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Comments on Ideal Ballooning

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Issued: January 1982

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COMMENTS ON IDEAL BALLOONING

by

R. Y. Dagazian and R. B. Paris

ABSTRACT

Ideal ballooning modes are investigated for the case of plane magnetized slab geometry. Toroidal effects are simulated by a gravitational acceleration periodically varying along magnetic field lines. High shear is shown to be very effective in stabilizing these modes even when field line curvature is most unfavorable to their stability.

I. INTRODUCTION

Ballooning modes can impose a serious limit on the amount of plasma that can be stably confined in a toroidal magnetic confinement device. The infinite conductivity, ideal, ballooning mode has been extensively studied in the literature for realistic plasma confinement configurations of the tokamak type.¹⁻³ All of these studies, to our knowledge, neglect the effects of compressibility and of strong shear. It is the purpose of this report to estimate the importance of these effects by employing the simple model of the magnetized plasma slab subject to a modulated gravity simulating varying toroidal curvature along magnetic field lines. We find that, in contrast to the real toroidal case, ideal ballooning modes in the plane slab are necessarily incompressible in the limit of large perpendicular wave numbers. On the other hand, strong magnetic shear is very effective in stabilizing these modes, even in the most adverse situations. Devices like reversed field pinches (RFP) can be stable for rather high values of kinetic to magnetic pressure ratio (β).

II. THE MODEL

We consider the case of the magnetized plasma slab equilibrium with magnetic shear. We write the equilibrium magnetic field as

$$\underline{B}^{(0)} = [0, B_y^{(0)}(x), B_z^{(0)}(x)] \quad , \quad (1)$$

where x , y , and z are Cartesian coordinates and no a priori assumption is made as to the relative magnitude of the two field components.

Toroidal curvature effects are simulated by introduction of a gravitational acceleration modulated along a given line of force

$$\underline{G} = G_0 \left[-\alpha + \cos \frac{2\pi}{L} z \right] \hat{e}_x \quad , \quad G_0 \equiv \beta_0 \frac{a}{R_c} \quad . \quad (2)$$

Here, R_c is the simulated radius of curvature of the magnetic field, a is a characteristic dimension of the plasma, and L is the connection length along the magnetic field lines. The parameter α simulates average curvature along these lines.

Taking the $\hat{e}_x \cdot \nabla_x \nabla_x$ and the $\hat{e}_\parallel = \underline{B}^{(0)} / |\underline{B}^{(0)}|$ components of the equation of motion together with the induction and continuity equations, we combine the normalized system of the linearized set of magnetohydrodynamic (MHD) equations, and the adiabatic state law into the form⁴

$$\rho^{(0)} \omega \left[-\nabla_x^2 v_x^{(1)} + \frac{\partial}{\partial x} \nabla \cdot \underline{v}^{(1)} \right] = -F^{(0)} \nabla_x^2 B_x^{(1)} + F^{(0)''} B_x^{(1)} - 2 \left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] p^{(1)} G(z) \quad , \quad (3)$$

$$\rho^{(0)} \omega v_\parallel^{(1)} = -\beta_0 \left[\nabla_\parallel p^{(1)} + \frac{1}{B^{(0)}} \frac{dp^{(0)}}{dx} B_x^{(1)} \right] \quad , \quad (4)$$

$$-\omega B_x^{(1)} + F^{(0)} v_x^{(1)} = 0 \quad , \quad (5)$$

and

$$\nabla \cdot \underline{v}^{(1)} = \Gamma_{v_x}^{(1)} - \frac{\omega_p^{(1)}}{\gamma_p^{(0)}} \quad (6)$$

Here symbols have their usual meanings and we have employed the normalization

$$\frac{v}{v_h} \rightarrow \underline{v} \quad , \quad \frac{t}{\tau_h} \rightarrow t \quad , \quad \frac{p}{p_0} \rightarrow p \quad , \quad \frac{\rho}{\rho_0} \rightarrow \rho \quad , \quad \frac{B}{B_0} \rightarrow \underline{B}$$

with

$$\tau_h \equiv a \left[\frac{\rho_0}{B_0} \right]^{1/2} \quad , \quad \beta_0 \equiv \frac{p_0}{B_0} \quad , \quad \Gamma \equiv - \frac{1}{\gamma} \frac{d}{dx} \ln p^{(0)}(x) \quad ,$$

$$F^{(0)} \equiv B_y^{(0)} \frac{\partial}{\partial y} + B_z^{(0)} \frac{\partial}{\partial z} \quad , \quad \text{and } F^{(0)''} \equiv \frac{d^2 B_y^{(0)}}{dx^2} \frac{\partial}{\partial y} + \frac{d^2 B_z^{(0)}}{dx^2} \frac{\partial}{\partial z} \quad .$$

B_0 , ρ_0 , and p_0 represent characteristic values, all distances are normalized to a and γ in the adiabatic index. The $\parallel$ subscript indicates the direction parallel to the equilibrium magnetic field, and the time dependence of modes is taken of the form $\exp(i\omega t)$. Finally, in deriving Eq. (3), constant mass density has been assumed.

We now consider ballooning modes of the twisted slicing form^{5,6}

$$f^{(1)}(x, y, z, t) = f(\delta x, z) \exp [ik(B_y^{(0)} z - B_z^{(0)} y) + i\omega t] \quad , \quad (7)$$

where $\delta \ll 1$ is an equilibrium-profile-dependent parameter and k is the wave number perpendicular to the field line

$$B_y^{(0)} z - B_z^{(0)} y = c_0(x) \quad . \quad (8)$$

Neglecting the variation of perturbed quantities on the scale of the equilibrium profile variation ($\delta \rightarrow 0$), we combine Eqs. (3) and (4) with Eq. (5) in the limit $k \rightarrow \infty$:

$$B_z^{(0)2} \frac{\partial}{\partial z} (1 + s^2 z^2) \frac{\partial}{\partial z} v_x^{(1)} - \omega^2 \rho^{(0)} (1 + s^2 z^2) v_x^{(1)} + 2G(z) \omega p^{(1)} = 0 \quad , \quad (9)$$

and

$$\rho^{(0)} \omega \nabla_{\parallel} v_{\parallel} = - \beta_0 \frac{B_z^{(0)2}}{B^{(0)2} z} \frac{\partial^2}{\partial z^2} [p^{(1)} - \gamma \frac{\Gamma p^{(0)}}{\omega} v_x^{(1)}] \quad . \quad (10)$$

Here

$$s = \frac{B_z^{(0)} \frac{dB_y^{(0)}}{dx} - B_y^{(0)} \frac{dB_z^{(0)}}{dx}}{B_z^{(0)} B^{(0)}} \quad ,$$

focusing on a field line with $c_0(x) = 0$.⁴

Furthermore, taking the divergence of the equation of motion we find in the same limit

$$B_{\parallel}^{(1)} = - \frac{\beta_0}{B^{(0)}} p^{(1)} \quad , \quad (11)$$

representing constant total pressure to leading order in k^2 .

From the flux-freezing condition, we have⁴

$$\nabla \cdot \underline{v}_{\perp}^{(1)} = \mathcal{L} \underline{v}_x^{(1)} - \omega B_{\parallel}^{(1)} / B^{(0)} \quad , \quad \mathcal{L} \equiv -\frac{1}{2} \frac{d}{dx} \ln B^{(0)^2} \quad . \quad (12)$$

Combining Eqs. (6), (10), (11), and (12) it is easy to show that

$$\left[\frac{\sigma}{1+\sigma} B_z^{(0)^2} \frac{\partial^2}{\partial z^2} - \omega^2 \right] \nabla \cdot \underline{v}^{(1)} = 0, \quad \sigma \equiv \frac{\gamma \beta_0}{B^{(0)^2}} \quad . \quad (13)$$

The trivial solution of Eq. (13) is the only physical solution; hence the modes we are concerned with are indeed incompressible in the large k^2 limit. In this limit, Eq. (9) describing ideal ballooning becomes simply

$$B_z^{(0)^2} \frac{\partial}{\partial z} (1 + s^2 z^2) \frac{\partial}{\partial z} \underline{v}_x^{(1)} - \omega^2 (1 + s^2 z^2) \underline{v}_x^{(1)} + 2\gamma \Gamma G(z) \underline{v}_x^{(1)} = 0 \quad . \quad (14)$$

Here we have normalized pressure, mass density, and magnetic fields to their local values, $p^{(0)} = \rho^{(0)} = B^{(0)} \equiv 1$.

The result derived in Eq. (13) does not hold in general toroidal geometry. Adopting the usual general toroidal coordinates (ψ, χ, ζ) such that

$$\hat{e}_{\psi} \equiv \frac{-\nabla p^{(0)}}{|\nabla p^{(0)}|} \quad , \quad \hat{e}_{\zeta} \equiv \frac{B_{\zeta}^{(0)}}{|B_{\zeta}^{(0)}|} \quad , \quad \hat{e}_{\chi} = \hat{e}_{\psi} \times \hat{e}_{\zeta}$$

and following the same procedure as above, we can show that

$$\omega^2 \underline{v}_{\parallel}^{(1)} = \gamma \beta_0 \nabla_{\parallel} \nabla \cdot \underline{v}^{(1)} - \frac{\beta_0}{h_{\psi}} \frac{d}{d\psi} p^{(0)} \kappa_1 \underline{v}_{\psi}^{(1)} \quad , \quad (15)$$

and

$$\nabla \cdot \mathbf{v}_\perp^{(1)} = -\sigma \nabla \cdot \mathbf{v}^{(1)} - 2\kappa_\psi \mathbf{v}_\psi^{(1)}, \quad (16)$$

where

$$\kappa_1 \equiv - \frac{1}{B^{(0)} h_\psi} \left[\frac{B_\chi^{(0)}}{h_\chi} \frac{\partial h_\psi}{\partial \chi} + \frac{B_\zeta^{(0)}}{h_\zeta} \frac{\partial h_\psi}{\partial \zeta} \right] = - (\hat{\mathbf{e}}_\parallel \cdot \nabla) \ln h_\psi,$$

and

$$\kappa_\psi \equiv - \frac{1}{B^{(0)2} h_\psi} \left[\frac{B_\chi^{(0)2}}{h_\chi} \frac{\partial h_\chi}{\partial \psi} + \frac{B_\zeta^{(0)2}}{h_\zeta} \frac{\partial h_\zeta}{\partial \psi} \right],$$

with the h_i denoting the metric coefficients. The term κ_ψ is the magnetic curvature of the configuration in the pressure gradient direction, while κ_1 is a curvature-related coefficient. Both κ_1 and κ_ψ vanish in the plane geometry case.

Combining Eqs. (15) and (16), we derive

$$(1 + \sigma) \left[\frac{\sigma}{1 + \sigma} B^{(0)2} \nabla_\parallel^2 - \omega^2 \right] \nabla \cdot \mathbf{v}^{(1)} = 2\omega^2 \kappa_\psi \mathbf{v}_\psi^{(1)} + \frac{\beta_0}{h_\psi} \frac{dp}{d\psi} \nabla_\parallel (\kappa_1 \mathbf{v}_\psi^{(1)}) \quad (17)$$

The gravitational acceleration $\underline{G}$ was introduced in Eq. (3) exactly to simulate the effects of κ_ψ in toroidal geometry. Although $\underline{G}$ does not occur in Eq. (13) derived for plane geometry, κ_ψ does enter the toroidal geometry expression, Eq. (17). The right-hand side of Eq. (17) is negligible for the case of a tokamak, making a contribution of $O(\beta_0/A^3)$ to the equation of motion. However, this is not the case for an RFP because $B_\theta^{(0)}/B^{(0)}$ can now be of $O(1)$.

For the case of zero shear, Eq. (14) can be put into the form of the Mathieu equation. Defining

$$\hat{a} \equiv -4\hat{R}^2(\omega^2 + 2\alpha\hat{G}_0) , \quad \hat{q} \equiv 4\hat{R}^2\hat{G}_0 , \quad \hat{G}_0 \equiv \gamma\beta_0\Gamma a/R_c , \quad (18)$$

and

$$\hat{R} \equiv L/2\pi B_z^{(0)} ,$$

we can find simple analytic results for the limits

$$\hat{q} \ll 1: \quad \omega^2 \approx 2\hat{G}_0^2 \hat{R}^2 - 2\alpha\hat{G}_0 \quad (19)$$

and

$$\hat{q} \gg 1: \quad \omega^2 \approx 2\hat{G}_0(1 - \alpha) - \hat{G}_0^{1/2}/\hat{R} . \quad (20)$$

When shear is finite, the disconnected mode approximation can be applied.⁷ By letting $\hat{z} \equiv sz$, Eq. (14) can now be put into the form

$$\frac{\partial}{\partial \hat{z}} (1 + \hat{z}^2) \frac{\partial}{\partial \hat{z}} v_x - [\hat{\lambda} + \hat{\gamma}^2 (1 + \hat{z}^2)] v_x = 0 , \quad (21)$$

where

$$\hat{\lambda} \equiv - \frac{2\hat{G}_0}{s^2 B_z^{(0)2}} \left[1 - \alpha + \frac{1}{2} \left(\frac{2\pi}{sL} \right)^2 \right]$$

and

$$\hat{\gamma}^2 \equiv \frac{\omega^2 + \hat{G}_0 \left(\frac{2\pi}{sL} \right)^2}{s^2 B_z^{(0)2}} .$$

Equation (21) is the zeroth-order oblate spheroidal equation. A discussion of its eigenvalues for solutions decaying in the limit $|\hat{z}| \rightarrow \infty$ is given in the Appendix. Here we merely state the principal results. There are two limits for which analytic estimates for the eigenvalues can be obtained:

(a) $\hat{\gamma}^2 \ll 1$. We find from Eq. (A-17) the largest even eigenvalue

$$\hat{\gamma}^2 \approx \frac{4u^2}{e^2} e^{-\pi/u} \exp \left[-\frac{4}{u} \phi(u) \right] , \quad (22)$$

where

$$u \equiv (-4\hat{\lambda} - 1)^{1/2} , \quad \phi(u) \equiv \arctan (e^{-\pi u/2}) .$$

For $u \rightarrow 0$, $\phi(u)$ can be expanded to find from Eq. (A-16) $\hat{\gamma}^2 \approx C^2 e^{-4\pi/u}$, $C = 16e^{\pi/2 - \bar{\gamma}}$ with $\bar{\gamma} = 0.5772\dots$ being Euler's constant, so that

$$\omega^2 \approx -\hat{G}_0 \left[\frac{2\pi}{sL} \right]^2 + C^2 \exp \left(\frac{-2\pi s B_z^{(0)}}{\{2\hat{G}_0 [1 - \alpha + \frac{1}{2} \left(\frac{2\pi}{sL} \right)^2] - \frac{1}{4} s^2 B_z^{(0)2}\}^{1/2}} \right) . \quad (23)$$

(b) $\hat{\gamma}^2 \gg 1$. In this case we find from Eq. (A-18)

$$-\hat{\lambda} = \hat{\gamma}^2 + \hat{\gamma} + \frac{1}{2} + O(\hat{\gamma}^{-1})$$

or

$$\omega^2 = 2\hat{G}_0(1 - \alpha) - sB_z^{(0)} \left\{ 2\hat{G}_0 \left[1 - \alpha + \frac{1}{2} \left(\frac{2\pi}{sL} \right)^2 \right] - \frac{1}{4} s^2 B_z^{(0)2} \right\}^{1/2}. \quad (24)$$

Numerical solution of Eq. (14) for the eigenvalue ω^2 poses no particular difficulty for the case of finite shear. For a treatment of this problem we followed essentially the procedure described in Ref. 4. It is found that the convergence of the iteration scheme is rapid for peaked modes but becomes slow as the shear parameter s approaches zero. However, for $s = 0$, exact solutions of Eq. (14) may be obtained from Mathieu-equation theory.

III. RESULTS AND DISCUSSION

The salient results of the present report are summarized in Fig. 1. The critical $\hat{G}_0$ beyond which instability arises has been numerically evaluated as a function of sL for a wide range of parameters obtainable in experiments. We have chosen $\hat{R} = 5$ as a representative value for any actual toroidal device. For tokamaks, $B_z^{(0)}$ is of $O(1)$; L is of the order of the aspect ratio A and $\hat{R} \sim Aq$, where q is the safety factor. In an RFP, L is in general shorter but $B_z^{(0)}$ is only a fraction of $B^{(0)}$. Positive values of α correspond to tokamaks, spheromaks, and possibly the near-axis regions of RFPs. We remind the reader

Fig. 1.

The critical $\hat{G}_0$ as a function of shear with α as the parameter. A possible set of conditions over the poloidal section of an RFP is indicated by a dashed line.

that this quantity is related to $V''(\psi)$. Negative values of α correspond to the region near the field-null in spheromaks and RFPs; average field line curvature is unfavorable to stability in such regions. It is seen that sizeable values for the critical β_0 can be obtained provided that the product sL is sufficiently high. However, in the limit of zero shear, it is seen that $\hat{G}_{\text{crit}}$ is nonzero only for positive values of α .

The dashed curve shows a possible set of equilibrium conditions as one moves from the inside toward the outside of an RFP. This device does exhibit a much larger variation in physical parameters as such a trajectory is followed.

Letting $\gamma\Gamma \equiv 1$ corresponds to setting $a = r_p$, the pressure gradient radius. Then $\hat{G}_0(\alpha, r_p, s, L)$ is just $\beta_0 r_p / R_c$. We note that both r_p and R_c can be large near the magnetic axis of a toroidal device. For the tokamak, R_c remains approximately constant throughout the discharge cross section and is roughly equal to the torus major radius R_0 . For the RFP this quantity varies from the major radius on axis to the minor radius at field reversal. It is difficult to specify the ratio r_p / R_c without a detailed equilibrium model. However, considering a pressure distribution of the form $\exp(-r^2/r_p^2)$, we can have roughly $\hat{G}_{\text{crit}} \sim \beta_0 r_p / R_0$ for a tokamak or the near-axis region of an RFP, and $\hat{G}_{\text{crit}} \sim \beta_0$ for the field reversal region of the latter device.

Figure 2 shows the dependence of ω^2 on shear with $\hat{G}_0$ as the parameter (the connection length is kept constant because we fix $\hat{R}$). It is seen that no matter how high $\hat{G}$ (and we have indeed allowed very large values of this quantity), a value of sL can be found beyond which the plasma is stable against ideal ballooning. Equations (22) and (23) are fairly accurate in predicting the cutoffs and the general features of the curves for the RFP-relevant set of parameters. However, Eq. (24) can accurately account only for the results shown in Fig. 2(b) involving high $\hat{G}_0$ values. Although the parameters relevant to lower β_0 values in the tokamak case, Fig. 2(c), do involve large $\hat{\gamma}^2$, the low values of sL lead to a breakdown of the disconnected mode approximation, which has been employed in deriving Eq. (21).

As we let $L \rightarrow \infty$, Eq. (23), unlike Eq. (24), remains valid and yields the familiar plane Suydam criterion for g-mode stability

$$D \equiv 2\hat{G}_0 (1 - \alpha) / s^2 B_z^{(0)2} - \frac{1}{4} < 0, \quad (25)$$

Fig. 2.

The dependence of the ideal ballooning growth rate on shear with $\hat{G}_0$ as the parameter. Favorable average field line curvature: (a) low $\hat{G}_0$; (b) high $\hat{G}_0$; (c) unfavorable line curvature with high shear.

evaluated at the worst curvature point on a given field line. Figure 3 shows the marginal stability curve representing the generalization of the plane Suydam

Fig. 3.

Marginal stability diagram for various sets of parameters. For a given large $\hat{R}$ the numerical curves corresponding to different α are indistinguishable. Observe that the disconnected mode approximation derived in the limit $\gamma^2 (\omega^2 = 0) \gg 1$ holds remarkably well even for small values of this quantity.

criterion to include ballooning. For $\hat{\gamma}^2(\omega^2 = 0) \gg 1$, an analytic expression can be obtained from Eq. (24) in the form

$$\hat{\gamma}^2(\omega^2 = 0) = \left[D - \frac{1}{4}\right]^2. \quad (26)$$

When $\hat{\gamma}^2(\omega^2 = 0) \ll 1$, we have to solve a transcendental equation (cf. Eq. (22)).

Figure 4 shows the growth rate as a function of $\hat{G}_0$ with sL as the parameter for (a) a tokamak and (b) an RFP-relevant case.

Figure 5 shows the dependence of $\hat{G}_{\text{crit}}$ on the parameter $\hat{R} = L/2\pi B_z^{(0)}$. Note that this quantity represents an effective connection length. The vanishing of $B_z^{(0)}$ then corresponds to $\hat{R} \rightarrow \infty$, which is precisely the g-mode limit of the present theory. This is the limit relevant to RFP stability near the field reversal surface. Modes then tend to be g-mode-like rather than ballooning-like on this surface. The critical $\hat{G}_0$ for instability is greatly enhanced for a modest reduction in $\hat{R}$; the growth rate decreases correspondingly.

Finally, Fig. 6 shows some examples of typical eigenfunctions for various discharge parameters. It is seen that for situations with high enough shear our modes tend to be highly peaked, thus justifying our disconnected mode approximation. When shear is low, the modes are broader about a given bad curvature point on the magnetic field line, but they exhibit undulations peaked on successive bad curvature points as infinity is approached.

Fig. 4.

Ideal ballooning growth rate as a function of $\hat{G}_0$ with shear as the parameter: (a) a tokamak and (b) an RFP-relevant case.

Fig. 5.

The critical $\hat{G}_0$ for ideal ballooning instability as a function of shear with α and R as the parameters.

Fig. 6.

Typical ideal ballooning eigenfunctions, (a) $\alpha = .05$, $sL = 0.5$, $\hat{G}_0 = .5 \times 10^{-2}$, $\omega^2 = .13 \times 10^{-3}$; (b) $\alpha = -1.5$, $sL = 25$, $\hat{G}_0 = .10$, $\omega^2 = .21 \times 10^{-1}$.

In conclusion, we have studied ideal ballooning modes simulating toroidal effects by means of a modulated gravitational acceleration along magnetic field lines in plane slab geometry with strong magnetic shear. It was shown that these modes are incompressible for the cases of plane geometry and for tokamaks, whereas strong curvature terms result in $\nabla \cdot \underline{v} \neq 0$ for an RFP-like configuration. From our results, it seems that even for the most severe choice of parameters for an RFP, this device can be easily stable against ideal ballooning. Shear is very effective in stabilizing these modes, and it works even when magnetic field line curvature is most unfavorable. The tokamak relies

mainly on favorable curvature to stabilize ideal ballooning; this results in general in lower β_{crit} limits for this device than for an RFP.

APPENDIX

EIGENVALUES OF THE OBLATE SPHEROIDAL EQUATION

I. BASIC PROPERTIES OF THE OBLATE SPHEROIDAL FUNCTIONS

We present the details of the calculation of the eigenvalues of the oblate spheroidal differential equation of order μ , expressed in the canonical form⁸

$$\frac{d}{dz} (1 - z^2) \frac{dy}{dz} + [\lambda + \gamma^2 (1 - z^2) - \mu^2 (1 - z^2)^{-1}] y = 0 \quad . \quad (A-1)$$

Equation (21) is then obtained from Equation (A-1) by putting $z = i\hat{z}$ and $\mu = 0$. The boundary conditions for the ballooning problem are that $y(\hat{z})$ be an even solution with $y(\hat{z}) \rightarrow 0$ as $\hat{z} \rightarrow \pm \infty$. For given λ, μ this constitutes an eigenvalue problem for the free parameter γ (or conversely, for given γ, μ the eigenvalue is λ).

Before determining the eigenvalues, we set down the necessary basic properties of the oblate spheroidal functions that we shall require. There are two limiting cases of Eq. (A-1) for which simple solutions may be given. When $|z^2| \gg 1$, Eq. (A-1) reduces to Bessel's equation with the solution

$$y = z^{-1/2} Z_{\nu+1/2}(\gamma z) \quad , \quad \nu = -\frac{1}{2} + \left(\lambda + \frac{1}{4}\right)^{1/2} \quad , \quad (A-2)$$

where Z denotes any Bessel function. When $\gamma^2 = 0$ (or when $\gamma^2 |z^2| \ll \lambda$), Eq. (A-1) reduces to Legendre's equation with the solutions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$.

These limiting cases suggest solutions of Eq. (A-1) in the form of expansions in terms of these functions.

A first set of oblate spheroidal functions involving an expansion in Bessel functions is

$$S_{\nu}^{\mu(j)}(z; \gamma) = (1 - z^{-2})^{-\mu/2} [A_{\nu}^{\mu}(\gamma)]^{-1} \sum_{r=-\infty}^{\infty} a_{2r} \psi_{\nu+2r}^{(j)}(\gamma z), \quad j = 1, 2, 3, 4 \quad (A-3)$$

where

$$\psi_{\nu}^{(1)} = \left[\frac{\pi}{2\gamma z} \right]^{1/2} J_{\nu+1/2}(\gamma z), \quad \psi_{\nu}^{(2)} = \left[\frac{\pi}{2\gamma z} \right]^{1/2} Y_{\nu+1/2}(\gamma z)$$

and

$$\psi_{\nu}^{(3,4)} = \left[\frac{\pi}{2\gamma z} \right]^{1/2} H_{\nu+1/2}^{(1,2)}(\gamma z)$$

respectively. The coefficients $a_{2r} \equiv a_{\nu, 2r}^{\mu}(\gamma)$ satisfy a three-term recurrence relation⁸ and are normalized such that $a_{\nu, 0}^{\mu}(0) = 1$ with

$$a_{\nu, 2r}^{\mu}(\gamma) = O(\gamma^{|2r|}) \quad (r \neq 0) \quad (A-4)$$

as $\gamma^2 \rightarrow 0$. The quantity $A_{\nu}^{\mu}(\gamma)$ is a normalizing factor chosen to be

$$A_{\nu}^{\mu}(\gamma) = \sum_{r=-\infty}^{\infty} (-)^r a_{2r}$$

so that as $z \rightarrow \infty$ in $|\arg(\gamma z)| < \pi$, $\psi_{\nu+2r}^{(j)} / \psi_{\nu}^{(j)} \sim (-)^r$, and $S_{\nu}^{\mu(j)}(z; \gamma)$ then possesses the simple asymptotic behavior

$$S_{\nu}^{\mu(j)}(z; \gamma) \sim \psi_{\nu}^{(j)}(\gamma z), \quad j = 1, 2, 3, 4; \quad z \rightarrow \infty \text{ in } |\arg(\gamma z)| < \pi. \quad (A-5)$$

The solutions in Eq. (A-3) may be shown to converge in $|z| > 1$.

A second set of oblate spheroidal functions satisfying Eq. (A-1), which involves the Legendre functions, is given by

$$Ps_{\nu}^{\mu}(z; \gamma) = \sum_{r=-\infty}^{\infty} (-)^r a_{2r} P_{\nu+2r}^{\mu}(z)$$

and

$$Qs_{\nu}^{\mu}(z; \gamma) = \sum_{r=-\infty}^{\infty} (-)^r a_{2r} Q_{\nu+2r}^{\mu}(z) , \quad (A-6)$$

where the coefficients a_{2r} are the same as in Eq. (A-3). These solutions converge everywhere (except possibly at $z = \pm 1$ and ∞).

Because Eq. (A-1) is a second order linear differential equation, there are only two linearly independent solutions and the different expansions in Eqs. (A-3) and (A-6) must be related. Only the relations relevant to the solution of the ballooning problem will be given here. From Eqs. (20) and (28) of Ref. 8 (Section 16.9) we have

$$\cos \pi \nu S_{\nu}^{\mu(3)}(z; \gamma) = e^{-\pi \nu i} S_{\nu}^{\mu(1)}(z; \gamma) - i S_{-\nu-1}^{\mu(1)}(z; \gamma) , \quad (A-7)$$

$$S_{\nu}^{\mu(1)}(z; \gamma) = \frac{1}{\pi} \sin \pi(\nu - \mu) e^{-(\nu+\mu+1)\pi i} \kappa_{\nu}^{\mu}(\gamma) Qs_{-\nu-1}^{\mu}(z; \gamma) , \quad (A-8)$$

and from standard properties of the Legendre functions

$$Qs_{-\nu-1}^{\mu}(z; \gamma) \sin \pi(\nu - \mu) = Qs_{\nu}^{\mu}(z; \gamma) \sin \pi(\nu + \mu) - \pi e^{\pi \mu i} \cos \pi \nu Ps_{\nu}^{\mu}(z; \gamma) . \quad (A-9)$$

The quantity $\kappa_{\nu}^{\mu}(\gamma)$ appearing in Eq. (A-8) is a constant (depending upon ν , μ and γ) known as the joining factor that links the $S_{\nu}^{\mu(i)}(z; \gamma)$ and $Qs_{-\nu-1}^{\mu}(z; \gamma)$ functions. Its value may be determined by expanding both sides of Eq. (A-8) in a Laurent double series in powers of z^2 and equating coefficients of like powers

of z^{2k} , $k = 0, \pm 1, \pm 2, \dots$. Because $\kappa_\nu^\mu(\gamma)$ is independent of k , it is necessary to consider only the coefficients corresponding to $k = 0$, to find⁸

$$\kappa_\nu^\mu(\gamma) = \frac{1}{2} \left(\frac{\gamma}{4}\right)^\nu \Gamma(1 + \nu - \mu) e^{\pi \nu i} [A_\nu^\mu(\gamma)]^{-1} \frac{\sum_{r=-\infty}^0 \frac{(-)^r a_{2r}}{(-r!) \Gamma(\nu + r + 3/2)}}{\sum_{r=0}^{\infty} \frac{(-)^r a_{2r}}{r! \Gamma(1/2 - \nu - r)}}. \quad (\text{A-10})$$

Combination of Eqs. (A-7) - (A-9) yields

$$\begin{aligned} S_\nu^{\mu(3)}(z; \gamma) &= \tilde{A} P_s^\mu_\nu(z; \gamma) + \tilde{B} Q_s^\mu_\nu(z; \gamma) \\ &= \sum_{r=-\infty}^{\infty} (-)^r a_{2r} [\tilde{A} P_{\nu+2r}^\mu(z) + \tilde{B} Q_{\nu+2r}^\mu(z)] , \end{aligned} \quad (\text{A-11})$$

where

$$\tilde{A} = e^{-2\pi \nu i} \kappa_\nu^\mu(\gamma)$$

and

$$\tilde{B} = -e^{-\pi \mu i} \frac{\sin \pi(\nu + \mu)}{\pi \cos \pi \nu} [e^{-2\pi \nu i} \kappa_\nu^\mu(\gamma) + i e^{\pi \nu i} \kappa_{-\nu-1}^\mu(\gamma)] . \quad (\text{A-12})$$

Equation (A-11) represents the analytic continuation of $S_\nu^{\mu(3)}(z; \gamma)$ inside the unit circle $|z| < 1$. Similar continuations may be obtained for the other Bessel function expansion solutions in Eq. (A-3), but these will not be required in this Appendix. This completes the summary of the properties of the solutions of Eq. (A-1) for the determination of the ballooning eigenvalues.

II. THE EIGENVALUE RELATION FOR γ IN TERMS OF μ AND λ

The desired solution behaving correctly at ∞ is $y = S_{\nu}^{\mu(3)}(ix; \gamma)$ for $x > 1$ (where $x \equiv \hat{z}$), because the Hankel function $H_{\nu}^{(1)}(ix)$ behaves like $K_{\nu}(x)$ when $x > 0$. The continuation of this solution onto the interval $0 \leq x \leq 1$ is given by Eq. (A-11). The definition of the Legendre functions in terms of Gauss hypergeometric functions in $|z^2| < 1$ [with $\text{Im}(z) > 0$ because $x > 0$] is

$$P_{\nu+2r}^{\mu}(z) = 2^{\mu} \pi^{1/2} (z^2 - 1)^{-\mu/2} [aF_1(z^2) + bzF_2(z^2)]$$

and

$$Q_{\nu+2r}^{\mu}(z) = 2^{\mu} \pi^{1/2} (z^2 - 1)^{-\mu/2} e^{\pi i \mu} [cF_1(z^2) + dzF_2(z^2)] ,$$

where

$$F_1(z^2) \equiv {}_2F_1\left(-\frac{1}{2}\nu - \frac{1}{2}\mu - r, \frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu + r; \frac{1}{2}; z^2\right) ,$$

$$F_2(z^2) \equiv {}_2F_1\left(\frac{1}{2} - \frac{1}{2}\nu - \frac{1}{2}\mu - r, 1 + \frac{1}{2}\nu - \frac{1}{2}\mu + r; \frac{3}{2}; z^2\right) ,$$

and

$$a = \frac{1}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\nu - \frac{1}{2}\mu - r\right) \Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2}\mu + r\right)} ,$$

$$b = \frac{-2}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu + r\right) \Gamma\left(-\frac{1}{2}\nu - \frac{1}{2}\mu - r\right)} ,$$

$$c = \frac{1}{2} e^{\pi i(\mu-\nu-1-2r)/2} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu + r)}{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\mu + r)},$$

$$d = e^{\pi i(\mu-\nu-2r)/2} \frac{\Gamma(1 + \frac{1}{2}\nu + \frac{1}{2}\mu + r)}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu + r)}.$$

The conditions for an even or odd solution, given by the vanishing of the coefficients of $zF_2(z^2)$ and $F_1(z^2)$ in the summand of Eq. (A-11), are consequently $\tilde{A}b + \tilde{B}de^{\pi\mu i} = 0$ and $\tilde{A}a + \tilde{B}ce^{\pi\mu i} = 0$ respectively, which become

$$\tilde{A} = \frac{1}{2} \tilde{B}(-)^r e^{\pi i(3\mu-\nu)/2} \Gamma(1 + \frac{1}{2}\nu + \frac{1}{2}\mu + r) \Gamma(-\frac{1}{2}\nu - \frac{1}{2}\mu - r)$$

$$= -\frac{1}{2} \pi \tilde{B} e^{\pi\mu i} \frac{e^{\pi i(\mu-\nu)/2}}{\sin \frac{1}{2}\pi(\mu + \nu)} \quad (\text{even}),$$

and

$$\tilde{A} = -\frac{1}{2} \pi \tilde{B} e^{\pi\mu i} \frac{e^{\pi i(\mu-\nu)/2}}{\cos \frac{1}{2}\pi(\mu + \nu)} \quad (\text{odd}).$$

Inserting the values of $\tilde{A}$ and $\tilde{B}$ from Eq. (A-12) then yields the conditions for $S_V^{\mu(3)}(ix; \gamma)$ and its continuation, Eq. (A-11), to be either an even or odd function of x , expressed in terms of the joining factor,

$$\kappa_{\nu}^{\mu}(\gamma) + e^{\pi i(\nu-1/2)} \left[\frac{1 \pm e^{\pi i(\mu+\nu)}}{1 \mp e^{\pi i(\mu-\nu)}} \right] \kappa_{-\nu-1}^{\mu}(\gamma) = 0, \quad \mu \pm \nu \neq 0, \pm 1, \pm 2, \dots, \quad (\text{A-13})$$

where the upper and lower signs correspond to the even and odd solutions respectively.

Equation (A-13) represents the eigenvalue relations for the even and odd eigenvalues, γ_e and γ_o , in terms of μ and λ (or, equivalently from Eq. (A-2), ν). In theory, Eq. (A-13) could be solved numerically provided the coefficients a_{2r} are known. Such a procedure would necessarily be iterative because the coefficients $a_{2r} \equiv a_{\nu, 2r}^{\mu}(\gamma)$ depend on γ .

III. THE EIGENVALUE RELATION FOR $\gamma^2 \ll 1$

When $\gamma^2 \ll 1$ we can approximately solve Eq. (A-13) by making use of Eq. (A-4) to find $A_{\nu}^{\mu}(\gamma) \approx 1$, and in turn from Eq. (A-10)

$$\kappa_{\nu}^{\mu}(\gamma) \approx \frac{1}{2} \left(\frac{\gamma}{4}\right)^{\nu} \Gamma(1 + \nu - \mu) e^{\pi i \nu} \frac{\Gamma(\frac{1}{2} - \nu)}{\Gamma(\frac{3}{2} + \nu)}, \quad (\text{A-14})$$

and similarly for $\kappa_{-\nu-1}^{\mu}(\gamma)$ with ν replaced by $-\nu-1$. Substitution of this expression for the joining factors into Eq. (A-13) and use of the reflection formula for the gamma function yields the approximate even and odd eigenvalue relations

$$\left(\frac{\gamma}{4}\right)^{2\nu+1} \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu - \nu)} \left[\frac{\Gamma(\frac{1}{2} - \nu)}{\Gamma(\frac{3}{2} + \nu)} \right]^2 + \frac{\left[\frac{\cos}{\sin} \right] \frac{1}{2} \pi(\mu - \nu)}{\left[\frac{\sin}{\cos} \right] \frac{1}{2} \pi(\mu + \nu)} = 0, \quad \text{}$$

respectively.

Defining $\nu = -\frac{1}{2} + \frac{1}{2}iu$, $u = (-4\lambda - 1)^{1/2}$ (because in the physical problem $\lambda < 0$), together with $\theta_1 = \arg \Gamma(\frac{1}{2} + \mu + \frac{1}{2}iu)$, $\theta_2 = \arg \Gamma(1 + \frac{1}{2}iu)$, we have

$$\exp i \left\{ u \log \frac{\gamma}{4} + 2 \theta_1 - 4 \theta_2 \right\} = \frac{\cos \frac{1}{2} \pi \left(\mu \pm \frac{1}{2} - \frac{1}{2} i u \right)}{\cos \frac{1}{2} \pi \left(\mu \pm \frac{1}{2} + \frac{1}{2} i u \right)}$$

$$= \exp i \left[\pi \left(\mu \pm \frac{1}{2} \right) - 2 \phi_{e,o} \right] ,$$

where

$$\phi_{e,o} = \pm \arctan \frac{\cos \pi \mu e^{-\pi u/2}}{1 \mp \sin \pi \mu e^{-\pi u/2}} ,$$

with $|\phi_{e,o}| < \frac{1}{2} \pi$ for $u > 0$ and the subscripts e,o referring to even and odd. The eigenvalues are then given by

$$\gamma_{e,o}^{(n)} \approx 4 \exp \frac{2}{u} \left(2 \theta_2 - \theta_1 + \frac{1}{2} \pi \mu - \phi_{e,o} - \left(n \mp \frac{1}{4} \right) \pi \right) , \quad (A-15)$$

where n is an integer.

Although the general case of finite μ may be applicable to a toroidal treatment of the RFP problem (with $\mu \neq 0$ resulting from finite $\nabla \cdot \tilde{v}$), we now confine our present discussion to the case $\mu = 0$. We hope a more complete discussion of the eigenvalue relation Eq. (A-13) for nonzero μ in a future publication. When $\mu = 0$, $\phi_{e,o} = \pm \arctan (e^{-\pi u/2})$ and because $\theta_1, \theta_2 \rightarrow 0$ as $u \rightarrow 0$, we see that for $\gamma \rightarrow 0$ we require $u \rightarrow 0$ with the admissible values of n being $n = 1, 2, \dots$. The eigenvalues $\gamma_{e,o}^{(n)}$, $n = 1, 2, \dots$ then form a decreasing sequence of values with zero as limit point. The largest eigenvalues (corresponding to the greatest growth rates in the physical problem) correspond to $n = 1$.

In the limit $u \rightarrow 0$, we find for $\mu = 0$

$$\gamma_{e,o}^{(n)} \approx 16 e^{-\tilde{\gamma} \pm \pi/2} e^{-2n\pi/u}, \quad n = 1, 2, \dots \quad (A-16)$$

where $\bar{\gamma} = 0.5772\dots$ is Euler's constant. For u somewhat larger than unity (but still such that $\gamma^2 \ll 1$), we have approximately for $\mu = 0$ $\theta_1 \sim \frac{1}{2}u \log \frac{1}{2}u - \frac{1}{2}u$, $\theta_2 \sim \frac{1}{2}u \log \frac{1}{2}u - \frac{1}{2}u + \frac{\pi}{4}$, so that

$$\gamma_{e,0}^{(n)} \approx \frac{2u}{e} \exp \left[-\frac{2\phi_{e,0}}{u} \pm \frac{\pi}{2u} \right] \exp \left[-2\left(n - \frac{1}{2}\right) \frac{\pi}{u} \right], \quad n = 1, 2, \dots \quad (\text{A-17})$$

It will be seen that the largest eigenvalue is the even eigenvalue $\gamma_e^{(1)}$.

The eigenvalue relation Eq. (A-15) for the even eigenvalues is in agreement with that given in Refs. 9 and 10 when $\mu = 0$. This is because the small γ^2 -approximation employed in Eq. (A-14) consists of taking only the first term in the expansions of $S_v^{\mu(3)}(ix; \gamma)$ and its continuation Eq. (A-11).

IV. THE EIGENVALUE RELATION FOR $\gamma^2 \gg 1$

Putting $x = \sinh \zeta$, $y = (\text{sech } \zeta)^{1/2} w(\zeta)$, where $z = ix$, Eq. (A-1) transforms into

$$\frac{d^2 w}{d\zeta^2} + \left[-\lambda - \gamma^2 \cosh^2 \zeta + \left(\mu^2 - \frac{1}{4} \right) \text{sech}^2 \zeta \right] w = 0.$$

For large values of γ^2 , we expect the solution $w(\zeta)$ to be highly damped away from $\zeta = 0$, and we have approximately

$$\frac{d^2 w}{d\zeta^2} + \left[\left(-\lambda - \frac{1}{4} - \gamma^2 + \mu^2 \right) - \left(\gamma^2 + \mu^2 - \frac{1}{4} \right) \zeta^2 \right] w = 0.$$

Expressing this equation in the canonical form of Weber's equation by putting $\xi = 2^{1/2} \left(\gamma^2 + \mu^2 - \frac{1}{4} \right)^{1/4} \zeta$, we find

$$\frac{d^2 w}{d\xi^2} + \left[A - \frac{1}{4} \xi^2 \right] w = 0$$

and

$$A = \frac{\frac{1}{2} \left(-\lambda - \frac{1}{4} - \gamma^2 + \mu^2 \right)}{\left(\gamma^2 + \mu^2 - \frac{1}{4} \right)^{1/2}} .$$

The lowest eigenvalue for even solutions decaying at $\xi = \pm \infty$ is consequently $A = \frac{1}{2}$ corresponding to the solution $w(\xi) = \exp \left(-\frac{1}{4}\xi^2 \right)$. Hence we have the approximate eigenvalue relation when $\gamma^2 \gg 1$,

$$-\lambda = \gamma^2 + \gamma - \mu^2 + \frac{1}{2} + O(\gamma^{-1}) . \quad (\text{A-18})$$

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