(s) = s + y(s) , \quad (47)$$

$Q(\xi)$ is a notation for a combination of functions, including $F(\xi)$.

$$Q(\xi) = \lambda[\xi F(\xi) + E A u_R(\xi)] \quad (48)$$

and C_1, C_2, C_3 , and C_4 are constants,

$$C_1 = W + w_e L, \quad C_2 = w - w_e, \quad C_3 = \frac{\ell_2^2 \ell_1}{\ell_2} \gamma A_s \lambda, \quad C_4 = \frac{\gamma A_s \ell_1}{\lambda(\ell_2 - \ell_1)} \quad (49)$$

Equation (45) is for the determination of $F(s)$ and Eq. (46) is for $z(s)$. These equations must be solved simultaneously. The next step outlined below solves for the level of free surface of the stemming when yielding begins between the pipe and the stemming material at the bottom of the pipe. This level of the free surface is denoted by s_0 . It will be determined from Eq. (46). When s is in the domain $0 \leq s \leq s_0 \leq L$ there is no yielding between the pipe and the stemming, $z(s)$ is either undefined or it can be assigned the value L , and $F(s)$ is given by Eq. (38) or (43). However, when s is in the range $s_0 \leq s \leq L$ there is yielding and Eqs. (45) and (46) must be solved simultaneously for $F(s)$ and $z(s)$. A suggested method for accomplishing this complicated calculation is given in Step 5 below. The curves $F(s)$ and $z(s)$ in the domain $0 \leq s \leq L$ are given in Fig. 5.

FIG. 5. Sketches of the curves $F(s)$ and $Z(s)$ in the domain $0 \leq s \leq L$.

Step 4: Determination of the level of the stemming surface s_0 when slipping between the stemming material and the pipe begins at the bottom of the pipe, $x = L$.

To find s_0 we must solve Eq. (46) for s when $z = L$ and $F(s)$ is given by Eq. (38) or (43). Thus we define the function

$$M_0(s) = F(s) \sinh \lambda(L-s) + [Q(s) - C_3] \cosh \lambda(L-s) - Q(L) - \int_s^L Q(\xi) \lambda \sinh \lambda(L-\xi) d\xi, \quad (50)$$

and note that $M_0(L) = -C_3$. Equation (46) is satisfied identically when $M_0(s) = C_4$, thus s_0 is defined as the value of s such that

when

$$M_0(s) = C_4$$

then

$$s = s_0. \quad (51)$$

The calculations of $M_0(s)$ should start at $s = L$ when $M_0(L) = -C_3$ and proceed in increments until $M_0(s) = C_4$, at which point s_0 will be determined.

Step 5: Determination of the functions $F(s)$ and $y(s)$ in the domain $0 \leq s \leq s_0 \leq L$.

The method suggested here for the simultaneous solution of Eqs. (45) and (46) may be primitive; it is possible that the software exists for the solution of this system of integral equations.

The method suggested is as follows. Starting at $s = s_0$, $z = L$, and $F(s)$ given by Eq. (38) or (43), the system of Eqs. (45) and (46) will be solved in step increments of magnitude q from $s = s_0$ to $s = 0$. At the n^{th} increment we set $z = z_n = L - qn$ and solve Eq. (46) for s_n using an estimate of the function $F(s)$ given by Eq. (45) in the previous step. The first estimate of $F(s)$ is given by Eq. (38) or (43) and it has been computed in Step 3. The notation $F_0(s)$ is used to denote the force at the top of the pipe when $s \leq s_0$.

To find s_n we must solve Eq. (46) for s when $z = z_n = L - qn$ and $F_0(s)$ is known for $0 \leq s_n \leq s \leq L$. Thus we define the function

$$M_n(s) = F_0(s) \sinh \lambda(L - nq - s) + [Q_0(s) - C_3] \cosh \lambda(L - nq - s) - Q_0(L - nq) - \int_s^{L-nq} Q_0(\xi) \lambda \sinh \lambda(L - nq - \xi) d\xi. \quad (52)$$

Equation (46) is satisfied identically when $M_n(s) = C_4$, thus s_n is defined as the value of s such that

when

$$M_n(s) = C_4$$

then

$$s = s_n. \quad (53)$$

The curve $z = z(s)$ can then be determined as the sequence of values $s_0, s_1, s_2, \dots, 0$ in one-to-one correspondence with the values $L, z_1, z_2, \dots, z_n$. This curve is illustrated in Fig. 5.

At each step in the calculation above the value of $F_0(s)$ at s_n must be determined. The value of $F(s)$ for $0 \leq s_0 \leq s \leq L$ is given by Eq. (38) or (43), thus $F(s_0)$ is known. To determine $F_0(s_n)$ Eqs. (46) and (48) are solved for $F_0(s)$ and we write

$$[\cosh \lambda(z_n - s_n) + \lambda s_n \sinh \lambda(z_n - s_n)] F_0(s_n) = C_1 + C_2 z_n + [C_3 - \lambda E A u_R(s_n)] \sinh \lambda(z_n - s_n) - \int_{s_n}^{z_n} Q_0(\xi) \lambda \cosh \lambda(z_n - \xi) d\xi, \quad (54)$$

where $z_n = L - qn$. The integral on the right will be based on the values of the $F_0(s)$ through s_{n-1} , with some estimate made for the value s_n . This calculation should be iterated with the solution for s_n to obtain both $F_0(s_n)$ and s_n to the desired accuracy. The curve $F_0(s)$ is illustrated in Fig. 5. It should be kept in mind that the portion of the curve in the domain $s_0 > s \geq L$, $F(s)$, is given by Eq. (38) or (43) and only that portion of the curve in the domain $0 \leq s \leq s_0$, $F_0(s)$ must be calculated by this iterative scheme.

This iterative scheme proceeds until the value of $s_n \leq 0$ occurs. The calculation is then terminated. The number of steps to achieve $s_n \leq 0$ is denoted by N , thus z_N is the value of z corresponding to $s = 0$.

Before proceeding to the next step compute $y(s) = z(s) - s$.

Prelude to Steps 7 and 8. It should be kept in mind that $T(x,s)$ represents the axial force in the pipe when there is no slipping at the pipe stemming interface. This situation pertains if s_0 does not exist in the interval 0 to L or if $0 \leq s_0 \leq s \leq L$. $T_0(x,s)$ represents the axial force in the pipe when there is slipping, $0 \leq s \leq s_0 \leq L$. Thus, except for the single situation when both $T(x,s)$ and $T_0(x,s)$ give the same answer, that is when $s = s_0$, the two calculations for the axial force, that is $T(x,s)$ and $T_0(x,s)$, are mutually exclusive. For any particular value of s either Step 7 or 8 applies, but not both.

Step 7: Determination of $T(x,s)$.

This is, of course, a more laborious calculation because $T(x,s)$ is a function of two variables. It is suggested that this calculation be done by fixing on a value of s and then determining the function of x , then incrementing in s and determining the new function of x , etc. One should start at $s = L$, of course. $T(x,s)$ is given by

$$T(x,s) = F(s) - wx \text{ for } 0 \leq x \leq s, \quad (55)$$

and

$$T(x,s) = \lambda E A u_R(s) \sinh \lambda(x-s) - \frac{q_3^2 q_1}{q_2} \lambda A_s \gamma \sinh \lambda(x-s) - wx + F(s) [\cosh \lambda(x-s) + \lambda s \sinh \lambda(x-s)] - \int_s^x [\xi F(\xi) + E A u_R(\xi)] \lambda^2 \cosh \lambda(x-\xi) d\xi \text{ for } s \leq x \leq L. \quad (56)$$

These results follow from Eqs. (8) and (34).

Step 8: Determination of $T_0(x,s)$.

This is an even more laborious calculation because $T_0(x,s)$ is a function of two variables and is specified by a different expression in three domains. $T_0(x,s)$ is given by

$$T_0(x,s) = F_0(s) - wx \text{ for } 0 \leq x \leq s, \quad (57)$$

$$T_0(x, s) = \lambda E A u_R(s) \sinh \lambda(x-s) - \frac{\ell_1^2 \ell_2}{\ell_2} \gamma A_1 \lambda \sinh \lambda(x-s) - w x + F_0(s) [\cosh \lambda(x-s) + \lambda s \sinh \lambda(x-s)] - \int_s^x [\xi F_0(\xi) + E A u_R(\xi)] \lambda^2 \cosh \lambda(x-\xi) d\xi \quad (58)$$

for $s \leq x \leq s+y$

$$T_0(x, s) = W + w_e(L-x) \quad \text{for } s+y \leq x \leq L. \quad (59)$$

These results follow from Eqs. (8), (22) and (34).

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APPENDIX I—THE SOLUTION WITHOUT THE APPROXIMATION

The approximation made in Sec. 4 simplified the solution, but it was not necessary to make the approximation to obtain the solution. Here the solution to the problem with no approximations is sketched. Appendix I-1 and Appendix I-2 parallel Secs. 5 and 6. The results of this Appendix support the assumed approximation because they show that the effects of the neglected term in Eq. (3) attenuate seven times more rapidly than the effects of the terms retained. It is also shown that there is no more formal difficulty in the solution of the equations without the approximation, although the algebra is more awkward.

I-1 THE SOLUTION IN THE LOWER PORTION OF THE STEMMED REGION WHEN SLIPPING OCCURS ON THE PIPE-STEMMING INTERFACE

We consider the problem in two parts. Here we consider the case when $\tau = -\mu_p K \sigma$ in the lower portion of the stemmed region. Then, in Sec. I-2, we consider the case when $\tau(x, s) = -\mu_p K \sigma(x, s)$ in the upper portion of the stemmed region. That is, we consider here the simpler problem of no slipping along the lower portion of the pipe stemming interface, and in the next section we use this result to obtain a solution to the problem in the no-slipping region above the slipping portion.

Let $s + y, y \geq 0$ denote the downhole distance at which slip first occurs, $0 \leq s + y \leq x \leq L$. Slip is assumed to occur along the pipe for x greater than $s + y$ and less than L , $s + y \leq x \leq L$. For these values of x , $\tau(x, s) = -\mu_p K \sigma(x, s)$ and Eqs. (2b) and (3) take the form

$$\frac{dT(x, s)}{dx} - \alpha \sigma + w = 0 \quad \text{for } s + y \leq x \leq L \quad (I-1)$$

and

$$\frac{d\sigma(x, s)}{dx} + \frac{N}{\ell_1} \sigma(x, s) - \gamma = 0 \quad \text{for } s + y \leq x \leq L \quad (I-2)$$

where

$$\alpha = 2\pi K \mu_p R_p \quad (I-3)$$

and

$$N \equiv 1 - \frac{R_p \mu_p}{R_h \mu_h} \quad (I-4)$$

The quantity N is equal to 1 when there is no pipe in the hole. In general $0 < N < 1$. The boundary conditions used for the solution of Eqs. (I-1) and (I-2) are

$$\underbrace{\sigma(x, s) \Big|_{x=s+y}}_1 = \sigma(s+y, s), \quad \underbrace{T(x, s) \Big|_{x=L}}_2 = W, \quad (I-5)$$

where W is the weight of the canister. The solution to the differential equation (I-2) using the boundary condition Eq. (I-5)₁ is

$$\sigma(x, s) = \frac{\gamma l_1}{N} + \left[\sigma(s + y, s) - \frac{\gamma l_1}{N} \right] e^{-(x-s-y)(N/l_1)}, \quad s + y \leq x \leq L. \quad (I-6)$$

The solution to the differential equation (I-1) subject to the boundary condition Eq. (I-5)₂ is

$$T(x, s) = W + \left(w - \frac{\sigma \gamma l_1}{N} \right) (L - x) - \frac{\alpha l_1}{N} \left[\sigma(y, s) - \frac{\gamma l_1}{N} \right] [e^{-(x-s-y)(N/l_1)} - e^{-(L-s-y)(N/l_1)}] \quad (I-7)$$

In the special case when the unslipped zone is very short, we may introduce the approximation $y \cong 0$ and observe that

$$\sigma(s, s) = 0, \quad (I-8)$$

then Eq. (I-6) becomes

$$\sigma(x, s) = \frac{\gamma l_1}{N} (1 - e^{-(N/l_1)(x-s)}), \quad s \leq x \leq L, \quad (I-9)$$

and the tension in the pipe is given by

$$T(x, s) = W + \left(w - \frac{\sigma \gamma l_1}{N} \right) (L - x) + \frac{\alpha l_1^2}{N^2} [e^{-(x-s)(N/l_1)} - e^{-(L-s)(N/l_1)}] \quad \text{for } s \leq x \leq L. \quad (I-10)$$

The tension in the pipe at the point $x = s$ for $y \cong 0$ is, from Eq. (I-10), given by

$$T(s, s) = W + \left(w - \frac{\sigma \gamma l_1}{N} \right) (L - s) + \frac{\alpha \gamma l_1^2}{N^2} [1 - e^{-(L-s)(N/l_1)}]; \quad (I-11)$$

thus from Eq. (8) the tension at the collar is

$$F(s) = W + wL - \frac{\sigma \gamma l_1}{N} (L - s) + \frac{\alpha \gamma l_1^2}{N^2} [1 - e^{-(L-s)(N/l_1)}], \quad (I-12)$$

and the tension in the region $0 \leq x \leq s$ is

$$T(x, s) = W + w(L - x) - \frac{\sigma \gamma l_1}{N} (L - s) + \frac{\alpha \gamma l_1^2}{N^2} [1 - e^{-(L-s)(N/l_1)}] \quad (I-13)$$

If there is no stemming in the hole ($\gamma = 0$ or $s = L$), then $T(x, L)$ is given by

$$T(x, L) = W + w(L - x), \quad \text{for } 0 \leq x \leq L. \quad (I-14)$$

Comparison of Eqs. (I-13) and (I-14) shows

$$T(x, s) = T(x, L) - \frac{\alpha \gamma \ell_1}{N} \left\{ L - s + \frac{\ell_1}{N} [1 - e^{-(L-s)(N/\ell_1)}] \right\}, \quad (I-15)$$

and since the second term is always nonpositive, it follows that $T(x, s)$ is less than $T(x, L)$. Thus, slipping along the entire length of the pipe-stemming interface reduces the pipe tension. This occurs because the friction force has been completely reversed and, instead of the pipe supporting stemming material by friction, the pipe is now being partially supported by the stemming material through friction.

I-2 THE SOLUTION IN THE UPPER PORTION OF THE STEMMED REGION WHERE SLIP DOES NOT OCCUR ON THE PIPE-STEMMING INTERFACE

In this case the system of differential equations (1), (2b), (3) and (5) can be reduced to a single third-order linear differential equation with constant coefficients for $u(x, s)$. To obtain this, we first substitute Eqs. (1) and (5) into Eq. (2b) and solve for $\sigma(x, s)$

$$\sigma(x, s) = \frac{EA\ell_2}{A_s \ell_3^2} [u(x, s) - u(x, x)] - \frac{\ell_2}{A_s} [EA u''(x, s) + w], \quad (I-16)$$

where the following notations have been introduced

$$\ell_2 = \frac{R_h^2 - R_p^2}{2\mu_p \beta_p k}, \quad \ell_3 = + \left(\frac{EA(R_h - R_p)}{2\pi R_p G} \right)^{1/2}, \quad A_s = \pi(R_h^2 - R_p^2). \quad (I-17)$$

The differential equation for $u(x, s)$ is obtained by substituting $\sigma(x, s)$ given by Eq. (I-16) into

$$\sigma'(x, s) + \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) \sigma(x, s) - \frac{EA}{\ell_3^2 A_s} [u(x, s) - u(x, x)] - \gamma = 0, \quad (I-18)$$

a result that is obtained by placing Eq. (5) into Eq. (3),

$$u''''(x, s) + \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) u''(x, s) - \frac{1}{\ell_3^2} [u'(x, s) - u'(x, x)] - \frac{1}{\ell_1 \ell_3^2} [u(x, s) - u(x, x)] + \frac{1}{\ell_4^2} = 0, \quad (I-19)$$

where

$$\frac{1}{\ell_4^2} = \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) \frac{w}{AE} + \frac{\gamma A_s}{\ell_2 EA}. \quad (I-20)$$

We are interested in the solution of the differential equation (I-19) in the region $s \leq x \leq s + y$, i.e., the stemmed region above the region where slipping has occurred. The solution of Eq. (I-19) in the case where no slipping occurs in the stemmed region is obtained from this solution by setting $s + y$ equal to L .

The functions $u(x, x)$ and $u'(x, x)$ are obtained from $u(x, s)$ for the region $0 \leq x \leq s$, Eq. (10), by evaluating $u(x, s)$ at $x = s$ and $u'(x, s)$ at $x = s$ and then setting $x = s$,

$$u(x, x) = \frac{F(x)}{EA} x - \frac{w}{2EA} x^2 + u_R(x), \quad u'(x, x) = \frac{F(x)}{EA} - \frac{w}{EA} x. \quad (I-21)$$

The differential equation governing $u(x,s)$ in the region $s \leq x \leq y$ is obtained by substituting Eq. (I-21) into Eq. (I-19),

$$u''''(x,s) + \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) u''(x,s) - \frac{1}{\ell_3^2} u'(x,s) - \frac{1}{\ell_1 \ell_3^2} u(x,s) = -\frac{1}{\ell_4^2} - \frac{u_R(x)}{\ell_1 \ell_3^2} + \frac{wx}{EA \ell_3^2} \left(1 + \frac{x}{2\ell_1} \right) - \frac{F(x)}{EA \ell_3^2} \left(1 + \frac{x}{\ell_1} \right) \quad (I-22)$$

The boundary conditions that apply to Eq. (I-22) at $x = s$ are that $u(x,s)$, $u'(x,s)$ and $u''(x,s)$ be continuous. Since $u(x,s)$ is given by Eq. (10) for $0 \leq x \leq s$, we can write

$$\begin{aligned} u(s,s) &= \frac{sF(s)}{EA} - \frac{w}{2EA} s^2 + u_R(s) , \\ u'(s,s) &= \frac{F(s)}{EA} - \frac{w}{EA} s , \\ u''(s,s) &= -\frac{w}{EA} . \end{aligned} \quad (I-23)$$

$u(x,s)$ is continuous at $x = s$ because the displacement is continuous, and $u'(x,s)$ is continuous because the axial force in the bar is continuous [see Eq. (1)]. To see that $u''(x,s)$ is continuous, place Eq. (1) in both Eqs. (2a) and (2b). The limit as $x \rightarrow s^-$ of $u''(x,s)$ is given by Eq. (2a) and the limit as $x \rightarrow s^+$ of $u''(x,s)$ is given by Eq. (2b). These two limits coincide since $\tau(x,x) = 0$.

The solution to the homogeneous part of Eq. (I-23) is

$$u_H(x,s) = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + a_3 e^{\lambda_3 x} , \quad (I-24)$$

where a_1 , a_2 , and a_3 are constants and λ_1 , λ_2 and λ_3 are the solutions to the cubic equation

$$\lambda^3 + \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) \lambda^2 - \frac{1}{\ell_3^2} \lambda - \frac{1}{\ell_1 \ell_3^2} = 0 . \quad (I-25)$$

It is easy to show that λ_1 , λ_2 , and λ_3 are related to ℓ_1 , ℓ_2 , and ℓ_3 by

$$\lambda_1 \lambda_2 \lambda_3 = \frac{1}{\ell_1 \ell_3^2} , \lambda_1 + \lambda_2 + \lambda_3 = -\frac{1}{\ell_1} - \frac{1}{\ell_2} , \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 = -\frac{1}{\ell_3^2} . \quad (I-26)$$

It can also be shown that since ℓ_1 , ℓ_2 and ℓ_3 are positive, the three roots λ_1 , λ_2 and λ_3 are all real and in general distinct. Furthermore, two are negative and one is positive. The solution to the homogeneous part of Eq. (I-22) subject to the boundary condition Eq. (I-23) is

$$\begin{aligned} u_H(x,s) &= \left[u_R(s) + \frac{1}{EA} \left(sF(s) - \frac{ws^2}{2} \right) \right] \frac{1}{\ell_1 \ell_3^2} g(x-s) + \frac{1}{EA} \left\{ [F(s) - ws] \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) - w \right\} \\ &\quad \times g'(x-s) + \frac{1}{EA} [F(s) - ws] g''(x-s) , \end{aligned} \quad (I-27)$$

where the function $g(x)$ is given by

$$g(x) = \frac{1}{\Delta} \left(\frac{\lambda_3 - \lambda_2}{\lambda_1} e^{\lambda_1 x} + \frac{\lambda_1 - \lambda_3}{\lambda_2} e^{\lambda_2 x} + \frac{\lambda_2 - \lambda_1}{\lambda_3} e^{\lambda_3 x} \right) \quad (I-28)$$

and where

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} \quad (I-29)$$

It is easy to verify that $g(x)$ satisfies the differential equation

$$g'''(x) + \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) g''(x) - \frac{1}{\ell_2^2} g'(x) - \frac{1}{\ell_1 \ell_2^2} g(x) = 0, \quad (I-30)$$

and the boundary conditions

$$g(0) = \ell_1 \ell_2^2, \quad g'(0) = 0, \quad g''(0) = 0, \quad g'''(0) = 1, \quad g'''' = -\frac{1}{\ell_1} - \frac{1}{\ell_2} \quad (I-31)$$

The function $g(x)$ plays an important role in the solution to this problem and we shall consider here its numerical value. If we take a typical emplacement, assuming

$$\begin{aligned} R_h &= 36 \text{ in.}, \quad R_p = 5 \text{ in.}, \quad A = 16 \text{ in.}^2, \quad k\mu_p \cong k\mu_h \cong \frac{1}{4}, \quad G = 2000 \text{ psi}, \\ E &= 30,000,000 \text{ psi}, \end{aligned} \quad (I-32)$$

then from Eqs. (4) and (I-17)

$$\ell_1 = 5.88 \text{ ft}, \quad \ell_2 = 42.36 \text{ ft}, \quad \ell_3 = 40.55 \text{ ft}. \quad (I-33)$$

When the values of Eq. (I-33) are placed in the cubic equation (I-25), the three roots are found to be

$$\begin{aligned} \lambda_1 &= -0.19439 \text{ ft}^{-1} = \frac{1}{5.14 \text{ ft}}, \\ \lambda_2 &= -0.02287 \text{ ft}^{-1} = \frac{1}{43.72 \text{ ft}}, \text{ and} \\ \lambda_3 &= +0.02326 \text{ ft}^{-1} = \frac{1}{43 \text{ ft}}, \end{aligned} \quad (I-34)$$

and from Eq. (I-29), Δ is given by

$$\Delta = 0.001722 \text{ ft}^{-3} = \frac{1}{580.69 \text{ ft}^3} \quad (I-35)$$

When the numerical values of Eqs. (I-34) and (I-35) are substituted into Eq. (I-28), we find

$$g(x) = \left(-137.8e^{-x/5.14} + 5526.6e^{-x/43.72} + 4282.1e^{+x/43} \right) \text{ft}^{-3}, \quad (\text{I-36})$$

where x is in feet.

With these preliminary results, the formal solution to the differential equation (I-22) can be written, subject to the boundary conditions Eq. (I-23)

$$u(x, s) = u_H(x, s) + \frac{\rho_1 \rho_3^2}{\rho_2 EA} A_s \gamma - \frac{wx^2}{2EA} - \frac{1}{EA \rho_3^2} \int_s^x g'(x - \xi) \left[\frac{EA u_R(\xi)}{\rho_1} + F(\xi) \left(1 + \frac{\xi}{\rho_1} \right) \right] d\xi, \quad (\text{I-37})$$

where $g'(x - \xi)$ is the Green's function and is the derivative of the function defined by Eq. (I-28), and $u_H(x, s)$ is given by Eq. (I-27). The solution (I-37) is a formal solution because the load at the collar $F(s)$ is not known. $F(s)$ will be determined below by matching the boundary conditions at $x = s + y$, that is to say at the interface between the region where slipping occurs and the region where slip does not occur. The boundary conditions are that $T(x, s)$ and $\sigma(x, s)$ are continuous at $x = s + y$. $T(x, s)$ in the region $s \leq x \leq s + y$ is obtained by substituting Eq. (I-37) into Eq. (I)

$$T(x, s) = EA u_H'(x, s) - wx - \frac{1}{\rho_3^2} \int_s^x g'(x - \xi) \left[\frac{EA}{\rho_1} u_R(\xi) + F(\xi) \left(1 + \frac{\xi}{\rho_1} \right) \right] d\xi, \quad (\text{I-38})$$

where the fact that $g'(0) = 0$ has been used. $\sigma(x, s)$ in the region $s \leq x \leq s + y$ is obtained by substituting Eq. (I-37) into Eq. (I-16)

$$\begin{aligned} \sigma(x, s) = & \rho_1 \gamma + \frac{EA \rho_2}{\rho_3^2 A_s} [u_H''(x, s) - \rho_3^2 u_H(x, s)] - \frac{\rho_2}{A_s \rho_3^4} \left\{ xF(x) + u_R(x) \right. \\ & \left. + \int_s^x [g'(x - \xi) - \rho_3^2 g'''(x - \xi)] \left[u_R(\xi) + F(\xi) \left(1 + \frac{\xi}{\rho_1} \right) \right] d\xi \right\}. \quad (\text{I-39}) \end{aligned}$$

This calculation is not recorded beyond this point. Formally, the calculation is identical with the one performed in Sec. 6 for the approximate case. The difficulty is the awkward and tedious algebra.

APPENDIX II—THE EFFECT OF TEMPERATURE ON THE DETERMINATION OF THE PIPE-STEMMING LOAD

Temperature distribution has a considerable effect on the support pipe tension. In the analysis of the effect of temperature on pipe tension, it can be assumed that the mechanical problem does not influence the thermal problem, although the converse is untrue. Thus, the determination of the temperature distribution along the pipe from the given or assumed thermal states of the pipe and stemming material can be solved independent of the mechanical problem of determining the support pipe tension. It is assumed that this has been done and the temperature θ is known as a function of downhole distance x and time t , $\theta = \theta(x, t)$. The temperature distribution $\theta = \theta(x, t_R)$ at some particular time is used as reference. The temperature distribution in the pipe will cause the pipe to change length. Let $u_T(x, t)$ denote the displacement at x due to the temperature change from time t_R , then

$$\frac{du_T}{dx} = \frac{\alpha}{1-2\nu} [\theta(x, t) - \theta(x, t_R)] \quad (II-1)$$

or

$$u_T = \frac{\alpha}{1-2\nu} \int_0^x [\theta(\xi, t) - \theta(\xi, t_R)] d\xi \quad (II-2)$$

where α is the coefficient of linear thermal expansion and ν is the Poisson's ratio of the pipe.

A complication of the mechanical problem caused by thermal effects is that time t now becomes an independent variable in the mechanical problem. Thus the axial force T depends upon x, s and t as does the displacement u , the shear stress τ and the vertical stress σ in the stemming material. If s is known as a function of time, then T is again a function of two independent parameters. Assume that $s = s(t)$ is known. Then we can write $T(x, t)$ for $T(x, s(t), t)$ and similarly for the other independent variables. With this notation Eq. (1) is replaced by

$$T(x, t) = EA \left\{ \frac{\partial u(x, t)}{\partial x} - \frac{\alpha}{1-2\nu} [\theta(x, t) - \theta(x, t_R)] \right\} \quad (II-3)$$

The equations governing the equilibrium of the pipe and stemming, Eqs. (2a), (2b) and (3), are unchanged except for independent variables

$$\frac{dT(x, t)}{dx} + w = 0, \quad \text{if } 0 \leq x \leq s(t), \quad (II-4a)$$

$$\frac{dT(x, t)}{dx} + 2\pi R_p \tau(x, t) + w = 0, \quad \text{if } s(t) \leq x \leq L, \quad (II-4b)$$

$$\frac{d\sigma(x, t)}{dx} + \frac{\sigma(x, t)}{\ell_1} + \frac{2R_p}{R_h^2 - R_p^2} \tau(x, t) - \gamma = 0, \quad \text{if } s(t) \leq x \leq L,$$

and

$$\sigma(x, t) = 0 \quad \text{if } 0 \leq x \leq s(t). \quad (II-5)$$

The constitutive equation specifying the shear stress at the pipe-stemming interface must be modified to account for the length changes caused by the temperature distribution. In the present situation

$$\tau(x, t) = \mu_p K \sigma(x, t) - \frac{G}{R_n - R_p} \left\{ u(x, t) + \frac{\alpha}{1 - 2\nu} \int_0^x [\theta(\xi, t) - \theta(\xi, t_R)] d\xi - u(x, t^*) \right\}, \quad (II-6)$$

where t^* is the time when s passed the station x , $t^* = s^{-1}(x)$, which can be obtained from the known function $s(t)$ by inversion.

Since $\theta(x, t)$ and $s(t)$ are assumed to be known, the solution of the combined thermal and mechanical problem to determine the collar load of the support pipe is not, in principle, any more difficult than the solution of the mechanical problem alone. The algebraic manipulations may be more complicated, but that will be a minor matter.