HA. 835

ANL-79-104

ANL-79-104

197
3-12-80

MASTER

**FINITE-ELEMENT METHOD
FOR ABOVE-CORE STRUCTURES**

by

J. M. Kennedy and T. B. Belytschko

BASE TECHNOLOGY

U of C-AUA-USDOE

ARGONNE NATIONAL LABORATORY, ARGONNE, ILLINOIS

**Prepared for the U. S. DEPARTMENT OF ENERGY
under Contract W-31-109-Eng-38**

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

The facilities of Argonne National Laboratory are owned by the United States Government. Under the terms of a contract (W-31-109-Eng-38) among the U. S. Department of Energy, Argonne Universities Association and The University of Chicago, the University employs the staff and operates the Laboratory in accordance with policies and programs formulated, approved and reviewed by the Association.

MEMBERS OF ARGONNE UNIVERSITIES ASSOCIATION

The University of Arizona	The University of Kansas	The Ohio State University
Carnegie-Mellon University	Kansas State University	Ohio University
Case Western Reserve University	Loyola University of Chicago	The Pennsylvania State University
The University of Chicago	Marquette University	Purdue University
University of Cincinnati	The University of Michigan	Saint Louis University
Illinois Institute of Technology	Michigan State University	Southern Illinois University
University of Illinois	University of Minnesota	The University of Texas at Austin
Indiana University	University of Missouri	Washington University
The University of Iowa	Northwestern University	Wayne State University
Iowa State University	University of Notre Dame	The University of Wisconsin-Madison

NOTICE

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government or any agency thereof, nor any of their employees, make any warranty, express or implied, or assume any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represent that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, mark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Printed in the United States of America
Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161

NTIS price codes
Printed copy: A04
Microfiche copy: A01

ANL-79-104

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

FINITE-ELEMENT METHOD
FOR ABOVE-CORE STRUCTURES

by

J. M. Kennedy and T. B. Belytschko*

Reactor Analysis and Safety Division

December 1979

DISCLAIMER

This book was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

*Northwestern University

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT.....	7
I. INTRODUCTION.....	7
II. MATHEMATICAL FORMULATION.....	10
A. Coordinate Systems and Nodes.....	10
B. Nomenclature and Coordinate Transformations.....	11
C. Equations of Motion.....	13
D. Elements: Corotational Procedure.....	14
E. Spring Element.....	15
F. Beam Element.....	17
G. Rigid-body Element.....	21
III. TIME INTEGRATION AND STABILITY.....	22
IV. RESULTS.....	29
A. Bailey Problem.....	29
B. Snap-through Problem.....	30
C. Elastic Response of UIS.....	34
D. Comparisons with SRI Scale-model Tests.....	41
APPENDIX: Stability Analysis of Newmark β Method with Damping..	45
ACKNOWLEDGMENTS.....	49
REFERENCES.....	49

LIST OF FIGURES

<u>No.</u>	<u>Title</u>	<u>Page</u>
1.	Loop-type Liquid Metal Fast Breeder Reactor	8
2.	Coordinate Systems and Nomenclature.	11
3.	Beam Element.	17
4.	Elastic Curved Beam	29
5.	Deflected Shape for Axially Loaded Curve Beam: (Case 1-1). . . .	30
6.	Deflected Shape for Axially Loaded Curve Beam: (Case 1-2). . . .	30
7.	Geometry of Shallow Circular Arch.	30
8.	Humphreys' Results for Dynamic Buckling of a Simply Supported Shallow Arch (Case 2-1)	31
9.	Results for Ten-element Model for Dynamic Buckling of a Simply Supported Shallow Arch, $p_0 = 0.200$ to 0.240 (Case 2-1)	32
10.	Results for Twenty-element Model for Dynamic Buckling of a Simply Supported Shallow Arch, $p_0 = 0.200$ to 0.240 (Case 2-1)	32
11.	Results for Ten-element Model for Dynamic Buckling of a Simply Supported Shallow Arch, $p_0 = 0.240$ to 0.270 (Case 2-1)	32
12.	Results for Twenty-element Model for Dynamic Buckling of a Simply Supported Shallow Arch, $p_0 = 0.240$ to 0.270 (Case 2-1)	32
13.	Humphreys' Results for Dynamic Buckling of a Clamped Shallow Arch (Case 2-2).	33
14.	Results for Twenty-element Model for Dynamic Buckling of a Clamped Shallow Arch (Case 2-2).	33
15.	Hoff's Results for Midpoint Column Deflection	34
16.	Hoff's Results for Compressive Force in Column	34
17.	One-quarter-symmetry Model	35
18.	Symmetric and Asymmetric Buckling Shapes of Support Columns.	36
19.	Full Model	36
20.	One-quarter-symmetry Model with Head Cover and Holddown Studs	36
21.	Ten-element Results for Midpoint Column Deflection.	37
22.	Ten-element Results for Compressive Force in Column.	38
23.	Twenty-element Results for Midpoint Column Deflection.	39
24.	Twenty-element Results for Compressive Force in Column.	39

LIST OF FIGURES

<u>No.</u>	<u>Title</u>	<u>Page</u>
25.	Comparison of Midpoint Column Deflection for Cases 3-4a and -9a	40
26.	Comparison of Compressive Force in Column for Cases 3-4a and -9a	40
27.	Schematic of SRI Experimental Apparatus for UIS Column Tests .	41
28.	Stress-Strain Curve for UIS Columns	42
29.	Pressure History for SRI Test ACS 4	42
30.	Pressure History for SRI Test ACS 3	42
31.	Column Axial Displacement: 661-MW·s Simulated Energy Release	43
32.	Column Axial Displacement: 992-MW·s Simulated Energy Release	43
33.	Deformed Shapes for SRI Tests ACS 2 and ACS 4.	44
34.	Deformed Shapes for SRI Tests ACS 3 and ACS 5.	44
35.	Deformed Shape for Finite-element Model: 661-MW·s Simulated Energy Release.	44
36.	Deformed Shape for Finite-element Model: 992-MW·s Simulated Energy Release	44

LIST OF TABLES

<u>No.</u>	<u>Title</u>	<u>Page</u>
I.	Elastic Curved Beam	29
II.	Elastic Shallow Circular Arch	31
III.	Dimensions and Material Properties.	35
IV.	Upper-internal-structure Models; Constant-velocity Loading of UIS	37
V.	Response of UIS Columns to Simulated Energy Releases.	43

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

FINITE-ELEMENT METHOD FOR ABOVE-CORE STRUCTURES

by

J. M. Kennedy and T. B. Belytschko

ABSTRACT

Three-dimensional finite-element models for the treatment of the nonlinear, transient response of a fast breeder reactor's above-core structures are described. For purposes of treating arbitrarily large rotations, node orientations are described by unit vectors and the deformable elements are treated by a corotational formulation in which the coordinate system is embedded in the elements. Deformable elements may be connected either to nodes directly or through rigid bodies. The time integration is carried out by the Newmark β method. These features have been incorporated to form the finite-element program SAFE/RAS (Safety Analysis by Finite Elements/Reactor Analysis and Safety Division). Computations are presented for semianalytical comparisons, simple scoping studies, and Stanford Research Institute (SRI) test comparisons.

I. INTRODUCTION

The upper internals, located above the core and below the head cover for a loop-type Liquid Metal Fast Breeder Reactor (LMFBR) shown in Fig. 1, may play a significant role in the hydrodynamics of an energy excursion by mitigating its response. A finite-element program, called SAFE/RAS (Safety Analysis by Finite Elements/Reactor Analysis and Safety Division), has been developed for this investigation. The upper internal structure (UIS) is essentially a massive, perforated rigid body connected to the reactor head by support columns with a cylindrical cross section. The model developed for the upper internals is a finite-element model capable of treating material nonlinearities and geometric nonlinearities arising from large displacements when subjected to dynamic loads. The time integration is carried out by the Newmark β method,¹ which is similar in stability and convergence properties to the central-difference method.² The finite-element model consists of three types of elements:

1. Spring elements.
2. Beam elements.
3. Rigid-body element or rigid linkage.

Fig. 1. Loop-type Liquid Metal Fast Breeder Reactor. ANL Neg. No. 900-79-682.

Spring elements are deformable elements with only axial stiffness. These elements have been incorporated for the purpose of modeling the hold-down bolts for the reactor head.

The beam elements have axial stiffness, torsional stiffness, and bending stiffnesses in two planes. These quantities may be related directly to deformation modes of the beam, or they may be related to the deformation modes indirectly through strain-displacement and stress-strain laws. Although the latter may at first appear a more rigorous approach to the treatment of elastic-plastic response, it is expected that the columns supporting the upper internals may undergo substantial changes in the shapes of their cross section; in this case it will be more convenient to treat the behavior of the element stiffness in terms of moment-curvature relations which account for the changes in flexural rigidity with change in the cross section.

The rigid-body elements have been incorporated in order to treat the upper internals and perhaps the head cover. Because of the very high stiffness of these structures, if they were treated as deformable, the stability limits for the explicit integration law would require a very small time step. Since the deformation of the upper internals should be negligible, it has therefore been decided to treat them as rigid bodies.

In the following section, we will describe the computational methods developed for this structural model. Each element is described in detail. The time-integration procedure is then described. It is an explicit Newmark method, and stability limits and methods for estimating these stability limits are given.

The concluding section describes some sample computations, which were made to validate the method. The first group of problems were elastic problems with large displacements for which semianalytical results were available. These results enabled us to check the capability of the program to handle geometric nonlinearities. The method performed quite well, though at times we encountered small discrepancies. These were attributed to the sensitivity of nonlinear problems.

Scoping studies were then made for a model of the above-core structures. The most important mechanism in their response was found to be the buckling of the support columns. Although we originally were concerned with the possibility of asymmetric buckling modes, the scoping studies show that, because of the large mass of the upper internals, the buckling pattern of the system is symmetric. In the scoping studies, our attention was confined to elastic-response calculations in which we were able to make comparisons with analytic solutions; these comparisons show that the analytical results can be duplicated quite well with a degree of nodalization that can be computed with acceptable CPU times. This finding was quite satisfying, inasmuch as the final objective of this program is to couple the upper-internals model with the ICECO code.³ If very fine nodalization were required in the upper-internals model, the computational expense associated with such coupled analysis would have been exorbitant. The elastic-plastic response behavior of the model has also been compared to the Stanford Research Institute (SRI) tests simulating highly energetic core-disruptive accidents (CDA's). The model predicts the magnitude of the axial displacement quite well.

II. MATHEMATICAL FORMULATION

A. Coordinate Systems and Nodes

In this finite-element model, each node is associated with six degrees of freedom--three translations and three rotations--so that complete three-dimensional structural-response phenomena can be studied. For the purpose of implementing the rigid-body elements, two types of nodes are defined:

1. Master nodes (primary nodes).
2. Slave nodes (secondary nodes).

Each slave node must be associated with a primary node and is considered to be rigidly attached to it; thus no degrees of freedom are associated with slave nodes. Each master node may be associated with an arbitrary number of slave nodes.

The displacements of the model are described by the displacements and rotations of the primary nodes. We denote the original position of each node I by X_{iI} , $i = 1$ to 3 representing the x , y , and z coordinates, respectively. The current position of a node, x_{iI} is then obtained by adding the displacements u_{iI} . That is,

$$x_{iI} = X_{iI} + u_{iI}. \quad (1)$$

The rotation of the master node is described by the orientation of three orthogonal unit vectors $\vec{b}_{1I}$, $\vec{b}_{2I}$, and $\vec{b}_{3I}$, which rotate with the node.

To describe the system, we will define three types of coordinate systems:

1. Fixed, global set of coordinates (x, y, z) , or x_i .
2. Body coordinates $(\bar{x}, \bar{y}, \bar{z})$; a set of body coordinates is associated with each primary node so that $\bar{x}$, $\bar{y}$, and $\bar{z}$ coincide with the unit vectors $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$, respectively, for each node.
3. Element coordinates $(\hat{x}, \hat{y}, \hat{z})$; a set of element coordinates is associated with each element, and the element coordinates are defined so that they rotate and translate with the element in a manner to be specified later. The $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes are associated with unit vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$, respectively, for each element.

These coordinate systems are shown in Fig. 2.

Fig. 2

Coordinate Systems and Nomenclature.
An element connected to a master node
at right, slave node at left, is shown.
ANL Neg. No. 900-79-469.

B. Nomenclature and Coordinate Transformations

The unit vectors $\vec{b}_{iI}$ and $\vec{e}_i$ immediately define the transformation laws for vector components between the various coordinate systems. Thus, if we consider a vector $\vec{v}$ with global components (v_x, v_y, v_z) , body-coordinate components $(\bar{v}_x, \bar{v}_y, \bar{v}_z)$ and element-coordinate components $(\hat{v}_x, \hat{v}_y, \hat{v}_z)$, we have the transformation

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \begin{bmatrix} b_{1x} & b_{2x} & b_{3x} \\ b_{1y} & b_{2y} & b_{3y} \\ b_{1z} & b_{2z} & b_{3z} \end{bmatrix} \begin{Bmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{v}_z \end{Bmatrix} = [GB]\{\bar{v}\}, \quad (2)$$

where b_{ix} , b_{iy} , and b_{iz} are the global components of the body vector i . The notation $[GB]$ for the transformation between global- and body-coordinate systems is designed to serve as a mnemonic device to simplify the understanding of subsequent equations. Similarly, the transformation of vector components in the global and element systems is given by

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \begin{bmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{bmatrix} \begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{Bmatrix} = [GE]\{\hat{v}\}, \quad (3)$$

where e_{ix} , e_{iy} , and e_{iz} are the global components of the element vector i . Since the matrices $[GB]$ and $[GE]$ are orthogonal, it also follows that

$$\{\bar{v}\} = [GB]^T\{v\} \quad (4a)$$

and

$$\{\hat{v}\} = [GE]^T\{v\}. \quad (4b)$$

Transformations between element and body components of a vector can then be obtained from the above to be

$$\{\hat{v}\} = [GE]^T[GB]\{\bar{v}\} = [EB]\{\bar{v}\}. \quad (5)$$

The above equations can be written in indicial notations as

$$\left. \begin{aligned} v_i &= \overline{GB}_{ij}\bar{v}_j = b_{ij}\bar{v}_j; & \bar{v}_i &= \overline{GB}_{ji}v_j = b_{ji}v_j; \\ v_i &= \overline{GE}_{ij}\hat{v}_j = e_{ij}\hat{v}_j; & \hat{v}_i &= \overline{GE}_{ji}v_j = e_{ji}v_j. \end{aligned} \right\} \quad (6)$$

Henceforth, the coordinate system in which a vector is expressed will always be designated by superscript bars designating nodal (body) components of a vector and hats designating element components of a vector. Thus, the components of a vector v in terms of the body coordinates of a rigid body I are denoted by $\bar{v}_{iI}$, with i equal 1 to 3 denoting $\bar{x}$, $\bar{y}$, and $\bar{z}$, respectively. Furthermore, the three Cartesian components of a vector in any of these systems will often be written as a column matrix, as illustrated in Eq. 2, for they are treated in that form in the computer program.

Time derivatives are denoted by superscript dots, so that velocities of the nodes are denoted by $\dot{u}_{iI}$ and the accelerations of the nodes are denoted by $\ddot{u}_{iI}$. The angular velocities and angular accelerations of a node are always expressed in terms of nodal coordinate components and are designated by $\bar{\omega}_{iI}$ and $\bar{\alpha}_{iI}$, respectively. The forces and moments at a node are denoted by f_{iI} and m_{iI} , respectively, and may be divided into externally applied forces and moments, f_{iI}^{ext} and m_{iI}^{ext} , and forces and moments due to the resistance of the deformable elements, f_{iI}^{int} and m_{iI}^{int} .

Three unit vectors, $\bar{b}_{1I}$, $\bar{b}_{2I}$, and $\bar{b}_{3I}$, are stored for each master node to describe the orientation of the node. This method thus employs nine numbers (three components of three vectors) to describe the three rotational degrees of freedom. Although this may appear at first to be a waste of computer storage, note that the alternative description of the orientations of the body nodes by Euler angles has serious shortcomings:

1. The Euler angles are not linearly independent for all their values.
2. The generalized moments corresponding to Euler angles are not easily interpreted in a physical sense.
3. The equations of motion and the transformations between body and global coordinates in terms of Euler angles are complex and computationally more involved than those developed here because they involve the evaluation of more trigonometric functions.

It would be possible to store only three of the nine components of three unit vectors, with the remaining components computed from the orthogonality and normality conditions of the unit vectors. This would be trivial for the three components of the third unit vector. However, the remaining three components would then have to be determined from equations involving square roots, and the signs of these components could not be readily determined. This leads to considerable programming complexity.

The inertial properties of the rigid bodies are described by the translational masses of the master nodes, ρ_I , and the principal moments of inertia of the rigid body (master node) $\bar{I}_{xI}$, $\bar{I}_{yI}$, and $\bar{I}_{zI}$. The angular momenta of the nodes are then given by

$$\bar{L}_{iI} = \bar{I}_{iI} \bar{\omega}_{iI} \text{ (no sums).} \quad (7)$$

The element nodal variables are extracted from the global nodal variables in the usual manner by a connectivity matrix $\ell_{AI}^{(e)}$, so that

$$u_{iA}^{(e)} = \ell_{AI}^{(e)} u_{iI}, \quad (8)$$

where $\ell_{AI}^{(e)} = 1$ if the A-th node of element e corresponds to primary node I ; otherwise, $\ell_{AI}^{(e)} = 0$. We will use the superscript e whenever there is any possibility of confusion between element and global quantities; otherwise, it will be omitted.

The matrix multiplication in Eq. 7 is not actually performed, but the above equation is useful for the theoretical developments that follow.

C. Equations of Motion

The formulation used here is basically a finite-element formulation, with the equations of motion developed from the principle of virtual work with the inertial forces included in a d'Alembert sense.

The equations of motion are then

$$\rho_I \ddot{u}_{iI} = f_{iI}^{\text{ext}} - f_{iI}^{\text{int}}, \quad (9a)$$

$$\bar{I}_{iI} \ddot{\omega}_{iI} + (\bar{I}_{kI} - \bar{I}_{jI}) \bar{\omega}_{jI} \bar{\omega}_{kI} = m_{iI}^{\text{ext}} - m_{iI}^{\text{int}} \quad (9b)$$

$$(i = 1, 2, 3; j = 2, 3, 1; k = 3, 1, 2),$$

$$f_{iI}^{\text{int}} = \sum_e \ell_{IA}^{(e)} f_{iA}^{(e)\text{int}}, \quad (9c)$$

and

$$m_{iI}^{int} = \sum_e \ell_{IA}^{(e)} m_{iA}^{(e)int}, \quad (9d)$$

where $f_{iA}^{(e)int}$ and $m_{iA}^{(e)int}$ are the element internal nodal forces and moments, respectively.

D. Elements: Corotational Procedure

The element relations are developed in a corotational framework which was presented for two-dimensional problems by Belytschko and Hsieh⁴ and for three-dimensional problems by Belytschko et al.⁵ In this procedure, a rigid, Cartesian-coordinate system $(\hat{x}, \hat{y}, \hat{z})$ is embedded in each element so that it rotates with the element. The displacement field of each element is then decomposed into rigid-body displacements and deformation displacements

$$u_i = u_i^{rig} + u_i^{def}. \quad (10)$$

It can then be shown that a corotational strain can be defined by

$$\hat{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial \hat{u}_i^{def}}{\partial \hat{x}_j} + \frac{\partial \hat{u}_j^{def}}{\partial \hat{x}_i} \right). \quad (11)$$

The important feature of a corotational formulation is the linearity of this strain-displacement equation. Other treatments of geometric nonlinearities, i.e., the nonlinearities arising from large displacements, usually use nonlinear strain-displacement relations. This adds significantly to the computational expense. The strain corresponds to the stretch matrix of continuum mechanics, so the normal strains correspond to "engineering" strains. For example,

$$\hat{\epsilon}_{11} = \frac{\ell_1 - \ell_1^0}{\ell_1^0}, \quad (12)$$

where ℓ_1^0 and ℓ_1 are the original and deformed lengths of a line segment aligned with the $\hat{x}_1$ coordinate.

The corotational stress $\hat{\sigma}_{ij}$ is defined as the conjugate of the corotational strain, so that

$$\dot{W}^{int} = \int_V \hat{\sigma}_{ij} \dot{\hat{\epsilon}}_{ij} dV. \quad (13)$$

The corotational stress in a case of uniaxial stress can be related to the Cauchy (physical) stress T_{11} by (see Belytschko and Glaum⁶)

$$\hat{\sigma}_{11} = \frac{A^{\text{new}}}{A^{\text{old}}} T_{11}. \quad (14)$$

Thus, the corotational stress in a uniaxial case is identical to engineering stress. Therefore, for any stress-strain laws used with these elements, the uniaxial stress-strain curve must be expressed in terms of engineering stress versus engineering strain.

Another feature of the corotational formulation used here is that the deformation modes are separated from the rigid-body modes. The deformation modes are nodal degrees of freedom or combinations of nodal degrees of freedom. Because of the linearity of Eq. 11, the corotational strains can be expressed linearly in terms of the deformation nodal displacements in the form

$$\{\hat{\epsilon}\} = [B]\{\hat{d}^{\text{def}}\}. \quad (15)$$

A set of conjugate nodal forces $\{\hat{f}^{\text{def}}\}$ can then be constructed so that work is given by

$$\dot{W} = \{\hat{d}^{\text{def}}\}^T \{\dot{\hat{f}}^{\text{def}}\}. \quad (16)$$

Substituting Eqs. 15 and 16 into Eq. 13 yields

$$\{\hat{d}^{\text{def}}\}^T \{\dot{\hat{f}}^{\text{def}}\} = \{\hat{d}^{\text{def}}\}^T \int_V [B]^T \{\hat{\sigma}\} dV. \quad (17)$$

Since Eq. 17 must hold for arbitrary $\{\hat{d}^{\text{def}}\}$, it follows that

$$\{\hat{f}^{\text{def}}\} = \int_V [B]^T \{\hat{\sigma}\} dV. \quad (18)$$

The advantage of using a corotational formulation with only deformation modes is that the integral in Eq. 18, which is usually evaluated numerically, is reduced in size (the matrix $[B]$ is smaller because it only relates to deformation modes) and involves only linear terms.

E. Spring Element

The spring element is used in the head-cover models for representing the hold-down bolts. It can be used for line elements wherever flexural effects are unimportant.

A spring element has two nodes, which for a generic element we will designate one and two. The corotational coordinate $\hat{x}$ is defined by always requiring it to pass through the two nodes. The $\hat{u}_x$ displacement is assumed to be linear in $\hat{x}$, so the strain is constant in the element.

The only deformation mode of the spring is the elongation δ of the element

$$\delta = l - l_0, \quad (19)$$

where l and l_0 are the current and undeformed lengths of the element. In Belytschko et al.,⁵ the formula given for δ is

$$\delta = \frac{1}{l + l_0} [2(x_{21}u_{x21} + y_{21}u_{y21} + z_{21}u_{z21}) + u_{x21}^2 + u_{y21}^2 + u_{z21}^2]. \quad (20)$$

The axial strain is then given by

$$\hat{\epsilon} = \frac{\delta}{l_0}. \quad (21)$$

The stress $\hat{\sigma}$ can be evaluated from $\hat{\epsilon}$ and its history by any constitutive law. The nodal force conjugate to $\hat{\delta}$ is the tension, $\hat{f}_{x2}$, so, by Eq. 18,

$$\begin{aligned} \hat{f}_{x2} &= \int_V \frac{1}{l} \hat{\sigma} dV_0 \\ &= A^{\text{old}} \hat{\sigma} = A^{\text{new}} T, \end{aligned} \quad (22)$$

where A is the cross-sectional area.

The remaining nodal forces are found by equilibrium and vector transformations. From equilibrium, it follows that

$$\hat{f}_{x1} = -\hat{f}_{x2} \quad (23)$$

and that

$$\hat{f}_{y1} = \hat{f}_{y2} = \hat{f}_{z1} = \hat{f}_{z2} = 0. \quad (24)$$

The global components may then be found by Eq. 3.

The corotational formulation for the spring is valid for arbitrarily large rotations and large strains. If the strains are large, however, as indicated previously, the stress-strain law must be the curve of engineering stress versus engineering strain.

F. Beam Element

The beam element is developed in terms of a corotational formulation as described by Belytschko et al.⁵ In a corotational formulation for beam element, displacements and rigid-body rotations of each element may be arbitrarily large, but the deformation of each element relative to the corotational coordinates must be small. This does not limit the formulation to small deformations, but imposes the requirement that the number of elements used in the model be sufficiently large so that the deformation of each element relative to its rigid, Cartesian corotational system be small.

Fig. 3. Beam Element. ANL
Neg. No. 900-79-468.

Standard engineering beam theory, often known as Euler-Bernoulli beam theory, will be used. This theory assumes that plane sections normal to the midline remain plane and normal. As a consequence, the midline of the beam at the nodes must rotate with the nodes; if two elements are connected in a straight line, the midline must remain continuous and continuously differentiable. Warping of the beam cross section will be neglected.

Consider a generic beam element with nodes 1 and 2 as shown in Fig. 3. The corotational coordinate system $(\hat{x}, \hat{y}, \hat{z})$ for the element is defined by requiring the $\hat{x}$ axis to always pass through the two nodes with its origin at node one. The $\hat{y}$ axis is defined by the average rigid body rotations about the $\hat{x}$ axis and the $\hat{z}$ axis is defined by its normality to the $\hat{x}$ - $\hat{y}$ plane.

The deformation displacements are then $\hat{u}_x^{\text{def}}$, $\hat{u}_y^{\text{def}}$, $\hat{u}_z^{\text{def}}$, and the rotation $\hat{\theta}_x^{\text{def}}$. From the definition of the corotational coordinate system, it follows that $\hat{u}_x^{\text{def}}$ vanishes at node 1, while $\hat{u}_y^{\text{def}}$ and $\hat{u}_z^{\text{def}}$ vanish at both nodes.

The deformation-displacement fields are chosen to be polynomials, and the lowest-order polynomials consistent with Euler-Bernoulli beam theory are used. Since the transverse displacements, $\hat{u}_y^{\text{def}}$ and $\hat{u}_z^{\text{def}}$, must be continuously differentiable, cubic polynomials are used for these deformation displacements. The axial displacement $\hat{u}_x^{\text{def}}$ and the rotation $\hat{\theta}_x^{\text{def}}$ need only be continuous, so a linear field is used. Thus,

$$\hat{u}_x^{\text{def}} = \hat{u}_{x2}^{\text{def}} \xi, \quad (25a)$$

$$\hat{u}_y^{\text{def}} = \ell(\xi - 2\xi^2 + \xi^3)\hat{\theta}_{z1} + \ell(-\xi^2 + \xi^3)\hat{\theta}_{z2}, \quad (25b)$$

$$\hat{u}_z^{\text{def}} = \ell(-\xi + 2\xi^2 - \xi^3)\hat{\theta}_{y1} + \ell(\xi^2 - \xi^3)\hat{\theta}_{y2}, \quad (25c)$$

$$\hat{\theta}_x^{\text{def}} = \hat{\theta}_{x21}\xi, \quad (25d)$$

and

$$\xi = x/\ell, \quad (25e)$$

where ℓ is the length of the element and $\hat{\theta}_{x21}$ the relative rotation between nodes two and one. The rotations $\hat{\theta}_{yI}$ and $\hat{\theta}_{zI}$ are defined by

$$\hat{\theta}_{zI} = - \frac{\partial \hat{u}_y^{\text{def}}(\hat{x}_I)}{\partial \hat{x}} \quad (26a)$$

and

$$\hat{\theta}_{yI} = \frac{\partial \hat{u}_z^{\text{def}}(\hat{x}_I)}{\partial \hat{x}}, \quad (26b)$$

$$I = 1, 2.$$

The deformation nodal displacements are then

$$\{\hat{d}^{\text{def}}\}^T = [\delta, \hat{\theta}_{x21}, \hat{\theta}_{y1}, \hat{\theta}_{z1}, \hat{\theta}_{y2}, \hat{\theta}_{z2}], \quad (27)$$

where δ is the elongation of the beam given by Eq. 20.

For computing $\hat{\theta}_{yI}$ and $\hat{\theta}_{zI}$ in the deformed configuration, nodal components of $\vec{e}_{1I}^0 \equiv \vec{r}_I$ and $\vec{e}_{2I}^0 \equiv \vec{s}_I$ in the undeformed configuration are stored for both nodes of the element. The nodal (body) components of the vectors $\vec{r}_I$ and $\vec{s}_I$ are considered invariant; therefore these vectors rotate with the nodes. Consequently, the angle between $\vec{r}$ and $\vec{e}_1$ in the deformed configuration gives the magnitude of the rotation of the midline relation to the corotational $\hat{x}$ axis of the element; if $\vec{r}$ and $\vec{e}_1$ are coincident in the deformed configuration, the deformation rotations $\hat{\theta}_y$ and $\hat{\theta}_z$ vanish for the element. For small deformations, the magnitudes of $\hat{\theta}_y$ and $\hat{\theta}_z$ may be found by

$$\hat{\theta}_y \vec{e}_2 + \hat{\theta}_z \vec{e}_3 = \vec{e}_1 \times \vec{r}. \quad (28)$$

The element components of $\vec{r}$ may be found in Eq. 4b. Expressing the right-hand side in element coordinates, we find

$$\hat{\theta}_y \vec{e}_2 + \hat{\theta}_z \vec{e}_3 = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \hat{r}_x & \hat{r}_y & \hat{r}_z \end{bmatrix} = -\hat{r}_z \vec{e}_2 + \hat{r}_y \vec{e}_3. \quad (29)$$

Thus, by equating components, we obtain

$$\hat{\theta}_y = -\hat{r}_z; \quad \hat{\theta}_z = \hat{r}_y. \quad (30)$$

The torsional deformation rotation is found by taking the cross-product of $\vec{s}_1$ with $\vec{s}_2$ and projecting the results on the unit vector along the $\hat{x}$ axis. This gives

$$\hat{\theta}_{xz1} = \vec{e}_1 \cdot (\vec{s}_1 \times \vec{s}_2) = \det \begin{bmatrix} 1 & 0 & 0 \\ \hat{s}_{1x} & \hat{s}_{1y} & \hat{s}_{1z} \\ \hat{s}_{2x} & \hat{s}_{2y} & \hat{s}_{2z} \end{bmatrix} = \hat{s}_{1y}\hat{s}_{2z} - \hat{s}_{2y}\hat{s}_{1z}. \quad (31)$$

For this case, the strains in a corotational formulation, Eq. 11, reduce to

$$\hat{\epsilon}_x = \frac{\partial \hat{u}_x^{\text{def}}}{\partial \hat{x}}. \quad (32)$$

For an Euler-Bernoulli beam, the assumption that normals to the mid-line remain straight and normal gives

$$\hat{u}_x^{\text{def}}(\hat{x}, \hat{y}, \hat{z}) = \hat{u}_x^{\text{def}}(\hat{x}, 0, 0) - \hat{y} \frac{\partial \hat{u}_z^{\text{def}}}{\partial \hat{x}} - \hat{z} \frac{\partial \hat{u}_y^{\text{def}}}{\partial \hat{x}}. \quad (33)$$

Substituting the above into Eq. 32 gives

$$\hat{\epsilon}_x = \frac{\partial \hat{u}_x^{\text{def}}(\hat{x}, 0, 0)}{\partial \hat{x}} - \hat{y} \frac{\partial^2 \hat{u}_z^{\text{def}}(\hat{x}, 0, 0)}{\partial \hat{x}^2} - \hat{z} \frac{\partial^2 \hat{u}_y^{\text{def}}(\hat{x}, 0, 0)}{\partial \hat{x}^2}. \quad (34)$$

For the purpose of determining the shear strains due to torsion, we consider polar coordinates r, θ in the $\hat{y}, \hat{z}$ plane. Then

$$2\hat{\epsilon}_{r\theta} = \frac{\partial \hat{u}_r^{\text{def}}}{\partial \theta} - \frac{\partial \hat{u}_\theta^{\text{def}}}{\partial r}. \quad (35)$$

We assume that $\hat{u}_r^{\text{def}} = 0$ and that $\hat{u}_\theta^{\text{def}}$ varies linearly with r ; this assumption is consistent only with cylindrical cross sections. Equations 25d and 35 and the above assumption, along with our neglect of warping, then give

$$\hat{\epsilon}_{r\theta} = \frac{\hat{\theta}_{x21} r}{l}. \quad (36)$$

Substituting Eqs. 25 into Eq. 34, we then obtain the strain-displacement equation for the element to be

$$\begin{Bmatrix} \hat{\epsilon}_x \\ \hat{\epsilon}_{r\theta} \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} 1 & 0 & \hat{z}(6\xi - 4) & \hat{y}(4 - 6\xi) & \hat{z}(6\xi - 2) & \hat{y}(2 - 6\xi) \\ 0 & r & 0 & 0 & 0 & 0 \end{bmatrix} \{\hat{d}^{\text{def}}\}. \quad (37)$$

The above relation defines the strain-displacement matrix $[B]$. If we use Eq. 18, it follows then that the deformation nodal forces are given by

$$\{\hat{f}^{\text{def}}\} = \int_V [B] \begin{Bmatrix} \hat{\sigma}_x \\ \hat{\sigma}_{r\theta} \end{Bmatrix} dV, \quad (38)$$

where V is the volume of the element and the deformation nodal forces are conjugate to the deformation nodal displacements in the sense that their scalar product gives work, so that

$$\{\hat{f}^{\text{def}}\}^T = [\hat{f}_{x2}, \hat{m}_{x2}, \hat{m}_{y1}, \hat{m}_{z1}, \hat{m}_{y2}, \hat{m}_{z2}]. \quad (39)$$

In order to minimize computational expense, it is often desirable to circumvent the volume integration of Eq. 38 by working in terms of generalized stresses

$$\hat{m}_x = \int_A r \hat{\sigma}_{r\theta} dA, \quad (40a)$$

$$\hat{m}_y = \int_A z \hat{\sigma}_x dA, \quad (40b)$$

and

$$\hat{m}_z = \int_A y \hat{\sigma}_x dA. \quad (40c)$$

These generalized stresses can then be directly related to the twist and curvatures.

G. Rigid-body Element

The rigid-body element or rigid linkage serves as a means of modeling comparatively inflexible portions of a structure. The availability of this feature is particularly important when the equations of motion are integrated by an explicit method, since the size of the stable time step is inversely proportional to the stiffness. Thus, including very stiff portions of the model as deformable would drastically reduce the stable time step. On the other hand, the use of rigid linkage has no adverse effects on the time step.

The rigid linkage can interconnect an arbitrary number of slave nodes with a master node. However, each rigid linkage may have only one master node, which must coincide with the center of mass of the rigid body.

To develop the modifications of the governing equations due to a rigid linkage, consider a rigid linkage with master node I and a generic slave node A. We note that the angular velocities of all points on a rigid body are identical, so that

$$\bar{\omega}_{jA} = \bar{\omega}_{jI}. \quad (41)$$

The nodal velocities of node A are given by

$$\dot{\bar{u}}_{iA} = \dot{\bar{u}}_{iI} + \Omega_{ij} \bar{\omega}_{jI}, \quad (42)$$

where

$$[\Omega] = \begin{bmatrix} 0 & \bar{z}_{IA} & -\bar{y}_{IA} \\ -\bar{z}_{IA} & 0 & \bar{x}_{IA} \\ \bar{y}_{IA} & -\bar{x}_{IA} & 0 \end{bmatrix} \quad (43)$$

and double subscripts, as usual, designate a difference in nodal coordinates.

It can be then be shown by the principle of virtual work that the total force at master node I is given by

$$f_{iI} = \sum f_{iA}, \quad (44)$$

where the summation is taken over all slave nodes. The moments at the master nodes are given by

$$\bar{m}_{iI} = \sum_A (\bar{m}_{iA} + \Omega_{ji} \bar{f}_{jA}). \quad (45)$$

Thus, the nodal forces at the master node are simply the sum of the nodal forces at the slave nodes, while in the nodal moments, an additional moment results because of the moment arm between the master and slave nodes.

III. TIME INTEGRATION AND STABILITY

In this investigation, time integration was performed by the Newmark β method with $\beta = 0$, which is an explicit time-integration method with stability and convergence properties that are almost identical to the central-difference method. The Newmark β method is sometimes more convenient than the central-difference method because all nodal variables and their time derivatives, i.e., velocities and accelerations, are defined at each time step, whereas the central-difference method defines velocities only at times equidistant between time steps, often called half-time steps.

In the Newmark β formulas, the velocities and displacements are updated in each time step by

$$u_{iI}^{j+1} = u_{iI}^j + \Delta t \dot{u}_{iI}^j + \frac{1}{2} \Delta t^2 \ddot{u}_{iI}^j \quad (46a)$$

and

$$\dot{u}_{iI}^{j+1} = \dot{u}_{iI}^j + \frac{1}{2} \Delta t (\ddot{u}_{iI}^j + \ddot{u}_{iI}^{j+1}), \quad (46b)$$

where Δt is the time step and the superscripts denote the time-step number.

For the rotational degrees of freedom, Eqs. 46 cannot be used directly because the orientations are described by the nodal (body) vectors $\vec{b}_{iI}$, and the rates of these vectors are not equivalent to the angular accelerations and velocities.

The counterpart of Eqs. 46 for the unit vector $\vec{b}$ is

$$\vec{b}_i^{j+1} = \vec{b}_i^j + \Delta t \frac{d\vec{b}_i^j}{dt} + \frac{1}{2} \Delta t^2 \frac{d^2 \vec{b}_i^j}{dt^2} \quad (47)$$

and

$$\frac{d\vec{b}_i^{j+1}}{dt} = \frac{d\vec{b}_i^j}{dt} + \frac{1}{2} \Delta t \left(\frac{d^2 \vec{b}_i^j}{dt^2} + \frac{d^2 \vec{b}_i^{j+1}}{dt^2} \right). \quad (48)$$

From vector analysis, it follows that

$$\frac{d\vec{b}_i}{dt} = \vec{\omega} \times \vec{b}_i \quad (49)$$

and

$$\frac{d^2 \vec{b}_i}{dt^2} = \vec{\omega} \times (\vec{\omega} \times \vec{b}_i) + (\dot{\vec{\omega}} \times \vec{b}_i). \quad (50)$$

To obtain the updated components of the unit vectors, we temporarily fix the $\bar{x}_i$ coordinates and consider a particular unit vector; we then dot Eq. 47 with the unit vector corresponding to the component of $\bar{b}_i$ to be updated, and substitute Eqs. 49 and 50 into the resulting expressions. For example, the updated $\bar{x}$ component of $\bar{b}_3$ is found by letting $i = 3$ in Eq. 47 and taking the scalar product with $\bar{b}_1$. After some simplification, this yields

$$\bar{b}_{3x}^{j+1} = \Delta t \bar{\omega}_y^j + \frac{1}{2} \Delta t^2 (\bar{\omega}_x^j \bar{\omega}_z^j + \bar{\alpha}_y^j). \quad (51)$$

Similarly,

$$\bar{b}_{3y}^{j+1} = -\Delta t \bar{\omega}_x^j + \frac{1}{2} \Delta t^2 (\bar{\omega}_y^j \bar{\omega}_z^j - \bar{\alpha}_x^j) \quad (52a)$$

and

$$\bar{b}_{1y}^{j+1} = \Delta t \bar{\omega}_z^j + \frac{1}{2} \Delta t^2 (\bar{\omega}_x^j \bar{\omega}_y^j + \bar{\alpha}_z^j). \quad (52b)$$

Normality and orthogonality of the unit vectors are used to find the remaining components. The rotations during a time step are assumed to be small so that second order-terms can be neglected. From normality, it follows that

$$\bar{b}_{3z}^{j+1} = \left[1 - (\bar{b}_{3x}^{j+1})^2 - (\bar{b}_{3y}^{j+1})^2 \right]^{1/2} \quad (53)$$

and if we assume that $\bar{b}_{1x}^{j+1} \approx 1$, orthogonality yields

$$\bar{b}_{1z}^{j+1} = -(\bar{b}_{3x}^{j+1} + \bar{b}_{1y}^{j+1} \bar{b}_{3y}^{j+1}) / \bar{b}_{3z}^{j+1} \quad (54)$$

Finally, the component $\bar{b}_{1x}^{j+1}$ can be found by the normality condition

$$\bar{b}_{1x}^{j+1} = \left[1 - (\bar{b}_{1y}^{j+1})^2 - (\bar{b}_{1z}^{j+1})^2 \right]^{1/2} \quad (55)$$

The vector $\bar{b}_2$ is then found by the cross-product.

The components of the $\bar{b}_1$ and $\bar{b}_3$ vectors given previously are in terms of the $\bar{x}_i$ coordinates (nodal coordinates) at time step j . These components are then transformed to global components by the transformation relationship, Eq. 2.

In explicit time integration, unless the time step is sufficiently small, numerical instabilities will render the results worthless. Although stability analyses cannot be performed for the fully nonlinear case, experience indicates that a linearized analysis gives useful estimates of the stable time step. The destabilizing effects of nonlinearities can usually be overcome by reducing

the time step that is used by 10-20% below the linear-stability limit. Although this does not always ensure a stable computation, stability can be checked after the computation by examining the energy balance as described at the end of this section.

To perform a linearized stability analysis, we begin with the linearized homogeneous equations of motion

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = 0. \quad (56)$$

The undamped frequencies of this equation are given by the eigenvalue problem

$$[K]\{X\} = \omega^2[M]\{X\}. \quad (57)$$

Transformation of Eq. 56 to the basis of the eigenvectors (natural modes) of Eq. 57 gives a set of uncoupled equations

$$\ddot{x} + 2\mu\omega\dot{x} + \omega^2x = 0, \quad (58)$$

where μ is the fraction of critical damping in the frequency ω . The equation of interest in stability analysis is that associated with the highest frequency, $\omega_{\max}$. A Fourier stability analysis of Eq. 58 with the Newmark β operator ($\beta = 0$) gives the stability limit

$$\Delta t \leq \frac{2}{\omega_{\max}} [(1 + \mu^2)^{1/2} - \mu], \quad (59)$$

where μ is the fraction of critical damping of the maximum frequency $\omega_{\max}$. This stability limit is identical to that of the central-difference method, but is based on the use of a backwards difference formula for the velocity of the form

$$\dot{u}^{j+1} = \frac{1}{\Delta t}(u^{j+1} - u^j). \quad (60)$$

Equations 46, i.e., a direct application of the Newmark β method, yield

$$\dot{u}^{j+1} = \frac{1}{2\Delta t}(u^{j+1} - u^{j-1}), \quad (61)$$

which has less desirable stability properties. This is elaborated in the appendix.

No provisions are made in the program for directly computing the maximum eigenvalue $\omega_{\max}$, for it would require considerable programming and computer time, and it would only be of marginal usefulness in nonlinear

problems because $[K]$ and $[M]$ can change dramatically with time. Instead, we have developed formulas for estimating $\omega_{\max}$. These are based on Rayleigh's theorem, which allows us to bound the maximum frequency of the system by the maximum frequency among all of the elements of the system (see Hughes et al.⁷), i.e.,

$$\omega_{\max} \leq \text{maximum} \left(\omega_{\max}^{(e)} \right); e = 1 \text{ to number of elements,} \quad (62)$$

where $\omega_{\max}^{(e)}$ is the maximum frequency of each element in the system.

The maximum frequency of any of the elements can be obtained exactly or approximated sufficiently accurately for our purposes. First we will consider a spring element with a cross-sectional area A , elastic modulus E , and length ℓ . The element stiffness and mass matrices, $[k]$ and $[m]$, respectively, are given by

$$[k] = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } [m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad (63)$$

where m_1 and m_2 are the lumped masses at the nodes. The maximum frequency, i.e., the maximum eigenvalue of $[k]\{X\} = \omega^2[m]\{X\}$, is given by

$$\omega_{\max}^2 = \frac{4AE}{(m_1 + m_2)\ell}. \quad (64)$$

Whenever the system consists of a system of springs connecting rigid bodies, all springs in parallel between any two rigid bodies should be summed to obtain the total spring constant between the two rigid bodies. For the purpose of using Eq. 64, half the lumped masses at each node of the spring element should be used. Although in principle the entire lumped masses at the two nodes of an element could be used to obtain a frequency for that element, note that the same mass cannot be ascribed to two elements in estimating the maximum frequency. Therefore, if all the mass is used in estimating the frequency for any one element, the adjacent elements will have no mass. This will increase the element frequency or even make it infinite. Thus Eq. 62 must be used judiciously with insight to attain the largest possible time step, but it cannot attain time steps greater than the actual stability limits.

When the lumped masses arise from the density of the element itself, then

$$m_1 = m_2 = \frac{1}{2}\rho A\ell, \quad (65)$$

where ρ is the density. Substituting these values into Eq. 64, we find

$$\omega_{\max}^2 = \frac{4E}{\rho\ell^2} = \frac{4c^2}{\ell^2} \quad (66)$$

or

$$\omega_{\max} = \frac{2c}{\ell}, \quad (67)$$

where c , given by

$$c^2 = \frac{E}{\rho}, \quad (68)$$

is the elastic wavespeed in the material. Substituting this result into Eq. 59, we obtain

$$\Delta t \leq \frac{\ell}{c} [(1 + \mu^2)^{1/2} - \mu] \quad (69)$$

so that, when $\mu = 0$, the stability limit on the time step corresponds to the time for the elastic wave to traverse the length of the element.

When an element with flexural stiffness connects two nodes, its effects on stability must be considered separately. For this purpose, it is sufficient to consider bending in one plane. The stiffness matrix (if the shear effect is neglected) $[k]$ then is (see Przemieniecki⁸)

$$[k] = \frac{EI}{\ell^2} \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ & 4\ell^2 & -6\ell & 2\ell^2 \\ & & 12 & -6\ell \\ \text{symmetric} & & & 4\ell^2 \end{bmatrix}, \quad (70)$$

and we will consider a mass matrix

$$[m] = \frac{\rho A \ell}{420} \begin{bmatrix} 210 & 0 & 0 & 0 \\ & \alpha \ell^2 & 0 & 0 \\ & & 210 & 0 \\ \text{symmetric} & & & \alpha \ell^2 \end{bmatrix}. \quad (71)$$

For $\alpha = 17.5$, which is the lumping we use whenever the mass arises from the density distribution of the element itself, the maximum frequency is

$$\omega_{\max}^2 = \frac{192c^2 I}{A \ell^4} \quad (72)$$

so that

$$\Delta t \leq 0.1443 \frac{\ell^2}{c} \sqrt{\frac{A}{I}}. \quad (73)$$

For a typical model, either the frequency associated with axial response, Eq. 64, or the flexural frequency, Eq. 72, may be the largest. In choosing the time step, we must use the largest frequency. In comparing Eq. 66 and 72, we can see that the axial frequencies increase inversely with ℓ , whereas the flexural frequencies increase inversely as ℓ^2 . Hence, for sufficiently refined models, the flexural frequency will govern the time step.

The frequency associated with a torsional mode is given by

$$\omega_{\max}^2 = \frac{4k^t}{\bar{I}_{x1} + \bar{I}_{x2}}, \quad (74)$$

where k^t is the torsional stiffness. This frequency should also be checked in obtaining a stability, but it is generally smaller than the axial frequencies.

To ensure post facto that the computations are stable, energy-balance checks were made as follows. The kinetic energy at each time step j is defined by

$$T^j = \frac{1}{2} \rho_I (\dot{u}_{iI}^j)^2 + \frac{1}{2} \bar{\omega}_{iI}^j \bar{I}_{iI} \bar{\omega}_I^j \quad (\text{sum on } i \text{ and } I). \quad (75)$$

The external work is defined by a trapezoidal integration in time, so that

$$\left. \begin{aligned} W^{\text{ext},0} &= 0 \\ \text{and} \\ W^{\text{ext},j} &= W^{\text{ext},j-1} + \frac{1}{2} (u_{iI}^j - u_{iI}^{j-1}) (f_{iI}^{\text{ext},j} + f_{iI}^{\text{ext},j-1}) \\ &\quad (\text{sum on } i \text{ and } I). \end{aligned} \right\} \quad (76)$$

The internal work is similarly defined by

$$\left. \begin{aligned} W^{\text{int},0} &= 0 \\ \text{and} \\ W^{\text{int},j} &= W^{\text{int},j-1} + \frac{1}{2} \sum_e \left[\hat{u}_{iA}^{(e),j} - \hat{u}_{iA}^{(e),j-1} \right] \left[\hat{f}_{iA}^{(e),j} + \hat{f}_{iA}^{(e),j-1} \right] \\ &\quad (\text{sum on } i \text{ and } A). \end{aligned} \right\} \quad (77)$$

Energy balance then requires that

$$W^{\text{int},j} + T^j - W^{\text{ext},j} < \epsilon(W^{\text{int},j} + T^j). \quad (78)$$

If ϵ is greater than about 0.05, a solution is considered unacceptable. The source of the energy error may be an arrested or incipient numerical instability, or excessive truncation error.

IV. RESULTS

To validate this finite-element method, we first compared it to analytical solutions or solutions obtained numerically by other procedures. Since buckling of the columns is considered an important response mechanism in the UIS, several problems with linear material behavior and buckling were selected.

A. Bailey Problem

The first problem is the dynamic bending of an elastic curved beam as shown in Fig. 4. One end was axially displaced toward the other at a prescribed constant velocity. Bailey⁹ previously solved this problem by directly deriving the governing equations and then solving the equations numerically. The problem was solved for two slenderness ratios, 66 and 144, where the slenderness ratio is defined as the ratio of element length, L , to the radius of gyration. Beam elements were used. A constant velocity was prescribed at the right-hand end, and the left-hand

Fig. 4. Elastic Curved Beam. ANL
Neg. No. 900-79-450.

end was pinned. The geometric, material computational parameters are summarized in Table I. Figures 5 and 6 compare the deflected shapes of the beam with Bailey's. The nondimensional time variable used is expressed in terms of $\bar{t}/\bar{L} = ct/L$, where c is the speed of sound in the material, $\bar{L} = L/r$, and r is the radius of gyration.

TABLE I. Elastic Curved Beam

	Case 1-1	Case 1-2
Slenderness Ratio, L/r	66	144
Dimensionless Forcing Velocity:		
$\bar{u}_x = \dot{u}_x/c$	2.25×10^{-4}	1.00×10^{-3}
Dimensionless Initial Deflection:		
$\bar{H} = H/r$	0.25	0.25
Young's Modulus	2.07×10^5 MPa	2.07×10^5 MPa
Density	8021 kg/m^3	8021 kg/m^3
Radius of Gyration	2.54 cm	2.54 cm
Area	81.07 cm^2	81.07 cm^2
Number of Elements	22	48
Time step, s	10.0×10^{-6}	10.0×10^{-6}

Fig. 5. Deflected Shape for Axially Loaded Curve Beam (Case 1-1). ANL Neg. No. 900-79-465.

Fig. 6. Deflected Shape for Axially Loaded Curve Beam (Case 1-2). ANL Neg. No. 900-79-466.

B. Snap-through Problem

The second problem is one of large deflections of a shallow circular elastic arch subjected to uniform dynamic-pressure loading (see Fig. 7). The arch is loaded by a uniform pressure which is a step function in time; no imperfections were considered in the arch shape. Humphreys¹⁰ obtained a solution for this problem by using a Galerkin approach in terms of the modes of the arch in space and then solving the temporal equation for the Galerkin parameters by an analog computer. The problem was solved for the conditions listed in Table II. The pressure is normal to the x axis and expressed in nondimensional form as $p_0 = R^2 q/h^2 E$, where E is the elastic modulus.

Fig. 7

Geometry of Shallow Circular Arch.
ANL Neg. No. 900-79-451.

TABLE II. Elastic Shallow Circular Arch

	Case 2-1	Case 2-2
Inner radius, R	38.100 cm	83.312 cm
Thickness, h	0.254 cm	0.254 cm
Depth, d	2.54 cm	2.54 cm
Length, L	13.231 cm	28.933 cm
Rise, H	0.579 cm	1.265 cm
Young's Modulus	2.07×10^5 MPa	2.07×10^5 MPa
Density	8021 kg/m^3	8021 kg/m^3
Half of subtended arch angle, β	10 deg	10 deg
Shape factor, $\lambda = \beta^2 R/h$	4.569	9.991
Number of elements	10 and 20	20
Time step, s	1.0×10^{-6}	1.0×10^{-6}

Case 1 is a simply supported shallow arch and was run for pressure loadings, p_0 , ranging from 0.200 to 0.270. Time histories of the displacements for Humphreys' results are shown in Fig. 8, where the deflection ratio = average deflection/average rise. Humphreys' results indicate that early dynamic instability occurs for values of p_0 between 0.210 and 0.215. For values less than 0.210, dynamic instability apparently does not occur. Once the arch snaps through, the arch oscillates about an equilibrium value displaced from its undeformed shape, and a further increase in loading will not appreciably change the maximum deflection, but will simply make the instability occur sooner.

Fig. 8

Humphreys' Results for Dynamic Buckling of a Simply Supported Shallow Arch (Case 2-1). ANL Neg. No. 900-79-461.

Results based on the three-dimensional beam element are given in Figs. 9-12 for 10- and 20-element models. These results generally agree with those of Humphreys, except that more sensitivity to dynamic instability was observed. There is some difference in the time at which instability occurs and in the prediction of complete snap-through for various values of p_0 between the 10- and 20-element models. Complete snap-through is not indicated by Humphreys in his discussion of the results. The sensitivity of the problem to nodalization is indicated by the discrepancy in p_0 , which was

0.200 and 0.205 for the 10- and 20-element models, respectively. However, if the 10-element model with $p_0 = 0.205$ and the 20-element model with $p_0 = 0.200$ are run further in time (TIME = 70.0), dynamic instability was found to occur in a form similar to that shown for the 10-element model with $p_0 = 0.200$.

Fig. 9

Results for Ten-element Model for Dynamic Buckling of a Simply Supported Shallow Arch, $p_0 = 0.200$ to 0.240 (Case 2-1). ANL Neg. No. 900-79-453.

Fig. 10

Results for Twenty-element Model for Dynamic Buckling of a Simply Supported Shallow Arch, $p_0 = 0.200$ to 0.240 (Case 2-1). ANL Neg. No. 900-79-456.

Fig. 11

Results for Ten-element Model for Dynamic Buckling of a Simply Supported Shallow Arch, $p_0 = 0.240$ to 0.270 (Case 2-1). ANL Neg. No. 900-79-458.

Fig. 12

Results for Twenty-element Model for Dynamic Buckling of a Simply Supported Shallow Arch, $p_0 = 0.240$ to 0.270 (Case 2-1). ANL Neg. No. 900-79-459.

In Case 2, a clamped shallow arch is subjected to pressure loading, p_0 , ranging from 0.160 to 0.400. Humphreys' results, with imperfections in the arch shape, are shown in Fig. 13. The results obtained here for a

20-element model are given in Fig. 14; no imperfections were included. Good agreement is seen for the two larger values of p_0 . A difference again was observed, 0.165 as opposed to 0.180, in the value of p_0 initializing dynamic instability. Of interest in the three-dimensional beam results is that three types of response were observed for the three values of p_0 . For $p_0 = 0.175$, dynamic instability does not occur; for $p_0 = 0.200$, complete snap-through occurs; and for $p_0 = 0.400$, the arch snaps through and oscillates about an equilibrium value.

Fig. 13
Humphreys' Results for Dynamic Buckling
of a Clamped Shallow Arch (Case 2-2).
ANL Neg. No. 900-79-470.

Fig. 14
Results for Twenty-element Model for Dy-
namic Buckling of a Clamped Shallow Arch
(Case 2-2). ANL Neg. No. 900-79-471.

These results indicate the complexity and difficulty of definitions associated with dynamic buckling. For an arbitrarily shaped initial mode of vibration, and with no internal damping, the coupling caused by the inherently nonlinear nature of these problems will eventually result in the concentration of energy in a critical mode. If this energy is sufficiently large, snap-through can occur. However, the time of snap-through, as shown by these results, depends on a variety of features, which include the initial impulse, the impulse distribution, and the model. The fact that the use of the three-dimensional beam element provides results that indicate this sensitivity and provide reasonable agreement with the work of others constitutes verification of this aspect of the code.

C. Elastic Response of UIS

In the third example, a 1/20th-scale model employed by SRI¹¹ in their tests was studied. However, only the elastic response was considered and the loading was simplified so that comparisons with semianalytic results of Hoff¹² were possible.

The purposes of these studies were fourfold:

1. To determine the required degree of nodalization.
2. To determine whether any initial imperfections are necessary to trigger buckling in this numerical procedure.
3. To study the sensitivity of the results to the magnitude of the imperfection.
4. To ascertain whether asymmetric buckling modes are possible.

Hoff reported results in the form of midpoint column deflection (Fig. 15) and column compressive force (Fig. 16) for a simply supported column compressed with a constant end velocity, $\dot{u}$. The Euler buckling load is defined as P_E , which is given as $P_E = \pi^2 EI / \ell^2$, where E is Young's Modulus, I the area moment of the cross section, and ℓ the column length. An initial curvature as defined by $y = a \sin(\pi x / \ell)$, was considered. The eccentricity is defined by $e = a / r$, where r is the radius of gyration. An additional parameter used to describe the column response is the buckling index. Columns are defined as dynamically similar if the nondimensional buckling index Ω and initial eccentricity are the same. If the prebuckling displacements are small, the buckling load only depends on Ω . The buckling index is expressed

Fig. 15

Hoff's Results for Midpoint
Column Deflection. ANL
Neg. No. 900-79-454.

Fig. 16

Hoff's Results for Compressive
Force in Column. ANL Neg.
No. 900-79-455.

as $\Omega = \pi^2 \epsilon_E^3 (\dot{u}/c)^2$ where the Euler strain is defined by $\epsilon_E = \pi^2 (r/l)^2$ with $\dot{u}$ the column end velocity and c the speed of sound in the column. Rewriting gives $\Omega = \pi^8 (\dot{u}/c)^2 (r/l)^6$; Ω is 2.25, while $e = 0.25$ for this problem.

The finite-element model consists of 10 (in some cases 20) beam elements for each column and a rigid body which represents the upper internal structure. A quarter-symmetric model is shown in Fig. 17. As can be seen, the beam elements are all interconnected by primary nodes, except at the junction with the upper internal structure (UIS), where the primary node is at the centroid of the UIS. The interconnection at the UIS is fully rigid. The material, geometric, and computational parameters are given in Table III.

Fig. 17

One-quarter-symmetry Model.
ANL Neg. No. 900-79-467.

TABLE III. Dimensions and Material Properties

Support Column-Nickel 200		Head Cover-533-B Class I Carbon Steel	
Wall thickness	0.127 cm	Diameter	40.958 cm
Column diameter	1.778 cm	Height	2.794 cm
Column length	20.320 cm	Young's Modulus	1.93×10^5 MPa
Young's Modulus	1.93×10^5 MPa	Mass	70 kg
Density	8888 kg/m ³		
UIS Mass-Aluminum		Holddown Studs-Steel	
Diameter	14.732 cm	Diameter	0.432 cm
Height	14.097 cm	Length	14.859 cm
Young's Modulus	7.17×10^4 MPa	Young's Modulus	1.93×10^5 MPa
Mass	5 kg	Density	7860 kg/m ³
Time Step for 10-element Model - 2.0×10^{-6} s			
Time Step for 20-element Model - 5.0×10^{-7} s			

The quarter-symmetric model is only applicable to symmetric buckling, which is shown in Fig. 18. For asymmetric buckling, also shown in Fig. 18, a full model as shown in Fig. 19 was used. At the head, the support column was clamped for the models shown in Figs. 17 and 19; in a few cases, we also considered a model of the head and holddown bolts as shown in Fig. 20. The head was modeled as a rigid body, the studs by spring elements connected to the head.

Fig. 18

Symmetric and Asymmetric Buckling
Shapes of Support Columns. ANL
Neg. No. 900-79-477,

Fig. 19

Full Model. ANL Neg.
No. 900-79-457.

Fig. 20

One-quarter-symmetry Model
with Head Cover and Holddown
Studs. ANL Neg. No. 900-79-478.

The cases considered in these studies are summarized in Table IV. First, a study was made (Case 3-1) to see if lateral motion (i.e., buckling) could be achieved in the absence of an initial imperfection by this computational technique. The response in the absence of an imperfection was found to be perfectly axial with no buckling, even for unrealistically large axial strains like 10%. Round-off error was not sufficient to trigger the buckling mode.

In the next studies, an initial imperfection was included by displacing the coordinates of the nodes from a straight line by the amount $y = a(1 - \cos 2\pi x/l)$ where $e = a/r$ is the eccentricity of the column and r is the radius of gyration. This corresponds to the shape of the lowest mode. Eccentricity values of 0.00435, 0.0435, and 0.435 were considered. Since the column has clamped-clamped end conditions, the Euler strain is $\epsilon_E = 4\pi^2(r/l)^2$, the Euler buckling load is $P_E = 4\pi^2 EI/l^2$, and the buckling index is $\Omega = 64\pi^2(\dot{u}/c)^2(r/l)^6$; Ω is 225 for these cases.

The results are given in Figs. 21 and 22 for Cases 3-2a, -2b, and -2c. Though Hoff's results are for a different boundary condition, buckling index,

and eccentricity, the results compare well qualitatively, and regardless of the magnitude of the imperfection, in this range of imperfection magnitudes, buckling always occurs.

TABLE IV. Upper-internal-structure Models; Constant-velocity Loading of UIS

Case	Model Symmetry	Initial Column Shape	Column Eccentricity	Column Elements	Boundary Condition
3-1	1/4	straight		10	fixed
3-2a	1/4	$y = a(1 - \cos \frac{2\pi x}{l})$	0.00435	10	fixed
3-2b	1/4	$y = a(1 - \cos \frac{2\pi x}{l})$	0.04350	10	fixed
3-2c	1/4	$y = a(1 - \cos \frac{2\pi x}{l})$	0.43500	10	fixed
3-3	1/4	straight		20	fixed
3-4a	1/4	$y = a(1 - \cos \frac{2\pi x}{l})$	0.00435	20	fixed
3-4b	1/4	$y = a(1 - \cos \frac{2\pi x}{l})$	0.04350	20	fixed
3-4c	1/4	$y = a(1 - \cos \frac{2\pi x}{l})$	0.43500	20	fixed
3-5	1/4	straight		10	head
3-6	1/4	$y = a(1 - \cos \frac{2\pi x}{l})$	0.00435	10	head
3-7	full	$y = a(1 - \cos \frac{2\pi x}{l})$	0.00435	10	fixed
3-8a	full	$y = a(1 - \cos \frac{\pi x}{l})$	0.00435	10	fixed
3-8b	full	$y = a(1 - \cos \frac{\pi x}{l})$	0.04350	10	fixed
3-8c	full	$y = a(1 - \cos \frac{\pi x}{l})$	0.43500	10	fixed
3-9a	full	$y = a(1 - \cos \frac{\pi x}{l})$	0.00435	20	fixed
3-9b	full	$y = a(1 - \cos \frac{\pi x}{l})$	0.04350	20	fixed
3-9c	full	$y = a(1 - \cos \frac{\pi x}{l})$	0.43500	20	fixed

Response mode is symmetric for all cases.

Fig. 21

Ten-element Results for Mid-point Column Deflection. ANL
Neg. No. 900-79-440.

Fig. 22

Ten-element Results for Compressive Force in Column. ANL Neg. No. 900-79-439.

It is worthwhile to make some observations about these results. The lateral motion of the column will be retarded by its mass inertia when subjected to a dynamic loading. This explains why the dynamic deflections initially lag behind those corresponding to nearly static or static loading as seen in Fig. 21. Once the column is sufficiently accelerated, it will cause the lateral deflections to increase rapidly. Finally, the deflections will exceed the static value about which further oscillations will follow. The lag is seen to decrease rapidly as the initial curvature increases.

The column compressive force shown in Fig. 22 corresponds directly to the lateral deflection. The compressive force increases linearly until the lateral deflection starts to increase rapidly, then oscillates about the mean value in opposite sign with the lateral oscillation. The superimposed oscillations are due to the longitudinal waves in the column.

For checking to see if the lowest buckling mode was attained, the shape of the deformed column was compared against the initial shape. For the three cases, the deformed shape was found to have the same form as the assumed initial shape, indicating that the lowest buckling mode was found.

Case 3-3 is the same as Case 3-1, except that 20 elements were used to model the column to check element-number sensitivity. Again it was observed that no lateral motion occurred.

In a further check of nodalization sensitivity, Cases 3-4a, -4b, and -4c were run; these are identical 3-2a, -2b, and -2c, except that 20-element models of the column were used. The results for the midpoint column deflection (Fig. 23) and the column compressive force (Fig. 24) indicate that the responses for the 20-element model are somewhat stiffer than the 10-element cases. This is manifested by the time offset in the lateral oscillations and the decrease in column compressive force. However, the magnitude of the discrepancy is small enough for engineering purposes to justify the 10-element models.

Fig. 23

Twenty-element Results for Mid-point Column Deflection. ANL
Neg. No. 900-79-441.

Fig. 24

Twenty-element Results for Compressive Force in Column. ANL
Neg. No. 900-79-436.

In the earlier cases, the head cover had been eliminated for simplification. Cases 3-5 and -6 include the head cover and hold-down bolts. Case 3-5 is initially straight, while Case 3-6 has an initial curvature with e equal to 0.00435. The results for Cases 3-5 and -6 are identical to Cases 3-1 and -2a, respectively, because the inertia of the head cover is so large that it experiences essentially no motion. Thus, when considering only loading on the UIS and/or the support column, we can neglect the head-cover response.

Case 3-7 is a modeling check in which a full model of Case 3-2a was used. The response of the four columns of the full model was identical to the response of the single column of the 1/4 model.

All the previous cases involved symmetric response because of forced symmetry. Asymmetric response, which entails a full model, was studied using an initial shape of the imperfection given by $y = a(1 - \cos \pi x/l)$. Eccentricity values e of 0.00435, 0.0435, and 0.435 were used. The UIS mass is found to be sufficiently large that it cannot be laterally displaced. By checking the deformed shape, we observed that the column attempts to reach the symmetric mode. However, since the initial shape of the column was not very close to the deformed shape, the lowest buckling mode was not found, but instead a higher load was carried by the columns. This is verified by the difference in the response of Cases 3-4a and -9a; the column midpoint deflection, and column compressive force are compared in Figs. 25 and 26, respectively.

Fig. 25

Comparison of Midpoint Column
Deflection for Cases 3-4a and
-9a, ANL Neg. No. 900-79-438.

Fig. 26

Comparison of Compressive Force
in Column for Cases 3-4a and
-9a, ANL Neg. No. 900-79-437.

D. Comparisons with SRI Scale-model Tests

For validating the structural model in the elastic-plastic range, its response was compared to experimental results obtained in SRI¹³ scale-model tests. Since analytic elastic-plastic buckling results are not available, this was the only possible means of validation. A schematic of this model is shown in Fig. 27.

Fig. 27. Schematic of SRI Experimental Apparatus for UIS Column Tests. ANL Neg. No. 900-79-460.

The model of the UIS consists of an aluminum model of the UIS and the four columns that suspend the UIS over the core. The columns are made of Ni-200 tubes that simulate the strength of the CRBR UIS columns. Short solid steel sections are welded to each end of the Ni-200 sections to provide secure attachment to the UIS and the head. The flow passages were plugged to simulate conditions in which upper-core materials have plugged these passages.

The finite-element model is depicted in Fig. 17. Based on the previous studies, only symmetric buckling was considered. The UIS is modeled as

a rigid body, the support columns by 20 beam-column finite elements. The dimensions are given in Table III, the stress-strain curve for the support-beam material in Fig. 28.

The pressure-time history for tests ACS 4 and ACS 3 are shown in Figs. 29 and 30, respectively. These pressure-time histories were provided by SRI by taking the difference between gauges P3 and P4 (see Fig. 27). They correspond to 661- and 992-MW's simulated energy releases. These pressure-time histories were applied to the finite-element model of the UIS and support columns. The axial displacements are compared with experimental results in Table V. Time histories on the axial displacements are shown in Figs. 31 and 32; for comparison, the final axial displacements measured in the experiments are also shown. As can be seen, the predicted axial displacement compares well with the experiment.

Fig. 28. Stress-Strain Curve for UIS Columns.
ANL Neg. No. 900-79-475.

Fig. 29. Pressure History for SRI Test ACS 4.
ANL Neg. No. 900-79-474.

Fig. 30
Pressure History for SRI Test ACS 3.
ANL Neg. No. 900-79-476.

TABLE V. Response of UIS Columns to Simulated Energy Releases

Simulation	CDA Energy Release, MW·s	Experimental Axial Displacement, cm	Computed Axial Displacement, cm
4-1	661	1.32 and 1.40	1.45
4-2	992	2.54 and 2.82	2.44

Fig. 31

Column Axial Displacement: 661-MW·s Simulated Energy Release. ANL Neg. No. 900-79-472.

Fig. 32

Column Axial Displacement: 992-MW·s Simulated Energy Release. ANL Neg. No. 900-79-473.

Figures 33 and 34 show photographs of the final deformed shape of the support columns, and Figs. 35 and 36 show the final shape of the finite-element model. The finite-element model replicates the salient feature of the experimental results. In the 661-MW·s simulation, the beam buckles in the lowest mode (a two-node mode); in the 992-MW·s simulation, the beam buckles in the second mode (a three-node mode).

ACS 2
19.7g 90/10 Explosive
Simulated Energy Release
= 661 MW-sec
Axial Displacement of Columns
= 0.132 cm

ACS 4
19.7g 90/10 Explosive
Simulated Energy Release
= 661 MW-sec
Axial Displacement of Columns
= 0.140 cm

ACS 3
29.5g 90/10 Explosive
Simulated Energy Release
= 992 MW-sec
Axial Displacement of Columns
= 2.82 cm

ACS 5
29.5g 90/10 Explosive
Simulated Energy Release
= 992 MW-sec
Axial Displacement of Columns
= 2.54 cm

Fig. 33. Deformed Shapes for SRI Tests ACS 2 and ACS 4. ANL Neg. No. 900-79-681.

Fig. 34. Deformed Shapes for SRI Tests ACS 3 and ACS 5. ANL Neg. No. 900-79-680.

Fig. 35. Deformed Shape for Finite-element Model: 661-MW·s Simulated Energy Release. ANL Neg. No. 900-79-462.

Fig. 36. Deformed Shape for Finite-element Model: 992-MW·s Simulated Energy Release. ANL Neg. No. 900-79-452.

The computations reproduce both the physics and the numerical values of the experiment quite well. Thus the applicability of this finite-element model for this magnitude of displacements is established.

APPENDIX

Stability Analysis of Newmark β Method with Damping

We consider here the stability analysis of the Newmark β method, with $\beta = 0$ and $\gamma = 1/2$ in the presence of damping. The intent of the analysis is to show that if the velocity used in the damping term is taken directly from the Newmark formulas, the stable time step is smaller than if a backwards difference is used for the velocities.

A standard Fourier or von Neumann stability analysis of the linear equations of motion will be considered. For this purpose, the homogeneous equations of motion can be treated in the uncoupled form

$$\ddot{x} + 2\mu\omega\dot{x} + \omega^2x = 0, \quad (\text{A.1})$$

where ω is a natural frequency of the system and μ is the fraction of critical damping in this frequency.

For the explicit Newmark β method with no artificial damping in the integrator ($\beta = 0$, $\gamma = 1/2$), the difference formulas are

$$x^{n+1} = x^n + \Delta t \dot{x}^n + \frac{1}{2}\Delta t^2 \ddot{x}^n \quad (\text{A.2})$$

and

$$\dot{x}^{n+1} = \dot{x}^n + \frac{1}{2}\Delta t(\ddot{x}^n + \ddot{x}^{n+1}), \quad (\text{A.3})$$

where a superscript designates the time step number. The discrete form of Eq. A.1 can be written as

$$\ddot{x}^n + 2\mu\omega\dot{x}^{n-1} + \omega^2x^n = 0. \quad (\text{A.4})$$

Note that the velocity here lags one time step. This is a necessity for economical explicit computations when the damping matrix is not diagonal. If the velocity $\dot{x}^n$ were used, as can be seen from Eq. A.6, the displacement x^{n+1} would enter through the damping term, which would require a matrix inversion of the damping matrix, even in an explicit solution. Use of Eqs. A.2 and A.3 shows that

$$\ddot{x}^n = \frac{x^{n+1} - 2x^n + x^{n-1}}{\Delta t^2} \quad (\text{A.5})$$

and

$$\dot{x}^n = \frac{x^{n+1} - x^{n-1}}{2\Delta t}. \quad (\text{A.6})$$

Substituting Eqs. A.5 and A.6 into Eq. A.4, we obtain

$$x^{n+1} + (\mu\omega\Delta t + \omega^2\Delta t^2 - 2)x^n + x^{n-1} - \mu\omega\Delta tx^{n-2} = 0. \quad (A.7)$$

In a Fourier stability analysis, we note that the solution to Eq. A.7 is of the form

$$x^n = e^{n\alpha\Delta t}, \quad (A.8)$$

which can be written as

$$x^n = \lambda^n; \lambda = e^{\alpha\Delta t}. \quad (A.9)$$

If we substitute Eq. A.9 into Eq. A.7 and factor out λ^{n-2} , we obtain the characteristic equation

$$\lambda^3 + (\mu\omega\Delta t + \omega^2\Delta t^2 - 2)\lambda^2 + \lambda - \mu\omega\Delta t = 0. \quad (A.10)$$

Stability of the numerical solution is then achieved if the modulus of the roots of the characteristic equation λ (λ may be complex) are all less or equal to unity.

Since the algebraic process of checking the stability conditions for the cubic equation, Eq. A.10, are quite tedious, we use the usual transformation

$$\lambda = \frac{1+z}{1-z}. \quad (A.11)$$

The region of stability in the complex λ plane is the unit circle with its origin at zero.

In the complex z plane, as defined by Eq. A.11, this corresponds to the entire left-hand plane. Thus, if Eq. A.11 is substituted into Eq. A.10, the stability requirement becomes that all roots of the polynomial equation expressed in terms of z must have negative or zero real parts. This substitution yields

$$(4 - \omega^2\Delta t^2)z^3 + (4 - 4\mu\omega\Delta t - \omega^2\Delta t^2)z^2 + (4\mu\omega\Delta t + \omega^2\Delta t^2)z + \omega^2\Delta t^2 = 0. \quad (A.12)$$

We will now use the method of Hurwitz matrices to determine the conditions under which all roots of Eq. A.12 have nonpositive real parts. According to this method, the roots of the polynomial of order n

$$\sum_{j=0}^n C_j z^{n-j} = 0 \quad (A.13)$$

have nonpositive real roots if $C_0 > 0$ and the leading principal minors of the matrix C_{ij}^* given by

$$\left. \begin{aligned} C_{ij}^* &= C_{2j-i} \\ \text{and} \\ C_{ij}^* &= 0 \text{ if } 2j - i < 0, 2j - i > n \end{aligned} \right\}, \quad (\text{A.14})$$

are all nonnegative. Among these conditions, the one that limits Δt is the third principal minor, which yields

$$\omega^2 \Delta t^2 + 2\mu \omega \Delta t - 2 \leq 0. \quad (\text{A.15})$$

Hence, it follows that

$$\Delta t \leq \frac{-\mu + \sqrt{\mu^2 + 2}}{\omega}. \quad (\text{A.16})$$

From the other conditions we obtain

$$\Delta t \leq \frac{2}{\omega}, \quad \mu \geq 0. \quad (\text{A.17})$$

Both Eqs. A.16 and A.17 must be met for stability. We obtain the governing condition when ω is the maximum frequency, $\omega_{\max}$. For a constant strain element

$$\omega_{\max} = \frac{2c}{\ell}, \quad (\text{A.18})$$

and we obtain the Courant-Lewy-Friedrichs condition when $\mu < 0$,

$$\Delta t \leq \frac{\ell}{c}, \quad (\text{A.19})$$

where ℓ is the characteristic length of the element and c is the elastic wave-speed of the material.

For Eq. A.16, the combination of μ and ω that gives the minimum value of the right-hand side must be chosen. This is usually $\omega_{\max}$ and the associated critical damping in this mode.

As $\mu \rightarrow 0$, Eq. A.16 gives

$$\Delta t \leq \frac{\sqrt{2}}{\omega_{\max}}, \quad (\text{A.20})$$

which is a more restrictive condition than Eq. A.18. If the analysis is repeated without damping ($\mu = 0$), then the characteristic equation, Eq. A.10, becomes quadratic and we obtain Eq. A.19.

Thus the treatment of damping with the Newmark velocity expression reduces the time step dramatically, even for small values of μ . This, of course, is highly undesirable, for it adds significantly to the computational expense.

This effect can be eliminated by using the following difference formula for the velocity

$$\dot{x}^{n-1} = \frac{x^n - x^{n-1}}{\Delta t} \quad (\text{A.21})$$

in place of Eq. A.6. The counterparts of Eqs. A.7, A.10, and A.12 are then

$$x^{n+1} + (2\mu\omega\Delta t + \omega^2\Delta t^2 - 2)x^n + (1 - 2\mu\omega\Delta t)x^{n-1} = 0, \quad (\text{A.22})$$

$$\lambda^2 + (2\mu\omega\Delta t + \omega^2\Delta t^2 - 2)\lambda + 1 - 2\mu\omega\Delta t = 0, \quad (\text{A.23})$$

and

$$(4 - 4\mu\omega\Delta t - \omega^2\Delta t^2)z^2 + (2\mu\omega\Delta t)z + \omega^2\Delta t^2 = 0, \quad (\text{A.24})$$

which yields

$$\Delta t \leq \frac{2}{\omega} \left(\sqrt{\mu^2 + 1} - \mu \right). \quad (\text{A.25})$$

This condition reduces to Eq. A.17 as $\mu \rightarrow 0$. Thus, the change in the velocity expression increases the stability limit substantially for small values of μ . The destabilizing effect of the back-dated Newmark velocity expression can thus be avoided by simply using a different form for the velocity expression in the damping terms.

ACKNOWLEDGMENTS

This work was performed in the Engineering Mechanics Section of the Reactor Analysis and Safety Division at Argonne National Laboratory, under the auspices of the U. S. Department of Energy.

REFERENCES

1. N. Newmark, *A Method of Computation for Structural Dynamics*, J. Eng. Mech. Div., Proc. ASCE, 67-94 (1959).
2. J. A. Duffey and H. L. Schreyer, *Simplified N-Dimensional Interaction Surfaces*, ASCE J. Struct. Div. 102, 1291-1298 (1976).
3. C. Y. Wang, *ICECO--An Implicit Eulerian Method for Calculating Fluid Transients in Fast-reactor Containment*, ANL-75-81 (Dec 1975).
4. T. Belytschko and B. J. Hsieh, *Nonlinear Transient Finite Element Analysis with Convected Coordinates*, Int. J. Num. Meth. Eng. 7, 255-271 (1973).
5. T. Belytschko, L. Schwer, and M. J. Klein, *Large Displacement, Transient Analysis of Space Frames*, Int. J. Num. Meth. Eng. 11, 64-84 (1977).
6. T. Belytschko and L. Glaum, *Applications of Higher Order Corotational Formulations for Nonlinear Finite Element Analysis*, Comput. Struct. 10, 175-182 (1979).
7. T. J. R. Hughes, K. Pister, and R. L. Taylor, *Implicit-Explicit Finite Elements in Nonlinear Transient Analysis*, Comp. Meth. Appl. Mech. Eng. 17-18, 159-182 (1979).
8. J. S. Przemieniecki, *Theory of Matrix Structural Analysis*, McGraw-Hill (1968).
9. H. R. Bailey, *Dynamic Bending of Elastic Columns*, ASCE J. Struct. Div. 89, 95-114 (1963).
10. J. S. Humphreys, *On Dynamic Snap Buckling of Shallow Arches*, AIAA J. 4(5), 878-886 (1966).
11. C. M. Romander and D. J. Cagliostro, *Structural Response of 1/20-Scale Models of the Clinch River Breeder Reactor to a Simulated Hypothetical Core Disruptive Accident*, SRI Report (Oct 1978).
12. N. J. Hoff, *The Dynamics of the Buckling of Elastic Columns*, J. Appl. Mech. 18, 68-74 (1951).
13. C. M. Romander, *Structural Response of 1/20-Scale Models of Above Core Structures to Increasingly Energetic Hypothetical Core Disruptive Accidents*, SRI Report (July 1979).

Distribution for ANL-79-104Internal:

W. E. Massey
 J. A. Kyger
 R. Avery
 L. Burris
 D. W. Cissel
 S. A. Davis
 B. R. T. Frost
 E. V. Krivanec
 R. J. Teunis
 C. E. Till
 R. S. Zono
 H. K. Fauske
 J. F. Marchaterre
 H. O. Monson
 S. H. Fistedis
 Y. W. Chang
 H. Y. Chu
 C. Fiala
 J. Gvildys
 R. F. Kulak
 M. T. Abdel-Moneim
 H. J. Petroski
 H. Schreyer
 C. Y. Wang
 W. Zeuch

T. Scale
 R. Timm
 I. Bornstein
 D. H. Cho
 A. DeVolpi
 C. E. Dickerman
 F. E. Dunn
 M. Epstein
 J. B. Heineman
 A. E. Klickman
 J. M. Kramer
 D. H. Lennox
 W. C. Lipinski
 P. A. Lottes
 C. E. Miller
 R. A. Noland
 D. R. Pedersen
 A. B. Rothman
 R. H. Sevy
 B. W. Spencer
 J. H. Tessier
 D. P. Weber
 H. U. Wider
 R. E. Henry

D. Rose
 A. J. Goldman
 R. Valentin
 C. K. Youngdahl
 G. Rosenberg
 R. W. Weeks
 H. Ahmed
 P. Chopra
 T. S. Wu
 M. Srinivasan
 C. Kot
 T. M. Mulcahy
 B. Hsieh
 Y. W. Shin
 P. Turula
 M. Wambsganss
 S. Majumdar
 S. Chen
 H. Halle
 G. Ridges
 J. M. Kennedy (20)
 A. B. Krisciunas
 ANL Contract File
 ANL Libraries (5)
 TIS Files (4)

External:

DOE-TIC, for distribution per UC-79p (197)
 Manager, Chicago Operations and Regional Office, DOE
 Chief, Office of Patent Counsel, DOE-CORO
 Director, Technology Management, DOE-CORO
 Director, DOE-RRT (2)
 President, Argonne Universities Association
 Reactor Analysis and Safety Division Review Committee:
 S. Baron, Burns and Roe, Inc.
 J. R. Dietrich, Combustion Engineering, Inc.
 W. Kerr, U. Michigan
 M. Levenson, Electric Power Research Inst.
 S. Levy, S. Levy, Inc.
 D. Okrent, U. California, Los Angeles
 N. C. Rasmussen, Massachusetts Inst. Technology