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**Magnetic Field, Vector Potential,
and Their Partial Derivatives
due to a Current-Carrying
Straight Wire of Finite Length**

D. K. Lee

OPERATED BY
UNION CARBIDE CORPORATION
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

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ORNL/CSD/TM-146

MAGNETIC FIELD, VECTOR POTENTIAL, AND THEIR PARTIAL DERIVATIVES
DUE TO A CURRENT-CARRYING STRAIGHT WIRE OF FINITE LENGTH

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ABSTRACT

A numerical study is made of a problem dealing with calculation of the magnetic field $\vec{B}$, the magnetic vector potential $\vec{A}$, and first partial derivatives of the $\vec{B}$ field and $\vec{A}$ vector components for two kinds of magnetic sources: 1) current-carrying straight wire of finite length and 2) current-carrying closed polygon. No restrictions are imposed on the type of the polygon and hence it need not be plane and can consist of arbitrary number of straight wires of arbitrary lengths. Separate consideration of the polygon case makes the numerical procedure more efficient than that based on the linear superposition of the straight-wire sources. All necessary quantities are derived analytically in simple and closed forms without use of any approximations and exact relations are utilized to the greatest possible extent to give an efficient algorithm. Thus, the involved numerical procedures are quite simple, fast, accurate, and straightforward. Results are given in four different coordinate systems (cartesian, cylindrical, spherical, and local toroidal) and results in any other orthogonal coordinates can be obtained with only a few minor modifications.

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1. FORMULATION IN CARTESIAN COORDINATE SYSTEM

The magnetic source under consideration is a straight wire with end points a and b , which carries current I from a to b , as shown in Fig. 1. All of the magnetic quantities are evaluated at point c , which is the observation point (or field point). In this section, explicit expressions are derived for the magnetic field $\vec{B}$, the magnetic vector potential $\vec{A}$, their magnitudes $|\vec{B}|$ and $|\vec{A}|$, and their partial derivatives in terms of coordinates of the three given points a , b , and c . Throughout this paper, superscripts a , b , and c will be used for points a , b , and c , respectively, and subscripts are used for vector components. The notation is simplified by using

$$\vec{x}^{\alpha\beta} = \vec{x}^{\alpha} - \vec{x}^{\beta}, \quad \vec{x} = \vec{x}^c, \quad \nabla = \partial/\partial\vec{x} = \partial/\partial\vec{x}^c.$$

The magnetic field $\vec{B}$ (also known as magnetic flux density or magnetic induction) can be evaluated from the Biot-Savart law¹

$$\vec{B}(\vec{x}) = k \int \vec{J}(\vec{\xi}) \times \frac{(\vec{x} - \vec{\xi})}{|\vec{x} - \vec{\xi}|^3} d\vec{\xi}, \quad (1)$$

where $\vec{J}(\vec{\xi})$ is the current density and $k = \mu_0/4\pi = 10^{-7}$ in the MKSA system of units and $k = c^{-1}$ in the Gaussian system, c being the speed of light. The integral in Eq. (1) can be integrated analytically (for the configuration shown in Fig. 1) with the result

$$|\vec{B}| = kI \left(\frac{1}{|\vec{x}^{ac}| \tan \theta^a} + \frac{1}{|\vec{x}^{bc}| \tan \theta^b} \right). \quad (2)$$

Since the direction of the $\vec{B}$ field is normal to the plane containing the three points a, b, and c, it follows that the components of $\vec{B}$ are given by

$$B_i = \frac{|\vec{B}| G_i}{|\vec{x}^{ac}| |\vec{x}^{bc}| \sin \theta^c} \quad (3)$$

where

$$G_i = x_j^{cb} x_k^{ac} - x_k^{cb} x_j^{ac} \quad (4)$$

with $i = 1, 2, 3$ and (i, j, k) representing the cyclic permutations of $(1, 2, 3)$. Subscripts 1, 2, and 3 denote x, y, and z components of a vector, respectively.

Next, expressions for $\nabla |\vec{B}|$ and ∇B_i can be obtained through a rather straightforward process of differentiation. The results are

$$\begin{aligned} \frac{\partial |\vec{B}|}{\partial x_j} = kI & \left(\frac{x_j^{ac}}{|\vec{x}^{ac}|^3 \tan \theta^a} + \frac{x_j^{bc}}{|\vec{x}^{bc}|^3 \tan \theta^b} \right. \\ & \left. - \frac{1}{|\vec{x}^{ac}| \sin^2 \theta^a} \frac{\partial \theta^a}{\partial x_j} - \frac{1}{|\vec{x}^{bc}| \sin^2 \theta^b} \frac{\partial \theta^b}{\partial x_j} \right), \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\partial B_i}{\partial x_j} = & \frac{B_i}{|\vec{B}|} \frac{\partial |\vec{B}|}{\partial x_j} + B_i \left(\frac{\vec{x}_j^{ac}}{|\vec{x}^{ac}|^2} + \frac{\vec{x}_j^{bc}}{|\vec{x}^{bc}|^2} - \cot \theta^c \frac{\partial \theta^c}{\partial x_j} \right) \\ & + \frac{|\vec{B}|}{|\vec{x}^{ac}| |\vec{x}^{bc}| \sin \theta^c} \frac{\partial G_i}{\partial x_j}, \end{aligned} \quad (6)$$

where

$$\frac{\partial G_i}{\partial x_j} = \epsilon_{ijk} x_k^{ab}, \quad i \neq k \neq j, \quad (7)$$

$$\epsilon_{ijk} = \begin{cases} 1, & (i, j, k) \text{ is cyclic permutation of } (1, 2, 3) \\ -1, & (i, j, k) \text{ is cyclic permutation of } (3, 2, 1) \\ 0, & i = j. \end{cases} \quad (8)$$

The expression for the derivative $\partial \theta^a / \partial x_j$ in Eq. (5) can be obtained from the relation

$$|\vec{x}^{bc}|^2 = |\vec{x}^{ab}|^2 + |\vec{x}^{ac}|^2 - 2 |\vec{x}^{ab}| |\vec{x}^{ac}| \cos \theta^a \quad (9)$$

and reduces to

$$\frac{\partial \theta^a}{\partial x_j} = \frac{\vec{x}_j^{ab}}{|\vec{x}^{ab}| |\vec{x}^{ac}| \sin \theta^a} - \frac{\vec{x}_j^{ac} \cot \theta^a}{|\vec{x}^{ac}|^2}. \quad (10)$$

The corresponding expression for $\partial \theta^b / \partial x_j$ can be derived in an analogous way, or simply by interchanging the superscripts a and b in Eq. (10):

$$\frac{\partial \theta^b}{\partial x_j} = \frac{-x_j^{ab}}{|\vec{x}^{ab}| |\vec{x}^{bc}| \sin \theta^b} - \frac{x_j^{bc} \cot \theta^b}{|\vec{x}^{ac}|^2} . \quad (11)$$

The third angle derivative appearing in Eq. (6) is found in a similar fashion by using

$$|\vec{x}^{ab}|^2 = |\vec{x}^{ac}|^2 + |\vec{x}^{bc}|^2 - 2|\vec{x}^{ac}| |\vec{x}^{bc}| \cos \theta^c , \quad (12)$$

but it has a somewhat different form

$$\frac{\partial \theta^c}{\partial x_j} = \frac{1}{|\vec{x}^{ac}| |\vec{x}^{bc}| \sin \theta^c} \left[x_j^{ac} + x_j^{bc} - \left(\frac{|\vec{x}^{ac}|}{|\vec{x}^{bc}|} x_j^{bc} + \frac{|\vec{x}^{bc}|}{|\vec{x}^{ac}|} x_j^{ac} \right) \cos \theta^c \right] . \quad (13)$$

We now turn to the evaluation of the magnetic vector potential $\vec{A}$ and partial derivatives of its components. With the choice of the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, the basic formula for $\vec{A}$ analogous to Eq. (1) is¹

$$\vec{A}(\vec{x}) = k \int \frac{\vec{J}(\vec{\xi})}{|\vec{x} - \vec{\xi}|} d\xi , \quad (14)$$

which can be integrated in the case of a straight wire to obtain

$$|\vec{A}| = kI \ln \left(\frac{\csc \theta^a + \cot \theta^a}{\csc \theta^b - \cot \theta^b} \right) , \quad (15)$$

$$A_i = \frac{-x_i^{ab}}{|\vec{x}^{ab}|} |\vec{A}| \quad . \quad (16)$$

Differentiation of Eqs. (15) and (16) yields

$$\frac{\partial |\vec{A}|}{\partial x_j} = -kI \left(\csc \theta^a \frac{\partial \theta^a}{\partial x_j} + \csc \theta^b \frac{\partial \theta^b}{\partial x_j} \right) , \quad (17)$$

$$\frac{\partial A_i}{\partial x_j} = \frac{-x_i^{ab}}{|\vec{x}^{ab}|} \frac{\partial |\vec{A}|}{\partial x_j} \quad . \quad (18)$$

We close the section with the examination of some of the important relations satisfied by $\vec{B}$ and $\vec{A}$:

$$\vec{\nabla} \times \vec{B} = 0 \quad , \quad (19)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad (20)$$

$$\vec{B} \cdot \vec{\nabla} |\vec{B}| = 0 \quad , \quad (21)$$

$$\vec{\nabla} \times \vec{A} = \vec{B} \quad , \quad (22)$$

$$\vec{\nabla} \cdot \vec{A} = 0 \quad , \quad (23)$$

$$\vec{B} \cdot \vec{\nabla} |\vec{A}| = 0 \quad . \quad (24)$$

These relations can be used in testing analytical and numerical results of $\vec{B}$ and partial derivatives of $\vec{B}$, $\vec{A}$, $|\vec{B}|$, and $|\vec{A}|$. Equation (19) is a special case of the general form $\vec{\nabla} \times \vec{B} = 4\pi k \vec{J}$, where $\vec{J}$ is the current density, which in our case vanishes at the observation point $\vec{x} = \vec{x}^c$. It is easily recognized that Eqs. (21) and (24) are consequences of the

special geometry involving rotational symmetry around the wire $\vec{x}^{ab}$, which implies $\vec{\nabla}|\vec{B}|$ and $\vec{\nabla}|\vec{A}|$ are perpendicular to $\vec{B}$. After somewhat lengthy but straightforward algebra, it can be shown that our derived expressions do indeed satisfy Eqs. (20)-(22) and (24), but they do not satisfy Eqs. (19) and (23). It must be remarked here that Eqs. (19) and (23) are based on the assumption that $\vec{\nabla} \cdot \vec{J} = 0$, which is not satisfied by the magnetic sources at the two end points of the wire. Therefore, in order to meet the requirement of $\vec{\nabla} \cdot \vec{J} = 0$ everywhere, it is necessary to have a closed-loop current configuration, in our case, by joining together at least three straight wires. Thus, Eqs. (19) and (23) are satisfied by a current loop (but not by a straight wire) and Eqs. (21) and (24) are satisfied by a straight wire (but not by a loop), while Eqs. (20) and (22) are valid in all cases.

2. ORTHOGONAL TRANSFORMATION

In many applications it is more convenient to use coordinates other than cartesian coordinates, such as cylindrical, spherical, and local toroidal. To simplify the analysis, we will not consider a general curvilinear coordinate system; only orthogonal systems will be studied here. Assume that the cartesian coordinates $\vec{x} = (x_1, x_2, x_3)$ are expressed as functions of other coordinates $\vec{x}' = (x'_1, x'_2, x'_3)$ so that $\vec{x} = \vec{x}(\vec{x}')$, or

$$x_i = x_i(x'_1, x'_2, x'_3) \quad . \quad (25)$$

While retaining the use of the subscripts i, j , and k for components of a vector in the original $\vec{x}$ -coordinate system, we will use Greek subscripts λ, μ , and ν to denote vector components and a prime for the vector in the new $\vec{x}'$ -coordinate system. Thus, $\nabla' \equiv \partial/\partial \vec{x}'$.

The basic quantities necessary for the transformation are the scale factors²

$$h_\lambda = \left| \frac{\partial \vec{x}}{\partial x'_\lambda} \right| = \left[\sum_{i=1}^3 \left(\frac{\partial x_i}{\partial x'_\lambda} \right)^2 \right]^{1/2} , \quad (26)$$

and the transition matrix $T = (t_{\lambda i})$, whose elements are the direction cosines given by

$$t_{\lambda i} = \frac{1}{h_\lambda} \frac{\partial x_i}{\partial x'_\lambda} \quad . \quad (27)$$

In most instances, the relations between the two coordinate systems are more simply expressed in the form of Eq. (25); that is, $\vec{x}$ as a function of $\vec{x}'$. On the other hand, if it is more convenient to use the inverse relation $\vec{x}' = \vec{x}'(\vec{x})$, or

$$x'_\lambda = x'_\lambda(x_1, x_2, x_3) \quad , \quad (28)$$

Eqs. (26) and (27) should be replaced by

$$h_\lambda = \left| \frac{\partial \vec{x}_\lambda}{\partial \vec{x}} \right|^{-1} = \left[\sum_{i=1}^3 \left(\frac{\partial x'_\lambda}{\partial x_i} \right)^2 \right]^{-1/2} \quad , \quad (29)$$

$$t_{\lambda i} = h_\lambda \frac{\partial x'_\lambda}{\partial x_i} \quad . \quad (30)$$

We represent the magnetic field $\vec{B}$ and the vector potential $\vec{A}$ with a general vector $\vec{F}$ and write the vector transformations $\vec{F}' = T\vec{F}$ and $\nabla' |\vec{F}'| = T \nabla |\vec{F}|$ in the component forms²

$$F'_\lambda = \sum_{i=1}^3 t_{\lambda i} F_i \quad , \quad (31)$$

$$\frac{\partial |\vec{F}'|}{\partial x'_\lambda} = h_\lambda \sum_{i=1}^3 t_{\lambda i} \frac{\partial |\vec{F}|}{\partial x_i} \quad , \quad (32)$$

The partial derivatives of the vector components then become

$$\begin{aligned}
\frac{\partial F'_\lambda}{\partial x'_\mu} &= \sum_{i=1}^3 \frac{\partial t_{\lambda i}}{\partial x'_\mu} F_i + \sum_{i,j=1}^3 t_{\lambda i} \frac{\partial x_j}{\partial x'_\mu} \frac{\partial F_i}{\partial x_j} \\
&= \sum_{i=1}^3 \frac{\partial t_{\lambda i}}{\partial x'_\mu} F_i + h_\mu \sum_{i,j=1}^3 t_{\lambda i} t_{\mu j} \frac{\partial F_i}{\partial x_j} ,
\end{aligned} \tag{33}$$

in which use is made of Eqs. (27) and (31). With F'_λ and $\partial F'_\lambda / \partial x'_\mu$, divergence and curl of $\vec{F}$ can be obtained, if necessary, in the new coordinate system from

$$\nabla' \cdot \vec{F}' = \frac{1}{h_1 h_2 h_3} \sum_{\lambda=1}^3 \frac{\partial}{\partial x'_\lambda} \left(h_1 h_2 h_3 \frac{F'_\lambda}{h_\lambda} \right) , \tag{34}$$

$$(\nabla' \times \vec{F}')_\lambda = \frac{1}{h_\mu h_\nu} \left[\frac{\partial}{\partial x'_\mu} (h_\nu F'_\nu) - \frac{\partial}{\partial x'_\nu} (h_\mu F'_\mu) \right] , \tag{35}$$

where (λ, μ, ν) represents cyclic permutations of $(1, 2, 3)$.

We now consider three commonly used orthogonal coordinate systems: cylindrical, spherical, and local toroidal. The third coordinate system, shown in Fig. 2, is of special interest in studies of plasmas in toroidal magnetic confinement devices, such as tokamaks, which often have magnetic sources of rectangular current loops. Here we list explicit expressions of Eqs. (25)–(27), (34), and (35), but results for $\partial t_{\lambda i} / \partial x'_\mu$ are not given since the derivation is trivial. Expressions for divergence and curl of a vector in cylindrical and spherical coordinates are usually found in many books dealing with vector analysis,² but they are given here to make the listing complete.

Primes will be dropped from quantities expressed in $\vec{x}$ coordinates, unless there could be any ambiguity.

Cylindrical coordinates $\vec{x} = (r, \phi, z)$

$$x = r \cos\phi, \quad y = r \sin\phi, \quad z = z, \quad (36)$$

$$h_r = 1, \quad h_\phi = r, \quad h_z = 1, \quad (37)$$

$$T = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (38)$$

$$\vec{\nabla} \cdot \vec{F} = \frac{F_r}{r} + \frac{\partial F_r}{\partial r} + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}, \quad (39)$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_r \left(\frac{1}{r} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) + \vec{e}_\phi \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) + \vec{e}_z \left(\frac{F_\phi}{r} + \frac{\partial F_\phi}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \phi} \right). \quad (40)$$

Spherical coordinates $\vec{x} = (r, \theta, \phi)$

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta, \quad (41)$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta, \quad (42)$$

$$T = \begin{pmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}, \quad (43)$$

$$\vec{\nabla} \cdot \vec{F} = \frac{2F_r}{r} + \frac{\partial F_r}{\partial r} + \frac{1}{r \sin\theta} \left(\cos\theta F_\theta + \sin\theta \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_\phi}{\partial \phi} \right), \quad (44)$$

$$\begin{aligned}
\nabla \times \vec{F} = & \frac{\vec{e}_r}{r \sin \theta} \left(\cos \theta F_\phi + \sin \theta \frac{\partial F_\phi}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi} \right) \\
& + \vec{e}_\theta \left(\frac{1}{r \sin \theta} \frac{\partial F_r}{\partial \phi} - \frac{F_\phi}{r} - \frac{\partial F_\phi}{\partial r} \right) \\
& + \vec{e}_\phi \left(\frac{F_\theta}{r} + \frac{\partial F_\theta}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \theta} \right) .
\end{aligned} \tag{45}$$

Local toroidal coordinates $\vec{x}' = (r, \theta, \phi)$. (See Fig. 2.)

Let $R = R_0 + r \cos \theta$, where R_0 is the major radius of the toroid.

$$x = R \cos \phi, \quad y = -R \sin \phi, \quad z = r \sin \theta, \tag{46}$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = R, \tag{47}$$

$$T = \begin{pmatrix} \cos \theta \cos \phi & -\cos \theta \sin \phi & \sin \theta \\ -\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ -\sin \phi & -\cos \phi & 0 \end{pmatrix}, \tag{48}$$

$$\nabla \cdot \vec{F} = \left(\frac{1}{r} + \frac{\cos \theta}{R} \right) F_r - \frac{\sin \theta}{R} F_\theta + \frac{\partial F_r}{\partial r} + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{1}{R} \frac{\partial F_\phi}{\partial \phi}, \tag{49}$$

$$\begin{aligned}
\nabla \times \vec{F} = & \vec{e}_r \left(-\frac{\sin \theta}{R} F_\phi + \frac{1}{r} \frac{\partial F_\phi}{\partial \theta} - \frac{1}{R} \frac{\partial F_\theta}{\partial \phi} \right) \\
& + \vec{e}_\theta \left(-\frac{\cos \theta}{R} F_\phi + \frac{1}{R} \frac{\partial F_r}{\partial \phi} - \frac{\partial F_\phi}{\partial r} \right) \\
& + \vec{e}_\phi \left(\frac{F_\theta}{r} + \frac{\partial F_\theta}{\partial r} - \frac{1}{r} \frac{\partial F_r}{\partial \theta} \right) .
\end{aligned} \tag{50}$$

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3. NUMERICAL PROCEDURE

With all analytic expressions for $\vec{B}$, $\vec{A}$, and their various partial derivatives given explicitly, their numerical evaluation is quite straightforward. A brief outline of the procedure is given here.

1. Input variables: cartesian coordinates of the three points a , b , c , and the current I .
2. Evaluate vectors $\vec{x}^{ab}$, $\vec{x}^{ac}$, $\vec{x}^{bc}$, and their magnitudes.
3. Evaluate $\cos\theta$, $\sin\theta$, $\cot\theta$, $\csc\theta$ for $\theta = \theta^a, \theta^b, \theta^c$ --Eqs. (9), (12), (52).
4. Evaluate $\partial\theta/\partial x_j$ for $\theta = \theta^a, \theta^b, \theta^c$, and $j = 1, 2, 3$ --Eqs. (10), (11), (13), (51).
5. Evaluate $|\vec{B}|$, $\vec{B}$, $\partial|\vec{B}|/\partial x_j$, $\partial G_1/\partial x_j$, $\partial B_1/\partial x_j$ --Eqs. (2), (3), (5)-(7), (19), (20).
6. If necessary, evaluate $|\vec{A}|$, $\vec{A}$, $\partial|\vec{A}|/\partial x_j$, $\partial A_1/\partial x_j$ --Eqs. (15)-(18), (22), (23).
7. If results are needed in non-cartesian orthogonal coordinates, evaluate h_λ , $t_{\lambda i}$, $\vec{B}'$, $\partial|\vec{B}'|/\partial x'_\mu$, $\partial B'_\lambda/\partial x'_\mu$, and if necessary, $\vec{A}'$, $\partial|\vec{A}'|/\partial x'_\mu$, $\partial A'_\lambda/\partial x'_\mu$ for $i, \lambda, \mu = 1, 2, 3$ --Eqs. (25)-(50).

We have already pointed out that some identities satisfied by $\vec{B}$ and $\vec{A}$ may be used to test the accuracy or correctness of the obtained analytical and numerical results. On the other hand, they can also be used to simplify the numerical processes. Here we discuss several such relations and point out how they can be employed.

- a) All of the nine partial derivatives $\partial B_i/\partial x_j$ can be obtained from Eq. (6), but it is far more efficient to evaluate only five of them from Eq. (6) and the rest using Eqs. (19) and (20). Similarly,

Eqs. (22) and (23) must be utilized for $\partial A_1 / \partial x_j$. One should, however, remember that Eqs. (19) and (23) are satisfied only by a closed current loop.

- b) Once $\partial \theta^a / \partial x_j$ and $\partial \theta^b / \partial x_j$ are evaluated from Eqs. (10) and (11), $\partial \theta^c / \partial x_j$ can be obtained simply from

$$\frac{\partial \theta^c}{\partial x_j} = - \frac{\partial \theta^a}{\partial x_j} - \frac{\partial \theta^b}{\partial x_j} , \quad (51)$$

which is an obvious consequence of $\theta^a + \theta^b + \theta^c = \pi$. Thus, Eq. (13) is not really needed.

- c) It is not necessary to evaluate the angles θ^a , θ^b , θ^c . Evaluation of $\cos \theta^a$ and $\cos \theta^c$ is carried out using Eqs. (9) and (12), respectively, and a similar relation can be used for $\cos \theta^b$. The sine relations

$$\frac{\sin \theta^a}{|\vec{x}^{bc}|} = \frac{\sin \theta^b}{|\vec{x}^{ac}|} = \frac{\sin \theta^c}{|\vec{x}^{ab}|} \quad (52)$$

give two of the sine functions, but it appears unavoidable that one of the three must come from $\sin \theta = (1 - \cos^2 \theta)^{1/2}$.

- d) Equation (52) is also useful in rewriting Eq. (3) as

$$B_i = \frac{G_i (\cos \theta^a + \cos \theta^b)}{|\vec{x}^{ac}|^2 |\vec{x}^{ab}| (1 - \cos^2 \theta^a)} , \quad (53)$$

which does not involve the sine or square-root function. Equation

(53) is, therefore, useful if one wishes to obtain only the $\vec{B}$ field.

- e) Equation (52) also shows that the absolute value of the first term on the right-hand side of Eq. (10) is the same as that of Eq. (11).

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4. SUBROUTINES

Four different but closely related subroutines have been written to calculate $\vec{B}$, $\vec{A}$, and partial derivatives of the $\vec{B}$ field and $\vec{A}$ vector components in cartesian coordinates. Although a subroutine for the straight wire is sufficient to handle a polygon loop, separate treatment of the loop makes it possible to improve the efficiency of the program [see remarks a) in Sec. 4 and 4) in this section]. Also, three transformation subroutines have been written for conversion of cartesian results into those in cylindrical, spherical, and local toroidal coordinates. Table I lists inputs and outputs for these subroutines. We make some useful remarks concerning them.

- 1) In Table I, $i, j = 1, 2, 3 = x, y, z$, $\partial/x_j = \partial/\partial x_j^c$, and $\ell = 1, 2, \dots, N$, where N is the number of straight wires in the loop. x_i^c is i^{th} component of the position vector for the observation point c and x_i^ℓ represents the i^{th} component for the ℓ^{th} corner point of the loop. $|\vec{x}^{\ell, \ell+1}|$ is the length of ℓ^{th} wire of the loop, where $|\vec{x}^{N, N+1}| \equiv |\vec{x}^{N, 1}|$.
- 2) Subroutine BWIRE is for a straight wire carrying current I from point a to point b . Subroutines BLOOP, BLOOPX, and BLOOPY are for a loop consisting of N straight wires, where the flow of the loop current follows the corner points of the loop in the order of $1, 2, \dots, N, 1$.
- 3) MKSA units are used in BWIRE and BLOOP, i.e., $k = \mu_0/4\pi = 10^{-7}$. If different units are used, the value of k should be changed.
- 4) Subroutine BLOOP consists basically of N applications of subroutine BWIRE, but it avoids any repeated evaluation of the same quantity.

For example, N calls of BWIRE for a loop evaluates the distance $|\vec{x}^c - \vec{x}^\ell|$ twice for $\ell = 1, 2, \dots, N$, which is expensive since the distance is calculated with the square-root function.

5) Many typical problems are expected to meet the following three conditions:

- (a) lengths of the wires $|\vec{x}^{\ell, \ell+1}|$ in the loop are fixed and known,
- (b) there is more than one current loop,
- (c) the magnetic vector potential and its partial derivatives are not needed.

In this case, efficiency can be improved by

- (a) using $|\vec{x}^{\ell, \ell+1}|$ and/or its functions as inputs,
- (b) postponing the operation of multiplication with kI until the last possible point outside the subroutine [see also remark 8)].
- (c) eliminating calculation of A_i and $\partial A_i / \partial x_j$.

Subroutine BLOOPX is the result of incorporating these three points into BLOOP. Subroutine BLOOPY differs from BLOOPX only in that it returns only B_i and does not evaluate $\partial B_i / \partial x_j$.

6) If the observation point $\vec{x}^c$ is colinear with any one of the wires, it is a singular point and no meaningful results can be obtained. No checks are made in any of the subroutines to see if this is the case.

- 7) Conversion of results obtained in cartesian coordinates into those in cylindrical, spherical, and local toroidal coordinates is accomplished by subroutines CYLIND, SPHERI, and TOROID, respectively. Each of these uses Eqs. (28), (31), and (33) to evaluate x'_λ , F'_λ , and $\partial F'_\lambda / \partial x'_u$ for given values of x_i , F_i , and $\partial F_i / \partial x_j$. Transformations to other orthogonal coordinate systems are equally straightforward, and treatment of a general curvilinear coordinate system is also possible but requires formulation beyond that given by Eqs. (25)-(35).
- 8) In problems involving many loops and/or wires, one should, of course, evaluate the total net result by linear superposition of individual contributions in the cartesian coordinates and then, if necessary, apply a transformation only once to obtain the final results in the desired coordinates. The same principle also applies to the operation which multiplies the results with the constant kI . (Subroutines BLOOPX and BLOOPY do not do this operation.)

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5. DESCRIPTION OF THE TEST RUN

As a test case, we consider a loop consisting of four straight wires, using the following values for current, observation point, and loop corner points:

$$I = 10^6, \quad x_1^c = 8 \cos(10i), \quad x_1^{\ell} = 5 \sin(2.2i + 3.3\ell),$$

where $i = 1, 2, 3$ and $\ell = 1, 2, 3, 4$. The numerical results are given in the table of TEST RUN OUTPUT, where CAR, CYL, SPH, and TOR represent cartesian, cylindrical, spherical, and local toroidal coordinates, respectively. $B(I)$, $DB(I)/DXJ$, $CURLB(I)$, and $DIV(B)$ denote B_i , $\partial B_i / \partial x_j$, $(\nabla \times \vec{B})_i$, and $\nabla \cdot \vec{B}$, respectively, with $I = i$ and $J = j$ representing components. Similar notation is used for the magnetic vector potential $\vec{A}$. The first set of cartesian results (first three lines with $I = 1, 2, 3$) is obtained by linear superposition of the results from BWIRE applied to the four wires of the loop. The second, third, and fourth sets in the cartesian coordinates are obtained, respectively, from BLOOP, BLOOPX, and BLOOPY.

As expected, results of these four sets all agree with each other. Divergences and curls of $\vec{B}$ and $\vec{A}$ were evaluated using Eqs. (34) and (35). Numbers of the order of 10^{-15} or less all represent zeros in reality and thus, it is seen that $\nabla \times \vec{B} = 0$, $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{A} = \vec{B}$, and $\nabla \cdot \vec{A}$ are indeed satisfied in all of the four coordinate systems. In fact, $\nabla \times \vec{B} = 0$, $\nabla \cdot \vec{B} = 0$ are used in subroutines BLOOP and BLOOPX to obtain four of the nine partial derivatives $\partial B_i / \partial x_j$ and also $\nabla \times \vec{A} = \vec{B}$ and $\nabla \cdot \vec{A} = 0$ are used in a similar way in BLOOP for $\partial A_i / \partial x_j$.

The execution times required for a single loop calculation (with $N = 4$) using BWIRE, BLOOP, BLOOPX, and BLOOPY are 2.66×10^{-4} sec, 2.28×10^{-4} sec, 1.36×10^{-4} sec, and 0.55×10^{-4} sec, respectively. This means that BWIRE is called four times while others are called only once.

REFERENCES

1. J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1975), Chap. 5.
2. P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York, 1953), Chap. 1.

Table I. Inputs and outputs of subroutines

Subroutine	Inputs	Outputs
BWIRE	I, x_i^a, x_i^b, x_i^c	$B_i, \partial B_i / \partial x_j, A_i, \partial A_i / \partial x_j$
BLOOP	I, N, x_i^c, x_i^ℓ	$B_i, \partial B_i / \partial x_j, A_i, \partial A_i / \partial x_j$
BLOOPX	$N, x_i^c, x_i^\ell, \vec{x}^{\ell, \ell+1} ^{-1}, \vec{x}^{\ell, \ell+1} ^2$	$(kI)^{-1} B_i, (kI)^{-1} \partial B_i / \partial x_j$
BLOOPY	$N, x_i^c, x_i^\ell, \vec{x}^{\ell, \ell+1} ^2$	$(kI)^{-1} B_i$
CYLIND	$x_i, F_i, \partial F_i / \partial x_j$	$x'_\lambda, F'_\lambda, \partial F'_\lambda / \partial x'_\mu$
SPHERI	$x_i, F_i, \partial F_i / \partial x_j$	$x'_\lambda, F'_\lambda, \partial F'_\lambda / \partial x'_\mu$
TOROID	$x_i, F_i, \partial F_i / \partial x_j$	$x'_\lambda, F'_\lambda, \partial F'_\lambda / \partial x'_\mu$

FIGURE CAPTIONS

Fig. 1. Configuration of current-carrying straight wire $\overline{ab}$ and observation point c .

Fig. 2. Local toroidal coordinate system (r, θ, ϕ) . The angle ϕ is measured from the positive x axis. The circle represents a cross-section of the torus with major radius R_0 and minor radius a .

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Fig. 1.

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Fig. 2.

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	CAR	CYL	SPH	TOR
DIV(E)	4.1633E-17	-8.6736E-19	0.	-4.3368E-19
DIV(A)	-2.5258E-15	0.	-1.7347E-18	-1.3010E-18

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