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LA-8138-MS

Informal Report

**A Finite Element Model for
Nonlinear Shells of Revolution**

MASTER

University of California

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A Finite Element Model for Nonlinear Shells of Revolution

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NOMENCLATURE

B - Boundary for shells (ends); also the B matrix is the strain displacement approximation matrix.

$\left. \begin{array}{l} C_s, C_{s\theta}, C_\theta, \\ G, \\ D_s, D_{s\theta}, D_\theta, \\ D_\lambda \\ T_s, T_{s\theta}, T_\theta \end{array} \right\}$ - Shell orthotropic material properties

$\bar{C}_s, \bar{C}_{s\theta}, \bar{C}_\theta, \bar{G}.$ - Orthotropic material properties

E_n, E_s, E_{sn} - Strains

k - Stiffness

F - Force

h - Two- and three-nodal point shape functions for shells, also shell thickness

H - Horizontal stress resultant and a matrix of shape functions

L - Length of shell

M_s - Meridional bending moment

$\bar{M}_s$ - Applied meridional bending moment

M_θ - Circumferential (hoop) bending moment

vi - Number of nodal points in shell element

N_s	- Meridional stress resultant
N_θ	- Circumferential (hoop) stress resultant
p	- Pressure on shell
Q	- Shear stress resultant
$\bar{Q}$	- Applied shear stress resultant
q	- Meridional variable ($ds = qd\xi$)
R	- Radius of curvature
S	- Shear on shell
s	- Meridional coordinate
u_n	- Normal displacement
u_r	- Radial displacement
u_s	- Meridional displacement
u_z	- Axial displacement
V	- Vertical stress resultant
z	- Axial coordinate

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j	
	Increment (or iteration)
$j+1$	

- o Designates that this quantity refers to the undeformed shell configuration

GREEK

- α - Thermal coefficient of expansion
- β - Rotation $\phi - \phi_0$
- γ - Shear deformation
- $\Gamma(\zeta)$ - Shear deformation as a function of thickness
- δ - Variational operator
- Δ - Incremental operator and temperature increment when used with α
- ϵ_s - Meridional membrane strain
- ϵ_θ - Circumferential (hoop) membrane strain
- ζ - Thickness variable
- θ - Circumferential (hoop) coordinate (1 radian)
- κ_s - Meridional bending strain
- κ_θ - Circumferential (hoop) bending strain
- ξ - Normalized meridional coordinate for shell element
- ϕ - Azimuthal angle for shell configuration

A FINITE ELEMENT MODEL FOR NONLINEAR SHELLS OF REVOLUTION

by

W. A. Cook

ABSTRACT

A finite element model for symmetrically loaded shells of revolution is described. The nonlinear geometric effects are accounted for by incrementing loads and iterating for equilibrium. The iteration process also allows for nonlinear materials. The shell model accounts for large strains, large rotations, and shear deformation. Three example problems demonstrate the versatility and accuracy of this model. An axially loaded cylinder and an internally pressurized spherical shell have large membrane strains, whereas a cylinder that deforms into a spherical shape has large rotations.

I. INTRODUCTION AND SUMMARY

Nuclear material shipping containers have shells of revolution as basic structural components (see Fig. 1). Analytically modeling the response of these containers to severe accident impact conditions requires a nonlinear shell-of-revolution model that accounts for both geometric and material nonlinearities. In this paper I present a finite element model for a nonlinear shell of revolution that will account for large displacements, large strains, large rotations, and nonlinear materials.

Fig. 1.
A typical nuclear material shipping container.

The nonlinear shell theory presented by Eric Reissner in Refs. 1 and 2 was used to develop our model and is described in Sec. II. Reissner's approach includes transverse shear deformation and moments turning about the middle surface normal.

This nonlinear shell model is developed using the virtual work principle associated with Reissner's equilibrium equations. First, the virtual work principle is modified for incremental loading; then it is linearized by assuming that the nonlinear portions of the strains are known. By iteration, equilibrium is then approximated for each increment. A benefit of this approach is that this iteration process makes it possible to use nonlinear material properties. The details of this method are presented in Sec. III.

To interface with finite element continuum models with two- and three-nodal-point boundaries, our model has both two- and three-nodal-point isoparametric elements. The two-nodal-point element is a conical element, whereas the three-nodal-point element includes curvature terms obtained from a quadratic curve fit of the geometry of the three nodal points. The displacements and meridional rotation are the basic field variables in the elements. For the conical element, these are approximated with linear functions; for the higher order element, these variables are approximated with quadratic functions. The finite element method as it was used in this incremental/iteration technique is described in Sec. IV.

Several analytical problems were studied using this model in the NONSAP computer code.^{3,4}

- A plate and a cylinder, both with shear loadings, and a portion of a hemisphere with an applied bending moment were analyzed. These loadings were all small, and the model converged quickly to the linear solutions.
 - A hemispherical shell was pressurized. The pressure was large enough to cause a normal displacement equal to the original radius of the shell.
 - An axially loaded cylinder was stretched to double its original length.
- The last two problems demonstrate the ability of the model to calculate large membrane displacements and nonlinear membrane strains.
- A cylindrical shell was deformed into a spherical shape. The loads required for this problem were calculated from the initial and final configurations and the equilibrium conditions.

This last exercise demonstrates the ability of the model to be used in analyzing problems with large rotations.

II. BASIC THEORY

Figures 2 and 3 show an incremental deformed shell of revolution. From these figures, the equilibrium equations can be derived as (see Appendix A)

$$\frac{d(r N_s)}{ds} + r Q \frac{d\phi}{ds} - N_\theta \cos\phi + r \bar{S} = 0 ,$$

$$\frac{d(r Q)}{ds} - r N_s \frac{d\phi}{ds} - N_\theta \sin\phi + r \bar{p} = 0 ,$$

and

$$\frac{d(r M_s)}{ds} + r N_s \sin\gamma - r Q \cos\gamma - M_\theta \cos\phi + P \sin\phi = 0 , \quad (1)$$

$$\text{where } \bar{S} = S \cos\gamma - p \sin\gamma \quad \text{and} \quad \bar{p} = S \sin\gamma + p \cos\gamma .$$

These equations include transverse shear deformation, γ , and moments turning about the surface normal, P . These terms are extensions to the usual thin shell theories and make it possible to model thicker shells.

If we define q and q_0 such that $ds = q d\xi$ and $ds_0 = q_0 d\xi$ (the subscript o refers that quantity to the undeformed geometry), then the stress resultants, bending moments, and loads transform as

$$N_s = \frac{r_0}{r} N_{s0} , \quad Q = \frac{r_0}{r} Q_0 , \quad M_s = \frac{r_0}{r} M_{s0} , \quad N_\theta = \frac{q_0}{q} N_{\theta 0} ,$$

$$P = \frac{q_0}{q} P_0 , \quad M_\theta = \frac{q_0}{q} M_{\theta 0} , \quad S = \frac{r_0 q_0}{r q} S_0 , \quad \text{and} \quad p = \frac{r_0 q_0}{r q} p_0 . \quad (2)$$

For the meridional bending strain

Fig. 2.
Shell of revolution.

Fig. 3.
Stress resultants, bending moments, and loads for a symmetrically loaded increment of a shell of revolution.

$$\kappa_{so} = \frac{d\phi}{ds_0} - \frac{d\phi_0}{ds_0} ,$$

and because $ds_0 = R_0 d\phi_0$,

$$\frac{d\phi}{ds} = \frac{q_0}{q} \left(\kappa_{so} + \frac{1}{R_0} \right) . \quad (3)$$

Also, the meridional strain

$$\epsilon_{so} = \frac{q - q_0}{q_0} .$$

Thus

$$q/q_0 = 1 + \epsilon_{so} , \quad (4)$$

and the equilibrium equations may be written in terms of undeformed quantities as

$$\begin{aligned} \frac{d(r_0 N_{so})}{ds_0} + r_0 Q_0 \left(\frac{1}{R_0} + \kappa_{so} \right) - N_{\theta 0} \cos\phi + r_0 \bar{S}_0 &= 0 , \\ \frac{d(r_0 Q_0)}{ds_0} - r_0 N_{so} \left(\frac{1}{R_0} + \kappa_{so} \right) - N_{\theta 0} \sin\phi + r_0 \bar{p}_0 &= 0 , \text{ and} \end{aligned} \quad (5)$$

$$\frac{d(r_0 M_{so})}{ds_0} + (r_0 N_{so} \sin\gamma - r_0 Q_0 \cos\gamma)(1 + \epsilon_{so}) - M_{\theta 0} \cos\phi + p_0 \sin\phi = 0 ,$$

where

$$\bar{S}_0 = S_0 \cos\gamma - p_0 \sin\gamma , \text{ and}$$

$$\bar{p}_0 = S_0 \sin \gamma + p_0 \cos \gamma .$$

When the force equilibrium equations are rotated from meridional and normal to horizontal and vertical, the Reissner equilibrium equations are obtained as

$$\frac{d(r_0 H_0)}{ds_0} - N_{\theta 0} + r_0 p_{H0} = 0 , \quad \text{and} \quad \frac{d(r_0 V_0)}{ds_0} + r_0 p_{V0} = 0 , \quad (6)$$

where

$$p_{H0} = S_0 \cos(\phi - \gamma) + p_0 \sin(\phi - \gamma) , \quad p_{V0} = -S_0 \sin(\phi - \gamma) + p_0 \cos(\phi - \gamma) ,$$

$$H_0 = Q_0 \sin \phi + N_{s0} \cos \phi , \quad \text{and} \quad V_0 = Q_0 \cos \phi - N_{s0} \sin \phi .$$

The virtual work as presented by Reissner in Refs. 1 and 2 is

$$\int_{L_0} \{N_{s0}\}^T \delta \{\hat{\epsilon}_{s0}\} r_0 ds_0 = \int_{L_0} \{p_{H0}\}^T \delta \{u_r\} r_0 ds_0 + \left[\{\bar{H}_0\}^T \delta \{u_r\} r_0 \right]_{B_0} , \quad (7)$$

where

$$\{N_{s0}\}^T = (N_{s0} \ N_{\theta 0} \ Q_0 \ M_{s0} \ M_{\theta 0} \ P_0) \text{ stress resultants and bending moments,}$$

$$\{\hat{\epsilon}_{s0}\}^T = (\hat{\epsilon}_{s0} \ \epsilon_{\theta 0} \ \hat{\gamma}_0 \ \kappa_{s0} \ \kappa_{\theta 0} \ \lambda_0) \text{ membrane and bending strains,}$$

$$\{p_{H0}\}^T = (p_{H0} \ p_{V0} \ 0) \text{ initial surface loads,}$$

$$\{u_r\}^T = (u_r \ u_z \ \beta) \text{ displacements and meridional rotation, and}$$

$$\{\bar{H}_0\}^T = (\bar{H}_0 \ \bar{V}_0 \ \bar{M}_{s0}) \text{ applied stress resultants and applied bending moment.}$$

- $\bar{H}_0$ and $\bar{V}_0$ are applied stress resultants, and $\bar{M}_{s0}$ is an applied bending moment.
- L_0 is the undeformed length of the neutral surface, and B_0 represents both boundaries at the ends of the shell.
- u_r and u_z are the vertical (radial) and horizontal (axial) displacements.
- β is the rotation $\phi - \phi_0$.
- δ is the variational operator.
- The variations of the strains are

$$\delta \hat{\epsilon}_{s0} = \cos \phi \frac{d(\delta u_r)}{ds_0} - \sin \phi \frac{d(\delta u_z)}{ds_0} - \frac{q}{q_0} \sin \gamma \delta \beta, \quad \delta \epsilon_{\theta 0} = \frac{\delta u_r}{r_0},$$

$$\delta \hat{\gamma}_0 = \sin \phi \frac{d(\delta u_r)}{ds_0} + \cos \phi \frac{d(\delta u_z)}{ds_0} + \frac{q}{q_0} \cos \gamma \delta \beta, \quad \delta \kappa_{s0} = \frac{d(\delta \beta)}{ds_0},$$

$$\delta \kappa_{\theta 0} = \frac{\cos \phi}{r_0} \delta \beta, \quad \text{and} \quad \delta \lambda_0 = -\frac{\sin \phi}{r_0} \delta \beta. \quad (8)$$

- The strains are

$$\hat{\epsilon}_{s0} = \cos \phi \frac{du_r}{ds_0} - \sin \phi \frac{du_z}{ds_0} + \cos \beta - 1 = (1 + \epsilon_{s0}) \cos \gamma - 1,$$

$$\text{where } \epsilon_{s0} = \frac{q - q_0}{q_0}, \quad \epsilon_{\theta 0} = \frac{r - r_0}{r_0} = \frac{u_r}{r_0},$$

$$\hat{\gamma}_0 = \sin \phi \frac{du_r}{ds_0} + \cos \phi \frac{du_z}{ds_0} + \sin \beta = (1 + \epsilon_{s0}) \sin \gamma, \quad \kappa_{s0} = \frac{d\beta}{ds_0},$$

$$\kappa_{\theta 0} = \frac{\sin \phi - \sin \phi_0}{r_0}, \quad \text{and} \quad \lambda_0 = \frac{\cos \phi - \cos \phi_0}{r_0}. \quad (9)$$

III. NONLINEAR SHELL-OF-REVOLUTION MODEL

The computational technique used to develop this nonlinear shell-of-revolution model consists of the following four steps.

- Modify the virtual work that satisfies equilibrium for incremental loadings.
- Linearize the virtual work so that it may be solved directly.
- Use the finite element method to approximate the linearized incremental virtual work. This step is identical to solving a linear problem.
- By iteration, approximate the original nonlinear virtual work. This iteration is necessary because the virtual work was linearized, and iteration ensures that equilibrium is satisfied. The iteration step is identical to the increment step except that the applied loads do not change. Also, this iteration step makes it possible to include nonlinear materials.

The incremental loads for the j increment are defined as

$$\{\Delta P_{H0}\}_j = \{P_{H0}\}_{j+1} - \{P_{H0}\}_j \quad \text{and} \quad \{\Delta \bar{H}_0\}_j = \{\bar{H}_0\}_{j+1} - \{\bar{H}_0\}_j . \quad (10)$$

The incremental displacements, the incremental stress resultants and bending moments, and the incremental membrane and bending strains for the j increment are defined as

$$\begin{aligned} \{\Delta u_r\}_j &= \{u_r\}_{j+1} - \{u_r\}_j , \quad \{\Delta N_{so}\}_j = \{N_{so}\}_{j+1} - \{N_{so}\}_j , \quad \text{and} \\ \{\Delta \hat{\epsilon}_{so}\}_j &= \{\hat{\epsilon}_{so}\}_{j+1} - \{\hat{\epsilon}_{so}\}_j . \end{aligned} \quad (11)$$

Thus the virtual work for the j increment can be written as

$$\int_{L_0} \left\{ \Delta N_{so} \right\}_j^T \delta \left\{ \Delta \hat{\epsilon}_{so} \right\}_j r_0 ds_0 = - \int_{L_0} \left\{ N_{so} \right\}_j^T \delta \left\{ \Delta \hat{\epsilon}_{so} \right\}_j r_0 ds_0$$

$$\int_{L_0} \{p_{Ho}\}_{j+1}^T \delta\{\Delta u_r\}_j r_0 ds_0 + [r_0 \{\bar{H}_0\}_{j+1} \delta\{\Delta u_r\}_j] B_0 \quad (12)$$

This equation can be linearized by assuming $(\Delta u_r)_j$, $(\Delta u_z)_j$, and $(\Delta \beta)_j$ are small. Thus the incremental strains are $(\phi)_j$, $(\gamma_0)_j$, and $(\epsilon_{so})_j$ are known from the last increment)

$$(\Delta \epsilon_{so})_j = \cos(\phi)_j \frac{d(\Delta u_r)_j}{ds_0} - \sin(\phi)_j \frac{d(\Delta u_z)_j}{ds_0} - (\gamma_0)_j (\Delta \beta)_j, \quad (\Delta \epsilon_{\theta 0})_j = \frac{(\Delta u_r)_j}{r_0},$$

$$(\Delta \gamma_0)_j = \sin(\phi)_j \frac{d(\Delta u_r)_j}{ds_0} + \cos(\phi)_j \frac{d(\Delta u_z)_j}{ds_0} + ((\epsilon_{so})_j + 1) (\Delta \beta)_j,$$

$$(\Delta \kappa_{so})_j = \frac{d(\Delta \beta)_j}{ds_0}, \quad (\Delta \kappa_{\theta 0})_j = \frac{\cos(\phi)_j}{r_0} (\Delta \beta)_j, \quad \text{and} \quad (\Delta \lambda_0)_j = -\frac{\sin(\phi)_j}{r_0} (\Delta \beta)_j. \quad (13)$$

Assume the following constitutive relations:

$$\{N_{so}\}_j = [D]_{sj} \{\epsilon_{so}\}_j + \{N_{soi}\}_j \quad \text{and} \quad \{\Delta N_{so}\}_j = [D]_{tj} \{\Delta \epsilon_{so}\}_j + \{\Delta N_{soi}\}_j, \quad (14)$$

where

$[D]_{sj}$ and $[D]_{tj}$ are the secant and tangent material matrix for the j increment. An orthotropic material would have the following form for the material matrix.

$$[D] = \begin{bmatrix} C_s & C_{s\theta} & 0 & T_s & T_{s\theta} & 0 \\ C_{s\theta} & C_\theta & 0 & T_{s\theta} & T_\theta & 0 \\ 0 & 0 & G & 0 & 0 & 0 \\ T_s & T_{s\theta} & 0 & D_s & D_{s\theta} & 0 \\ T_{s\theta} & T_\theta & 0 & D_{s\theta} & D_\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & D_\lambda \end{bmatrix} \quad (15)$$

Lacking a good feel for D_λ , I decided to use $D_\lambda = D_\theta$, because P_o is a bending moment that uses hoop stress as does $\kappa_{\theta o}$. The other terms of the material matrix are discussed in Appendixes B, C, and D. $\{N_{soi}\}_j$ and $\{\Delta N_{soi}\}_j$ are the initial stress resultants and initial bending moments (thermal stresses can be included through these matrices).

These are defined as

$$\left\{N_{soi}\right\}_j^T = \left((N_{soi})_j \quad (N_{\theta oi})_j \quad (Q_{oi})_j \quad (M_{soi})_j \quad (M_{\theta oi})_j \quad (P_{oi})_j \right), \quad \text{and} \quad (16)$$

$$\left\{\Delta N_{soi}\right\}_j^T = \left((\Delta N_{soi})_j \quad (\Delta N_{\theta oi})_j \quad (\Delta Q_{oi})_j \quad (\Delta M_{soi})_j \quad (\Delta M_{\theta oi})_j \quad (\Delta P_{oi})_j \right). \quad (17)$$

Thus the incremental virtual work for the j increment can be written as

$$\begin{aligned} \int_{L_o} \left\{\Delta \hat{\epsilon}_{so}\right\}_j^T [D]_{tj} \delta \left\{\Delta \hat{\epsilon}_{so}\right\}_j r_o ds_o = & - \int_{L_o} \left(\left\{N_{so}\right\}_j^T + \left\{\Delta N_{soi}\right\}_j^T \right) \delta \left\{\Delta \hat{\epsilon}_{so}\right\}_j r_o ds_o \\ & + \int_{L_o} \left\{p_{Ho}\right\}_{j+1}^T \delta \left\{\Delta u_r\right\}_j r_o ds_o + \left[r_o \left\{\bar{H}_o\right\}_{j+1}^T \delta \left\{\Delta u_r\right\}_j \right] B_o. \end{aligned} \quad (18)$$

Again, the iteration step is the same as the incremental step except the loads $\{p_{Ho}\}_{j+1}$ and $\{\bar{H}_o\}_{j+1}$ do not change. Also, the material matrix $[D]_{tj}$ can change for each iteration.

IV. FINITE ELEMENT APPROXIMATION

The finite element approximation for this shell of revolution can be either two- or three-nodal point elements.

Thus the approximations for displacements and the rotation are

$$\Delta u_r = \sum_{i=1}^m h_{mi} a_{ri} , \quad \Delta u_z = \sum_{i=1}^m h_{mi} a_{zi} , \quad \text{and} \quad \Delta \beta = \sum_{i=1}^m h_{mi} a_{\beta i} , \quad (19)$$

where m is the number of nodal points in this element and h_{mi} are the shape functions. For two nodal point elements,

$$h_{21} = \frac{1}{2} (1 + \xi) \quad \text{and} \quad h_{22} = \frac{1}{2} (1 - \xi) . \quad (20)$$

For three nodal point elements,

$$h_{31} = \frac{1}{2} \xi (1 + \xi) , \quad h_{32} = -\frac{1}{2} \xi (1 - \xi) , \quad \text{and} \quad h_{33} = 1 - \xi^2 . \quad (21)$$

For isoparametric elements, r_0 and z_0 are approximated with the same shape functions.

$$r_0 = \sum_{i=1}^m h_{mi} r_{0i} = r_0(\xi) , \quad \text{and} \quad z_0 = \sum_{i=1}^m h_{mi} z_{0i} = z_0(\xi) ; \quad (22)$$

then

$$q_0 = q_0(\xi) = \left[\left(\frac{dr_0}{d\xi} \right)^2 + \left(\frac{dz_0}{d\xi} \right)^2 \right]^{1/2} = \left[\left(\sum_{i=1}^m \frac{dh_{mi}}{d\xi} r_{0i} \right)^2 + \left(\sum_{i=1}^m \frac{dh_{mi}}{d\xi} z_{0i} \right)^2 \right]^{1/2} . \quad (23)$$

Thus $r_0 ds_0$ is $r_0(\xi) q_0(\xi) d\xi$ where $-1 \leq \xi \leq 1$. In matrix notation

$$\{\Delta u_r\}_j^T = [H(\xi)] \{a_{r1}\}_j , \quad (24)$$

where

$$\{\Delta a_{r1}\}_j^T = \left((a_{r1})_j \quad (a_{z1})_j \quad (a_{\beta 1})_j \dots (a_{\beta m})_j \right) \quad \text{and}$$

$$H(\xi) = \begin{bmatrix} h_{m1} & . & . & . & h_{mm} & 0 & 0 \\ 0 & . & . & . & 0 & h_{mm} & 0 \\ 0 & . & . & . & 0 & 0 & h_{mm} \end{bmatrix} .$$

Also

$$\{\Delta \hat{\varepsilon}_{so}\}_j^T = [B]_j \{a_{r1}\}_j , \quad (25)$$

where $[B]_j$ is obtained using the linearized form of $\{\Delta \hat{\varepsilon}_{so}\}_j$ and the finite element approximations for $\{\Delta u_r\}_j$.

(26)

$$[B]_j = \begin{bmatrix} \frac{\cos(\phi)_j}{q_0} \frac{dh_{m1}}{d\xi} & . & . & \frac{\cos(\phi)_j}{q_0} \frac{dh_{mm}}{d\xi} & -\frac{\sin(\phi)_j}{q_0} \frac{dh_{mm}}{d\xi} & -(\hat{\gamma}_0)_j h_{mm} \\ \frac{h_{m1}}{r_0} & . & . & \frac{h_{mm}}{r_0} & 0 & 0 \\ \frac{\sin(\phi)_j}{q_0} \frac{dh_{m1}}{d\xi} & . & . & \frac{\sin(\phi)_j}{q_0} \frac{dh_{mm}}{d\xi} & \frac{\cos(\phi)_j}{q_0} \frac{dh_{mm}}{d\xi} & [(\hat{\varepsilon}_{so})_j + 1] h_{mm} \\ 0 & . & . & 0 & 0 & \frac{1}{q_0} \frac{dh_{mm}}{d\xi} \\ 0 & . & . & 0 & 0 & \frac{\cos(\phi)_j}{r_0} h_{mm} \\ 0 & . & . & 0 & 0 & -\frac{\sin(\phi)_j}{r_0} h_{mm} \end{bmatrix}$$

With the finite element approximations presented, the incremental virtual work may be written

$$\sum_{k=1}^K \{a_{r1}\}_{jk}^T [k]_{jk} \{a_{r1}\}_{jk} - \{F\}_{jk}^T \{a_{r1}\}_{jk} = 0 , \quad (27)$$

where

K is the number of elements used in the problem;

$$\left\{ a_{r1} \right\}_{jk}^T \quad \text{is} \quad \left\{ a_{r1} \right\}_j^T \quad \text{for the } k \text{ element}$$

$$[k]_{jk} = \left(\int_{-1}^1 [B]_j^T [D]_{tj} [B]_j r_o(\xi) q_o(\xi) d\xi \right)_k$$

and is the stiffness matrix for the k element; and

$$\begin{aligned} \{F\}_{jk}^T = & \left(\int_{-1}^1 \{p_{Ho}\}_{j+1}^T [H(\xi)] r_o(\xi) q_o(\xi) d\xi \right. \\ & - \int_{-1}^1 \left(\{N_{so}\}_j^T + \{\Delta N_{soi}\}_j^T \right) [B]_j r_o(\xi) q_o(\xi) d\xi \\ & \left. + (\text{Boundary applied forces and moments}) \right)_k \end{aligned}$$

and is the force vector for the k element.

V. VERIFICATION PROBLEMS

In this section I describe the problems I used to check this model.

Grafton and Strome developed a finite element shell-of-revolution code in the early sixties. In their paper (Ref. 5) they checked their linear code with several example problems. I used the nonlinear model described in this report to solve three of these.

- (1) A circular plate with a shear load on the boundary of an interior hole and the outer boundary fixed. This problem was modeled with 21 equally spaced nodal points and is illustrated in Fig. 4.

Fig. 4.
Circular plate with shear loading.

- (2) A cylindrical shell with a shear load on one end and the other end fixed. This problem used 25 nodal points with a very fine spacing next to the shear load and the space between nodal points increasing to a coarse spacing next to the fixed boundary edge. This problem is illustrated in Fig. 5.

- (3) A 60° segment of a hemisphere. At one boundary edge the radial coordinate was one half of the radius of the hemisphere. This boundary edge was loaded with an applied moment, and the other boundary was fixed. Again, the spacing of nodal points was very fine next to the applied moment and increasing to coarse next to the fixed boundary.

There were 29 nodal points as illustrated in Fig. 6.

This model solved all three linear problems in a nonlinear mode with exact integration and converged to the linear solutions using both two- and three-nodal-point elements. It also solved all three linear problems in a linear mode (one increment and no iterations). The three-nodal-point elements were accurate with both exact or reduced integration, whereas the two-nodal-point elements were only accurate with reduced integration or exact integration with many more nodal points than those illustrated in Figs. 4, 5, and 6. Reference 6 compares the solution of these three problems using this model with other models.

The next two example problems were solved to check whether this model could effectively calculate large membrane strains. See Appendix E for theoretical calculations for these two problems.

- (1) An axially loaded cylinder, which is shown in Fig. 7. This problem, which degenerates to a linear solution, can be solved with one increment and does not require any equilibrium iteration. Only four nodal points were used for this problem, and the answers were exact.
- (2) A pressure-loaded hemisphere, which is shown in Fig. 8. This problem was modeled with 9 nodal points and solved with one load increment and one equilibrium iteration. The maximum error in the displacements was 0.42%. Additional iterations changed the answers only slightly.

Fig. 5.
Cylindrical shell with shear loading.

Fig. 6.
Hemispherical shell with an applied bending moment.

Fig. 7.
Axially loaded cylindrical problem.

Fig. 8.
Hemisphere with thermal and pressure loadings.

These problems were solved with both two- and three-nodal-point elements and demonstrate the ability of this model to calculate large membrane strains.

The last problem solved was to load a cylinder such that it deforms into a portion of a hemisphere. Appendix E describes the load calculations that were required for this problem. This problem is shown in Fig. 9 and demonstrates the ability of this model to calculate large rotations. This problem was modeled with 21 nodal points and was solved with one load increment. See Fig. 10 for a graphical illustration of the error in the rotation β vs number of iterations. The error in the rotation was much higher than the errors in the displacements. When I used a shear modulus G calculated from $\frac{E}{2(1+\nu)}$ for this problem, the solution diverged in the iteration portion of the solution scheme for both two- and three-nodal-point elements. By increasing the shear modulus to $G = 40E$, the problem converged for two-nodal-point elements. I also found that

- (1) if γ 's are set to zero, the solution scheme diverges during iteration;
- (2) if γ 's are set to zero after the first four iterations, the iteration scheme converges to the correct answers; and
- (3) If γ 's are set to their theoretical values, the solution scheme again diverges during iteration.

Thus, for this problem, γ 's are very small and have no effect on the final answers but are necessary for the solution scheme to converge during iteration. This is typical; as the thickness decreases, shear deformations have less effect, but the conditioning for iteration is poorer. Improved iteration schemes (Ref. 7) do exist, and I expect that these would enhance the performance of this model.

VI. CONCLUSIONS AND RECOMMENDATIONS

The finite element nonlinear shell model described in this report was developed to solve

- linear problems,
- problems with large membrane strains,
- problems with large rotations, and
- problems with nonlinear materials.

As shown in Sec. V, this model accurately solves linear problems, large membrane problems, and large rotations problems. However, the iteration scheme

Fig. 9.
Cylindrical shell deforms into a spherical shape.

Fig. 10.
Rotation error (percent) vs iterations.

does not always converge for large rotations. In Ref. 8 this model was used for a nonlinear material problem and severe dynamic loads, and again stability problems halted the solution.

Thus I recommend that improved iteration schemes be investigated for this model such as those described in Ref. 7. Until that is done, I suggest that a large shear modulus be used to improve the stability of the solution scheme.

Also, remember that these nonlinear large rotations problems (50° rotation) are very difficult problems, and many researchers are searching for general techniques to solve these nonlinear problems.

One of the basic assumptions used in shell theory is to assume zero normal stresses. This assumption is valid when the normal pressure is small compared to the material moduli. This assumption was checked for the example problems illustrated in Figs. 8 and 9 (see Appendix E). In both cases, the error was less than 5%. Eliminating this assumption is very similar to adding the thermal strain effect and could easily be done on all problems except those where continuum elements are effecting the normal pressure on the shell. Then the programming would be quite involved.

APPENDIX A EQUILIBRIUM EQUATIONS

There are three equilibrium equations for a symmetrically loaded shell of revolution with shear deformation. These are equilibrium along the meridional coordinate s , the normal coordinate n , and rotational equilibrium.

From Figs. 2--3 and A-1--A-3, the meridional equilibrium equation is

$$\begin{aligned}
 & (N_s + \Delta N_s)(r + \Delta r) \Delta \theta \cos(\gamma + \frac{\Delta \gamma}{2} + \frac{\Delta \phi}{2}) - N_s r \Delta \theta \cos(\gamma + \frac{\Delta \gamma}{2} - \frac{\Delta \phi}{2}) \\
 & + (Q + \Delta Q)(r + \Delta r) \Delta \theta \sin(\gamma + \frac{\Delta \gamma}{2} + \frac{\Delta \phi}{2}) - Q r \Delta \theta \sin(\gamma + \frac{\Delta \gamma}{2} - \frac{\Delta \phi}{2}) \quad (A-1) \\
 & - 2 N_\theta \Delta s \sin(\frac{\Delta \theta}{2}) \cos(\phi - \gamma + \frac{\Delta \phi}{2} - \frac{\Delta \gamma}{2}) + S (r + \frac{\Delta r}{2}) \Delta \theta \Delta s = 0 .
 \end{aligned}$$

Fig. A-1.

Meridional cross-sectional view of incremental element (θ is constant in Fig. 2).

Fig. A-2.

Normal view of incremental element.

Fig. A-3.

Circumferential cross-sectional view of incremental element (ϕ is a constant in Fig. 2).

As $\Delta\phi - \Delta\gamma$ becomes small and $\Delta\theta$ becomes small,

$$\cos(\gamma + \frac{\Delta\gamma}{2} + \frac{\Delta\phi}{2}) \doteq \cos\gamma - \frac{(\Delta\gamma + \Delta\phi)}{2} \sin\gamma, \quad (A-2)$$

$$\sin(\gamma + \frac{\Delta\gamma}{2} + \frac{\Delta\phi}{2}) \doteq \sin\gamma + \frac{(\Delta\gamma + \Delta\phi)}{2} \cos\gamma,$$

$$\cos(\phi - \gamma + \frac{\Delta\phi}{2} - \frac{\Delta\gamma}{2}) \doteq \cos(\phi - \gamma) - \frac{(\Delta\phi - \Delta\gamma)}{2} \sin(\phi - \gamma), \text{ and } \sin(\frac{\Delta\theta}{2}) \doteq \frac{\Delta\theta}{2}$$

Substitute Eq. (A-2) into Eq. (A-1) and ignore terms with more than two incremental quantities, as these will go to zero in the limit.

$$\begin{aligned} & - N_s r \sin\gamma \Delta\phi \Delta\theta + (N_s \Delta r + r \Delta N_s) \Delta\theta \cos\gamma + Q r \cos\gamma \Delta\phi \Delta\theta \\ & + (Q \Delta r + r \Delta Q) \Delta\theta \sin\gamma - N_\theta \cos(\phi - \gamma) \Delta\theta \Delta s + S r \Delta\theta \Delta s = 0. \end{aligned} \quad (A-3)$$

Dividing this equation by $\Delta s \Delta\theta$:

$$\begin{aligned} & \frac{(N_s \Delta r + r \Delta N_s)}{\Delta s} \cos\gamma - N_s r \frac{\Delta\phi}{\Delta s} \sin\gamma + \frac{(Q \Delta r + r \Delta Q)}{\Delta s} \sin\gamma + Q r \frac{\Delta\phi}{\Delta s} \cos\gamma \\ & - N_\theta \cos(\phi - \gamma) + r S = 0. \end{aligned} \quad (A-4)$$

In the limit, this equation can be written as

$$\frac{d(r N_s)}{ds} \cos\gamma - N_s r \frac{d\phi}{ds} \sin\gamma + \frac{d(rQ)}{ds} \sin\gamma + Q r \frac{d\phi}{ds} \cos\gamma - N_\theta \cos(\phi - \gamma) + r S = 0. \quad (A-5)$$

Also, from Figs. 2--3 and A-1--A-3, the normal equilibrium equation is

$$\begin{aligned}
& N_s r \Delta \theta \sin\left(\gamma + \frac{\Delta \gamma}{2} - \frac{\Delta \phi}{2}\right) - (N_s + \Delta N_s)(r + \Delta r) \Delta \theta \sin\left(\gamma + \frac{\Delta \gamma}{2} + \frac{\Delta \phi}{2}\right) \\
& + (Q + \Delta Q)(r + \Delta r) \Delta \theta \cos\left(\gamma + \frac{\Delta \gamma}{2} + \frac{\Delta \phi}{2}\right) - Q r \Delta \theta \cos\left(\gamma + \frac{\Delta \gamma}{2} - \frac{\Delta \phi}{2}\right) \quad (A-6) \\
& - 2 N_\theta \Delta s \sin\left(\frac{\Delta \theta}{2}\right) \sin\left(\phi - \gamma + \frac{\Delta \phi}{2} - \frac{\Delta \gamma}{2}\right) + p(r + \frac{\Delta r}{2}) \Delta \theta \Delta s = 0 .
\end{aligned}$$

In a manner similar to the meridional equilibrium equation, this normal equilibrium equation can be written as

$$\begin{aligned}
& - \frac{d(r N_s)}{ds} \sin \gamma - N_s r \frac{d\phi}{ds} \cos \gamma + \frac{d(r Q)}{ds} \cos \gamma - Q r \frac{d\phi}{ds} \sin \gamma - N_\theta \sin(\phi - \gamma) \\
& + r p = 0 . \quad (A-7)
\end{aligned}$$

The meridional equilibrium Eq. (A-5) and the normal equilibrium Eq. (A-7) can be rotated by the angle γ to give

$$\begin{aligned}
& - \frac{d(r N_s)}{ds} + r Q \frac{d\phi}{ds} - N_\theta \cos \phi + r \bar{S} = 0 \text{ and } \frac{d(r Q)}{ds} - r N_s \frac{d\phi}{ds} - N_\theta \sin \phi + r \bar{p} = 0 \\
& \quad (A-8)
\end{aligned}$$

$$\text{where } \bar{S} = S \cos \gamma - p \sin \gamma \quad \text{and} \quad \bar{p} = S \sin \gamma + p \cos \gamma .$$

The rotational equilibrium equation is derived using Figs. 2--3 and A-1--A-4 as

$$\begin{aligned}
& (N_s + \Delta N_s)(r + \Delta r) \Delta \theta \sin\left(\gamma + \frac{\Delta \gamma}{2} + \frac{\Delta \phi}{2}\right) R \sin\left(\frac{\Delta \phi}{2} - \frac{\Delta \gamma}{2}\right) \\
& + N_s r \Delta \theta \sin\left(\gamma + \frac{\Delta \gamma}{2} - \frac{\Delta \phi}{2}\right) R \sin\left(\frac{\Delta \phi}{2} - \frac{\Delta \gamma}{2}\right) \\
& - (Q + \Delta Q)(r + \Delta r) \Delta \theta \cos\left(\gamma + \frac{\Delta \gamma}{2} + \frac{\Delta \phi}{2}\right) R \sin\left(\frac{\Delta \phi}{2} - \frac{\Delta \gamma}{2}\right) \quad (A-9)
\end{aligned}$$

Fig. A-4.

Meridional sectional view of M_θ and P for an incremental conical element (θ is constant).

$$- Q r \Delta \theta \cos(\gamma + \frac{\Delta \gamma}{2} - \frac{\Delta \phi}{2}) R \sin(\frac{\Delta \phi}{2} - \frac{\Delta \gamma}{2}) \\ + (M_s + \Delta M_s)(r + \Delta r) \Delta \theta$$

$$- M_s r \Delta \theta - 2 M_\theta \sin(\frac{\Delta \theta}{2}) \Delta s \cos(\phi + \frac{\Delta \phi}{2}) \\ + 2 P \sin(\frac{\Delta \theta}{2}) \Delta s \sin(\phi + \frac{\Delta \phi}{2}) = 0 .$$

Again, using the approximation for $\Delta \phi - \Delta \gamma$ and $\Delta \theta$ becoming small angles (Eq. A-2), and because

$$R \sin(\frac{\Delta \phi}{2} - \frac{\Delta \gamma}{2}) = \Delta s ,$$

Eq. (A-9) in the limit becomes

$$\frac{d(r M_s)}{ds} + r N_s \sin \gamma - r Q \cos \gamma - M_\theta \cos \phi + P \sin \phi = 0 . \quad (A-10)$$

APPENDIX B

INTEGRATION THROUGH THE THICKNESS

In this appendix the rationale for integration through the thickness is explained. Consider the following constitutive equations.

$$\sigma_{s0} = \bar{C}_s E_{s0} + \bar{C}_{s\theta} E_{\theta 0} ,$$

$$\sigma_{\theta 0} = \bar{C}_{s\theta} E_{s0} + \bar{C}_\theta E_{\theta 0} , \text{ and} \quad (B-1)$$

$$\tau_o = \bar{G} E_{s\theta 0} .$$

σ_{s0} , $\sigma_{\theta 0}$, and τ_0 are Lagrangian, the meridional, hoop, and shearing stresses (Ref. 9). E_{s0} , $E_{\theta 0}$, and $E_{s\theta 0}$ are the meridional, hoop, and shearing strains. $\bar{C}_s$, $\bar{C}_\theta$, $\bar{C}_{s\theta}$, and $\bar{G}$ are orthotropic material properties.

The strains are (see Appendix C)

$$E_{s0} = \frac{\epsilon_{s0} + \zeta \kappa_{s0}}{1 + \zeta \frac{1}{R_0}} ,$$

$$E_{\theta 0} = \frac{\epsilon_{\theta 0} + \zeta \kappa_{\theta 0}}{1 + \zeta \frac{\sin \phi_0}{r_0}} , \text{ and} \quad (B-2)$$

$$E_{s\theta 0} = \frac{\gamma_0}{1 + \zeta \frac{1}{R_0}} .$$

The stress resultants are defined as (see Appendix D)

$$N_{s0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{s0} \left(1 + \zeta \frac{\sin \phi_0}{r_0} \right) d\zeta ,$$

$$N_{\theta 0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta 0} \left(1 + \zeta \frac{1}{R_0} \right) d\zeta , \text{ and} \quad (B-3)$$

$$Q_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_0 \left(1 + \zeta \frac{\sin \phi_0}{r_0} \right) d\zeta .$$

Note, $\frac{1}{R_0} = \frac{d\phi_0}{ds_0}$.

Similarly the bending moments are defined as

$$M_{s0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{s0} \zeta \left(1 + \zeta \frac{\sin \phi_0}{r_0} \right) d\zeta , \text{ and}$$

(B-4)

$$M_{\theta 0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta 0} \zeta \left(1 + \zeta \frac{1}{R_0} \right) d\zeta .$$

Thus

$$N_{s0} = C_s \hat{\epsilon}_{s0} + C_{s\theta} \epsilon_{\theta 0} + T_s \kappa_{s0} + T_{s\theta} \kappa_{\theta 0} ,$$

$$N_{\theta 0} = C_{s\theta} \hat{\epsilon}_{s0} + C_{\theta} \epsilon_{\theta 0} + T_{s\theta} \kappa_{s0} + T_{\theta} \kappa_{\theta 0} ,$$

(B-5)

$$Q_0 = G \hat{\gamma}_0 ,$$

$$M_{s0} = T_s \hat{\epsilon}_{s0} + T_{s\theta} \epsilon_{\theta 0} + D_s \kappa_{s0} + D_{s\theta} \kappa_{\theta 0} , \text{ and}$$

$$M_{\theta 0} = T_{s\theta} \hat{\epsilon}_{s0} + T_{\theta} \epsilon_{\theta 0} + D_{s\theta} \kappa_{s0} + D_{\theta} \kappa_{\theta 0} ,$$

where

$$C_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{C}_s \left(1 + \zeta \frac{\sin \phi_0}{r_0} \right) \left(1 + \zeta \frac{1}{R_0} \right)^{-1} d\zeta ,$$

$$C_{s\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{C}_{s\theta} d\zeta \quad ,$$

$$C_{\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{C}_{\theta} \left(1 + \zeta \frac{1}{R_0}\right) \left(1 + \zeta \frac{\sin\phi_0}{r_0}\right)^{-1} d\zeta \quad ,$$

$$D_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{C}_s \zeta^2 \left(1 + \zeta \frac{\sin\phi_0}{r_0}\right) \left(1 + \zeta \frac{1}{R_0}\right)^{-1} d\zeta \quad ,$$

$$D_{s\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{C}_{s\theta} \zeta^2 d\zeta \quad ,$$

(B-6)

$$D_{\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{C}_{\theta} \zeta^2 \left(1 + \zeta \frac{1}{R_0}\right) \left(1 + \zeta \frac{\sin\phi_0}{r_0}\right)^{-1} d\zeta \quad ,$$

$$G = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{G} \left(1 + \zeta \frac{\sin\phi_0}{r_0}\right) \left(1 + \zeta \frac{1}{R_0}\right)^{-1} d\zeta \quad ,$$

$$T_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \zeta \bar{C}_s \left(1 + \zeta \frac{\sin\phi_0}{r_0}\right) \left(1 + \zeta \frac{1}{R_0}\right)^{-1} d\zeta \quad ,$$

$$T_{s\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{C}_{s\theta} \zeta \, d\zeta \quad , \text{ and}$$

$$T_{\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{C}_{\theta} \zeta \left(1 + \zeta \frac{1}{R_0}\right) \left(1 + \zeta \frac{\sin \phi_0}{r_0}\right)^{-1} d\zeta \quad .$$

Note that

- (1) for plastic material, $\bar{C}_s$, $\bar{C}_{\theta}$, and $\bar{C}_{s\theta}$ may vary with ζ , and
- (2) for isotropic material,

$$\bar{C}_s = \frac{E}{1 - \nu^2} = \bar{C}_{\theta} \quad , \quad \text{and} \quad \bar{C}_{s\theta} = \nu \bar{C}_s \quad .$$

For thin shells

$$\zeta \frac{\sin \phi_0}{r_0} \ll 1 \quad \text{and} \quad \zeta \frac{1}{R_0} \ll 1 \quad . \quad (B-7)$$

For thick shells

$$\left(1 + \zeta \frac{1}{R_0}\right)^{-1} \doteq 1 - \zeta \frac{1}{R_0} \quad \text{and} \quad \left(1 + \zeta \frac{\sin \phi_0}{r_0}\right)^{-1} \doteq 1 - \zeta \frac{\sin \phi_0}{r_0} \quad . \quad (B-8)$$

The extension to thick shells has not yet been made in the computer code discussed in Ref. 8.

APPENDIX C NONLINEAR STRAINS

In this appendix we will derive general strains in terms of the shell theory strains (membrane strains and bending strains). This exercise will demonstrate some of the limitations of this theory. Reference 1 discusses general strains for the small strain condition.

For this discussion we will need geometric relations for three separate states. These are illustrated with Figs. C-1--C-3. They are as follows.

(1) Undeformed state (Fig. C-1):

$$\Delta s_0 \Delta \xi = q_0 \Delta \xi = R_0 \Delta \phi_0 ,$$

$$\Delta s_0(\zeta) = (1 + \zeta \frac{1}{R_0}) \Delta s_0 ,$$

$$\frac{\Delta r_0}{\Delta s_0} = \cos \phi_0 ,$$

$$\frac{\Delta z_0}{\Delta s_0} = - \sin \phi_0 , \tag{C-1}$$

$$r_0(\zeta) = r_0 + \zeta \sin \phi_0 ,$$

$$z_0(\zeta) = z_0 + \zeta \cos \phi_0 ,$$

$$\Delta r_0(\zeta) = \Delta r_0 + \zeta \Delta \phi_0 \cos \phi_0 , \text{ and}$$

$$\Delta z_0(\zeta) = \Delta z_0 - \zeta \Delta \phi_0 \sin \phi_0 .$$

(2) Deformed state (Fig. C-2):

$$\Delta s = q \Delta \xi = R(\Delta \phi - \Delta \gamma) ,$$

$$\frac{\Delta r(\zeta)}{\Delta s(\zeta)} = \cos(\phi - \Gamma(\zeta)) ,$$

Fig. C-1.
Meridional cross-sectional view of incremental element (original geometry).

Fig. C-2.
Meridional cross-sectional view of incremental element (deformed geometry).

Fig. C-3.
Meridional cross-sectional view of incremental element (deformed geometry except for shear deformation).

$$\frac{\Delta z(\zeta)}{\Delta s(\zeta)} = -\sin(\phi - \Gamma(\zeta)) \quad ,$$

$$\frac{\Delta r}{\Delta s} = \cos(\phi - \gamma) \quad ,$$

$$\frac{\Delta z}{\Delta s} = -\sin(\phi - \gamma) \quad ,$$

(C-2)

$$r(\zeta) = r + \zeta \sin\phi \quad ,$$

$$z(\zeta) = z + \zeta \cos\phi \quad ,$$

$$\Delta r(\zeta) = \Delta r + \zeta \Delta\phi \cos\phi \quad , \text{ and}$$

$$\Delta z(\zeta) = \Delta z - \zeta \Delta\phi \sin\phi \quad .$$

(3) Deformed state without shear deformation (Fig. C-3):

$$\Delta \bar{s} = \bar{q} \Delta \xi = \bar{R} \Delta \phi \quad \text{and}$$

$$\Delta \bar{s}(\zeta) = \left(1 + \zeta \frac{1}{\bar{R}}\right) \Delta \bar{s} = \left(1 + \zeta \frac{\Delta \phi}{\Delta \bar{s}}\right) \Delta \bar{s} \quad (C-3)$$

From Figs. C-2 and C-3 we can see that

$$\Delta \bar{s} = \Delta s \cos\gamma \quad \text{and} \quad \Delta \bar{s}(\zeta) = \Delta s(\zeta) \cos\Gamma(\zeta) \quad . \quad (C-4)$$

Because

$$\Delta \bar{s}(\zeta) = \left(1 + \zeta \frac{\Delta \phi}{\Delta \bar{s}}\right) \Delta \bar{s} \quad , \quad \Delta s(\zeta) = \frac{1}{\cos\Gamma(\zeta)} \left(\cos\gamma + \zeta \frac{\Delta \phi}{\Delta \bar{s}}\right) \Delta s \quad . \quad (C-5)$$

Consider the following strains with respect to the deformed configuration.

$$E_{\theta} = \frac{r(\zeta) - r_0(\zeta)}{r(\zeta)} ,$$

$$E_s = \frac{\Delta \bar{s}(\zeta) - \Delta s_0(\zeta)}{\Delta \bar{s}(\zeta)} , \text{ and} \quad (C-6)$$

$$E_{sn} = \tan \Gamma(\zeta) .$$

From Eqs. (C-1) and (C-2), $r(\zeta) = r + \zeta \sin \phi$ and $r_0(\zeta) = r_0 + \zeta \sin \phi_0$,
the hoop strain is

$$E_{\theta} = \frac{\left(\frac{r - r_0}{r} \right) + \zeta \left(\frac{\sin \phi - \sin \phi_0}{r} \right)}{1 + \zeta \frac{\sin \phi}{r}} = \frac{\epsilon_{\theta} + \zeta \kappa_{\theta}}{1 + \zeta \frac{\sin \phi}{r}} . \quad (C-7)$$

The meridional strain E_s can be written as (from Eq. (C-6))

$$E_s = 1 - \frac{\Delta s_0(\zeta)}{\Delta \bar{s}(\zeta)} . \quad (C-8)$$

From Eqs. (C-1), (C-4), and (C-5), the meridional strain (C-8) can be written
as

$$E_s = 1 - \left(\frac{\frac{q_0}{q} + \zeta \frac{\Delta \phi_0}{\Delta s}}{\cos \gamma - \zeta \frac{\Delta \phi}{\Delta s}} \right) ; \quad (C-9)$$

or

$$E_s = \frac{\cos \gamma - \frac{q_0}{q} + \zeta \left(\frac{\Delta \phi}{\Delta s} - \frac{\Delta \phi_0}{\Delta s} \right)}{\cos \gamma - \zeta \frac{\Delta \phi}{\Delta s}} = \frac{1 - \frac{q_0}{q \cos \gamma} + \zeta \left(\frac{\Delta \phi}{\Delta s} - \frac{\Delta \phi_0}{\Delta s} \right)}{1 - \zeta \frac{1}{R}} = \frac{\epsilon_s + \zeta \kappa_s}{1 + \zeta \frac{1}{R}} . \quad (C-10)$$

Next, we will determine a relation for the shearing strain E_{sn} . From the following (C-2) equations,

$$\begin{aligned}\Delta r(\zeta) &= \Delta s(\zeta) \cos(\phi - \Gamma(\zeta)) \quad , \\ \Delta z(\zeta) &= -\Delta s(\zeta) \sin(\phi - \Gamma(\zeta)) \quad , \\ \Delta r &= \Delta s \cos(\phi - \gamma) \quad , \text{ and} \\ \Delta z &= -\Delta s \sin(\phi - \gamma) \quad ,\end{aligned}\tag{C-11}$$

the following equation can be derived

$$\sin \Gamma(\zeta) = \frac{\Delta s}{\Delta s(\zeta)} \sin \gamma \quad .\tag{C-12}$$

Substituting Eq. (C-5) into Eq. (C-12) gives

$$E_{sn} = \tan \Gamma(\zeta) = \frac{\sin \gamma}{\cos \gamma + \zeta \frac{\Delta \phi}{\Delta s}} = \frac{\tan \gamma}{1 + \zeta \frac{\Delta \phi}{\cos \gamma \Delta s}} \quad .\tag{C-13}$$

From Eq. (C-3), $\Delta \bar{s} = \cos \gamma \Delta s$, and from Eq. (C-3), $\frac{1}{R} = \frac{\Delta \phi}{\Delta \bar{s}}$.

Thus Eq. (C-13) can be written as

$$E_{sn} = \frac{\tan \gamma}{1 + \zeta \frac{1}{R}} = \frac{\gamma}{1 + \zeta \frac{1}{R}} \quad .\tag{C-14}$$

Consider the following strains with respect to the undeformed configuration.

$$E_{\theta 0} = \frac{r(\zeta) - r_0(\zeta)}{r_0(\zeta)} \quad \text{and} \quad E_{s0} = \frac{\Delta \bar{s}(\zeta) - \Delta s_0(\zeta)}{\Delta s_0(\zeta)} \quad .\tag{C-15}$$

From Eqs. (C-1) and (C-2), $r(\zeta) = r + \zeta \sin\phi$ and $r_0(\zeta) = r_0 + \zeta \sin\phi_0$, the hoop strain is

$$E_{\theta 0} = \frac{\left(\frac{r - r_0}{r_0}\right) + \zeta \left(\frac{\sin\phi - \sin\phi_0}{r_0}\right)}{1 + \zeta \frac{\sin\phi_0}{r_0}} = \frac{\epsilon_{\theta 0} + \zeta \kappa_{\theta 0}}{1 + \zeta \frac{\sin\phi_0}{r_0}} . \quad (C-16)$$

Next, we will consider the meridional strain E_{s0} . Substituting an equation from Eq. (C-4) into the equation for E_{s0} in Eq. (C-15),

$$\frac{\Delta s(\zeta)}{\Delta s_0(\zeta)} \cos\Gamma(\zeta) = 1 + E_{s0} , \text{ and thus}$$

$$\Delta s(\zeta) = \frac{(1 + E_{s0})}{\cos\Gamma(\zeta)} \Delta s_0(\zeta) = \frac{(1 + E_{s0})}{\cos\Gamma(\zeta)} \left(1 + \zeta \frac{1}{R_0}\right) \Delta s_0 . \quad (C-17)$$

Also, from Eqs. (C-5), (C-1), and (C-2),

$$\Delta s(\zeta) = \frac{1}{\cos\Gamma(\zeta)} \left(\cos\gamma + \zeta \frac{\Delta\phi}{\Delta s}\right) \Delta s \quad \text{and} \quad \frac{\Delta s}{\Delta s_0} = \frac{q}{q_0} . \quad (C-18)$$

Thus

$$\Delta s(\zeta) = \frac{1}{\cos\Gamma(\zeta)} \left(\cos\gamma + \zeta \frac{\Delta\phi}{\Delta s}\right) \frac{q}{q_0} \Delta s_0 . \quad (C-19)$$

From Eqs. (C-17) and (C-19),

$$(1 + E_{s0}) \left(1 + \zeta \frac{1}{R_0}\right) = \left(\cos\gamma + \zeta \frac{\Delta\phi}{\Delta s}\right) \frac{q}{q_0} ;$$

$$\text{thus } E_{s0} = \frac{\frac{q}{q_0} \cos\gamma - 1 + \zeta \left(\frac{\Delta\phi}{\Delta s_0} - \frac{\Delta\phi_0}{\Delta s_0}\right)}{1 + \zeta \frac{1}{R_0}} = \frac{\hat{\epsilon}_{s0} + \zeta \kappa_{s0}}{1 + \zeta \frac{1}{R_0}} . \quad (C-20)$$

Last, we will determine a relation for the shearing strain E_{sno} . From Eqs. (C-7), (C-10), (C-16), and (C-20),

$$\epsilon_{\theta 0} = \frac{r}{r_0} \epsilon_{\theta} \quad ,$$

$$\hat{\epsilon}_{so} = \frac{q \cos \gamma}{q_0} \hat{\epsilon}_s \quad ,$$

(C-21)

$$\kappa_{so} = \frac{q \cos \gamma}{q_0} \kappa_s \quad , \text{ and}$$

$$\kappa_{\theta 0} = \frac{r}{r_0} \kappa_{\theta} \quad .$$

Also, from Ref. 2,

$$\hat{\gamma}_0 = \frac{q}{q_0} \sin \gamma \quad .$$

(C-22)

Thus, from Eq. (C-14) we can show

$$\hat{\gamma}_0 = \frac{q \cos \gamma}{q_0} \hat{\gamma} \quad .$$

(C-23)

An inspection of Eqs. (C-20) and (C-14) shows that

$$E_{sno} = \frac{\hat{\gamma}_0}{1 + \zeta \frac{1}{R_0}} \quad .$$

(C-24)

APPENDIX D STRESS RESULTANTS AND BENDING MOMENTS

The stress resultants and bending moments as shown in Fig. A-1 are defined as

$$N_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_s \left(1 + \zeta \frac{\sin \phi}{r} \right) d\zeta ,$$

$$N_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta \left(1 + \zeta \frac{1}{R} \right) d\zeta ,$$

$$Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau \left(1 + \zeta \frac{\sin \phi}{r} \right) d\zeta , \tag{D-1}$$

$$M_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_s \zeta \left(1 + \zeta \frac{\sin \phi}{r} \right) d\zeta , \text{ and}$$

$$M_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta \zeta \left(1 + \zeta \frac{1}{R} \right) d\zeta ,$$

where

$\cos\Gamma(\zeta) = \cos\gamma$ in Eq. (C-5), and

$\frac{1}{R} = \frac{d\phi}{d\bar{s}}$, as shown in Fig. C-3.

The stresses σ_s , σ_θ , and τ are Cauchy Stresses (also called Eulerian or true stresses, see Ref. 9).

From Eq. (2)

$$N_{so} = \frac{r}{r_0} N_s, \quad Q_o = \frac{r}{r_0} Q, \quad M_{so} = \frac{r}{r_0} M_s, \quad N_{\theta o} = \frac{q}{q_0} N_\theta, \quad \text{and} \quad M_{\theta o} = \frac{q}{q_0} M_{\theta o}. \quad (D-2)$$

Let the Lagrangian strains σ_{so} , $\sigma_{\theta o}$, and τ_o be defined as (see Ref. 9)

$$\begin{aligned} \sigma_{so} (r_0 + \zeta \sin\phi_0) &= \sigma_{\bar{s}} (r + \zeta \sin\phi), \\ \sigma_{\theta o} q_0 \left(1 + \frac{\zeta}{R_0}\right) &= \sigma_\theta q \left(1 + \frac{\zeta}{R}\right), \quad \text{and} \\ \tau_o (r_0 + \zeta \sin\phi_0) &= \tau (r + \zeta \sin\phi). \end{aligned} \quad (D-3)$$

When Eq. (D-3) is substituted into Eq. (D-1) and Eq. (D-1) is substituted into Eq. (D-2), Eqs. (B-3), and (B-4),

$$N_{so} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{so} \left(1 + \zeta \frac{\sin\phi_0}{r_0}\right) d\zeta,$$

$$N_{\theta 0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta 0} \left(1 + \zeta \frac{1}{R_0} \right) d\zeta ,$$

$$Q_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_0 \left(1 + \zeta \frac{\sin \phi_0}{r_0} \right) d\zeta . \quad (D-4)$$

$$M_{s0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{s0} \zeta \left(1 + \zeta \frac{\sin \phi_0}{r_0} \right) d\zeta , \text{ and}$$

$$M_{\theta 0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta 0} \zeta \left(1 + \zeta \frac{1}{R_0} \right) d\zeta .$$

The thickness h is the thickness in the deformed configuration.

APPENDIX E

NONLINEAR GEOMETRIC PROBLEMS

Axially Loaded Cylinder

Consider the cylinder shown in Fig. 7. What are the displacements for an axially loaded cylinder? From the equilibrium equation Eq. (5),

$$\frac{d(r_0 N_{s0})}{ds_0} = 0 \quad , \quad N_{\theta 0} = 0 \quad , \quad Q_0 = 0 \quad , \quad M_{s0} = 0 \quad , \quad M_{\theta 0} = 0 \quad , \quad \text{and } P_0 = 0 \quad . \quad (E-1)$$

Also, there is no shear deformation. Thus

$$r_0 N_{s0} = \text{constant}. \quad (E-2)$$

From Eq. (2) of Sec. II,

$$r_0 N_{s0} = r N_s = F, \quad \text{or} \quad N_{s0} = \frac{F}{r_0} \quad (\Delta\theta = 1 \text{ radian}). \quad (E-3)$$

The constitutive equations for these stress resultants give

$$N_{s0} = \frac{F}{r_0} = C_s \epsilon_{s0} + C_{s\theta} \epsilon_{\theta 0} \quad , \quad \text{and} \quad (E-4)$$

$$N_{\theta 0} = 0 = C_\theta \epsilon_{s0} + C_{\theta\theta} \epsilon_{\theta 0} \quad ,$$

$$\text{where } \epsilon_0 = \frac{q}{q_0} - 1 \quad , \quad \text{and} \quad \epsilon_{\theta 0} = \frac{u_r}{r_0} \quad .$$

Solving these equations for ϵ_{s0} and $\epsilon_{\theta 0}$ gives

$$\epsilon_{\theta 0} = \frac{-C_{s\theta} F}{r_0 (C_s C_\theta - C_{s\theta}^2)} \quad , \quad (E-5)$$

$$\epsilon_{s0} = \frac{C_\theta F}{r_0 (C_s C_\theta - C_{s\theta}^2)} \quad .$$

Because $\frac{q}{q_0} = \frac{\ell}{\ell_0}$,

u_r and ℓ can be solved for as

$$u_r = \frac{-C_{s\theta} F}{C_s C_\theta - C_{s\theta}^2} \quad \text{and} \quad \ell = \frac{\ell_0 C_\theta F}{r_0 (C_s C_\theta - C_{s\theta}^2)} + \ell_0 . \quad (\text{E-6})$$

For $F = 10^6$ lb ,

$r_0 = 10$ in. ,

$\ell_0 = 5$ in. ,

$E = 10^6$ psi ,

$\nu = 0.25$,

and $h = 0.1$, the materials are

$$C_s = C_\theta = \frac{Eh}{1 - \nu^2} = 1.06667 \cdot 10^5 \text{ lb/in.} , \text{ and}$$

$$C_{s\theta} = \frac{\nu Eh}{1 - \nu^2} = .266667 \cdot 10^5 \text{ lb/in.}$$

Thus

$$u_r = -2.5 \text{ in.} \quad \text{and} \quad \ell = 10 \text{ in.} \quad (\text{E-7})$$

Expansion of a Hemisphere

Consider the hemisphere in Fig. 8. What pressure is required to double the size of the hemisphere? From Fig. 8, $u_s = 0$, $u_n = c$, and $\phi = \phi_0$. Thus the membrane and bending strains are

$$\epsilon_{s\theta} = \frac{q}{q_0} - 1 = \frac{a + c}{a} - 1 = \frac{c}{a} , \quad \epsilon_{\theta\theta} = \frac{c \sin \phi_0}{a \sin \phi_0} = \frac{c}{a} , \quad (\text{E-8})$$

$\gamma = 0$ (no shear deformation), $\kappa_{s0} = 0$, $\kappa_{\theta 0} = 0$, and $P_0 = 0$.

The constitutive equations for this membrane problem of isotropic material can be written as (includes thermal strains)

$$N_{s0} = \frac{Eh}{1 - \nu^2} \{ \epsilon_{s0} - \alpha\Delta + \nu (\epsilon_{\theta 0} - \alpha\Delta) \} , \text{ and} \quad (E-9)$$

$$N_{\theta 0} = \frac{Eh}{1 - \nu^2} \{ \epsilon_{\theta 0} - \alpha\Delta + \nu (\epsilon_{s0} - \alpha\Delta) \} .$$

Because $\epsilon_{s0} = \epsilon_{\theta 0}$,

$$N_{s0} = N_{\theta 0} = \frac{Eh}{1 - \nu} (c/a - \alpha\Delta) .$$

Again from Fig. 8

$$r_0 = a \sin \phi_0 \quad \text{and} \quad R_0 = a , \quad (E-10)$$

and thus the equilibrium equation, Eq. (5), gives

$$-\frac{r_0}{R_0} N_{s0} - N_{\theta 0} \sin \phi_0 + r_0 p_0 = 0 , \quad (E-11)$$

which results in

$$p_0 = \frac{N_{s0} + N_{\theta 0}}{a} = \frac{2 Eh}{a(1 - \nu)} (c/a - \alpha\Delta) . \quad (E-12)$$

$$\text{Also } p = p_0 \frac{r_0 q_0}{r q} = p_0 \frac{a^2}{(a + c)^2} \quad (E-13)$$

for $a = 12.0 \text{ in.}$,
 $c = 12.0 \text{ in.}$,
 $E = 10^6 \text{ psi}$,
 $\nu = .25$,
 $h = .4 \text{ in.}$, and
 $\alpha\Delta = .25 \text{ (thermal strain)}$, the loads are
 $p_0 = .66667 \times 10^5 \text{ psi}$, and
 $p = .16667 \times 10^5 \text{ psi}$.

Cylinder Deforms Into a Spherical Shape

For this problem we will assume a cylinder deforms into a spherical shape, and then we calculate the loads that would be necessary to do this. Thus from Fig. 9

$$\phi_0 = \pi/2 \quad , \quad r = a \sin(\phi - \gamma) \quad , \quad z = a \cos(\phi - \gamma) \quad , \quad (E-14)$$

$$s = a(\phi - \gamma) - a(\phi_i - \gamma_i) \quad , \quad \text{and} \quad \phi = \frac{s}{a} + \gamma + (\phi_i - \gamma_i) \quad ;$$

$$\text{also} \quad \frac{q}{q_0} = \frac{a(\pi/2 - \phi_i + \gamma_i)}{r_0} \quad (\text{a constant}) \quad .$$

Thus the membrane strains

$$\hat{\epsilon}_{s0} = \frac{q}{q_0} \cos\gamma - 1 \quad , \quad \epsilon_{\theta 0} = \frac{a \sin(\phi - \gamma)}{r_0} - 1 \quad , \quad \text{and} \quad \gamma_0 = \frac{q}{q_0} \sin\gamma \quad . \quad (E-15)$$

The bending strains are

$$\kappa_{s0} = \frac{d\beta}{ds_0} = \frac{q}{q_0} \frac{d\beta}{ds} = \frac{q}{q_0} \left(\frac{d\phi}{ds} \right) = \frac{q}{q_0} \left(\frac{1}{a} + \frac{d\gamma}{ds} \right) \quad , \quad (E-16)$$

$$\text{where} \quad \beta = \phi - \phi_0 \quad , \quad \kappa_{\theta 0} = \frac{\sin\phi}{r_0} - \frac{1}{r_0} \quad , \quad \text{and} \quad \lambda_0 = \frac{\cos\phi}{r_0} \quad .$$

For an isotropic material (except for shear modulus G), the stress resultants are

$$N_{s0} = C \left\{ \frac{q}{q_0} \cos \gamma + \frac{\nu a}{r_0} \sin(\phi - \gamma) - (1 + \nu)(1 + \alpha \Delta) \right\} , \quad (E-17)$$

$$N_{\theta 0} = C \left\{ \frac{\nu q}{q_0} \cos \gamma + \frac{a}{r_0} \sin(\phi - \gamma) - (1 + \nu)(1 + \alpha \Delta) \right\} , \quad \text{and}$$

$$Q_0 = \bar{G} \frac{q}{q_0} \sin \gamma ,$$

where

$$C = \frac{Eh}{1 - \nu^2} \quad \text{and} \quad \bar{G} = Gh .$$

The bending moments are

$$M_{s0} = D \left\{ \frac{q}{q_0} \left(\frac{1}{a} + \frac{d\gamma}{ds} \right) + \nu \left(\frac{\sin \phi}{r_0} - \frac{1}{r_0} \right) \right\} , \quad (E-18)$$

$$M_{\theta 0} = D \left\{ \frac{\nu q}{q_0} \left(\frac{1}{a} + \frac{d\gamma}{ds} \right) + \frac{\sin \phi}{r_0} - \frac{1}{r_0} \right\} ,$$

$$P_0 = D \frac{\cos \phi}{r_0} ,$$

where

$$D = \frac{Eh^3}{12(1 - \nu^2)} .$$

When these quantities are substituted into the equilibrium equations, Eq. (5), the rotational equation gives

$$B_1 \frac{d^2 \gamma}{ds^2} + B_2 \gamma + B_3 = 0 \quad , \quad (E-19)$$

where

$$B_1 = D r_o \left(\frac{q}{q_o} \right)^2 \quad ,$$

$$B_2 = (C - \bar{G}) r_o \left(\frac{q}{q_o} \right)^2 - C r_o \frac{q}{q_o} (1 + \nu)(1 + \alpha \Delta) + \nu Ca \left(\frac{q}{q_o} \right) \sin \left(\frac{s}{a} + \phi_i - \gamma_i \right)$$

$$- \frac{D\lambda}{r_o} \sin \left(\frac{s}{a} + \phi_i - \gamma_i \right) \quad ,$$

$$B_3 = \frac{D\lambda}{r_o} \cos \left(\frac{s}{a} + \phi_i - \gamma_i \right) \quad , \quad \text{and}$$

γ is assumed small.

The boundary conditions for this equation are

$$(1) \quad \gamma = 0 \quad \text{at} \quad \phi = \pi/2 \quad , \quad \text{and}$$

$$(2) \quad \frac{d\gamma}{ds} = 0 \quad \text{at} \quad \phi = \pi/2 \quad (\text{symmetry boundary condition}). \quad (E-20)$$

Equation (E-19) can be solved using a finite difference scheme as follows.

$$B_1 \left(\frac{\gamma_{i+1} - 2\gamma_i + \gamma_{i-1}}{(\Delta s)^2} \right) + \gamma_i B_2 + B_3 = 0 \quad (E-21)$$

where $[i]$ is a counter for the incremental.

$$\gamma_{i+1} = 2\gamma_i - \gamma_{i-1} - \left(\frac{\Delta s}{B_1}\right)^2 (\gamma_i B_2 + B_3) \text{ or, if } \left| B_1 \frac{d^2\gamma}{ds^2} \right| \ll B_1 \gamma + B_3, \quad (E-22)$$

then $\gamma_i B_2 - B_3 = 0$, and

$$\gamma_i = \frac{B_3}{B_2}. \quad (E-23)$$

With the γ known, the two equilibrium equations, Eq. (6), give the loads required to deform a cylinder into a spherical shape. These are

$$p_{H_0} = \sin^2 \phi \left(\frac{q}{q_0}\right)^2 \left\{ \frac{C}{a} + (C - \bar{G}) \frac{d\gamma}{ds} \right\} + \gamma \cos \phi \left(\frac{q}{q_0}\right)^2 \left\{ -\frac{\bar{G}}{a} + (C - \bar{G}) \frac{d\gamma}{ds} \right\} \\ - \sin \phi \left(\frac{q}{q_0}\right) C(1 + \nu) \left\{ \frac{1}{a} + \frac{d\gamma}{ds} \right\} - \frac{\nu C}{r_0} \left(\frac{q}{q_0}\right) \cos(\phi - \gamma) \cos \phi \quad (E-24)$$

$$+ \frac{\nu C}{r_0} \left(\frac{q}{q_0}\right) a \sin(\phi - \gamma) \sin \phi \left\{ \frac{1}{a} + \frac{d\gamma}{ds} \right\} + \frac{\nu C}{r_0} \left(\frac{q}{q_0}\right) \cos \gamma + \frac{Ca}{r_0^2} \sin(\phi - \gamma)$$

$$- (1 - \nu) \frac{C}{r_0} - \left(\frac{q}{q_0}\right) \sin \phi \left\{ \frac{1}{a} + \frac{d\gamma}{ds} \right\} (1 + \nu) \alpha \Delta C - \frac{C}{r_0} (1 + \nu) \alpha \Delta$$

and

$$p_{V_0} = \cos^2 \phi \left(\frac{q}{q_0}\right)^2 \left\{ \frac{C}{a} + (C - \bar{G}) \frac{d\gamma}{ds} \right\} + \gamma \sin \phi \left(\frac{q}{q_0}\right)^2 \left\{ \frac{\bar{G}}{a} + (\bar{G} - C) \frac{d\gamma}{ds} \right\}$$

$$- \cos \phi \left(\frac{q}{q_0}\right) C(1 + \nu) \left\{ \frac{1}{a} + \frac{d\gamma}{ds} \right\} + \frac{\nu C}{r_0} \left(\frac{q}{q_0}\right) \cos(\phi - \gamma) \sin \phi \quad (E-25)$$

$$+ \frac{\nu C a}{r_0} \left(\frac{q}{q_0} \right) \sin(\phi - \gamma) \cos \phi \left\{ \frac{1}{a} + \frac{d\gamma}{ds} \right\} + C \alpha \Delta (1 + \nu) \left(\frac{q}{q_0} \right) \cos \phi \left\{ \frac{1}{a} + \frac{d\gamma}{ds} \right\} .$$

The problem solved used

$$a = 5 ,$$

$$\ell_0 = 4 ,$$

$$\alpha \Delta = 0 \text{ (thermal strain),}$$

$$\nu = .25 ,$$

$$E = 10^5 , \text{ and}$$

$$G = 40 E .$$

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