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**An Efficient Beam-Column Finite Element
Including
Application to Thermoelastic Deformation**

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AN EFFICIENT BEAM-COLUMN FINITE ELEMENT INCLUDING APPLICATION TO THERMOELASTIC DEFORMATION

by

Frederick D. Ju

ABSTRACT

During the study of the structural design of the core of the Gas Cooled Fast Reactor (GCFR), there is evidence of undue frictional interaction between fuel rod claddings and their spacer grids. In cases for which thermal bowing of the rods is sufficient to cause two-sided contact between the rods and their supporting grids (and thus a tendency to "lock up") subsequent rod deformation caused by the axial thermal expansion would result in excessive buckling of the rods between locked supports. The numerical solution is best served by the finite element method. Yet, there is a large number of fuel rods in the reactor, of which each is multiply-supported by spacers. Hence, an efficient finite element is needed that would model each section between supports with not more than three elements.

The new efficient finite element is developed, beginning with the principle of virtual work. The finite element equations are developed into a standard form of

$$\underline{K} \underline{a} = \underline{f}$$

where the stiffness matrix is nonlinear and asymmetric. The formulation may also be symmetrized to suit certain solvers, although the matrix remains nonlinear. The formulation proves to be extremely efficient in terms of the number of elements. In general, with only two elements highly accurate results can be achieved. Examples of applications are shown comparing the numerical results to these of the analytical solution of some standard beam columns.

1.0 INTRODUCTION

The work described in this report arose from a need to determine thermo-elastic deformations and forces developed in nuclear reactor fuel rod bundles particularly for thermal loading occurring during upset conditions. A fuel rod bends and buckles at spans between a series of spaced support under some combined external and thermal loadings. The rod can be modeled as a beam-column on multiple supports. The finite element method is considered most appropriate for the computer solution of a large system of such rods. Furthermore, because of the number of supports for a single rod, economy on the required number of elements will be a contributing factor.

The development of the beam-column element first appeared in a 1966 Jet Propulsion Lab report by Martin.¹ Subsequently, several texts on the finite element method presented the beam-column elements, such as: Martin and Carey,² Prezemieniecki³ and Zienkiewicz.⁴ More elaborate and general treatment was available in the Massachusetts Institute of Technology report by Bolourchi and Bathe⁵ as an addendum to the ADINA finite element code. In both the first two references, the stiffness matrix is derived from second derivatives of the strain energy with respect to the spatial coordinates. Such approximation is permissible only when the quadratic terms in the strain energy function dominate; while the nonlinear effect is essentially perturbative. Prezemieniecki, though following a different approach, basically employed the assumption of Martin. Zienkiewicz separated the membrane and the bending strains at an early stage. As a result, the bending effect of the membrane force in an element is only secondary. It is to be pointed out that Zienkiewicz developed the element only for a plate; and the example showed the result of covering the plate with 32 elements. The ADINA finite element is formulated in three-dimensional space and takes into account nonlinear material properties. It also takes into account the short beam effect by allowing the constant plane cross section to change its angle with respect to the neutral plane of the beam. However, as a result of simplifications for symmetric stiffness matrix, it takes over 20 elements for a single span of a beam column. Other noteworthy literature includes the formulation by Belytschko, et al.⁶ and the subsequent application to a problem of a reactor structure by Kennedy and Belytschko.⁷ There is some concern for the need of a large number of elements per span and the implication this requirement has on the Euler-Bernoulli's condition for short elements.

In the current development the following conditions will be observed in order to simplify the algorithm in the numerical analysis. The material shall be linearly elastic. The deformation is moderately large according to von Karman's postulation. Two or three elements should be adequate to cover a single span of the beam column. For a slender beam theory, not applicable for short elements, Euler-Bernoulli's conditions will be used; i.e., the cross-sectional plane remains undeformed and orthogonal to the neutral plane. Planar deformation is assumed and considered adequate for plane and axi-symmetrical problems.

The method of approach is based on the virtual work principle, which is considered most appropriate for nonlinear problems. After development of the beam-column element for the general loading state, the thermal load is then incorporated.

The resulting finite element was tested against some known solutions of the beam column. Both the single element model and the two-element model were employed to check the convergence of the finite element solutions toward the analytical ones.

The advantage of the development of the new beam-column element is demonstrated thus: (1) it is consistent with the structural theory of the beam column, (2) it requires sparingly few elements between supports, (3) results improve on the accuracy as compared to other structural elements such as that proposed by Martin.

2.0 FORMULATION OF THE BEAM-COLUMN ELEMENT

Because the mathematical model of the element is based on the structural theory of the beam column, we must stipulate that for every element the following conditions are satisfied: that the cross section is uniform; that its characteristic dimension (d) is of small order to the element length (l); that the slope is of small order of one everywhere in the element; and that the material properties are uniform with respect to the axial coordinate for one element.

Furthermore, it is stipulated that there is no material nonlinearity, yet the geometrical nonlinearity will be emphasized. Hence, the strain energy in general will not be a quadratic function of the spatial coordinates. This condition will govern the method in deriving the stiffness matrix for the element.

2.1 Kinematic and Kinetic Configurations

The kinematic and the loading states of a finite element is uniquely defined by the displacement vector and the force vector at its nodes, $\underline{u}$ and $\underline{f}$, respectively. Their components are shown in their respective positive directions in Figs. 1, 2, and 3. Figure 1 shows the coordinates, the nodal designation (I,J), the material and geometrical properties. The axial coordinate x shall coincide with the centroidal axis. The J node in the practice is usually the (I+1) node, though not necessarily. EA and EI shall be respectively the axial and the flexural stiffness of the element. Figures 2 and 3 show the components of the displacement and the force vectors as pairs of triplets (u,w,θ) or (S,R,M) , respectively the axial, the lateral and the angular measures.

Fig. 1. Coordinates, nodal designation (I,J), material and geometric properties of a finite element.

Fig. 2. Components of the displacement and force vectors (u,w,θ) .

Fig. 3. Components of the displacement and force vectors (S,R,M).

2.2 Displacement Field and Shape Functions

Following the practice of many successful finite element codes, we shall select shape functions that are simple but adequate to satisfy the nodal constraints. For lateral displacement, since both the displacement and its derivative, $w(x)$ and θ , are defined at the node, a cubic function in x will be used, while a linear function will be sufficient for the axial displacement of particle along the centroidal axis. Hence, the shape function $N(x,y)$ is defined as

$$\begin{aligned} \tilde{N}^T(x,y) &= [\tilde{N}_u(x,y) \quad \tilde{N}_w(x,y)]^T = \begin{bmatrix} \tilde{N}_u(x,y)^T \\ \tilde{N}_w(x,y)^T \end{bmatrix} \\ &= \begin{bmatrix} (1-\xi) \xi & \frac{6}{l} y \xi (1-\xi) & -y(1-\xi)(1-3\xi) & -\frac{6}{l} y \xi (1-\xi) & y \xi (2-3\xi) \\ 0 & 0 & (1-\xi)^2 (1+2\xi) & (1-\xi)^2 \xi & \xi^2 (3-2\xi) & -l(1-\xi) \xi^2 \end{bmatrix}. \end{aligned} \quad (1)$$

The displacement field in the element is defined uniquely by the nodal displacement vector through the shape function, as

$$\begin{Bmatrix} u(x,y) \\ w(x) \end{Bmatrix} = \begin{Bmatrix} \tilde{N}_u^T \tilde{a} \\ \tilde{N}_w^T \tilde{a} \end{Bmatrix} = \tilde{N}^T \tilde{a}. \quad (2)$$

where

$$\tilde{a}^T = [u_I \quad u_J \quad w_I \quad \theta_I \quad w_J \quad \theta_J].$$

2.3 Equilibrium Equation

For a system in equilibrium, the principle of virtual work must hold regardless of the linearity of the system, that is

$$\delta U = \delta U^{(i)} + \delta U^{(e)} = 0 \quad (3)$$

where the internal virtual work is the negative virtual change of strain energy. We may therefore determine at first the strain energy U_ϵ to derive the internal virtual work; that is, $\delta U^{(i)} = -\delta U_\epsilon$.

With the postulation of the small strain (not small displacement) theory and by considering residual stress separately, we come to the Hookian Law*

$$\sigma_{ij} = c_{ijkl} \epsilon_{kl}.$$

Because of the geometrical nonlinearity

$$\epsilon_{kl} = e_{kl} + \eta_{kl}$$

$$\text{where } e_{kl} = \frac{1}{2} (u_{k,l} + u_{l,k}) \quad \text{and} \quad \eta_{kl} = \frac{1}{2} u_{j,k} u_{k,l}.$$

The global strain energy is

$$U_\epsilon = \int_V \sigma_{ij} d\epsilon_{ij} dV.$$

Hence

$$\begin{aligned} -\delta U^{(i)} &= \delta U_\epsilon = \int_V \sigma_{ij} \delta \epsilon_{ij} dV \\ &= \left\{ \int_V c_{ijkl} e_{kl} \delta e_{ij} dV + \int_V c_{ijkl} \eta_{kl} \delta e_{ij} dV \right. \\ &\quad \left. + \int_V \sigma_{ij} \delta \eta_{ij} dV \right\}. \end{aligned} \quad (4)$$

*Summation convention is used and applied to subscripts of Roman minuscules.

We shall find the internal virtual work in the expression of

$$\delta U^{(i)} = - \delta \underline{a}^T \underline{K} \underline{a} . \quad (5)$$

Equation (5) combined with the known external virtual work expression

$$\delta U^{(e)} = \delta \underline{a}^T \underline{f}$$

is substituted into Eq. (3) to yield

$$\delta \underline{a}^T (-\underline{K} \underline{a} + \underline{f}) = 0.$$

Since the virtual displacement is arbitrary, the equilibrium equation thus results as

$$\underline{f} = \underline{K} \underline{a}. \quad (6)$$

$\underline{K}$ is of course the stiffness matrix, be it a constant or otherwise. It is noticed that in the derivation there is no differentiation of the strain energy density function.

2.4 Element Stiffness Matrix

It is observed in Eq. (4) that the first term on the right hand side contains only the linear portion of the strain tensor, while both of the last two terms contain the nonlinear portion of the strain tensor. We may express Eq. (4) accordingly

$$\delta U = \delta U_L + \delta U_N^{(1)} + \delta U_N^{(2)}, \quad (7)$$

where the subscripts L and N denote the linear and the nonlinear portions of the virtual change in strain energy respectively.

The general expression of Eq. (4) is further simplified, with the structural theory of beam column. There, the only nontrivial strain component is the longitudinal fiber strain $\epsilon_{xx} = e_{xx} + \eta_{xx}$. The linear portion is

Eq. (2)

$$e_{xx} = \partial_x u(x, y) = \partial_x \underline{N}_u(x, y)^T \underline{a} = \underline{a}^T \partial_x \underline{N}_u$$

and the nonlinear portion is

$$\begin{aligned} &= \frac{1}{2} \left[\partial_x u^2 + \partial_x w^2 \right] = \frac{1}{2} \left[\left(\partial_x \underline{N}_u^T \underline{a} \right)^2 + \left(\partial_x \underline{N}_w \underline{a} \right)^2 \right] \\ &= \frac{1}{2} \underline{a}^T \left[\partial_x \underline{N}_u \partial_x \underline{N}_u^T + \underline{N}_w' \underline{N}_w'^T \right] \underline{a} . \end{aligned} \quad (8)$$

Their virtual variations are

$$\left. \begin{aligned} \delta e_{xx} &= \delta \underline{a}^T \partial_x \underline{N}_u \\ \delta \eta_{xx} &= \delta \underline{a}^T \left[\partial_x \underline{N}_u \partial_x \underline{N}_u^T + \underline{N}_w' \underline{N}_w'^T \right] \underline{a} . \end{aligned} \right\} \quad (9)$$

The elastic coefficient c_{ijkl} becomes simply the Young's Modulus E. When Eq. (8) and (9) are substituted in Eq. (4) or a term thereof, there will be volume integrals of the products of shape functions of varying complexity.

1⁰ - For the linear term in Eq. (5)

$$\delta U_L = \int_V E e_{xx} \delta e_{xx} dV = \delta \underline{a}^T \left[E \ell \int_0^\ell \int_A \partial_x \underline{N}_u \partial_x \underline{N}_u^T dA dx \right] \underline{a} .$$

We shall define $\underline{K}_L$ as

$$\underline{K}_L = E \ell \int_0^\ell \int_A \partial_x \underline{N}_u \partial_x \underline{N}_u^T dA d\xi = \frac{EA}{\ell} \begin{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \underline{0} \\ \underline{0} & \rho^2 \begin{bmatrix} 12 & & \\ 6\ell & 4\ell^2 & \text{sym} \\ -12 & -6\ell & 12 \\ 6\ell & 2\ell^2 & -6 & 4\ell^2 \end{bmatrix} \end{bmatrix}$$

where ρ is the element slenderness ratio ($\rho^2 = EI/EA\ell^2$).

2⁰ - For the first nonlinear term in Eq. (5)

$$\delta U_N^{(1)} = E \int_V \eta_{xx} \delta e_{xx} dV = \delta \underline{a}^T \left[\frac{1}{2} E \ell \int_0^\ell \int_A \partial_x \underline{N}_u \underline{a}^T (\partial_x \underline{N}_u^T \partial_x \underline{N}_u^T + \underline{N}_w' \underline{N}_w'^T) dA d\xi \right] \underline{a} .$$

We now define the term inside the brackets as $\underline{K}_N^{(1)}$. After integration, we have, with $\Delta u = u_J - u_I$

$$\begin{aligned}
\mathbf{K}_N^{(1)} = & \frac{\Delta u}{2\ell} \mathbf{K}_L + \frac{EA}{2\ell} \left(\rho^2 \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{A}} \\ \underline{\underline{A}}^T & \underline{\underline{0}} \end{bmatrix} \left[\frac{6(w_I - w_J)}{\ell} + 3(\theta_I + \theta_J) \right] + \rho^2 \ell (\theta_I - \theta_J) \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{B}} \\ \underline{\underline{B}}^T & \underline{\underline{0}} \end{bmatrix} \right. \\
& \left. + \frac{w_I - w_J}{10\ell} \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{C}} \\ \underline{\underline{C}} & \underline{\underline{0}} \end{bmatrix} + \frac{\theta_I + \theta_J}{20} \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{A}} \\ \underline{\underline{A}} & \underline{\underline{0}} \end{bmatrix} + \frac{\ell(\theta_I - \theta_J)}{12} \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{B}} \\ \underline{\underline{B}} & \underline{\underline{0}} \end{bmatrix} \right), \quad (10)
\end{aligned}$$

where

$$\underline{\underline{A}} = \begin{bmatrix} -2 & -\ell & 2 & -\ell \\ 2 & \ell & -2 & \ell \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad \underline{\underline{C}} = \begin{bmatrix} -12 & -\ell & 12 & -\ell \\ 12 & \ell & -12 & \ell \end{bmatrix}.$$

3⁰ - In the second nonlinear term, the fibre stress is

$$\sigma_{xx}(x, y) = \sigma(x, y) = \frac{S}{A} + \frac{M_R(x)y}{I}$$

where S is the axial force in the beam element, tensile being positive, $M_R(x)$ is the resistant moment in the element such that

$$M_R = \left[m_I \left(1 - \frac{x}{\ell} \right) \quad m_J \left(\frac{x}{\ell} \right) \right] - S \left[w(x) - w_I \right] = \begin{bmatrix} (1 - \xi) & -\xi \end{bmatrix} \begin{bmatrix} m_I \\ m_J \end{bmatrix} - S \left(\underline{\underline{N}}_W^T \underline{\underline{a}} - w_I \right).$$

The first term represents the bending moment due to the beam effect alone, excluding the bending from the column effect in that element. It is the result of the lateral load, the external moment and the thermal moment. Its distribution in the element is approximated by a linear function. The last term gives the column effect. $[m_I \quad m_J]$ are the resistant moments at the leading and terminal ends of the element due to the beam effect only. The last term in Eq. (5) thus becomes

$$U_N^{(2)} = \int_V \sigma_{xx} \delta \eta_{xx} dV = \delta \underline{\underline{a}}^T \left[\int_V \sigma (\partial_x \underline{\underline{N}}_U \partial_x \underline{\underline{N}}_U^T + \underline{\underline{N}}_W' \underline{\underline{N}}_W'^T) dV \right] \underline{\underline{a}}.$$

The expression inside the brackets is designated $\mathbf{K}_N^{(2)}$. The expression is omitted herewith, since it will appear in the overall stiffness matrix. Based on Eq. (5), we obtain

$$\underline{\underline{K}} = \underline{\underline{K}}_L + \underline{\underline{K}}_N^{(1)} + \underline{\underline{K}}_N^{(2)}. \quad (11)$$

The stiffness matrix is rearranged by nodes. $\underline{K}$ is thus written as

$$\begin{aligned}
 \underline{K}^I = & \frac{(EI)^I}{\ell_I^3} \left(1 + \frac{u_J - u_I}{2\ell_I} + \frac{S^I}{(EA)^I} \right) \begin{bmatrix} \rho_I^{-2} & & & & & \text{sym} \\ 0 & 12 & & & & \\ 0 & 6\ell_I & 4\ell_I^2 & & & \\ -\rho_I^{-2} & 0 & 0 & \rho_I^{-2} & & \\ 0 & -12 & -6\ell_I & 0 & 12 & \\ 0 & 6\ell_I & 2\ell_I^2 & 0 & -6\ell_I & 4\ell_I^2 \end{bmatrix} + \frac{S^I}{30\ell_I} \begin{bmatrix} 0 & & & & & \text{sym} \\ 0 & 36 & & & & \\ 0 & 3\ell_I & 4\ell_I^2 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -36 & -3\ell_I & 0 & 36 & \\ 0 & 3\ell_I & -\ell_I^2 & 0 & -3\ell_I & 4\ell_I^2 \end{bmatrix} \\
 & + \frac{1}{2\ell_I^2} \left[6 \left(\frac{(EI)^I}{\ell_I^2} - \frac{S^I}{5} \right) (w_I - w_J) + 3 \left(\frac{(EI)^I}{\ell_I} - \frac{S^I \ell_I}{30} \right) (\theta_I + \theta_J) + (n_I^I + m_J^I) \right] \begin{bmatrix} 0 & & & & & \text{sym} \\ -2 & 0 & & & & \\ -\ell_I & 0 & 0 & & & \\ 0 & 2 & \ell_I & 0 & & \\ 2 & 0 & 0 & -2 & 0 & \\ -\ell_I & 0 & 0 & \ell_I & 0 & 0 \end{bmatrix} \\
 & + \frac{1}{2\ell_I} \left[S^I (w_I - w_J) + \left(\frac{(EI)^I}{\ell_I} - \frac{S^I \ell_I}{6} \right) (\theta_I - \theta_J) + (m_I^I - m_J^I) \right] \begin{bmatrix} 0 & & & & & \text{sym} \\ 0 & 0 & & & & \\ -1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \\
 & + (EA)^I \left(\frac{w_I - w_J}{20\ell_I^2} \begin{bmatrix} 0 & -12 & -2 & 0 & 12 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 2 & 0 & -12 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\theta_I + \theta_J}{40\ell_I} \begin{bmatrix} 0 & -2 & -2 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\theta_I - \theta_J}{24} \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \quad (12)
 \end{aligned}$$

The corresponding displacement and force vectors are

$$\underline{a}^T = [u_I \ w_I \ \theta_I \ u_J \ w_J \ \theta_J]$$

$$\underline{f}^T = [S_I \ R_I \ M_I \ S_J \ R_J \ M_J].$$

2.5 Symmetrical Expression of the Stiffness Matrix

The nonsymmetrical portion of $\underline{K}^I$ in Eq. (12) may be demonstrated to be essentially the effect of flexure on the axial load of the beam column. It was shown in Eq. (12) the nonsymmetrical matrices affect the axial forces

only. The last three terms of Eq. (12) thus relate the flexural components of the displacement vector to the axial component of the force vector. Hence, if we define $\Delta \underline{S}$ as

$$\Delta \underline{S} = \begin{Bmatrix} \Delta S_I \\ \Delta S_J \end{Bmatrix} = (EA)^I \left(\frac{w_I - w_J}{20\ell_I^2} \underline{C} + \frac{\theta_I + \theta_J}{40\ell_I} \underline{A} + \frac{\theta_I - \theta_J}{24} \underline{B} \right) \begin{Bmatrix} w_I \\ \theta_I \\ w_J \\ \theta_J \end{Bmatrix}.$$

Evidently, $\Delta S_J = -\Delta S_I = \Delta S^I$ and

$$\Delta S^I = \frac{(EA)^I}{120\ell_I^2} \left[72(w_I - w_J)^2 + 12\ell_I(w_I - w_J)(\theta_I + \theta_J) + 3\ell_I^2(\theta_I + \theta_J)^2 + 5\ell_I^2(\theta_I - \theta_J)^2 \right]. \quad (13)$$

If we consider the contraction of the element in the axial direction due to the flexure $(w_I, \theta_I, w_J, \theta_J)$, the contraction λ^I is

$$\begin{aligned} \lambda^I &= \int_0^{\ell_I} \left(\sqrt{1+(w')^2} - 1 \right) dx \approx \frac{1}{2} \int_0^{\ell_I} w'^2 dx = \frac{1}{2} \underline{a}^T \int_0^{\ell_I} \underline{N}_w' \underline{N}_w'^T dx \underline{a} \\ &= \frac{1}{120\ell_I} \left[72(w_I - w_J)^2 + 12\ell_I(w_I - w_J)(\theta_I + \theta_J) + 3\ell_I^2(\theta_I + \theta_J)^2 + 5\ell_I^2(\theta_I - \theta_J)^2 \right] = \frac{\Delta S^I_I}{(EA)^I}. \end{aligned} \quad (14)$$

The physical interpretation of the nonsymmetrical terms may thus be inferred by Eq. (14) to give the axial force of the beam column due to flexure when both ends are prevented from free contraction. We may very well choose to remove the effect during computation and add on the forces later, whereas

$$S_I^* = S_I + \Delta S^I, \quad S_J^* = S_J - \Delta S^I$$

and

$$\underline{f}_*^T = \begin{bmatrix} S_I^* & R_I & M_I & S_J^* & R_J & M_J \end{bmatrix}.$$

The equilibrium equation Eq. (12) thus becomes

$$\underline{f}_* = \underline{K}_S \underline{a} \quad (15)$$

$$\begin{aligned}
K_S^I &= \frac{(EI)^I}{\ell^3 I} \left(1 + \frac{u_J - u_I}{2\ell_I} + \frac{S^I}{(EA)^I} \right) \begin{bmatrix} \rho_I^{-2} & & & & & \\ 0 & 12 & & & & \\ 0 & 6\ell_I & 4\ell_I^2 & & & \\ -\rho_I^{-2} & 0 & 0 & \rho_I^{-2} & & \\ 0 & -12 & -6\ell_I & 0 & 12 & \\ 0 & 6\ell_I & 2\ell_I^2 & 0 & -6\ell_I & 4\ell_I^2 \end{bmatrix} \text{sym} + \frac{S^I}{30} \begin{bmatrix} 0 & & & & & \\ 0 & 36 & & & & \\ 0 & 3\ell_I & 4\ell_I^2 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -36 & -3\ell_I & 0 & 36 & \\ 0 & 3\ell_I & -\ell_I^2 & 0 & -3\ell_I & 4\ell_I \end{bmatrix} \text{sym} \\
&+ \frac{1}{2\ell_I^2} \left[6 \left(\frac{(EI)^I}{\ell_I^2} - \frac{S^I}{5} \right) (w_I - w_J) + 3 \left(\frac{(EI)^I}{\ell_I} - \frac{S^I \ell_I}{30} \right) (\theta_I + \theta_J) + (m_I^I + m_J^I) \right] \begin{bmatrix} 0 & & & & & \\ -2 & 0 & & & & \\ -\ell_I & 0 & 0 & & & \\ 0 & 2 & \ell_I & 0 & & \\ 2 & 0 & 0 & -2 & 0 & \\ -\ell_I & 0 & 0 & \ell_I & 0 & 0 \end{bmatrix} \text{sym} \\
&+ \frac{1}{2\ell_I} \left[S^I (w_I - w_J) + \left(\frac{(EI)^I}{\ell_I} - \frac{S^I \ell_I}{6} \right) (\theta_I - \theta_J) + (m_I^I - m_J^I) \right] \begin{bmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ -1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.
\end{aligned}
\tag{16}$$

It is noticed that in the stiffness matrix for both nonsymmetrical and the symmetrical expressions, there are the nodal displacement terms ($u_I, u_J, w_I, w_J, \theta_I, \theta_J$) and the element resistant forces (S_I^I, m_I^I, m_J^I). These terms may or may not be given. Consequently, some numerical iteration scheme is necessary to derive the solution. The displacement terms are directly solved at each iteration, while the element resistant forces must be computed or updated from the nodal forces. Let's denote, for the I^{th} element, the preceding element as the $(I-1)^{th}$, the terminal node as the $(I+1)^{th}$

node. Then, whenever the resistant forces in the preceding element are determined, we shall have

$$S^I = S^{I-1} - S_I$$

$$m_I^I = -m_I^{I-1} + M_I - S^{I-1}(w_I - w_{I-1}) = M_I^I \quad (17)$$

$$m_{I+1}^I = -m_I^I + R_{\ell_I}^I = M_{I+1}^I + S^I(w_{I+1} - w_I)$$

where

$$R^I = R^{I-1} + R_I,$$

and at the leading element $R^1 = R_1$, $S^1 = -S_1$, and $m_1^0 = 0$.

3.0 THERMAL LOAD

The beam-column element developed herewith is referred to a system of mechanical forces. Consistent with the structural theory, the thermal load may be incorporated therein to apply to the thermoelastic structural problems. A beam column under the thermal load is essentially equivalent to one subject to the mechanical load resulting from constraints to free thermal expansions. In the linear structural member such as the beam column, the thermal strain generates two effects. The mean value of the thermal strain over the cross-sectional area results in the axial expansion. Its gradient across the cross section effectively bends the member as if there were an exterior moment.

The axial expansion comes from the areal mean of the temperature field over the cross-sectional area, with the coefficients of thermal expansion and the Young's moduli as weighting functions. Free axial thermal expansion is ignored in the present formulation. Column effect results only when free axial thermal expansion is restricted. Such restriction is treated as prescribed axial displacements at a pair of nodes.

The thermal strain gradient is input as a thermal moment. It could result from either a temperature gradient or a difference in coefficients of thermal expansion across the beam. Because the effect is different from that of the axial thermal expansion, it is discussed separately.

3.1 Axial Thermal Deformation

It is postulated at first that the change in length of the element will be of small order to the length. The effect of restriction to thermal expansion

is similar to that of displacing one end toward the other by the same amount as the thermal expansion.

Let the axial thermal strain ϵ_T be constant between a pair of locked nodes I and J. The thermal strain is referred to a datum temperature, which in general is the temperature at which the pair of nodes first become locked. We shall label ϵ_{Ts} as the thermal strain of the support and ϵ_{Tb} as the thermal strain of the beam. If the support has much higher axial stiffness than the beam, we may express the displacement of the J^{th} node in terms of that of the I^{th} node as:

$$u_J = u_I + (\epsilon_{Ts} - \epsilon_{Tb})\ell + u_J^O \quad (18)$$

where u_J^O is the relative axial displacement of J with respect to I at the datum state.

Equation (18) may be generalized for more flexible support by introducing the stiffness ratio

$$\gamma = (EA)_b / (EA)_s \quad (19)$$

then,

$$u_J = u_I + (\epsilon_{Ts} - \epsilon_{Tb})\ell / \left[1 + \gamma \left(\frac{1 + \epsilon_{Ts}}{1 + \epsilon_{Tb}} \right) \right] + u_J^O. \quad (20)$$

For a relatively rigid support, that is $\gamma = 0$; Eq. (20) reduces to Eq. (18).

It is noticed that if u_I is an unknown, u_J may not be determined to yield a displacement prescribed boundary condition. It is therefore practical to compute the axial displacement of all locked supports sequentially along a beam prior to the use of the method. In case that I and J span over many elements of varying thermal strain, the average value is used as

$$\epsilon_{TJ\text{ave}} = \sum_i \epsilon_{Ti} \ell_i / \sum_i \ell_i. \quad (21)$$

3.2 Thermal Moment

Let the centroidal axis be the coordinate axis for the beam and the gradient of the thermal strain be linear in the principal direction of the cross-sectional area. The thermal moment at the cross-section is then

$$M_T(x) = EI \delta \epsilon_T. \quad (22)$$

The incorporation of the thermal moment may begin with replacing its axial distribution function with a piece-wise linear distribution, such that the thermal moment over a single element is linear. The entire thermal moment function is then defined by the set of step increments across the nodes $\{\Delta M_{TI}\}$ and the set of linear increments over the elements $\{\Delta M_T^I\}$, with the stipulation that $\Delta M_{TI} = M_{TI}^1$ (thermal moment at node 1 for the element 1) (see Fig. 4).

Fig. 4. Thermal moment at node 1 for the element 1.

We shall define the set of nodal moment $\{M_I^*\}$ due to exterior constraints. The adjusted nodal moments $\{M_I\}$ are then

$$M_I = M_I^* + \Delta M_{TI}. \quad (23)$$

For known exterior moments the incremental nodal thermal moments are added as by Eq. (23) for the values used in the computation. For the determined values of nodal moments, the incremental nodal thermal moments should be subtracted to obtain the values of moments resulted from exterior constraints.

The set of dual element moments $\{m_I^I, m_{I+1}^I\}$ is adjusted accordingly; such that Eq. (17) becomes

$$m_I^I = -m_I^{I-1} + M_I - S^{I-1}(w_I - w_{I-1})$$

$$m_{I+1}^I = -(m_I^I + \Delta M_{TI}^I) + R^I \varrho_I. \quad (24)$$

4.0 APPLICATION TO A THERMALLY LOADED BEAM COLUMN WITH FIXED PIVOTED SUPPORTS

Figure 5 illustrates the freebody of a beam column, which is supported by pivots at two fixed points A and B. The thermal load applied on the beam is described by an average uniform temperature of T_0 and a gradient ΔT across the beam. The mechanical effect on the beam is an equivalent moment M_T due to the thermal gradient and an axial force S as a result of restraint from axial thermal expansion. The problem is to determine the axial force and the elastica. With given coefficient of thermal expansion, the thermal moment and the axial thermal strain are respectively

$$M_T = \int_A E \alpha T(y) y dA = \int_A E \alpha (T_0 + \Delta T y) y dA = EI \alpha \Delta T \quad (25)$$

and

$$\epsilon_T = \alpha T_0.$$

The analytical solution of the problem may be readily obtained, which is to be used for comparison with the approximate solutions obtained by the finite element method. The analytical solution is

$$\eta(\xi) = -\kappa \left[\sin \beta - \sin \langle \beta \xi \rangle - \sin \langle \beta(1 - \xi) \rangle \right] / \beta^2 \sin \beta \quad (26)$$

where

$$\xi = \frac{x}{L}, \quad \eta(\xi) = \frac{w}{L}, \quad \beta^2 = \frac{SL^2}{EI}, \quad \kappa = \frac{M_T L}{EI} = \alpha \Delta T L.$$

The end extension comes from three sources: that from thermal expansion $\lambda_T (= \epsilon_T L)$; that from the axial load $\lambda_S (= -SL/AE)$; and that from the flexure of elastica λ_w such that

$$\begin{aligned} \lambda_w &= - \int_0^L \left[\sqrt{1 + w'^2} - 1 \right] dx \approx - \frac{1}{2} \int_0^L w'^2 dx = - \frac{L}{2} \int_0^1 \eta'(\xi)^2 d\xi \\ &= - \frac{L \kappa^2}{2\beta^3 \sin^2 \beta} (1 - \cos \beta)(\beta - \sin \beta). \end{aligned}$$

Fig. 5. Freebody of a beam column.

For fixed pivoted-ends problem, we have

$$\lambda_T + \lambda_S + \lambda_w = 0,$$

which results in

$$\epsilon_T - \rho^2 \beta^2 - \frac{\kappa^2}{2\beta^3 \sin^2 \beta} (1 - \cos \beta) (\beta - \sin \beta) = 0. \quad (27)$$

Equation (27) may not be solved in close form for β (thus S). But numerical solutions may be obtained for specific values of ρ , κ and ϵ_T . In other words, numerical values of the axial load may be obtained with specific value given for the beam geometry, material and the temperature field.

If the previous problem is altered to stipulate that the ends may not rotate more than a fixed value θ_0 , the bending moment at the pivot will be a combination of the thermal moment M_T and the support moment M_0 (see Fig. 6 for the freebody). The analytical solution may be shown as

$$\eta(\xi) = \frac{\theta_0}{\beta} \sin \langle \beta \xi \rangle - \frac{\bar{\kappa}}{\beta^2} (1 - \cos \langle \beta \xi \rangle)$$

such that

$$\theta_0 \sin \beta - \frac{\bar{\kappa}}{\beta} (1 - \cos \beta) = 0.$$

The crucial displacement compatibility condition becomes

$$\epsilon_T - \rho^2 \beta^2 - \frac{\theta_0^2}{2\beta} \frac{(\beta - \sin \beta)}{(1 - \cos \beta)} = 0. \quad (28)$$

Fig. 6. Bending moment at the pivot with the combination of thermal moment M_T and the support moment M_O .

4.1 Single Element Model

In the present analysis, the beam column is modeled by a single finite element, that is, $L = \ell$. The given conditions will be

$$w_I = w_J = 0 \text{ and } M_I = -M_J = EI\ell(\Delta T) = M = m_I = -m_J.$$

An initial algebraic analysis would result in

$$R_I = R_J = 0 \text{ and } \theta_I = -\theta_J = \theta,$$

which could be posulated from the symmetry condition. The simplified equations reduce to three nontrivial ones, as

$$\left. \begin{aligned} \bar{S} - \Delta \bar{S} &= (1 + \frac{\Delta \bar{u}}{2} + \bar{S} \rho^2) \Delta \bar{u} \rho^{-2} + 2\theta^2 + 2\bar{M}\theta - \frac{\bar{S}}{3} \theta^2 \\ \bar{M} &= 2\theta \left[1 + \Delta \bar{u} + \bar{S} \left(\frac{1}{12} + \rho^2 - \frac{\Delta \bar{u}}{12} \right) \right] / (1 - \Delta \bar{u}) \\ \bar{S} &= \frac{1}{6} \rho^{-2} \theta^2 \end{aligned} \right\} \quad (29)$$

where nondimensionalized variables are used; which are

$$\bar{S} = \frac{S\ell^2}{EI}, \quad \bar{M} = \frac{M\ell}{EI}, \quad \bar{u} = \frac{u}{\ell} \text{ and } \bar{w} = \frac{w}{\ell}.$$

Equation (29) may be combined to result in a cubic equation in θ and solved for given data on beam and temperature field: ρ^2 , $\bar{M}$ and $\epsilon_T (= -\Delta \bar{u})$. The last expression is equivalent to letting the beam expanding an axial thermal strain of ϵ_T then compressing it back to the original position. The displacement at the midspan is

$$\bar{w}\left(\frac{1}{2}\right) = N_w\left(\frac{1}{2}\right)^t \bar{a} = \frac{\theta}{4} \quad \text{or} \quad w\left(\frac{1}{2}\right) = \frac{1}{4} \theta L. \quad (30)$$

For the problem with restricted end rotation, same set of Eqs. (29) and (30) are used. Except, in the latter case, the unknowns in Eq. (29) are $(\bar{S}, \bar{M})$ instead of (S, θ) . The disadvantage of the single element model is evidenced by Eq. (30); that is, whatever the values of ϵ_T (or Δu) are, the midspan deflection is a constant for a fixed value of θ .

4.2 Two-Element Model

Figure 7 shows the beam column divided into two equal segments of finite elements (i.e., $L = 2\ell$).

Fig. 7. Beam column divided into two equal segments of finite elements.

The given conditions will be

$$w_I = w_K = 0; \quad m_I = -m_K = M; \quad M_J = 0; \quad \text{and} \quad S_J = 0.$$

To save on the algebra, we may impose the symmetry conditions; that

$$S_K = -S_I = S; \quad \theta_I = -\theta_K = \theta; \quad \theta_2 = 0; \quad R_I = R_J = R_K = 0$$

and

$$u_J - u_I = u_K - u_J = \Delta u \quad w_J = w.$$

Using the symmetry formulation and the dimension less parameters we may obtain three sets of equations:

$$\left. \begin{aligned} F(\bar{S}, \theta, \bar{w}) &= (1 - \Delta\bar{u} + 2\phi) \bar{S} - \rho^2 (1 + \frac{\Delta\bar{u}}{2}) \Delta\bar{u} - \bar{M} \theta - \rho^2 \phi - 4 = 0 \\ G(\bar{S}, \theta, \bar{w}) &= (6\alpha + \beta) \theta - 12(\alpha + \beta) \bar{w} = 0 \\ H(\bar{S}, \theta, \bar{w}) &= 4(\alpha + \frac{1}{3}\beta) \theta - (6\alpha + \beta) \bar{w} - (1 - \Delta\bar{u}) \bar{M} = 0 \end{aligned} \right\} \quad (31)$$

where

$$\bar{S} = \frac{S\ell^2}{EI}; \quad \bar{M} = \frac{M\ell}{EI}; \quad \Delta\bar{u} = \frac{\Delta u}{\ell}; \quad \bar{w} = \frac{w}{\ell};$$

and

$$\left. \begin{aligned} \alpha(\bar{S}) &= 1 + \bar{S}\rho^2 + \Delta\bar{u}; \quad \beta(\bar{S}) = (1 - \Delta\bar{u})\bar{S}/10 \\ \phi(\theta, \bar{w}) &= (18\bar{w}^2 - 3\bar{w}\theta + 2\theta^2)/30 \\ \psi(\theta, \bar{w}) &= 6\bar{w}^2 - 6\bar{w}\theta + 2\theta^2 \end{aligned} \right\} \quad (32)$$

With a set of given data on the beam and the temperature field, $\{\rho^2, \bar{M}, \Delta\bar{u}(= -\epsilon_T)\}$, Eq. (31) may be solved for the axial load S , the midspan deflection $w_j = w_\ell$ and the pivot rotation θ . The same set of equations is applied for the solution of a restricted-pivot-rotation problem. The given data would then be $(\rho^2, \theta, \Delta\bar{u})$ for the solution set $(\bar{S}, \bar{M}, \bar{w})$.

4.3 Numerical Results

The single element model ($L = \ell$) and the two element node ($L = 2\ell$) are then applied to a beam column with

$$\Delta T = 17 \times 10^3 \text{ }^\circ\text{C/M}, \quad EI = 33.3 \text{ N} \cdot \text{M}^2$$

for two different lengths $L = 0.15 \text{ M}$ and 0.3 M . Pertinent parameters are listed in Table I for the two different beam lengths.

The thermal strain temperature T_0 shall indicate the average temperature rise after the beam is first restrained from free axial expansion. Data are obtained for the temperature rises

$$T_0 = 20^\circ\text{C}, 50^\circ\text{C}, 100^\circ\text{C}, 200^\circ\text{C}, 300^\circ\text{C}, 400^\circ\text{C}$$

corresponding to thermal strains

$$\epsilon_T = 1.8\text{E-}4, 4.5\text{E-}4, 9\text{E-}4, 1.8\text{E-}3, 2.7\text{E-}3, 3.6\text{E-}3.$$

TABLE I
PARAMETERS FOR $L = 0.15M$ and $L = 0.3 M$

L (M)	$\rho^2 = [r_g/\ell]^2$	$\bar{M}_T = \alpha \Delta T \ell$	EI/ℓ^2 (N)	ℓ (M)	
0.15	5.9358E-4	1.1475E-2	5.92E3	0.075	$L = 2\ell$
	1.48395E-4	2.295E-2	1.48E3	0.15	$L = \ell$
0.30	1.48395E-4	2.295E-2	1.48E3	0.15	$L = 2\ell$
	3.709875E-3	4.59E-2	3.7E2	0.30	$L = \ell$

For the case of restricted end rotations, two end rotations are used;
 $\theta_o = 0.015$ rad and 0.02 rad. The support moments are obtain as

$$M_o = M_T - M = (\bar{M}_T - \bar{M})EI/\ell.$$

Figures 8 through 11 and Table II show the comparison of the finite element models against the exact solution in Eqs. (26), (27), and (28).

5.0 DISCUSSION

The new element for the beam column has the advantage of simulating the actual beam with only two elements with reasonable accuracy. It is therefore particularly suitable for the multiple span beams, especially when there are a large number of them.

The single element simulation is adequate for a beam with unrestricted ends. However, when end restrictions are present, the cubic shape function is not adequate in describing the elastica. Large errors result in deformation.

At large thermal strains the axial load approaches the Euler's load. Numerical instability is evident in some numerical analysis methods. Convergence is fairly rapid with simple updating methods up to $\epsilon_T = 9E-4$ ($T_o = 100^\circ C$).

The results are compared with the symmetrical stiffness method (as presented by Martin) in Table III.

Fig. 8. Both ends free pivot column.
($L = 0.15$ m)

Fig. 9. Both ends free pivot column. ($L = 0.30$ m).

Fig. 10. Restricted end rotation, 0.015 Rad. ($L = 0.15$ m).

Fig. 11. Restricted end rotation, 0.015 Rad. ($L = 0.30$ m).

TABLE II
COMPARISON OF SOLUTIONS: SINGLE ELEMENT: $L = l = 0.15$ M and 0.3 M
TWO ELEMENTS: $L = 2l = 0.15$ M and 0.3 M

1. Both ends hinged -- unrestrained rotation.

$T_0 [^{\circ}\text{C}]$	20		100		200		400		
$l [\text{M}]$	0.15	0.3	0.15	0.3	0.15	0.3	0.15	0.3	
$S [10^4 \text{N}]$	0.150 0.153 0.152	0.057 0.061 0.0570	0.730 0.820 0.7931	0.230 0.276 0.2342	1.18 1.36 1.186	0.28 0.338 0.2786	1.32 1.51* 1.315	0.306 0.368* 0.3055	exact $L = l$ $L = 2l$
$\delta [10^{-2}]$	1.26 1.257 1.259	2.65 2.66 2.646	2.3 2.14 2.263	5.61 6.10 5.627	5.2 5.05 5.172	8.25 9.36 8.287	9.8 10.95 9.796	11.9 13.9 11.984	exact $L = l$ $L = 2l$
$w [10^{-3} \text{M}]$	0.48 0.47 0.482	2.05 2.00 2.051	0.95 0.803 0.958	4.8 4.58 4.883	2.3 1.9 2.347	7.4 7.0 7.422	4.5 4.1 4.558	10.9 10.4 10.955	exact $L = l$ $L = 2l$

Notes: *S value larger than Euler's load. Element too stiff.

2. End rotation restricted: $M_T = 5.1$ NM, $L = 150$ mm, $S_g = 5.84\text{E-}4$ N

$T_0 [^{\circ}\text{C}]$	50		100		200		300		400		
$\theta_0 [\text{Rad}]$	0.015	0.02	0.015	0.02	0.015	0.02	0.015	0.02	0.015	0.02	
$S [10^4 \text{N}]$	0.4076 0.411 0.407	* 0.382 0.376	0.8512 0.859 0.8503	0.8160 0.830 0.8152	1.733 1.75 1.730	1.687 1.72 1.684	2.602 2.64 2.593	2.54 2.615 2.53	3.439 3.53 3.421	3.340 3.50 3.323	exact $L = l$ $L = 2l$
$w [10^{-3} \text{M}]$	0.597 0.563 0.597	* 0.75 0.79	0.641 0.563 0.6414	0.850 0.75 0.8501	0.757 0.563 0.7574	1.00 0.75 1.000	0.932 0.563 0.933	1.221 0.75 1.222	1.219 0.563 1.22	1.566 0.75 1.570	exact $L = l$ $L = 2l$
$M [\text{NM}]$	5.06 5.11* 5.05	* 6.96* 6.910*	3.11 3.42 3.118	4.35 4.71 4.373	-1.61 0.05 -1.546	-1.78 0.215 -1.692	-8.10 -3.32 -7.920	-10.03 -4.28 -9.806	-17.88 -6.68 -17.50	-21.89 -8.77 -21.45	exact $L = l$ $L = 2l$

Note: *The moment is larger than the thermal moment, then the mathematical model does not apply.

3. $L = 300$ mm, $S_g = 1.46\text{E-}4$ N

$T_0 [^{\circ}\text{C}]$	50		100		200		300		
$\theta_0 [\text{Rad}]$	0.015	0.020	0.015	0.020	0.015	0.020	0.015	0.020	
$S [10^4 \text{N}]$	0.3906 0.411 0.3912	0.3510 0.382 0.359	0.777 0.859 0.7778	0.713 0.83 0.715	1.176 1.75* 1.1809	1.10 1.72* 1.1038	1.274 2.64* 1.279	1.216 2.615* 1.220	exact $L = l$ $L = 2l$
$w [10^{-3} \text{M}]$	1.462 1.125 1.462	1.890 1.50 1.8906	2.168 1.125 2.169	2.67 1.50 2.67	4.910 1.125 4.916	5.22 1.50 5.236	7.38 1.125 7.411	7.56 1.50 7.599	exact $L = l$ $L = 2l$
$M [\text{NM}]$	-0.289 0.243 -0.2684	0.211 0.615 0.2280	-6.69 -3.12 -6.5737	-7.03 -3.87 -6.92	-28.08 -9.85 -27.93	-27.4 -12.8 -27.24	-46.5 -16.6 -46.50	-45.1 -21.8 -45.00	exact $L = l$ $L = 2l$

Note: *S value larger than the Euler force. The implication is that the finite element model is too stiff.

TABLE III
COMPARISON OF ELEMENTS

M - Martin's formulation - symmetric stiffness matrix.

N - New formulation - including the effect of nonsymmetric stiffness matrix.

Case: Restricted end rotation θ .

Thermal moment $M_T = 5.1$ N M.

End moment $M \leq M_T + M_0$ (pivot moment) ≥ 0 .

$L = 150$ mm, $S_E = 5.84E-4$ N.

1. $\theta = 0.015$ rad.

$T_O [^{\circ}C]$	50		100		200		300		400		
$\alpha T_O [1]$	4.54E-4		9E-4		1.8E-3		2.7E-3		3.6E-3		
	(M)	(N)	(M)	(N)	(M)	(N)	(M)	(N)	(M)	(N)	
	0.4076		0.851		1.73		2.60		3.44		exact
$S [10^4]$	0.411 0.409	0.411 0.407	0.860 0.856	0.859 0.850	1.76 1.75	1.75 1.73	2.65 2.64	2.64 2.59	3.54 3.52	3.53 3.42	$L = 1$ $L = 2$
	5.06		3.11		-1.61		-8.10		-17.88		exact
$M [NM]$	5.88 5.08	5.11 5.05	5.04 4.93	3.42 3.11	3.35 2.95	0.051 -1.55	1.68 0.75	-3.32 -7.92	-0.002 -1.8	-6.69 -17.5	$L = 1$ $L = 2$
	0.597		0.641		0.758		0.930		1.37		exact
$w(\frac{1}{2}) [10^{-3}M]$	0.563 0.597	0.563 0.597	0.563 0.60	0.563 0.641	0.563 0.645	0.563 0.757	0.563 0.70	0.563 0.933	0.563 0.76	0.563 1.22	$L = 1$ $L = 2$

2. $\theta = 0.020$ rad.

	*		0.816		1.687		2.54		3.34		exact
$S [10^4N]$	0.382 0.379	0.382 0.376	0.831 0.824	0.830 0.815	1.73 1.71	1.72 1.684	2.62 2.60	2.61 2.53	3.52 3.48	3.50 3.32	$L = 1$ $L = 2$
	*		4.35		-1.78		-10.03		-21.89		exact
$M [NM]$	7.9 7.87	6.96 6.91	6.8 6.66	4.71 4.37	4.55 4.05	0.215 -1.69	2.31 1.11	-4.28 -9.8	0.07 -2.25	-8.77 -21.4	$L = 1$ $L = 2$
	*		0.850		1.00		1.22		1.57		exact
$w(\frac{1}{2}) [10^{-3}M]$	0.75 0.77	0.75 0.79	0.75 0.80	0.75 0.850	0.75 0.86	0.75 1.00	0.75 0.928	0.75 1.22	0.75 1.01	0.75 1.57	$L = 1$ $L = 2$

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APPENDIX A - NOMENCLATURE

A	Cross-sectional area of the beam column
$\underline{a}$	Displacement vector of dimension (3 x number of nodes)
E	Young's modulus
$e_{k\ell}$	Linear portions of the strain tensor
EA	Axial stiffness
EI	Flexural stiffness
$\underline{f}$	Force vector of dimension (3 x number of nodes)
$\underline{f}_*$	Adjusted force vector for symmetrical stiffness matrix
I	Area moment of inertia
I, J	Superscripts for identifying elements } Element number correspond- Subscripts for identifying node points } ing to the node number initiating the element
$\underline{K}$	Stiffness matrix
$\underline{K}_S$	Stiffness matrix, symmetrical form
L	Any length parameter (e.g., span between supports)
ℓ	Length of an element
M_I	Exterior moment applied to the column at I^{th} node
M_T	Thermal moment, linear within an element
ΔM_T^I	Increase of thermal moment across an element 'I'
ΔM_{TI}	Increase of thermal moment across I^{th} node
m^I	Resistant moment of I^{th} element, linear in the element
m_I^I	Its value at the initiating end, by I^{th} node
(NOTE: The above resistant moments do not include the effect due to axial force.)	
$\underline{N}_u(x, y)$	Shape function for the axial displacement, $u(x, y)$
$\underline{N}_w(x, y)$	Shape function for the lateral displacement, $w(x, y)$

$\mathbf{0}$	Null matrix or sub-matrix
R_I	Exterior lateral force at I^{th} node
R_J^I	Cross shear of I^{th} element by J^{th} node
S_I	Exterior axial force at I^{th} node
S^I	Axial force in I^{th} element
S_E	Euler's load = $\phi^2 EI/L_2$ (L = length of column)
T	Subscript - thermal Superscript - transpose of a matrix
T_I	Average temperature at I^{th} node
ΔT_I	Temperature gradient at I^{th} node
$u(x,y)$	Axial displacement in an element
u_I	Axial displacement in I^{th} node
$w(x,y)$	Lateral displacement at I^{th} node
x,y	Axial and lateral coordinates in an element
α	Coefficient of thermal expansion
ϵ	Virtual variation
ϵ_{kl}	Strain tensor
ϵ_t	Thermal strain = αT (axial)
ϕ	Stability parameter, $S_E = \phi^2 EI/L_2$
η_{kl}	Nonlinear portion of the strain tensor
ρ	Element slenderness ratio: $\rho^2 = EI/EA\ell^2$
σ_{kl}	The stress tensor
σ	The fiber stress in beam-column
θ_I	Rotation (or slope) of beam at I^{th} node
∂_i	Partial derivative w.r.t. i^{th} coordinate
$(\bar{\quad})$	Bar over a quantity denotes the non-dimensional expression of the quantity. The mathematical relationship with the original quantity will be shown in the text.