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By

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NOVEMBER 1983

MASTER

PLASMA
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PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

PREPARED FOR THE U.S. DEPARTMENT OF ENERGY,
UNDER CONTRACT DE-AC02-76-CO0-3073.

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RESISTIVE STABILITY OF THE CYLINDRICAL SPHEROMAK

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ABSTRACT

The growth rates for resistive instabilities in a straight circular cylinder with spheromak profiles are computed by using two complementary methods. The first method employs boundary layer analysis and asymptotic matching, most valid for values of the magnetic Reynolds number $S \gtrsim 10^5$. The second method solves the full linearized resistive MHD equations as an initial value problem, utilizing zone packing around the mode rational surface. Resolution requirements limit this to $S \lesssim 10^7$. The results from these two methods agree to better than 1 in 10^3 in the overlap region $10^7 \gtrsim S \gtrsim 10^5$. A scan of parameter space reveals that for parabolic q -profiles, the least unstable configurations have $q_0 R/a \sim 0.67$. The Hall term in Ohm's Law is easily incorporated into both methods. Recalculating the resistive MHD growth rates in the presence of this term shows that the resistive interchange mode is completely stabilized for a large enough value of the ion cyclotron time.

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1. INTRODUCTION

The spheromak configuration has recently received considerable attention in both experimental and theoretical studies. The theoretical stability studies have divided mostly into two types. One class [1-3] of papers utilizes "Taylor Analysis" to examine a special class of force free equilibria in which the current density is everywhere proportional to the magnetic field, $\vec{j} = \lambda \vec{B}$. The stability boundaries obtained using this method apply to both ideal and finite resistivity instabilities, but only for this special form of equilibrium. The second class of papers examines a larger class of equilibrium profiles, but restricts consideration to ideal MHD instabilities. In particular, Ref. [4] examines a wide range of parameter space for both the cylindrical and the toroidal spheromak geometries. It found that the ideal MHD stability results for the cylindrical spheromak were qualitatively similar to those for the toroidal spheromak, i.e., toroidal effects were relatively unimportant in determining ideal MHD stability. In exception to this were a small number of new ideal MHD instabilities which are present in the torus but not in the cylinder. These were the $m = 1$, $n = 1$ internal kink mode which exists in toroidal geometry whenever the safety factor at the magnetic axis exceeds unity. Also in toroidal geometry, the presence of the required externally generated vertical field causes the spheromak to be unstable to $n = 2$ bending modes and $n = 1$ modes of either the shift and/or tilt polarity for the wall removed far enough.

In the present paper we study the resistive stability of the spheromak configurations parameterized as in Ref. [4]. Since the numerical evaluation of resistive MHD instability growth rates is considerably more involved than is the calculation of ideal growth rates, we limit the present study to the consideration of the stability of straight circular cylindrical configurations

with spheromak profiles. We explain in Sec. VI why we expect the stability results obtained here to be a good approximation to the stability of the toroidal spheromak.

The physical problem we address is defined by the starting equations in Sec. II. We then describe two different numerical methods for solving these equations in Secs. III and IV. The asymptotic matching method employs boundary layer analysis to solve the eigenmode equation separately near and far from the mode rational surface, and to match the solutions in the overlap regions. The initial value method solves the full set of linear resistive equations everywhere by evolving the time dependent equations to isolate the fastest growing eigenmode. The change in spatial scale lengths near the rational surface is accommodated by using nonuniform grid spacing. A new decomposition of the velocity field allows for the implementation of an accurate implicit time integration method which makes this method feasible.

In Sec. V we discuss under what conditions each of these two methods is applicable, and we present comparisons demonstrating the agreement in the parameter regimes of overlapping validity. In Section VI we scan the parameter space of possible spheromak configurations by computing contours of the resistive growth rates in the continuous space formed by $(nq_0, na/R)$ for parabolic profiles with several values of the central pressure p_0 and the poloidal mode number m . In Sec. VII we recalculate the growth rate and eigenfunctions of the $m = 1$ resistive interchange mode with the Hall term included in Ohm's Law. The results of this paper are summarized in the final section.

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II. THE CYLINDRICAL SPHEROMAK

The periodic cylindrical spheromak has proved to be a useful model for locating and understanding parameter regimes of internal mode stability of the spheromak configuration. Using standard (r, θ, z) cylindrical coordinates, we define a periodicity length in the z -direction of $L = 2\pi/k$, where k is the analog of the reciprocal major radius in a toroidal device, $k \equiv 1/R$. The equilibrium equation in this geometry is

$$\frac{dp}{dr} + \frac{d}{dr} \frac{1}{2} B_z^2 + \frac{B_\theta}{r} \frac{d}{dr} r B_\theta = 0. \quad (1)$$

The safety factor profile, $q(r) \equiv r k B_z(r) / B_\theta(r)$, is taken to be of the form $q(r) = q_0(1-r^2)$, where the plasma boundary is defined by $r = 1$. As in previous work, we parameterize the pressure by introducing the quantity α such that $0 < \alpha < 1$ and $dp/dr = -\alpha r B_z^2 / 8(q'/q)^2$. For $\alpha = 0$ the equilibrium is force free, while for $\alpha = 1$ the equilibrium is everywhere marginally stable to ideal interchange modes. The equilibrium fields for such a system are given by

$$B_\theta(r) = r B_{\theta 0} \exp \left\{ - \int_0^r \frac{-(\alpha/8k^2) r q'^2 + 2r + d/dr (q^2/2k^2)}{(q^2/k^2 + r^2)} dr \right\}, \quad (2a)$$

$$B_z(r) = \frac{q B_\theta(r)}{rk}, \quad (2b)$$

$$p(r) = -\frac{\alpha}{8} \int_0^r r B_z^2(r) \left(\frac{q'}{q} \right)^2 dr. \quad (2c)$$

Thus, in addition to the overall scale factor $B_{\theta 0}$, the equilibrium profiles are seen to depend only on two parameters, the pitch q_0/k and the pressure factor α .

We seek to determine the linear stability of such equilibria to resistive magnetohydrodynamic perturbations. That is, if ξ is the linearized Lagrangian displacement of a fluid element which grows in time as $\xi \sim e^{\gamma t}$, these resistive modes are governed by the equations

$$\rho_0 \gamma^2 \xi = (\nabla \times \hat{B}_0) \times \delta \hat{B} + (\nabla \times \delta \hat{B}) \times \hat{B}_0 - \nabla \delta p, \quad (3a)$$

$$\delta \hat{B} = \nabla \times (\xi \times \hat{B}_0 - \frac{1}{\gamma} \nabla \times \delta \hat{B}), \quad (3b)$$

$$\delta p = -\xi \cdot \nabla p_0 - \frac{5}{3} p_0 \nabla \cdot \xi. \quad (3c)$$

A normal mode is obtained by imposing boundary conditions and treating Eq. (3) as an eigenvalue problem for the growth rate γ .

III. THE ASYMPTOTIC MATCHING METHOD

The asymptotic matching method entails dividing the solution region into two parts, an inner resistive layer in which the variation of the equilibrium quantities can be neglected and an outer hydromagnetic region in which the transport terms and inertia can be neglected. The solution method that we describe here should be totally general and strictly valid in the limit of vanishing resistivity. We do not restrict attention to either current driven (tearing) modes or pressure driven (resistive interchange) modes, but treat them both on an equal footing. In general, the eigenmodes found by the matching method will not have a definite parity about the rational surface, but will be a superposition of the odd and even modes which fit the boundary conditions.

3a. Outer Region

The outer region equations, valid to the lowest order in the resistivity, are obtained from Eq. (3) by letting $\eta \rightarrow 0$, assuming that the growth rate γ varies as a fractional power of η . Following Newcomb [5], we eliminate the surface components of the displacement, ξ_θ and ξ_z , in terms of the radial component ξ_r , which has the spatial dependence $\xi_r(r)e^{i(m\theta - nkz)}$. We have

$$\frac{d}{dr} \left(f \frac{d}{dr} \xi_r \right) - g \xi_r = 0, \quad (4)$$

where

$$f = \frac{r B_\theta^2 (m - nq)^2}{m^2 + n^2 k^2 r^2}, \quad (5)$$

$$g = \frac{n^2 k^2 r B_\theta^2}{m^2 + n^2 k^2 r^2} \left[-\frac{q}{4} \left(\frac{q'}{k} \right)^2 + \left(1 + \frac{m^2 - 1}{n^2 k^2 r^2} \right) (m - nq)^2 + 2 \frac{(n^2 q^2 - m^2)}{m^2 + n^2 k^2 r^2} \right]. \quad (6)$$

The outer region equation, Eq. (4), may have one or more singular points, r_s , where f vanishes, or where

$$m - nq(r_s) = 0. \quad (7)$$

The outer region equations cannot be continued across these singular points. A Frobenius expansion of Eq. (4) about the singular points r_s , Eq. (7), shows that the two independent solutions on either side of r_s , ξ_r^+ and ξ_r^- are given by

$$\begin{aligned} \xi_r^\pm = C^\pm [|r-r_s|^{-1/2 - (-D_I)}]^{1/2} & \{ 1 + \xi_1^l (r-r_s) + \xi_2^l (r-r_s)^2 + \dots \} \\ & + \Delta^\pm |r-r_s|^{-1/2 + (-D_I)} \{ 1 + \xi_1^s (r-r_s) + \xi_2^s (r-r_s)^2 + \dots \}. \end{aligned} \quad (8)$$

Here, $D_I = -2p'r_s/B_0^2 R^2 q'^2 - 1/4 = (\alpha-1)/4$, Δ^+ and Δ^- are the ratios of the small to the large solution to the right and the left of the singular surfaces, C^+ and C^- are the corresponding amplitudes of the large solution, and $\xi_n^{l,s}$ are coefficients of the series expansion for the independent large and small solutions, expressible in terms of equilibrium quantities evaluated at the singular surface (see [6]).

The outer region solutions obtained from Eq. (4) cannot be continued across the singular points r_s . Rather, Eq. (4) is numerically integrated separately in each subinterval bounded by the origin $r = 0$, a singular point r_s , or the boundary $r = 1$. Thus, if there are n singular points r_{s1} , r_{s2} , ..., r_{sn} , there would be $n+1$ subintervals $[0, r_{s1})$, (r_{s1}, r_{s2}) , ..., $(r_{sn}, 1]$. In order to perform the matching with the inner region solutions, what is needed are the ratios $\Delta^\pm$ at each singular surface for the solution that contains only the small solution (or is zero) at the adjacent singular point (or endpoint). If the interval is bounded by two singular points, then the coefficients of the large solution $C^\pm$ are also needed to complete the matching. It is useful to define the quantity

$$\Delta' \equiv \Delta^+ + \Delta^-.$$

This is the conventional Δ' which indicates the resistive stability of a zero pressure configuration, i.e., if $\Delta' < 0$ and $p = 0$ everywhere the configuration

is stable to all resistive instabilities, and if $\Delta' > 0$ and $p = 0$, the configuration is unstable to tearing modes.

3b. Inner Region

In the resistive inner region in the vicinity of a singular surface r_s , the equilibrium quantities can be taken as constants. A rescaling of the independent variable r and the growth rate γ leads to the resistive inner region equations⁷ which are valid near $r = r_s$

$$\frac{d^2}{dx^2} \Psi = U \cdot \Psi + V \cdot \frac{d\Psi}{dx} \quad (9)$$

Here x is the scaled distance from the singular surface, $\Psi(x) = [\Psi(x), \Xi(x), T(x)]$ is a vector containing the perturbed magnetic field and the fluid displacement in the direction across the magnetic surfaces, and the perturbed magnetic field in the direction of the equilibrium magnetic field. The coefficient matrices are defined by

$$U = \begin{bmatrix} Q & -xQ & 0 \\ -x/Q & x^2/Q & -(E+F)/Q^2 \\ -x/Q & -(G+KE)Q & x^2/Q + (G+KF)Q \end{bmatrix} \quad (10)$$

$$V = \begin{bmatrix} 0 & 0 & H \\ -H/Q^2 & 0 & 0 \\ HKQ & 0 & 0 \end{bmatrix} \quad (11)$$

Here Q is the scaled growth rate and E, F, G, H, K , defined in Ref. [7], are constants characterizing the equilibrium at the singular surface. For the circular cylindrical configuration being considered here, we have

$$\Psi(x) = (ir_s/nB_0 q' L_r) \hat{B}_r, \quad \Xi(x) = \xi_r, \quad T(x) = (-2Ba/R^2 q'^2 B_0^2) \hat{B}_\parallel,$$

$$Q = \gamma/Q_r, \quad x = (r-r_s)/L_r, \quad E + F = -2p'r_s/(B_\theta^2 R^2 q'^2),$$

$$F = p'^2 r_s^2 / (B_\theta^4 R^2 q'^2), \quad K = 1/F, \quad G = \frac{3}{5} B^2/p, \quad H = 0.$$

$$L_r = [\rho_0 \eta^2 r_s^2 / (\eta^2 B_\theta^2 q'^2)]^{1/6}, \quad Q_r = [\eta \eta^2 B_\theta^2 q'^2 / (\rho_0 r_s^2)]^{1/3}.$$

Since equation (9) is parity conserving in x , the general solution is a superposition of modes odd in $\Xi(x)$

$$\Psi'(0) = \Xi(0) = T(0) = 0 \quad (\text{odd modes}) \quad (12)$$

and modes even in $\Xi(x)$

$$\Psi(0) = \Xi'(0) = T'(0) = 0 \quad (\text{even modes}) \quad (13)$$

Because of this symmetry, we need only consider solutions for $x > 0$. Eq. (12) or (13) represents three boundary conditions for the 6th order system, Eq. (9). Since the equations are homogeneous, a fourth condition is taken up by an overall normalization constant. The remaining two conditions are that the solution not diverge exponentially at infinity. To be more precise, an asymptotic analysis shows that at large x both the even and the odd solutions have the asymptotic form:

$$\begin{aligned}
 \Psi(x) \sim & x^{-1/2 - (-D_1)^{1/2}} T_1 \\
 & + \Delta x^{-1/2 + (-D_1)^{1/2}} T_2 \\
 & + E_1^- \exp(-x^2/2Q^{1/2}) x^{s_+} T_3 \\
 & + E_2^- \exp(-x^2/2Q^{1/2}) x^{s_-} T_4 \\
 & + E_1^+ \exp(+x^2/2Q^{1/2}) x^{s_+} T_5 \\
 & + E_2^+ \exp(+x^2/2Q^{1/2}) x^{s_-} T_6.
 \end{aligned} \tag{14}$$

Here

$$\begin{aligned}
 s_{\pm} = & -\frac{1}{2} + \frac{1}{4} \sigma Q^{3/2} \{1 + G + K(F + H^2)\} \\
 & \pm \frac{1}{4} \{Q^3 \{(G + KF - 1)^2 + KH^2 [KH^2 + 2(G + KF + 1)]\} \\
 & + 4[(G - KE - 1)D_R + H^2]\}^{1/2},
 \end{aligned}$$

$\sigma = +1$ for $T_{3,4}$ and -1 for $T_{5,6}$, $D_R = E + F + H$, and the T_i through T_6 are asymptotic series in decreasing powers of x , for example

$$T_1 = \left\{ \begin{array}{cccc} x & + & \psi_{11}/x & + & \psi_{12}/x^3 & + \dots \\ 1 & + & \xi_{11}/x^2 & + & \xi_{12}/x^4 & + \dots \\ 1 & + & \gamma_{11}/x^2 & + & \gamma_{12}/x^4 & + \dots \end{array} \right\}, \tag{15a}$$

$$T_2 = \left\{ \begin{array}{cccc} x & + & \psi_{21}/x & + & \psi_{22}/x^3 & + \dots \\ 1 & + & \xi_{21}/x^2 & + & \xi_{22}/x^4 & + \dots \\ 1 & + & \gamma_{21}/x^2 & + & \gamma_{22}/x^4 & + \dots \end{array} \right\}, \quad (15b)$$

etc. The coefficients of these series are obtained from recursion relations, described in Ref. [8].

The boundary conditions at infinity can now be stated precisely. For x large, the solution must match onto the asymptotic form of Eq. (14) with zero coefficient for the exponentially growing terms, i.e.,

$$E_1^+ = E_2^+ = 0 \quad . \quad (16)$$

Note that since the series in Eq. (15) are asymptotic but nonconvergent, the larger the value of x , the more accurate can be the imposition of this boundary condition.

Note also that since the exponentially decaying terms being multiplied by E_1^- and E_2^- die away rapidly at large x , they play no role in the matching to the outer region solutions. It is only the ratios Δ of the small to the large power-like solutions in Eq. (14) that enter into the matching. At a particular rational surface with the equilibrium quantities E, F, G, H, K held fixed, the odd mode ratios, Eq. (12), and the even mode ratios, Eq. (13), are functions only of the normalized growth rate Q . In order to facilitate the matching, we define the two unscaled ratios as follows:

$$\Delta^o(Q) = L_R^{-2(-D_I)^{1/2}} \Delta \quad (\text{odd modes}) \quad ; \quad (17a)$$

$$\Delta^e(Q) = L_R^{-2(-D_I)^{1/2}} \Delta \quad (\text{even modes}) \quad . \quad (17b)$$

Several accurate numerical methods for computing these functions by integrating Eq. (9) subject to the appropriate boundary conditions, Eq. (16) and either Eq. (12) or Eq. (13) are described in Ref. [8]. Here we assume that $\Delta^o(Q)$ and $\Delta^e(Q)$ are well-defined, computable, single-valued functions of Q once the equilibrium quantities E, F, G, H, K are given.

3c. Matching

Once the outer region equation, Eq. (4), has been solved in each subinterval and the inner region equation, Eq. (9), has been solved in the vicinity of each rational surface, the solutions can be matched together to determine the eigenmode structure and the growth rate. Figure 1 illustrates the most straightforward case with only one rational surface present.

If we denote by c_1 and c_4 the overall multiplicative constants for the outer solution to the left and the right of the singular surface, and by c_2 and c_3 the constants multiplying the even and odd inner region solutions, the conditions that the large and the small asymptotic solutions remain continuous when going from the inner to the outer region to the right and the left of the singular point are

large solution:

$$c_1 = c_2 - c_3 \quad (\text{left}) \quad ; \quad (18a)$$

$$c_4 = c_2 + c_3 \quad (\text{right}) \quad ; \quad (18b)$$

small solution:

$$c_1 \Delta^- = c_2 \Delta^e(Q) - c_3 \Delta^o(Q) \quad (\text{left}) \quad ; \quad (18c)$$

$$c_4 \Delta^+ = c_2 \Delta^e(Q) + c_3 \Delta^o(Q) \quad (\text{right}) \quad . \quad (18d)$$

The solubility condition on Eq. (18) gives the dispersion relation, most naturally expressed as the determinant.

$$D(Q) \equiv \begin{bmatrix} \Delta^+ - \Delta^e(Q) & [\Delta^+ - \Delta^o(Q)] \\ \Delta^- - \Delta^e(Q) & -[\Delta^- - \Delta^o(Q)] \end{bmatrix} = 0 \quad . \quad (19)$$

Equation (19) is solved using standard root finding techniques by repeatedly evaluating the functions $\Delta^e(Q)$ and $\Delta^o(Q)$.

We now examine under what conditions we expect the asymptotic matching method to provide accurate results. The validity of this method lies in the existence of an overlap region where both the inner (resistive) region asymptotic expansion for the displacement,

$$\xi_r(x) \sim |x|^s [1 + \xi_{11}/x^2 + \dots] \quad , \quad (14)'$$

and the outer (ideal) region asymptotic expansion for the displacement,

$$\xi_r \sim |x - r_s|^s [1 + \xi_1(x - r_s) + \dots] \quad , \quad (8)'$$

are valid. Here $x = (r - r_s)/L_r$ and the exponent $s = -1/2 \pm [-D_r]^{1/2}$. Since only the first terms in the two asymptotic series (14)' and (8)' match onto each other, the existence of an overlapping region of validity requires

$$L_r(\xi_{11})^{1/2} \ll 1/\xi_1 .$$

Now ξ_1 is a coefficient independent of the growth rate and the resistivity, dependent only on the equilibrium parameters evaluated at the singular surface⁶. On the other hand, ξ_{11} does depend on the scaled growth rate $Q = \gamma/Q_r$. Depending on the magnitude of Q , ξ_{11} takes one of the limiting forms given in [8]:

$$\xi_{11} = Q^2 A_1 , \quad Q \gg 1 ,$$

$$\xi_{11} = Q^{-1/2} A_2 , \quad Q \ll 1 ,$$

where A_1 and A_2 again are equilibrium coefficients independent of the growth rate and the resistivity. Treating the coefficients ξ_1 , A_1 , A_2 of order unity, the criterion for validity becomes

$$\tau_A \gamma \ll 1 , \quad Q \gg 1 ,$$

$$\gamma \gg \eta/r_s^2 , \quad Q \ll 1 ,$$

where $\tau_A = r_s(4\pi\rho_0)^{1/2}/B$ is the Alfvén time scaled to the singular surface position. These inequalities will always be satisfied for the resistivity η small enough, provided that the growth rate γ scales like a fractional power of the resistivity.

3d. Two rational surfaces

Figure 2 illustrates the case when two rational surfaces are present r_{s1} and r_{s2} . Here we denote by c_1 , c_4 , c_5 , and c_8 the four multiplicative constants for the outer region solutions to the left of c_1 , between c_4 and c_5 , and to the right of c_8 the singular surfaces. The outer region solution between r_{s1} and r_{s2} that is multiplied by c_4 is such that it contains only the "small" solution at r_{s2} (with a multiplier Γ_2). Similarly, the solution being multiplied by c_5 contains only the small solution at r_{s1} (with a multiplier Γ_1). As indicated in Fig. 2, the normalization of all these piecewise portions of the outer solution is such that before being multiplied by the c 's, the coefficients of the "large" solutions on either side of each rational surface are unity. As further indicated in Fig. 2, we let c_2 , c_3 , c_6 , c_7 be the multipliers of the inner region even and odd solutions at surfaces r_{s1} and r_{s2} that are so normalized that the coefficient of the large solution on the positive x side is unity. Continuity at the four matching points now gives the eight conditions.

large solution:

$$c_1 = c_2 - c_3 \quad , \quad (20a)$$

$$c_4 = c_2 + c_3 \quad , \quad (20b)$$

$$c_5 = c_6 - c_7 \quad , \quad (20c)$$

$$c_8 = c_6 + c_7 \quad , \quad (20d)$$

small solution:

$$c_1 \Delta_1^- = c_2 \Delta_1^e(Q) - c_3 \Delta_1^o(Q) \quad , \quad (20e)$$

$$c_4 \Delta_1^+ + c_5 \Gamma_1 = c_2 \Delta_1^e(Q) + c_3 \Delta_1^o(Q) \quad , \quad (20f)$$

$$c_5 \Delta_2^- + c_4 \Gamma_2 = c_6 \Delta_2^e(Q) - c_7 \Delta_2^o(Q) \quad , \quad (20g)$$

$$c_8 \Delta_2^+ = c_6 \Delta_2^e(Q) + c_7 \Delta_2^o(Q) \quad , \quad (20h)$$

The solubility condition on Eq. (19) gives the dispersion relation.

$$\begin{vmatrix} \Delta_1^- - \Delta_1^e(Q) & -\Delta_1^- + \Delta_1^o(Q) & 0 & 0 \\ \Delta_1^+ - \Delta_1^e(Q) & \Delta_1^+ - \Delta_1^o(Q) & \Gamma_1 & -\Gamma_1 \\ \Gamma_2 & \Gamma_2 & \Delta_2^- - \Delta_2^e(Q) & -\Delta_2^- + \Delta_2^o(Q) \\ 0 & 0 & \Delta_2^+ - \Delta_2^e(Q) & \Delta_2^+ - \Delta_2^o(Q) \end{vmatrix} = 0 \quad (21)$$

It is evident how the matching conditions are generalized should there exist more than two rational surfaces.

IV. THE INITIAL VALUE METHOD

We describe here an alternative technique for computing the most unstable eigenmode and growth rate of a circular cross-section cylindrical configuration, the initial value method. The initial value form of the

linearized resistive MHD equations [i.e., Eq. (3) with γ replaced by $\partial/\partial t$] are integrated forward in time with helical symmetry imposed. Finite difference methods are utilized in the radial direction and in time. The full resistive equations are solved everywhere, both near and away from the mode rational surface. This method is made feasible by utilizing "zone packing" near the mode rational surface in the radial direction, and by the choice of a representation for the velocity and the magnetic field that leads to finite difference equations that can be solved implicitly over the fast wave time scales.

If (r, θ, z) are standard cylindrical coordinates, we introduce a helical coordinate by

$$\tau \equiv m\theta - nkz. \quad (22)$$

Defining unit vectors in the radial, helical, and symmetry directions by

$$\hat{r} = \nabla r, \quad (23a)$$

$$\hat{\tau} = \nabla \tau / |\nabla \tau|, \quad (23b)$$

$$\hat{e} = \hat{r} \times \hat{\tau}, \quad (23c)$$

we can represent the magnetic field and fluid velocity by the variables

$$\vec{B} = \frac{\hat{e}}{h} \times \nabla \chi(r, \tau, t) + \frac{\hat{e}}{h} g(r, \tau, t). \quad (24a)$$

$$\vec{v} = \frac{\hat{e}}{h} \times \nabla A(r, \tau, t) + \frac{\hat{e}}{h} \omega(r, \tau, t) + \nabla \Omega(r, \tau, t). \quad (24b)$$

Here χ is the helical magnetic flux, g is the helical magnetic field, A is the helical stream function for the velocity, ω is the helical velocity, Ω is the helical velocity potential, and $h = (m^2 + n^2 k^2 r^2)^{1/2}$. The above representation of magnetic field and velocity variables is a generalization to compressible flow of the incompressible equations in Ref. [9] for studying the nonlinear development of the ideal internal kink mode in a cylinder.

We linearize assuming a single helicity $m = 1$ perturbation

$$\chi = \chi_0(r) + \chi_1(r, t) \cos \tau, \quad (25a)$$

$$g = g_0(r) + g_1(r, t) \cos \tau, \quad (25b)$$

$$\omega = \omega_1(r, t) \sin \tau, \quad (25c)$$

$$A = A_1(r, t) \sin \tau, \quad (25d)$$

$$\Omega = \Omega_1(r, t) \cos \tau, \quad (25e)$$

$$p = p_1(r, t) \cos \tau, \quad (25f)$$

where

$$\chi_0(r) = \int_0^r (mB_\theta - nkrB_z) dr \quad (26a)$$

and

$$g_0(r) = mB_z + nkr B_\theta. \quad (26b)$$

To first order in the perturbed quantities, the resistive MHD field evolution equations become

$$\frac{\partial}{\partial t} \chi_1 - \psi_0 \left(A_1 - r \frac{\partial \Omega_1}{\partial r} \right) = \eta (L \chi_1 + \frac{2mnk}{h^2} g_1), \quad (27)$$

$$\frac{\partial}{\partial t} (g_1 + \frac{2mnk}{h^2} x_1) + g_0 D \Omega_1 - \frac{h^2}{r} \left(\frac{g_0}{2} \right)' (A_1 - r \frac{\partial \Omega_1}{\partial r}) - \psi_0 \omega_1 = \eta L g_1, \quad (28)$$

where $\psi_0 \equiv (\chi_0'/r)$, prime denotes d/dr , and we define the radial operators

$$L \equiv \frac{h^2}{r} \frac{\partial}{\partial r} \frac{r}{h^2} \frac{\partial}{\partial r} - \frac{h^2}{r^2}, \quad (29a)$$

$$D \equiv \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{h^2}{r^2}. \quad (29b)$$

The equation for the helical velocity becomes

$$\frac{\partial}{\partial t} \omega_1 = \left(\frac{g_0'}{r} \right) \chi_1 - \psi_0 g_1. \quad (30)$$

Taking the curl of the momentum equation, we obtain the following equation for the velocity stream function:

$$\frac{\partial}{\partial t} L A_1 = \left(\frac{h^2}{r} \sigma_0' - \psi_0 L \right) \chi_1 - nk F_0 g_1, \quad (31)$$

where

$$\sigma_0 \equiv \frac{1}{r} \left[\left(\frac{r^2}{h^2} \psi_0 \right)' + \frac{2mnkx}{h^4} g_0 \right], \quad (32a)$$

and

$$F_0 \equiv \frac{2}{h^2} (m \psi_0 + nk g_0). \quad (32b)$$

Taking the divergence of the momentum equation, we obtain the following equation for the velocity potential:

$$\frac{\partial}{\partial t} D\Omega_1 + Dp_1 + \frac{\psi_0}{h^2} L\phi_1 + \frac{g_0}{h^2} Lg_1 \quad (33)$$

$$= \lambda_1 \chi_1 + \lambda_2 \frac{\partial}{\partial r} g_1 + \lambda_3 g_1 + \lambda_4 \frac{\partial}{\partial r} \phi_1 + \lambda_5 \phi_1 ,$$

where

$$\phi_1 \equiv r \frac{\partial}{\partial r} \chi_1 , \quad (34)$$

and the λ_i are equilibrium functions given by

$$\lambda_1 = \frac{1}{r^2} (2h\sigma_0 - mF_0) , \quad (35a)$$

$$\lambda_2 = -\frac{2}{h^2} (g'_0 + \frac{mnkr}{h^2} \psi_0) , \quad (35b)$$

$$\lambda_3 = -\frac{1}{r} \left(\frac{r}{h^2} g'_0 \right)' + \frac{2mnk}{h^2} \left[\frac{2nk}{h^4} (nkr^2 \psi_0 + mg_0) - \sigma_0 \right] , \quad (35c)$$

$$\lambda_4 = \frac{2}{rh^2} \left[\psi_0 + \frac{mnk}{h^2} g_0 - \frac{h^2}{r} \sigma_0 \right] , \quad (35d)$$

$$\lambda_5 = -\frac{\sigma'_0}{r} + 2n^2 \frac{k^2}{h^2} \sigma_0 - 4 \left(\frac{nk}{h^2} \right)^3 (nkr^2 \psi_0 + mg_0) . \quad (35e)$$

The pressure equation is

$$\frac{\partial}{\partial t} p_1 - \left(\frac{p'_0}{r} \right) (A_1 - r \frac{\partial}{\partial r} \Omega_1) + \frac{5}{3} p_0 D\Omega_1 = 0 . \quad (36)$$

By taking the divergence and curl of the momentum equation, the MHD characteristics are partially separated. The compressible magnetoacoustic wave motion is removed from the quantity A_1 and the incompressible shear Alfvén wave motion is removed from the quantity Ω_1 . As discussed in Ref. [10], this separation of MHD time scales leads to dramatic improvements in the accuracy and stability of the numerical solution.

Equations (27), (28), (30), (31), (33), and (36) are integrated using finite difference techniques in both radius and time. To resolve the singular behavior of resistive MHD eigenfunctions efficiently, we employ a nonuniform radial mesh which has grid points closely spaced in the neighborhood of the singular surface and further separated in regions away from the singular layer. We demand that the grid spacing be "mildly nonuniform" so that the regular equal spacing central difference formulas are approximately second order accurate. The radial domain is divided into five regions. They are the singular region, two ideal regions, and two transition regions. The singular point, r_s , is centered in the singular region. One ideal region extends outward from the origin, and the other extends inward from the wall. The grid spacing is uniform in the singular and ideal regions with the spacing in the singular region being much smaller than the spacing in the ideal regions. Connecting the singular region to the two ideal regions are the transition regions. Here, successive grid spacing increases geometrically from the grid spacing in the singular region to the grid spacing in the respective ideal region. By "mildly nonuniform" we mean that the geometric increase in spacing between successive grid points in the transition regions is less than 5%.

In order to calculate growth rates and eigenfunctions of resistively unstable modes, the above equations must be advanced to a time which is orders of magnitude larger than any of the typical MHD wave time scales. Hence an

explicit time advancement scheme is impractical, especially since the fast magnetoacoustic wave travels across the mode rational surface where computational zones are packed closely together. Therefore, an implicit time advancement scheme is employed. All equations are advanced simultaneously using the generalized Crank-Nicholson scheme. Time steps are used which are large enough to calculate the resistive growth rates in a reasonable amount of computer time but small enough to obtain accurate results. Since the MHD waves produce high frequency, small amplitude stable oscillations about the lower frequency motion of the growing instability, taking large time steps is, in actuality numerically averaging over the stable oscillations. The calculation proceeds until the quantity $[(\partial^2 \chi_1 / \partial t^2) / \chi_1]$ evaluated at the singular surface converges to within some desired tolerance. The square root of this converged quantity is the growth rate.

V. COMPARISON

We present here a comparison of the growth rates obtained by the asymptotic matching method and the initial value method for calculating the growth rates of resistive instabilities, as the value of the resistivity and the pressure of the equilibrium profiles are varied. Excellent agreement between these methods in a large region of overlapping validity leads us to substantial confidence in both methods. The equilibrium configuration used in this comparison is characterized by the pitch on axis $q_0 R/a = 0.8$. At zero pressure, $\alpha = 0$, this equilibrium is tearing mode stable. Adding pressure, $\alpha > 0$, gives rise to resistive instabilities which are predominantly pressure driven. The particular mode considered is $m = 1$, $n = 4$, giving the singular surface position $r_s \approx 0.61$.

The results of this comparison are summarized in Figs. 3 and 4. Here we

see that the two methods calculate growth rates which agree to better than 1 in 10^3 in the parameter regions, $9 \times 10^6 > S > 10^4$ with $\alpha = 0.7$, and $0.1 < \alpha < 0.7$ with $S = 10^6$. This agreement is an excellent demonstration of the accuracy of the two methods.

Several other features of Fig. 3 are worth mentioning. First, we see that both methods obtain the asymptotic scaling, $\gamma \sim S^{-1/3}$, given by Coppi, Greene, and Johnson¹¹ for the slow resistive interchange mode in the limit of vanishingly small resistivity. This fact is another indication of the accuracy of the two methods being considered. Second, the range of validity of the boundary layer analysis in resistivity space is demonstrated. We see that the boundary layer analysis is most accurate for $S > 10^4$ for the particular equilibrium and mode numbers considered. This value may change a bit as the equilibrium parameters and mode numbers are varied. Finally, both methods show that for $S < 10^4$ the growth rate becomes a decreasing function of the resistivity.

The range of usefulness of the initial value method is limited to instabilities which have growth rates that are large enough to be computed in the available computer time and with eigenfunctions that can be resolved with only a few hundred radial grid points. As such we have limited the initial value calculations to $S < 9 \times 10^6$ in Fig. 3 and to $\alpha > 0.1$ in Fig. 4. However, the asymptotic matching method can analyze modes with extremely small growth rates and narrow resistive layer widths. This capability of the asymptotic matching method is used in Fig. 4 to demonstrate some new physics. Here we see that for very small values of the central pressure, $p_0/B_0^2 < 10^{-3}$, a regime emerges in which the growth rate of the resistive interchange mode in a cylinder is considerably smaller than previously believed. This stabilization, noted in Refs. [7] and [12], is due to coupling

to the external tearing stable region.

In Fig. 5 is a radial profile of the radial displacement ξ as obtained by the asymptotic matching method. Here $q_0 R/a = 2.0$, $\alpha = 0.5$, $m = 1$, $n = 4$, and $S = 10^6$. The dotted curve is the solution to the Euler equation in the ideal plasma region, and the solid curve is the solution to the resistive boundary layer equations in the vicinity of the mode rational surface. These two solutions are asymptotically matched according to the procedure described in Sec. III. The corresponding profile of ξ obtained by the initial value method is indistinguishable to the eye from the curve shown in Fig. 5.

VI. STABILITY RESULTS

Reference [4] summarizes the ideal MHD stability properties of different spheromak profiles covering a large region of the cylindrical spheromak parameter space by noting that with the parameterization introduced here, for a given value of the pressure parameter α and the poloidal mode number m , stability only depends on the quantities nq_0 and nk , where n is the toroidal mode number, q_0 is the central safety factor, and k is the inverse aspect ratio. This reduction of the number of parameters needed to specify an equilibrium configuration allowed for the identification of ideal MHD stable operating regimes by mapping out stability boundaries in the continuous space formed by nq_0 and nk for various values of α and the wall radius with $m = 1$. In Fig. 6 we present four such diagrams for the more general case which includes resistive instabilities. Here growth rate contours are mapped out for $m = 1$ and 3, $\alpha = 0.5$ and 0.01, and resistivity $\eta = 10^{-6}$.

In these diagrams the growth rate corresponding to a particular contour is linearly proportional to the integer label with the growth rate increasing with the integer value. In general, each diagram consists of three types of

regions, one which is stable to both ideal and resistive modes, one which is unstable to ideal modes, and one which is stable to ideal modes but unstable to resistive modes. These three regions are separated by the dashed curves. The resistive MHD unstable region is further divided into two types of regions which are distinguished according to the sign of Δ' . These two types of regions are separated by the dotted-dashed curves. The stability properties of a given configuration are obtained by drawing a straight line through the origin with slope $a/(q_0 R)$ and then plotting the points along this line corresponding to the axial mode numbers n . The configuration $q_0 = 0.4$ and $R/a = 2$ is illustrated.

The well known fact¹¹ that any finite pressure cylindrical configuration with negative pressure gradient everywhere is always unstable to resistive modes when the singular surface is inside of the plasma is evident in Fig. 6 because the resistive unstable regions do not surround any stable regions. In a zero pressure configuration, i.e., $\alpha = 0$, the $\Delta' = 0$ curves separate tearing mode stable and unstable regions with the growth rate contours lining up parallel to the $\Delta' = 0$ curves on the unstable side. However, when pressure is introduced, the contours cross the $\Delta' = 0$ curves and the tearing mode stable region becomes unstable to pressure driven resistive modes.

Since all configurations are unstable, according to the diagrams in Fig. 6, then the optimal value of q_0 at which to operate is one for which the configuration line crosses the contours with the smallest growth rates. Careful examination of Figs. 6(a)-(d) reveals that, for both $\alpha = 0.5$ and 0.01 , the least unstable configurations have $q_0 R/a \sim 0.67$.

To see how the resistive growth rates depend on resistivity, we plot in Fig. 7 the natural logarithm of the growth rate vs the base 10 logarithm of the resistivity for various values of q_0/k and of the pressure parameter α .

with $m = 1$. The range of resistivity considered is $10^{-9} < \eta < 10^{-3}$.

First consider the solid curves in Fig. 7, i.e., those for which $q_0/k = 0.8$. These curves correspond to the point labeled $n = 4$ on the configuration line $q_0 = 0.4$ and $R/a = 2$ illustrated in Fig. 6. Note that this mode lies inside of the $\Delta' = 0$ curves in Fig. 6 so that at zero pressure it is stable. This fact is consistent with the results shown in Fig. 7, where it is seen that the growth rate decreases several orders of magnitude, over the range of resistivity considered, as α is lowered from 0.50 to 0.01. Another point to notice about the solid curves in Fig. 7 is that they deviate from the behavior given by pure boundary layer analysis, i.e., $\gamma \sim \eta^{1/3}$ for pressure driven modes in the limit of vanishingly small η , at smaller values of η as α is lowered. When considered as a function of η the growth rate increases as $\eta^{1/3}$, as η increases from zero, then eventually peaks at some value $\eta_{\max}$, and becomes a decreasing function of η for $\eta > \eta_{\max}$. The value of $\eta_{\max}$ decreases as the pressure parameter α is lowered, for instance, $\eta_{\max} = 10^{-4}$ for $\alpha = 0.50$ and $\eta_{\max} = 10^{-7}$ for $\alpha = 0.01$. This deviation from the results of pure boundary layer analyses illustrates that the resistive interchange mode behaves nonlocally in certain regions of parameter space so that properly matching the inner and outer solutions is necessary.

Next, consider the dashed curves in Fig. 7, i.e., those corresponding to $q_0/k = 2$. The configuration line for $q_0/k = 2$ passes entirely through a tearing mode unstable region in Fig. 6 so that at zero pressure the configuration is unstable. The dashed curves in Fig. 7 show that the finite pressure corrections to the growth rate of a purely current driven mode is relatively independent of resistivity over the range of η considered.

We now apply the initial value method to study the effect that finite fluid compression has on the growth rates of the resistive interchange mode in

the cylindrical spheromak. This is easily accomplished since the incompressible initial value equations are a subset of the full set of equations and are obtained by setting $\Omega_1 = 0$ in Eqs. (27) and (28), and dropping Eqs. (33) and (36). In Fig. 8 is plotted the growth rate versus resistivity of the resistive interchange unstable configuration, $q_0 R/a = 0.8$, $\alpha = 0.7$, $m = 1$, $na/R = 2.4$, from both the incompressible and compressible sets of equations. The qualitative dependence of the growth rate on resistivity is the same in both cases and is as described previously for Fig. 3. Quantitatively, finite fluid compression always has the effect of slightly increasing the growth rate. This increase is insignificant in the limit of vanishingly small resistivity and is somewhat more pronounced for large resistivity.

In Fig. 9 we plot the eigenfunction profiles corresponding to various points in Fig. 3 as obtained by the initial value method. Moving from left to right are the helical flux χ_1 , the helical field g_1 , the radial velocity u_{r1} , and the helical velocity ω_1 , for a given value of the magnetic Reynold's number S . Moving from top to bottom the value of S increases from 10^3 to 10^6 . Recall that the configuration considered in Fig. 3 is unstable to resistive interchange modes. Note the increased localization of the eigenfunctions about the mode rational surface as S is increased. This characteristic of the eigenfunctions is assumed in the derivation of the boundary layer equations¹¹ and is shown here to be a valid one. The magnitude of ω_1 is seen to be much larger than the magnitude of u_{r1} , indicating that the fluid flow for the resistive interchange mode is mostly parallel to the magnetic field, a property given by the boundary layer analysis.¹¹ Finally, we call attention to the distinct change in structure of the helical flux, χ_1 , in the vicinity of the mode rational surface as S is increased. For this

reason the constant flux approximation cannot be used in the boundary layer analysis of resistive interchange modes as is done for tearing modes.

VII. HALL EFFECT MHD

In this section we describe the effect that inclusion of the Hall term into the resistive MHD model has on the growth rate and eigenfunctions of the resistive interchange mode in the cylindrical spheromak. A boundary layer analysis has suggested that these terms should have a strong stabilizing influence.¹³ However, the eigenfunctions were seen to become less localized and highly oscillatory in the presence of the Hall term, so that the validity of a purely local analysis becomes questionable. In order to properly treat the connection of the resistive inner region solutions to the ideal outer region solutions, we have modified the asymptotic matching code and the linear initial value resistive MHD code, described in Secs. III and IV, to include the Hall term in Ohm's Law.

The form of Ohm's Law which includes the drift, or Hall term, is given in cgs units by

$$\vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta \vec{J} + \frac{m_i}{e} \frac{d\vec{v}}{dt} + \frac{1}{ne} \nabla p_i, \quad (37)$$

where m_i is the ion mass, p_i is the ion pressure, and $d/dt \equiv \partial/\partial t + \vec{v} \cdot \nabla$ is the full convective derivative. We assume, as in Ref. [13], that the ion temperature is small enough so that the ion pressure gradient term can be ignored in the above equation. Inserting the resulting equation into Faraday's Law and converting to rationalized cgs units gives the following equation for determining the perturbed field quantities:

$$\frac{\partial}{\partial t} [\delta \vec{B} + \Lambda \nabla \times \vec{v}] = \nabla \times [\vec{v} \times \vec{B}_0 - \eta \nabla \times \delta \vec{B}] \quad (38)$$

The strength of the Hall term is measured by the quantity $\Lambda \equiv (\omega_{ci} t_A)^{-1}$ where $\omega_{ci} \equiv eB(0)/m_i c$ is the ion cyclotron frequency, $t_A \equiv (4\pi\rho_0)^{1/2} a/B(0)$ is the Alfvén time, $B(0)$ is the axial magnetic field strength, and ρ_0 is the volume averaged plasma density. Generalizing the initial value method to include the Hall terms is easily accomplished after writing the field equations in the above form. The effect of finite Λ is to impart an imaginary part to the growth rate and eigenfunctions of resistive instabilities.

In the asymptotic matching method, the outer region equations, Eq. (4), remain unchanged with the addition of the Hall term. The inner region coefficient matrix in Eq. (9) now becomes complex, $\underline{U} = \underline{U}_R + i\underline{U}_I$, with

$$\underline{U}_R = \begin{bmatrix} Q & -xQ & 0 \\ -x/Q & x^2/Q & -(E+F)/Q^2 \\ -x/Q + KQ_*^2 x & -(G - KE)Q & x^2/Q + (G + KF)Q + KQ_*^2 x^2 \end{bmatrix} \quad (39a)$$

$$\underline{U}_I = \begin{bmatrix} Q_* & 0 & -Q_* x \\ -xQ_*/Q^2 & 0 & x^2 Q_*/Q^2 \\ KQ_* Q x & -KQ_* Q x^2 & (E+F)KQ_* \end{bmatrix} \quad (39b)$$

where $Q_* = 1/2(V_A \omega_{ci}^{-1}/L_R)(E+F)Rq'$ and $V_A = a/\tau_A$. The solution of these equations and the matching to the outside solution proceed as before, with the

main difference being that the exponentially decaying terms in Eq. (14) become $\sim E^{-\exp(-\sigma x^2)}$ with

$$\sigma = -\frac{1}{2} \left[1/Q + KQ_*^2/4 \right]^{1/2} \pm Q_* K^{1/2}/4 \quad (40)$$

The solutions will now have complex Q for $Q_* \neq 0$. For $\text{Re } Q \ll \text{Im } Q$, Q_* , Eq. (40) leads to highly oscillatory solutions that decay on a length scale much greater than the distance needed to match the powerlike solutions of Eq. (9).

First, we reconsider the resistive interchange mode unstable configuration illustrated in Fig. 6, i.e., $q_0 R/a = 0.8$, with $\alpha = 0.7$. The complex growth rate, $\gamma_R + i \gamma_I$, and eigenfunctions are calculated as a function of Λ for $m = 1$, $na/R = 2$, and resistivity $\eta = 10^{-5}$. Figure 9 shows the growth rates calculated by both codes. The real part is seen to decrease with increasing Λ until complete stabilization occurs at a critical value of $\Lambda_{\text{crit}} \approx 0.15$. The imaginary part is an increasing function of Λ and, at the point of stabilization, has risen to nearly three times the value of γ_R at $\Lambda = 0$. Results from both codes agree well over the range of Λ considered. Small differences at larger values of the Hall parameter are due to the fact that the eigenfunctions are becoming less localized, so that the conditions for validity of the asymptotic matching method begin to break down.

The real and imaginary parts of the radial displacement, obtained by the initial value code, are shown in Figs. 10(a)-(c) for various values of the Hall parameter. As Λ is increased the scale length of the eigenfunctions remains essentially unchanged in the vicinity of the mode rational surface. However, short wavelength oscillations develop in the transition regions between the resistive layer and the ideal regions. For large enough Λ these

oscillations spread across the ideal regions until they begin to interact with the plasma boundaries.

As a final application of the asymptotic matching method, we scan three configurations which are unstable to the $m = 1$ resistive interchange mode and calculate, as a function of nk , the critical value of the Hall parameter at which complete stabilization occurs. The three configurations considered are $q_0 R/a = 0.8, 1.0$, and 1.2 with $\alpha = 0.7$, and $\eta = 10^{-5}$. The configuration lines corresponding to these three values of $q_0 R/a$ pass, almost entirely, through the region between the $\Delta' = 0$ curves in the diagram corresponding to Fig. 7(a) but with $\alpha = 0.7$. The majority of the region between the $\Delta' = 0$ curves is sampled by these three configurations.

The results of this scan are shown in Fig. 11. Here we see that, for the configurations considered, the pressure driven modes are all stabilized for $\Lambda \gtrsim 0.20$, i.e., if ω_{ci}^{-1} is about 20% of the Alfvén time t_A . In terms of the diagram corresponding to Fig. 6(a), but for $\alpha = 0.7$, these results have the effect of excluding most of the growth rate contours from the region between the $\Delta' = 0$ curves if $\Lambda \gtrsim 0.20$. Also, it is clear from Fig. 11 that, for a given aspect ratio R/a , higher n modes and higher q_0 configurations are stabilized at smaller values of the Hall parameter.

A preliminary study of the Hall term effects on $m = 1$ tearing modes in the cylindrical spheromak has revealed no substantial stabilizing influence; however, more work in this direction is in progress and will be the subject of another publication. We therefore conclude this section by saying that the Hall term does indeed have a strong stabilizing influence on the resistive interchange mode in a cylinder, as was seen by a pure boundary layer analysis,¹³ and, depending on the size of $[\omega_{ci} t_A]^{-1}$, has the effect of excluding most of the growth rate contours from the region between the $\Delta' = 0$ curves in diagrams such as those seen in Fig. 6.

VIII. SUMMARY

Two complementary methods were described for calculating the growth rates and eigenfunctions of resistive instabilities in the cylindrical spheromak, the asymptotic matching method and the initial value method. The accuracy of these methods is demonstrated by comparing growth rates obtained by each method in a region of parameter space where they are both valid. Growth rates agree to better than 1 in 10^3 .

The asymptotic matching method is extremely fast and efficient since it employs boundary layer analysis, and thereby allows us to map out contours of ideal and resistive MHD growth rates in the continuous space formed by nq_0 and nk . Doing this reveals that, over a large range of the pressure parameter ($0.01 < \alpha < 0.5$), the least unstable cylindrical spheromak configurations have a pitch on axis of $q_0 R/a \sim 0.67$. Until the present work, such extensive studies of resistive instabilities in plasma confinement devices have not been feasible, even in the reduced parameter space employed here. The reason is that resistive MHD growth rates and eigenfunctions have been calculated in the past by initial value codes which are expensive due to the small growth rates being computed. However, with the development of a numerical method to solve the boundary layer equations and matching conditions, it is now possible to calculate resistive MHD growth rates and eigenfunctions quickly and accurately. The asymptotic matching method has enabled us to map out contours of resistive and ideal MHD growth rates in the space formed by nq_0 and nk for various values of m , α , and the resistivity η .

Both methods demonstrate the nonlocal behavior of the resistive interchange mode when the resistivity $\eta \geq \eta_{\max}$ where $\eta_{\max}$ is proportional to the pressure parameter α . This behavior was shown in Fig. 3 and Fig. 7 where deviation from the results of pure boundary layer analysis,¹¹ i.e., $\gamma \sim \eta^{1/3}$,

is seen. Also, finite fluid compression is seen to have an insignificant, although destabilizing, effect on the resistive interchange mode in the limit of vanishingly small resistivity.

Finally, the Hall term in Ohm's Law is easily incorporated into both methods. The strong stabilizing influence that this term has on the resistive interchange mode, as predicted by a pure boundary layer analysis,¹³ is verified. A scan of parameter space reveals that, for configurations having pitch on axis in the range $0.8 < q_0 R/a < 1.2$ and pressure parameter $\alpha = 0.7$, corresponding to an axial beta value of approximately 6%, the resistive interchange mode is completely stabilized if the Hall parameter $[\omega_{ci} t_A]^{-1} \gtrsim 20\%$.

ACKNOWLEDGMENTS

The authors are grateful to R. C. Grimm, M. Yamada, and J. L. Johnson for reviewing parts of this work and for several useful discussions concerning the linear theory of resistive instabilities, and to C. M. Ryu, D. Hewett, and J. Freidberg for discussing their unpublished numerical results.

This work was supported by United States Department of Energy Contract No. DE-AC02-76-CHO-3073.

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FIGURE CAPTIONS

- FIG. 1 Even (ξ^e) and odd (ξ^o) inner region solutions, and outer region solutions, ξ^+ and ξ^- , for the case with only one rational surface present.
- FIG. 2 Inner region solutions, $\xi_1^{e,o}$ and $\xi_2^{e,o}$, and outer region solutions, $\xi_1^\pm$ and $\xi_2^\pm$, for the case with two rational surfaces present.
- FIG. 3 Growth rate versus resistivity from asymptotic matching and initial value methods for the resistive interchange unstable case $q_0 R/a = 0.8$, $\alpha = 0.7$, $m = 1$, $na/R = 2$. Growth rates from the two methods agree to better than 1 in 10^3 in the region $9 \times 10^6 > S > 10^4$.
- FIG. 4 Growth rate versus central pressure from asymptotic matching and initial value methods for the case $q_0 R/a = 0.8$, $m = 1$, $n = 4$, $S = 10^6$. Growth rates from the two methods agree to better than 1 in 10^3 in the region corresponding to $0.1 < \alpha < 0.7$.
- FIG. 5 Radial profile of the radial displacement ξ from the asymptotic matching method for the case $q_0 R/a = 2.0$, $\alpha = 0.5$, $m = 1$, $n = 4$, $S = 10^6$. The outer region solutions (dotted curves) match smoothly onto the inner region solution (solid curve) in the overlap region.

FIG. 6 Growth rate contours in the continuous space formed by nq_0 and na/R as obtained by the asymptotic matching method. Growth rates are proportional to the integer labels. Stable, ideal MHD unstable, and resistive MHD unstable regions are distinguished by dashed curves. The $\Delta' = 0$ curves indicate tearing mode stability boundaries at zero pressure. $m = 1$ in (a) and (b), $m = 3$ in (c) and (d), $\alpha = 0.5$ in (a) and (c), and $\alpha = 0.01$ in (b) and (d).

FIG. 7 Growth rate versus resistivity for $m = 1$ and various values of q_0/k and α . Solid curves are tearing mode stable at zero pressure, dashed curves are tearing mode unstable at zero pressure.

FIG. 8 Effect of finite fluid compression on the resistive interchange mode. Here, $q_0R/a = 0.8$, $\alpha = 0.7$, $m = 1$, and $na/R = 2.4$.

FIG. 9 Eigenfunction profiles corresponding to various points in Fig. 3.

FIG. 10 Real and imaginary parts of the growth rate as a function of the Hall parameter for the resistive interchange mode $q_0R/a = 0.8$, $\alpha = 0.7$, $m = 1$, $na/R = 2$, and $S = 10^5$.

FIG. 11(a) Real and imaginary parts of the radial displacement for $\Lambda = 0.02$.

FIG. 11(b) Real and imaginary parts of the radial displacement for $\Lambda = 0.10$.

FIG. 11(c) Real and imaginary parts of the radial displacement for $\Lambda = 0.14$.

FIG. 12 Stabilizing value of the Hall parameter as a function of na/R for various values of the pitch q_0R/a and $\alpha = 0.7$.

Fig. 1

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Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11a

Fig. 11b

Fig. 11c

Fig. 12

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