MASTER

Monte Carlo Simulations
of Lattice Gauge Theories

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The lattice formulation of gauge theories and its relevance for elementary particle systems have been the subject of other lectures given at this Institute. The powerful computational method provided by Monte Carlo simulations has also been treated in some detail. Very stringent limitations of space allow me to present only a schematic summary of results, which have been obtained in this framework at the Brookhaven National Laboratory.¹⁻⁷

In a sense, a Monte Carlo simulation creates inside the computer a small specimen of some type of matter, which the physicist then subjects to a variety of experiments. Obvious technological constraints limit the size of the sample rather severely: for instance, in a typical computation with a four-dimensional system, the lattice side would extend for ten sites. Still, this implies a number of statistical variables equal to 40,000, a rather impressive collection of spins, which gives confidence that at least the bulk properties of the material will be determined without excessive errors.

I shall describe Monte Carlo simulations done for four-dimensional lattice gauge systems, where the gauge group is one of the following: i) $U(1)$; ii) $SU(2)$; iii) Z_N , i.e., the subgroup of $U(1)$ consisting of the elements $e^{\frac{2\pi i n}{N}}$ with integer n and N ; iv) the eight-element group of quaternions, Q ; v) the 24- and 48-element subgroups of $SU(2)$, denoted by $\tilde{T}$ and $\tilde{O}$, which reduce to the rotation groups of the tetrahedron and the octahedron when their centers, Z_2 , are factored out. All of these groups can be considered subgroups of $SU(2)$ and a common normalization has been used for the action: if a plaquette is bounded by links with spins u_1, u_2, u_3, u_4 , the product

$u = u_1 u_2 u_3 u_4$ (as well as the u_i 's) may be represented in the form $\cos\theta + i \sin\theta \vec{\sigma} \cdot \hat{n}$ and an action $S_{\square} = 1 - \cos\theta$ is associated to the plaquette. The total action is $S = \sum_{\square} S_{\square}$ and quantum averages are constructed with the measure $e^{-\beta S}$.

$U(1)$ is the group relevant to the lattice formulation of quantum electrodynamics; the $SU(2)$ model is the prototype of non-Abelian gauge theories which are currently used to describe strong and unified interactions. The models with discrete groups Z_N and Q , $\tilde{T}$, $\tilde{O}$, beyond being interesting per se, give information about the corresponding systems with continuum groups $U(1)$ and $SU(2)$.

In this lecture I will relate about the following types of Monte Carlo experiments, which I believe have produced the most interesting results: i) simulations of a thermal cycle, where the temperature of the system is varied slightly every few Monte Carlo iterations and the internal energy, $E = \langle S_{\square} \rangle$, is measured. Hysteresis cycles, due to a lengthening of the relaxation time, give then a clue to possible phase transitions. ii) Mixed-phase runs, where several Monte Carlo iterations are done at a few temperatures near a phase transition starting with a lattice which is half ordered and half disordered. The curves giving internal energy versus number of iterations display rather characteristic patterns, according to whether the transition is of the first order, with a discontinuity in the internal energy, or of a continuous type. In first-order transitions, after a few iterations in which the two halves of the system approach one a stable and the other a metastable configuration, the stable phase expands until it overtakes the whole lattice and one notices a drift of the internal energy to either one of the different values E_+ , E_- , according to whether

$\beta < \beta_c$ or $\beta > \beta_c$. In higher order transitions the curves for E change continuously as β is varied. iii) Measurements of averages of Wilson factors, $W = \frac{1}{2} \text{Tr} (u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_N i_1})$, for loops of different shape, typically rectangular loops of sides m and n . $\langle W \rangle$ approaches zero exponentially as the size of the loop is increased and behaviors $-\ln \langle W \rangle \approx A$ or $-\ln \langle W \rangle \approx L$, where A and L are the area enclosed by the loop and its perimeter, are taken as a sign that the system confines or, respectively, does not confine the charges of the group.⁸

Thermal cycles for the models with discrete gauge groups are displayed in Figs. 1 a-i (from Refs. 2 and 5). One notices hysteresis cycles in all of these graphs. The Z_2, Z_3 and Z_4 models are self-dual and show a single hysteresis loop around a temperature coinciding with the self-duality point. Very likely these systems have a two-phase structure and undergo a phase transition at $\beta_{s.d.}$. The qualitative features of the thermal cycles change in the models with $N > 4$. The Z_5 system already shows a slant and a partial closing of the hysteresis loop that hints to two separate phase transitions; but a three-phase structure becomes dramatically evident in the models with gauge groups Z_6 and Z_8 . The sequence of Z_N models appears thus characterized by the following properties. For low N these systems have a two-phase structure, which develops into a three-phase structure for $N > 4$.⁹ Of the two phase transitions, one seems to stabilize at a finite value of β , the other moves toward $\beta = \infty$ (zero temperature) as N increases. Measurements of $-\ln \langle W \rangle$ indicate that the high temperature phase is confining, the other two phases are not.² The curve $E = E(\beta)$ in the intermediate phase is fitted reasonably well with spin-wave excitations. All of this is consistent with what one expects of the $U(1)$ model, a theory which on a four-dimensional lattice should possess a high-temperature confining

phase and a low-temperature phase of spin-wave excitations, giving origin to ordinary QED in the continuum limit. $E(\beta)$, as determined for the $U(1)$ -system with Monte Carlo computations, is shown in Fig. 2.²

All the models with discrete non-Abelian groups display a single hysteresis loop in their thermal cycles at temperatures which decrease as the order of the group increases. There is no sign of the emergence of a spin-wave phase. The comparison of Figs. 1-f and 1-i is particularly revealing: the Z_8 - and $\tilde{O}$ -models have the same action gap (i.e. difference in action between a plaquette in its lowest state of excitation and an unexcited one) and Z_8 is contained as a subgroup in $\tilde{O}$. Still, whereas the Abelian system has a clear three-phase structure, only two phases are apparent in its non-Abelian counterpart. Once again, the discrete non-Abelian models display a behavior consistent with what one expects of the $SU(2)$ -model. A direct Monte Carlo study of this system indicates the existence of a single, confining phase.^{3,4} The internal energy curve for the $SU(2)$ -model overlaid with the thermal cycle of the $\tilde{O}$ -model is presented in Fig. 3.

Figs. 4a - g illustrate the results of mixed-phase simulations (from Refs. 2,5). One notices a drift toward different values E_+ and E_- in all the figures but Fig. 4-e (Z_6 -model at the phase transition with lowest β), which strongly suggests that the phase transitions in the models with gauge groups Z_2 , Z_3 , Z_4 , Q , $\tilde{T}$ and $\tilde{O}$ are of the first order. The critical temperatures may then be read off the diagrams; in particular, for the three self-dual Abelian systems $\beta_c = \beta_{s.d.}$ within the experimental error.

The values measured for $-\ln \langle W \rangle$ for the $\tilde{O}$ -model are reproduced in the table.⁵ The average is over all space-like loops of sides m and n and 12 Monte Carlo iterations. Only values <5 are reported, because $-\ln \langle W \rangle > 5$

means that $\langle W \rangle$ itself is so small that it cannot be distinguished from statistical fluctuations. It is particularly interesting to compare the numbers in the diagonals $m+n = \text{const.}$ They correspond of course to loops which differ only in area. The presence of an area term is clearly noticed for all $\beta < \beta_c$ (β_c is determined to be $\beta_c \approx 3.21 \pm 0.01$ from refined mixed-phase computations), but for $\beta > \beta_c$ $-\ln \langle W \rangle$ becomes suddenly independent of A . A determination of the string tension (the coefficient in the area term) may also be attempted but finite size effects (the lattice extends for 8 sites in all spatial directions, 10 sites in the temporal one, with periodic boundary conditions) make it uncertain up to a factor of 2.

The values of $-\ln \langle W \rangle$ for the square loops agree with previous measurements of the same quantity for the SU(2)-model⁴ almost up to the phase transition. A plot of the string tension versus β for the SU(2)-system, which reveals an interesting passage between a strong coupling regime to the behavior predicted by asymptotic freedom, is reproduced in Fig. 5 (from Ref. 4).¹⁰

These and other numerical results have provided us with extremely important information about the properties of lattice gauge theories, often confirming theoretical conjectures and expectations. The experiments performed up to now do not exhaust the set of relevant computations that can be made, on the contrary, there is a wealth of quantities which may still be measured, but they have certainly established that Monte Carlo computations, in spite of the difficulties connected with the large dimensionality, can be an invaluable tool in the investigation of quantum gauge systems.

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TABLE

The quantity $-\ln W$ for different values of β and rectangular loops of sides m, n , in the $\tilde{O}$ model.

$\beta=1$	$m=$	1	2	3	4	5	6	7	8
$n=1$		1.42	2.79	4.10					
$\beta=1.5$	$m=$	1	2	3	4	5	6	7	8
$n=1$		1.02	2.02	3.01	4.06				
	2		4.11						
$\beta=2$	$m=$	1	2	3	4	5	6	7	8
$n=1$		.70	1.37	2.03	2.69	3.34	4.03	4.59	4.99
	2		2.69	4.17					
$\beta=2.5$	$m=$	1	2	3	4	5	6	7	8
$n=1$		.42	.78	1.11	1.43	1.75	2.08	2.41	2.75
	2		1.35	1.87	2.37	2.90	3.40	3.93	4.34
	3			2.57	3.14	3.77	4.39	4.70	
	4				3.94	4.52	4.87		
$\beta=3$	$m=$	1	2	3	4	5	6	7	8
$n=1$		.30	.54	.76	.98	1.20	1.41	1.63	1.84
	2		.90	1.21	1.51	1.81	2.11	2.41	2.71
	3			1.56	1.91	2.25	2.62	2.90	3.07
	4				2.27	2.62	2.98	3.26	3.23
	5					2.99	3.28	3.59	3.38
	6						3.65	3.88	3.51
	7							3.89	3.05
	8								1.41

[illegible][illegible]

References

1. M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. Lett. 42, 1390 (1979).
2. M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. D20, 1915 (1979).
3. M. Creutz, Phys. Rev. Lett. 43, 553 (1979).
4. M. Creutz, BNL Preprint #26847, 1979.
5. C. Rebbi, BNL Preprint #27203, 1980.
6. Monte Carlo computations for the SU(2) model have also been done by K. Wilson. See Cornell preprint, 1979, and lectures presented at this Institute.
7. Interesting results for models with matter fields or with dimensionality different than 4 have been obtained at BNL by M. Creutz, BNL Preprint # 26587 (1979), G. Bhanot and M. Creutz, BNL Preprint # 27276 (1980), and G. Bhanot and B. Freedman, in preparation.
8. K. G. Wilson, Phys. Rev. D10, 2445 (1974).
9. For theoretical considerations see:
S. Elitzur, R. B. Pearson and J. Shigemitsu, Phys. Rev. D19, 3698 (1979);
D. Horn, M. Weinstein, and S. Yankielowicz, *ibid.* 19, 3715 (1979);
A. Guth, A. Ukama and P. Windey, Princeton Univ. preprint, 1979.
10. For studies of the string tension based on different methods, see:
J. Kogut, R. B. Pearson and J. Shigemitsu, Phys. Rev. Lett. 43, 484 (1979);
C. G. Callan, R. Dashen and D. J. Gross, Phys. Rev. D20, 1979 and
Princeton Univ. preprint, 1980.

Figure Captions

- Fig. 1 Thermal cycles for models with gauge group: a) Z_2 ; b) Z_3 ; c) Z_4 ; d) Z_5 ; e) Z_6 ; f) Z_8 ; g) Q ; h) $\tilde{T}$; i) $\tilde{O}$.
- Fig. 2 Internal energy versus inverse temperature for the $U(1)$ model.
- Fig. 3. Internal energies of the models with gauge groups $SU(2)(0)$ and $\tilde{O}$ (+ and x).
- Fig. 4 Results of mixed-phase simulations. The gauge groups and inverse temperatures are as follows:
- a) Z_2 ; $\beta = 0.41, 0.42, 0.43, 0.44, 0.45, 0.46, 0.47$.
 - b) Z_3 ; $\beta = 0.64, 0.65, 0.66, 0.67, 0.68, 0.69, 0.7$.
 - c) Z_4 ; $\beta = 0.85, 0.86, 0.87, 0.88, 0.89, 0.9, 0.91$.
 - d) Z_6 ; $\beta = 0.97, 0.98, 0.99, 1, 1.01, 1.02, 1.03$.
 - e) Q ; $\beta = 1.23, 1.25$ and 1.27 .
 - f) $\tilde{T}$; $\beta = 2.1, 2.15, 2.2$.
 - g) $\tilde{O}$; $\beta = 3, 3.1, 3.2, 3.3, 3.4, 3.5$ and 3.6 .
- Fig. 5 String tension versus inverse temperature for the $SU(2)$ -model.

Fig. 1. a.-b.

Fig. 1. c.-d.

Fig. 1. e.-f.

Fig. 1. g.-h.

Fig. 1.i.

Fig. 2

Fig. 3

Fig. 4. a.-b.

Fig. 4. c.-d.

Fig. 4. e.-f.

Fig. 4.g.

Fig. 5