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Masses and Mixing Angles in SU(5) Gauge Model

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Abstract

We obtain Georgi and Jarlskog mass relations $m_\mu/m_e = 9m_s/m_d$, $m_b = m_\tau$ above the grand unification mass $M = 10^{15}$ GeV with two $\underline{5}$'s and one $\underline{45}$ Higgs representations of SU(5) and a discrete symmetry. In the lowest order, the Kobayashi-Maskawa angles are found to be $s_2 = -(m_c/m_t)^{\frac{1}{2}}$ and $s_3 = -(m_u/m_t)^{\frac{1}{2}}/s_1$, where s_1 is the sine of the Cabibbo angle. The CP violation is considered and the b-quark decays predominantly into c-quarks with lifetime of $\tau_b \approx 10^{-13}$ sec. for $m_t = 25$ GeV.

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The SU(5) grand unified model of Georgi and Glashow¹ that unifies the color SU(3) and the electroweak SU(2) $\times$ U(1) gauge model has generated a great deal of interest. In this model the left-handed fermions for each family are assigned to the $\underline{5}^*$ and $\underline{10}$ representations in which case the choice of the representation of the Higgs fields (that generate the fermion masses) are $\underline{5}$ and $\underline{45}$. It was suggested² that the lepton-quark mass relations above the grand unification mass $M = 10^{15}$ GeV should be the following^{#1}

$$m_\tau = m_b, \quad m_\mu/m_e = 9m_s/m_d, \quad (1)$$

so that the mass renormalization^{3,4} leads to $m_\tau \approx m_b/3$ and the other relation remains unchanged at low energies. Equation (1) was obtained with the aid of three $\underline{5}$'s and one $\underline{45}$ of Higgs. The question of strong CP violation was addressed in a similar model and no violation was found in the tree approximation and the violation was shown to be computable in higher order.⁵

We present here an SU(5) gauge model, supplemented by a discrete symmetry, that uses two $\underline{5}$'s and one $\underline{45}$ of Higgs to generate the fermion masses. We obtain the mass relations (1) and expressions for the Kobayashi-Maskawa⁶ mixing angles s_2 and s_3 in terms of the quark mass ratios and the Cabibbo angle. The weak CP violating phase is taken into account and no strong CP violation is found in the tree approximation.

Let the three $\underline{10}$ and $\underline{5}$ multiplets of SU(5) corresponding to the three families of left-handed fermions be denoted by ψ_{iL}^{ab} and χ_{jR}^c , and the two $\underline{5}$'s and one $\underline{45}$ of Higgs by ϕ_k^d and Σ_{id}^{ef} , respectively. The indices a, b, c, d, e, and f are SU(5) indices and run 1-5, i and j refers to the number of families and runs 1-3 and k runs 1-2. The most general Yukawa couplings of fermions and Higgs bosons in SU(5) are given by

$$\begin{aligned} \mathcal{L}_Y = & f_{ij}^k \bar{\psi}_{iL}^{ab} \chi_{jR}^a \phi_k^b + g_{ij} \bar{\psi}_{iL}^{ab} \chi_{jR}^c \sum_c^{ab} + h_{ij}^k \bar{\psi}_{iL}^{abT} \gamma^0 \psi_{jL}^{cd} \phi_k^e \epsilon_{abcde} \\ & + d_{ij} \bar{\psi}_{iL}^{abT} \gamma^0 \psi_{jL}^{cd} \sum_c^{ef} \epsilon_{abdef} + \text{h.c.} \end{aligned} \quad (2)$$

In Eq.(2), $\psi^T \gamma^0 = \bar{\psi}^c$ where ψ^c is the charge conjugate field, the letter T means transpose, ϵ_{abcde} is the completely antisymmetric five index tensor, and a sum over repeated indices is implied.

In order to restrict the coupling parameters, the following discrete symmetry is imposed,

$$\begin{aligned} \psi_{1L} &\rightarrow \eta^{2m} \psi_{1L}, \quad \psi_{2L} \rightarrow \eta^{2n} \psi_{2L}, \quad \psi_{3L} \rightarrow \eta^{(m+n)} \psi_{3L}, \\ \chi_{1R} &\rightarrow \eta^{-(m-7n)} \chi_{1R}, \quad \chi_{2R} \rightarrow \eta^{(m+5n)} \chi_{2R}, \quad \chi_{3R} \rightarrow \eta^{6n} \chi_{3R}, \\ \phi_1 &\rightarrow \eta^{(m-5n)} \phi_1, \quad \phi_2 \rightarrow \eta^{-2(m+n)} \phi_2, \quad \Sigma \rightarrow \eta^{-(m+3n)} \Sigma, \end{aligned} \quad (3)$$

where m and n are arbitrary numbers and $|\eta^i| = 1$. The resulting Yukawa coupling is

$$\begin{aligned} \mathcal{L} = & (A' \bar{\psi}_{1L}^{ab} \chi_{2R}^a + A \bar{\psi}_{2L}^{ab} \chi_{1R}^a + B \bar{\psi}_{3L}^{ab} \chi_{3R}^a) \phi_1^b + \frac{1}{2} C \bar{\psi}_{2L}^{ab} \chi_{2R}^c \sum_c^{ab} \\ & + \frac{1}{4} (D \bar{\psi}_{1L}^{abT} \gamma^0 \psi_{2L}^{cd} + E \bar{\psi}_{3L}^{abT} \gamma^0 \psi_{3L}^{cd}) \phi_2^e \epsilon_{abcde} \\ & + \frac{1}{12} F \bar{\psi}_{2L}^{abT} \gamma^0 \psi_{3L}^{cd} \sum_c^{ef} \epsilon_{abdef} + \text{h.c.} \end{aligned} \quad (4)$$

At the level of $SU(3) \times SU(2)$ symmetry, we expect there are three Higgs multiplets (which are $SU(3)$ color singlets and $SU(2)$ weak doublets) that generate masses corresponding to the familiar three generations. In the Lagrangian (4) we

do have three Higgs fields, two $\tilde{5}$'s and one $\tilde{45}$. Note that the term with the coefficient F does not contribute to the lepton masses.

After spontaneous symmetry breaking, the fermion mass matrices M^U , M^D , and M^ℓ for charge $2/3$ quarks $\psi_1^0 = (u_o, c_o, t_o)$, charge $-1/3$ quarks $\psi_2^0 = (d_o, s_o, b_o)$, and leptons $\psi_\ell^0 = (e_o, \mu_o, \tau_o)$ take the form, respectively,

$$M^U = \begin{pmatrix} 0 & v_2 D e^{-i\alpha_2} & 0 \\ v_2 D e^{-i\alpha_2} & 0 & vF \\ 0 & vF & v_2 E e^{-i\alpha_2} \end{pmatrix}, \quad (5)$$

$$M^D = \begin{pmatrix} 0 & v_1 A' e^{i\alpha_1} & 0 \\ v_1 A e^{i\alpha_1} & vC & 0 \\ 0 & 0 & v_1 B e^{i\alpha_1} \end{pmatrix}, \quad (6)$$

$$M^\ell = \begin{pmatrix} 0 & v_1 A' e^{i\alpha_1} & 0 \\ v_1 A e^{i\alpha_1} & -3vC & 0 \\ 0 & 0 & v_1 B e^{i\alpha_1} \end{pmatrix}, \quad (7)$$

where the vacuum expectation values (TeV) are $\langle \phi_i^5 \rangle = v_i e^{i\alpha_i}$ and $\langle \Sigma_b^{a5} \rangle = v(\delta_b^a - 4\delta^{a4} \delta_{b4})$, $i=1-2$, $a,b=1-4$. We note from Eqs. (5), (6), and (7) that for M^U , equal coupling parameters are associated with the same VeV, and for M^D and M^ℓ the same situation holds when $A=A'$ (which is approximately the case as we shall see later). The effect is that the mass matrices are independent of the scale i.e., the size of the VeV's. One can eliminate the phases in the mass matrices (5) and (6) by a suitable redefinition of the charge $2/3$ and $-1/3$ quark states so that $\text{Arg}[\text{Det} M^U \text{Det} M^D] = 0$ and hence there is no strong CP

violation in the tree approximation.^{‡2}

Since M^U and M^D are not hermitian, we diagonalize the square of the mass matrices $M^U M^{U\dagger}$ and $M^D M^{D\dagger}$ given by

$$M^U M^{U\dagger} = \begin{pmatrix} D^2 & 0 & DF e^{-i\alpha_2} \\ 0 & D^2 + F^2 & EF e^{i\alpha_2} \\ DF e^{i\alpha_2} & EF e^{-i\alpha_2} & E^2 + F^2 \end{pmatrix}, \quad (8)$$

$$M^D M^{D\dagger} = \begin{pmatrix} A'^2 & A'C e^{i\alpha_1} & 0 \\ A'C e^{-i\alpha_1} & A^2 + C^2 & 0 \\ 0 & 0 & B^2 \end{pmatrix}, \quad (9)$$

where the vacuum expectation values v_1 , v_2 , and v are suppressed. The characteristic equation for $M^U M^{U\dagger}$ with eigenvalues (m_u^2, m_c^2, m_t^2) yields

$$m_u^2 + m_c^2 + m_t^2 = 2(D^2 + F^2) + E^2,$$

$$m_u^2 m_c^2 + m_c^2 m_t^2 + m_t^2 m_u^2 = 2D^2 E^2 + (D^2 + F^2)^2,$$

$$m_u m_c m_t = D^2 E, \quad (10)$$

and for $M^D M^{D\dagger}$ with eigenvalues (m_d^2, m_s^2, m_b^2) yields

$$m_d^2 + m_s^2 = A'^2 + A^2 + C^2,$$

$$m_d^2 m_s^2 = A^2 A'^2,$$

$$m_b^2 = B^2. \quad (11)$$

The solution of Eq.(10) corresponding to the eigenvalues for the masses $(m_u, -m_c, m_t)$ of Eq.(5) with $\alpha_2 = 0$ is

$$D = [m_u m_c m_t / m_t - m_c + m_u]^{\frac{1}{2}}, \quad E = m_t - m_c + m_u,$$

$$F = [m_u m_c + m_c m_t - m_t m_u - (m_u m_c m_t / m_t - m_c + m_u)]^{\frac{1}{2}}. \quad (12)$$

There are four unknowns A, A', B , and C in Eq.(11). We shall find that A' is determined by the Cabibbo angle θ_c , m_d , and m_s .

The mass matrices (8) and (9) are diagonalized respectively by the following unitary transformations $\psi_1^0 = U_1 \psi_1$ and $\psi_2^0 = U_2 \psi_2$

$$U_1 = \begin{pmatrix} N_1 D / m_u & -N_2 D / m_c & N_3 D / m_t \\ N_1 e^{2i\alpha_2} & N_2 e^{2i\alpha_2} & N_3 e^{2i\alpha_2} \\ (N_1 F / m_u - E) e^{i\alpha_2} & -(N_2 F / m_c + E) e^{i\alpha_2} & (N_3 F / m_t - E) e^{i\alpha_2} \end{pmatrix}, \quad (13)$$

where $N_i = [1 + (D/m_i)^2 + (F/m_i - E)^2]^{-\frac{1}{2}}$, $m_i = (m_u, -m_c, m_t)$,

$$U_2 = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta e^{-i\alpha_1} & \cos \theta e^{-i\alpha_1} & 0 \\ 0 & 0 & e^{-i\alpha_1} \end{pmatrix}, \quad (14)$$

where

$$\tan^2 \theta = (A'^2 - m_d^2) / (m_s^2 - A'^2). \quad (15)$$

The charged current coupled to physical quarks are

$J_{\mu L} = \bar{\psi}_{1L}^0 \gamma_\mu \psi_{2L}^0 = \bar{\psi}_{1L} \Gamma \gamma_\mu \psi_{2L}$ where phase transformations $\psi_{1L} \rightarrow Q_1 P_1 \psi_{1L}$ and $\psi_{2L} \rightarrow Q_2 P_2 \psi_{2L}$ are introduced to bring Γ to the form

$$\Gamma = Q_1^\dagger P_1^\dagger U_1^\dagger U_2 P_2 Q_2$$

$$= Q_1^\dagger \begin{pmatrix} (N_1 D/m_u) \cos \theta - N_1 \sin \theta e^{-i\delta} & (N_1 D/m_u) \sin \theta + N_1 \cos \theta e^{-i\delta} & (N_1 F/m_u - E) \\ -(N_2 D/m_c) \cos \theta e^{i\delta} - N_2 \sin \theta & -(N_2 D/m_c) \sin \theta e^{i\delta} + N_2 \cos \theta & -(N_2 F/m_c + E) e^{i\delta} \\ (N_3 D/m_t) \cos \theta e^{i\delta} - N_3 \sin \theta & (N_3 D/m_t) \sin \theta e^{i\delta} + N_3 \cos \theta & (N_3 F/m_t - E) e^{i\delta} \end{pmatrix} Q_2, \quad (16)$$

where

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\delta} & 0 \\ 0 & 0 & e^{-i\delta} \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\beta} \end{pmatrix},$$

and $\delta = \alpha_1 + 2\alpha_2$, $\beta = -\alpha_1 - \alpha_2$, and the phase transformations Q_1 and Q_2 are defined shortly.

The leading approximation to Eq.(16) with the aid of (11) and (12) is

$$\Gamma = Q_1^\dagger \begin{pmatrix} \cos \theta - (m_u/m_c)^{\frac{1}{2}} \sin \theta e^{-i\delta} & \sin \theta + (m_u/m_c)^{\frac{1}{2}} \cos \theta e^{-i\delta} & -(m_u/m_t)^{\frac{1}{2}} \\ -\sin \theta - (m_u/m_c)^{\frac{1}{2}} \cos \theta e^{i\delta} & \cos \theta - (m_u/m_c)^{\frac{1}{2}} \sin \theta e^{i\delta} & -(m_c/m_t)^{\frac{1}{2}} e^{i\delta} \\ -\sin \theta (m_c/m_t)^{\frac{1}{2}} & (m_c/m_t)^{\frac{1}{2}} \cos \theta & e^{i\delta} \end{pmatrix} Q_2. \quad (17)$$

Let us define the Cabibbo angle θ_c as

$$\cos \theta - (m_u/m_c)^{\frac{1}{2}} \sin \theta e^{-i\delta} = \cos \theta_c e^{i\rho},$$

$$\sin \theta + (m_u/m_c)^{\frac{1}{2}} \cos \theta e^{-i\delta} = \sin \theta_c e^{i\sigma},$$

(18)

where ρ and σ are arbitrary phases. Then the choice of

$$Q_1 = \begin{pmatrix} e^{i\rho} & 0 & 0 \\ 0 & e^{-i\sigma} & 0 \\ 0 & 0 & e^{-i\sigma} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i(\rho-\sigma)} & 0 \\ 0 & 0 & e^{i\rho} \end{pmatrix}, \quad (19)$$

yields the Kobayashi-Maskawa form Γ

$$\Gamma = \begin{pmatrix} c_1 & s_1 & s_1 s_3 \\ -s_1 & c_1 & s_2 e^{i\delta_c} \\ s_1 s_2 & -c_1 s_2 - s_3 e^{i\delta_c} & e^{i\delta_c} \end{pmatrix}, \quad (20)$$

where

$$s_1 = \sin\theta_c, \quad s_2 = \sin\theta_2 = -(m_c/m_t)^{\frac{1}{2}} + (m_u/m_t)^{\frac{1}{2}} \cot\theta_c e^{i\delta_c},$$

$$s_3 = \sin\theta_3 = -(m_u/m_t)^{\frac{1}{2}} (\sin\theta_c)^{-1}, \quad \delta_c = \rho + \sigma + \delta. \quad (21)$$

Note that in this approximation θ_c depends on an unknown constant A' as given by Eq.(15). If we put the familiar expression $\tan^2\theta = (m_d/m_s)$ in Eq.(15) then $A' = A = (m_d m_s)^{\frac{1}{2}}$ with the aid of Eq.(11), while the experimental value $\tan\theta = 0.23$ gives $A = A'$ to within 10%. The lepton mass matrix can be handled in a similar way. The combination of the equations corresponding to (11) [that follow from (7)], (11), and the result $C^2 \gg A^2, A'^2$ lead to the mass relations (1).

The CP violation in $K^0 - \bar{K}^0$ system is proportional to $\text{Im } \Gamma_{31}^* \Gamma_{32}$ of (20) and is expressible in terms of the mixing angles and δ_c by⁷

$$|\epsilon| = 2 |c_1 c_2 c_3 s_2 s_3 \sin\delta_c| [c_2^2 (m_t^2/m_t^2 - m_c^2) \ln(m_t^2/m_c^2) - c_2^2 + s_2^2 (m_t^2/m_c^2)]. \quad (22)$$

We obtain from Eqs.(20)-(22)

$$|\epsilon| = 2(m_c m_s m_u / m_d m_t^2)^{\frac{1}{2}} [\ln(m_t^2 / m_c^2) - 1 + (m_t / m_c)] |\sin \delta_c| = 0.5 |\sin \delta_c|, \quad (23)$$

with $m_s / m_d = 20$, $m_u = 4$ MeV, $m_c = 1.5$ GeV and $m_t = 25$ GeV. The experimental value $\epsilon = 2 \times 10^{-3}$ is much smaller than (23) unless $\sin \delta_c$ is very small.

We note from (20) and (21) that $\Gamma(b \rightarrow u + X) / \Gamma(b \rightarrow c + X) = m_u / m_c = 2.7 \times 10^{-3}$ so that b-quarks decay predominantly into c-quarks. The b-quark lifetime τ_b can be obtained from $\tau_b = \tau_\mu (m_\mu / m_b)^5 / (m_c / m_t)$, where $\tau_\mu = 2.2 \times 10^{-6}$ sec. is the muon lifetime of mass m_μ . One obtains $\tau_b \cong 2 \times 10^{-13}$ sec. for $m_t = 25$ GeV.

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#1 The actual mass relations given in Ref. 2 are $m_\tau = m_b$, $m_\mu = 3m_s$, and $m_e = m_d/3$.

The present model also leads to these three mass relations.

#2 We thank R. N. Mohapatra for a discussion on this point.