LINEAR PROGRAMMING - A MANAGEMENT TOOL

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LINEAR PROGRAMMING - A MANAGEMENT TOOL

THESIS

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TABLE OF CONTENTS

	Page
LIST OF TABLES	v
LIST OF ILLUSTRATIONS	vi
Chapter	
I. INTRODUCTION	1
Origin and Background of Linear Programming	
Statement of the Problem	
Delimitations	
Sources of Data	
Procedure	
Related Studies	
Definition of Terms	
Treatment of Data	
II. OPERATIONS RESEARCH	12
Management and the Scientific Method	
Methods and Techniques of Operations Research	
Linear Programming - A Management Tool	
III. METHODS OF LINEAR PROGRAMMING	25
The Simplex Method	
The Transportation Problem	
Steps in Solving Problem	
Trial and Error Method	
The Stepping Stone Method	
Mutually Preferred Method	
Degeneracy	
The Input - Output Method	
VAM - Vogel's Approximation Solution	
Transportation Program Solution	
IV. SOME APPLICATIONS OF LINEAR PROGRAMMING	50
Military Applications	
Industrial Applications	
Allocation of Scarce Raw Materials	
Between Two Plants	

Chapter	Page
Agriculture	
The Paper Industry	
Samples of Industrial Management Problems	
The General Transportation Problem	
Economic Analysis	
Additional Applications of Linear Programming	
V. OPTIMUM BLEND PROBLEM	79
The Kreck Packing Company Case	
Computer Solutions to Blending Problems	
Shadow Price--Sensitivity Analysis	
VI. SUMMARY	92
Requirements of Linear Programming	
Linear Programming - A Management and Its Applications	
BIBLIOGRAPHY	99

LIST OF TABLES

Table	Page
I. Leather Goods Problem	26
II. Possible Combinations of Time and Hide	27
III. Surplus Merchandise	34
IV. Merchandise Demand	34
V. Mileage	35
VI. Computer Output	44
VII. Computer Output	46
VIII. Computer Output	48
IX. Vapor Pressure and Unit Cost of Components	59
X. Raw Materials for All-Meat Frankfurter	81
XI. Taste and Content Constraints for All-Meat Frankfurter	82
XII. Simplex Matrix of Frankfurter Mixture for the Kreck Packing Company of Dallas, Texas	86
XIII. Computer Output	90

LIST OF ILLUSTRATIONS

Figure	Page
1. The Information-Decision Cycle	22
2. Polygon ABCD Showing the Limits of the Resources Available for Production	28
3. Polygon ABCD Showing Isoprofit Lines and the Point of Optimum Production	30
4. Polygon ABCD Showing Point of Maximum Profit	31
5. Distance - Demand - Supply	37
6. Trial and Error Solution	38
7. The Northwest Corner Method Solution	40
8. The Mutually Preferred Method Solution	41
9. Block Diagram	55
10. Operations Research Center through Linear Programming can help optimize the three M's of industry: <u>Management</u> , <u>Manufacturing</u> and <u>Marketing</u>	96

CHAPTER I

INTRODUCTION

In relatively recent years a new tool of economic theory, mathematical programming, has been developing in the hands of some applied mathematicians, economic theorists, and Operations Research men. There has been an enormous amount of literature written on the subject.

Management's interest has been stimulated considerably but the pursuit of this interest is often difficult and unrewarding to the manager.

Mathematical programming is one of the most valuable techniques of Operations Research available to management. It is concerned with minimizing or maximizing an objective function subject to a set of restrictions upon the variables of the system.

Linear programming is a special type of mathematical programming in which the objective function and the restrictions are all linear functions; i. e., functions in which the exponent of the variable is zero or one, and in which there are no products of variables.

The concept, because of its pragmatic flavor, has been applied to solve problems in military strategy, gasoline blending, government taxation, agriculture, manufacturing

processes, and even in navigation. Most of these applications have been quite successful.

Although linear programming is an objective technique, it is completely dependent on the subjective factors of human personality and skill. The greatest risk in linear programming solutions is that quantitative values are sometimes impossible to measure accurately. This total reliance on the reasoning and the evaluating capacity of the research analyst and his particular talents for assembling data into a mathematical model makes linear programming more an art than a science. By this it is meant that different analysts may have different opinions as to the value of critical data.

Origin and Background of Linear Programming

The origin and background of linear programming has been in the field of econometrics and research into economic relationships. Its roots go back to 1874 and the works of the mathematical economist Leon Walras. In his work Elements d'Economie Politique (Elements of Political Economy) Walras showed that the price of any number of commodities at a single time can be determined by solving simultaneously the correct number of equations in terms of the number of unknowns for which a solution is sought. At the time, the concept was revolutionary, and today it is recognized as a contribution to economics. It was this first attempt to solve problems of scarcity by stating problem conditions in equation form that provides the connection between Walras and linear programming. The method of solving the equations for solution by L P is completely different from that used by Walras, however.¹

¹Robert O. Ferguson and Lauren F. Sargent, Linear Programming: Fundamentals and Applications (New York, 1958), p. 6.

The Walrasian method of economics tries to interpret the entire system at once. The advantage in the use of linear programming is that it can be focused on a particular problem at a time and overcome some of the weaknesses inherent in the general equilibrium analysis. Both forms of analysis are complementary to each other. The Walrasian analysis will clarify the objectives further while linear programming will give the practical insight for the actual solution of economic activities.

The Walrasian system is indeed a perfectly general analysis of static equilibrium. The problem considered is general enough: the use of given and limited resources to achieve as far as possible certain specified objectives,² allowing for all interrelations in the system.²

Statement of the Problem

To be "100 per cent correct" management can no longer, as in earlier times, rely solely on the basis of experience, perception, intuition and understanding to integrate the activities of a big business toward the planned goals. Linear programming can help in the decision-making process by enforcing sheer objectivity in judging variables that may affect results.

The future of linear programming as an industrial technique depends largely on the attitude of management. It is important that management at all levels should understand

²R. G. D. Allen, Mathematical Economics, London, (St. Martin's, 1957), p. 565.

the tools available, the principles behind them, and the work they can do to simplify the job of management.

The purpose of this thesis is to integrate the most up-to-date information on the subject of linear programming into a comprehensive and understandable treatise for the consideration of management. The value of this study, then, is determined by the effectiveness of its presentation so that management may grasp an ample understanding of the subject.

Delimitations

This study places particular emphasis on linear programming--its methodology and applications.

Reference is made, nevertheless, to other types of mathematical programming and appropriate sources.

Due to their mathematical complexity, no attempt is made here to explain other modifications of mathematical programming theory such as parametric linear programming, sensitivity analysis, and non-linear programming.

Sources of Data

The primary sources of data in this study were obtained from library research at North Texas State University Library, Denton, Texas; Southern Methodist University Library, Dallas, Texas; and the University of Texas Library, Austin, Texas; also from interviews with the managers of the Kreck Packing Company, Dallas, Texas.

Procedure

After the problem was selected for study, recommendations were sought for sources of data. The next step involved the preparation of an outline and the procurement of the books suggested. The books were located at the libraries of North Texas State University, Southern Methodist University, and the University of Texas, as well as from the personal libraries of C. L. Littlefield, Chairman of the Management Division, School of Business Administration, North Texas State University and Rafael B. Cruz, Mathematical Programmer, Scientific Division, Service Bureau Corporation, a subsidiary of International Business Machines Corporation, Dallas, Texas.

After the introductory chapter was prepared, secondary data were accumulated by taking notes on the material found in the recommended books.

Related Studies

Michael C. Forster, "Optimum Linear Systems," thesis, M. S. in Engineering, Southern Methodist University, June, 1960.--This study presents a method for analyzing and designing ballistic missile tracking systems. An integral equation for optimum time-varying linear system is derived for non-stationary inputs by use of mathematical techniques.

Marvin Wayne Bahnman, "Application of the Dynamic Programming Concept to Engineering Problems," thesis, M. S. in Aeronautical Engineering, Southern Methodist University,

April, 1961.---This thesis presents a careful study of mathematical techniques with emphasis on their applications to the field of engineering. Particular attention is given to the problem of rocket staging.

Raymond L. Gallahue, Jr., "Linear Programming - An Application to a Transportation Problem," an engineering report, Southern Methodist University, 1958.---This study introduces the transportation method of linear programming by the Northwest Corner Method, the Mutually Preferred Method, and the Vogel's Approximation Method. The report is limited to linear programming techniques involving simple mathematics.

Definition of Terms

Mathematical model.---A representation of an operation, organization or business for the purpose of performing an analysis by means of mathematical formulae.

Mathematical programming.---A technique devised to optimize, i. e., maximize or minimize, an objective function subject to certain restrictions upon the variables in the objective function.

Optimize: $f(x_1, x_2, x_3, \dots, x_n)$

$g_1(x_1, x_2, \dots, x_n)$

$g_2(x_1, x_2, \dots, x_n)$

Subject to:

.

.

.

$g_m(x_1, x_2, \dots, x_n)$

Linear programming.--A special type of mathematical programming in which the objective function and the restrictions are all linear functions; i. e., functions in which the exponents of the variables is zero or one and in which there are no products nor variables.

$$\text{Optimize: } f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n c_i x_i$$

$$\text{Subject to: } \sum_{i=1}^n a_{ki} x_i = b_k, \quad k = 1, 2, \dots, m.$$

Degeneracy.--Degeneracy is said to occur in linear programming problems when a program can be divided into two or more groups whose combined capacities satisfy the requirements of the solution. This leads to an unbounded solution. To correct this problem, dummy variables are added to the equations. The purpose of these dummy variables will be explained further in the study.

Feasible solution.--A set of values satisfying a set of constraints.

Slack variables.--Any inequation can be made into an equation by adding a new non-negative "slack variable" which represents the difference or slackness between the two orders of the equation.

Basic solution.--A solution in which there is exactly one non-zero variable for each constraint.

Artificial variables.--Variables added to an equation to simplify its solution. These variables are non-negative,

do not belong in the system, and will be suppressed. To any constraint not containing a slack variable nor having one with a coefficient of one, a non-negative artificial variable is added. The purpose for this use will be explained later in the theoretical considerations.

Simplex tableau.--A matrix representation of a system to be solved by linear programming.

The concept of linearity.--The term linearity is an abstraction meaning direct relationships between a set of given factors. In a special case $y = f(x)$, the relationship is linear because a given change in x causes a constant change to the value of y . Linear equations generally are expressed as $f(x) = a + b(x)$ or $y = a + b(x)$, where: $f(x)$ is the dependent variable, x is the independent variable; a is the numerical constant (the intercept); b is the numerical constant called the slope. The objective of linearity as referred to here, relates to the assumption of proportionality of inputs and outputs in each elementary productive activity and the assumption carrying out two or more activities is the sum of the results of these separate activities.

As an example of linearity, a change in the number of production workers will bring a proportional change in output and profit margin. Thus the ratio of the number of workers to output and the ratio of the number of workers to profit margin will remain the same.

The concept of duality.--In a profit maximization problem, an optimum solution is developed. Any variance from it will result in a loss of profit. It should be understood that a maximum profit means a negative loss and vice versa. The algebraic models for maximization and minimization are directly related. This relationship is called duality and is explained by Danzig³ in his Dual Theorem.

The duality theorem may be used in solving linear programming problems. With certain interpretation the solutions of a minimization problem are found in the final simplex tableau which solves the maximization problem. This affords a saving of time in calculations required, since one may set up for solution by the simplex method that problem which will give the least number of rows in the basis.⁴

When an optimum solution of a linear programming problem exists, the dual of the problem has also the same solution. In production, for example, the optimization may be worked out from the side of output as a maximization of the output with given cost, but it may be thought of as a minimization of cost to produce a given amount of output. This is the same thing as the saddle point of the minimax theorem of a

³George B. Dantzig, L. P. Ford and D. R. Fulkerson, "A Primal-Dual Algorithm for Linear Programs," Linear Inequalities and Related Systems (Princeton, N. J., 1956), pp. 171-180.

⁴Albert Charnes, W. W. Cooper and A. Henderson, Introduction to Linear Programming (New York, 1953), p. 33.

two person zero-sum game. Here, the paying player tries to minimize his payment while the receiving player tries to maximize his collection.

Basis.--A collection of the variables, including slack variables, which contain exactly as many variables as there are constraints.

Iteration.--Each of a chain of trials tending to approach an optimum solution.

Treatment of Data

The data are presented in this study in five chapters: Chapter I includes the introduction to linear programming, the statement of the problem, delimitations, sources of data, procedure, related studies, definition of terms, and organization of data.

Chapter II introduces linear programming as an Operations Research technique and contains also the relationship between management and the scientific method, and linear programming as a management tool.

Chapter III presents the most common methods of linear programming: the Simplex, the Transportation, and the Input-output Methods. It explains some basic requirements and types of problems that can be solved through these different methods.

Chapter IV is concerned with linear programming as applied to industry, agriculture, economic analysis, and presents specific cases.

Finally, Chapter V presents an actual case study. Its purpose was to minimize the cost of the blend for an all-meat frankfurter for the Kreck Packing Company of Dallas, Texas, by the simplex method of linear programming.

CHAPTER II

OPERATIONS RESEARCH

Operations Research may be considered an approach to problem-solving, a state of mind, an idea that accepts no limit on the scope of the problems it can handle nor on the analytical techniques it may employ to solve these problems.

It is the application of the method most used by the sciences to the study of operating problems in business and government. The objective is to have a quantitative understanding of the principal factors of operations, and the factors affecting results, so that management may have a sounder basis for decision-making.

Operations Research can help management by: helping to understand the business as a whole in simple terms, thus it is possible to avoid making decisions which might improve a segment of the business and hamper another segment or the work of the whole organization; providing the executive with a wide choice of actions; pointing out the risks involved in each action as well as the margin of error; and the resources and efforts which are necessary; designing an adequate communication flow that insures that all related parts work together smoothly and at maximum effectiveness; telling the executive who should know what and when; and predicting future outcomes and helping to draw out long-range goals.

Compiling the necessary information is the initial step in research. Once the information is collected, it is subject to quantitative study and measurement and expressed in definite form, either mathematically or in terms of importance.

The practice of Operations Research relies on the model, the measure of effectiveness, the role of experimentation, and the necessity for decision.

The model is an early representation of the operation on paper. The critical aspects are implicated in mathematical or symbolic expressions. The accounting books of a corporation are a model for they represent the corporation on paper. It provides measures of the rate of flow, the values produced and the performance achieved.

The construction of a model develops an operational theory of the firm. The application of mathematics to models of business operations is considered to be the most important contribution of Operations Research. Here, the main concern is with the measurable aspects of operations, the objectives sought, the alternative courses of action, and the variables. All of these are expressed mathematically. They are mathematical models. This involves making equations that illustrate costs, profits and interrelationships, and then, by the use of matrix algebra or differential calculus, find the maxima or minima functions as needed.

The measure of effectiveness indicates the extent of the realization of the company goal. This must be expressed in

quantitative terms and the critical factors must be measurable before the operation can be expressed mathematically. The return on investment is an excellent measure of the effectiveness or of the performance of the company.

Testing the theories, first for validity and then for result, insures management of a sounder basis for decision-making. This is the role of experimentation.

A mathematically correct solution may prove to be unrealistic. It is possible that a computer may work out a sales plan which would disregard public concern. Experimentation by house-to-house canvassing may reveal whether a plan would work in practice or not. This would test the value of the sales plan. The value of the theory can be tested also by comparing retrospectively with past information.

The final decision is made by management. The Operations Research men submit their results to management who assumes the responsibility to weigh the quantitative answers in the light of tradition, environment, morale, public relations, etc.

In the decision-making process, management simulates, theoretically, the results, quality of service, profit or cost under each of the decision alternatives.

Since World War II, new techniques have evolved for use in the selection of alternative courses of action. These new techniques are based on the scientific method and some advanced mathematics.

The scientific management movement involves statistical decisions and Operations Research methods. The roots for statistical decision-making can be traced back to the 1650's when the compilation and use of figures for the purposes of government finance and public taxation first were used. Up to the 1890's the statistician was merely a gatherer of numerical information. Karl Pearson¹ was the first to devise an accurate statistical method for making accurate statements from processed data.

R. A. Fisher² expanded and revised Pearson's pioneering methods and devised new techniques for collecting representative data. These included the proper design of experiments and random sampling.

Statistics as a technique was centered in the biological sciences in England. Later the emphasis changed to industrial application in the United States. Then the purpose of the statistical method was expanded to cover the selection of the course of action. Statistical decision-making is actually all the statistical methods collected into a single comprehensive system and directed toward the unified objective of decision-making.³

The objective of the statistical decision is to provide a comprehensive and practical answer to the problem of finding

¹Irwin D. Bross, "Statistical Decision-Making," Harvard Business Review, XXXII (September-October, 1954), 133-134.

²Ibid.

³Ibid.

the optimum course of action that will meet the given requirements and will satisfy best the objective in mind.

The use of statistical techniques in the decision process will insure a higher degree of accuracy than would be possible otherwise.

No research or evaluation method relieves the executive of his responsibility for making the actual selection of an alternate course of action nor for following up the effectiveness of the decision.

In many cases the decision process consists of a selection of objectives rather than a selection of alternative courses of action to achieve an already planned objective, whether to spend available funds to build a modern and attractive building or to spend the money in the development of a new product. The former may be prestige, the latter may be expansion of the line.

Decision-making is a process composed of five distinct phases: 1) defining the problem; 2) analyzing the problem; 3) developing alternative solutions; 4) deciding upon the best solution; and 5) converting the decision into effective action.⁴

To arrive at a definition of the problem there is the need of deciding what the critical factor or factors are. This usually involves differentiating between symptoms and

⁴Peter F. Drucker, The Practice of Management (New York, 1954), p. 353.

causes. Elimination of unacceptable courses of action is in itself a prerequisite to decision-making.

Analyzing the problem involves the finding and the classifying of facts. Classification requires an evaluation of the futurity of the decision, and its effect upon the company objectives and functions. Classification impels the manager to view the problem from an all-inclusive point of view, thus making certain that the decision contributes to the long-range company plan.

Management ingenuity should next develop alternative courses of action, thus following the scientific method which will be discussed later in the chapter.

Finding the best solution requires a careful consideration of the risks and possible results of each course of action.

Action, once its course is selected, requires understanding by the executors of the decision as to what management wants to do and what changes are necessary.

Management and the Scientific Method

The manager of earlier times was in charge of a small and simple enterprise. He conducted his business according to his own ideas and personal experience. As business grew to the great financial enterprise of today, the profession of management became more complex. Business became a team effort in which contributions of each individual were integrated with

those of other individuals in a group which in turn were integrated even further to other contributions of other groups.

Planning generally was conducted on a day-to-day, week-to-week basis. Today many businesses employ whole staffs consisting of engineers, mathematicians, economists and management personnel to engage in planning the future operations of the organization for as long as ten years and sometimes for a longer period of time.

Control in the business, ascertaining that the operations be carried out as planned and that they be aimed at satisfying the objectives, is as important as planning itself.

Methods and Techniques of Operations Research

Operations Research is not limited to any specific analytical methods or techniques. The following are representative of Operations Research: game theory, queueing theory, and mathematical programming.

Game theory.--As in most Operations Research problems, an optimum solution is searched, the selection of which is to be made by management. In game theory a complex situation takes place, since the actions and reactions of the opponents, i. e., competitors, etc., are taken into account mutatis mutandi. The consequences of a decision by one party, therefore, depend not only on it but also upon the action which the opponent may choose from among his alternatives.

John von Neumann⁵ introduced game theory as a mathematical technique in 1928. Since then, it has been a topic of interest among economists, mathematicians and many scientifically-minded executives. Presently zero sum games are plotted linearly, the opposing interests are faced with losses equal to gains. The theory employs a game matrix in which all of the possible courses of action are plotted and from which the course which will minimize the possibility of losing is selected.

Game theory has been applied to military strategy, weapons defense systems and in the designing of optimum advertising campaigns.

Queueing theory.---Queueing theory makes use of the laws of probability to relate input demand to output capacity; for example, a production line where a machine worker attaches two segments of a product unit together, if the input demand is greater than the output capacity, a queue or waiting line is formed. The key to this theory is that the units arrive at the mouth of the input demand at random, and that over a given time the same number of units arrive for service. The average time of arrival is computed as well as the standard deviation from the service time. A mathematical formula yields the average waiting time under existing conditions.

⁵ John von Neumann and Oscar Morgenstern, Theory of Games and Economic Behavior (Princeton, N. J., 1947), p. 1.

If an improvement is needed it can be substituted into the formula to eliminate or reduce waiting time.

Mathematical programming.---Mathematical programming is one of the most valuable systems of Operations Research available to management.

Linear programming is a technique of mathematical programming used in the efficient allocation of resources. The limits and demands of available resources and the criteria for allocation (optimization of profits and costs), are expressed in algebraic terms and equations which show the "boundaries" within which an optimum solution must be found. All the possible solutions between boundaries should be considered by management as alternate courses of action.

The methodology of linear programming evaluates these alternatives and always chooses the optimum solution to the problem. The fundamental limitations of linear programming are the need for linear proportionality in terms of the equations, the volume of arithmetical calculations and a high degree of certainty in the evaluation of variables.

Linear Programming - A Management Tool

If management is the process of converting information into action, the success of decision-making depends on the quality and quantity of the information available and upon the conversion itself. By this it is meant that the success of management depends on what use is made of the information

available, the speed of action, the consideration given to the proper sources of information in view of the desired objectives.

The success of an industrial company depends on its management and, as already pointed out, the success of management depends on the information available. For there to be adequate action, management must be able to see the interactions between flow of information, materials, money, manpower, and capital equipment. These are flow systems, the way that management juggles them, interlocks them, and changes them, is the basis for anticipating decisions, policies, organization and choice of investments.

The manager is a converter of information into managerial instructions that eventually lead to action; this action may create new information, thus a cycle is formed. (See Fig. 1)

The growing application of the "scientific method" and of mathematics to business operations has meant that more and better information has been available faster for management to act.

The present time seems to be the beginning of a technical revolution in business administration. Since the 1950's the trend has been toward more effective factual information as a basis of decision-making in all management levels.

In recent years mathematicians have worked out a number of new procedures which make it possible for management to solve a wide variety of important company problems much faster, more easily, and more accurately than ever before.

Fig. 1--The Information-Decision Cycle

These procedures have sometimes been called "linear programming" Mathematical programming is not just an improved way of getting certain jobs done. It is in every sense a new way. It is new in the sense that double-entry bookkeeping was new in the Middle Ages, or that mechanization in the office was new earlier in this century, or that automation in the plant is new today. Because mathematical programming is so new, the gap between the scientist and the businessman, between the researcher and the user has not yet been bridged. Mathematical programming has made the news, but few businessmen really understand how it can be of use in their own companies.⁶

Linear programming has been the longest stride toward effective handling of facts to yield compiled information on which to base decision-making. The ideal decision-making situation has two basic types of information present: first, management should know how much manpower, capital, floor space, equipment, and inventory are available; he should have an accurate measure of his productive resources in terms of demand; second, management should also know how much profit can be obtained from using a unit of these resources for each available opportunity. Actual conditions usually are far from these ideal conditions.

Although linear programming was developed to help solve problems of using scarce resources as effectively as possible, it helps to solve the data and information problem of management as well. In doing so, it provides a way of organizing

⁶Alexander Henderson and Robert Schlaifer, "Mathematical Programming Better Information for Better Decision Making," Statistical Decision Series, Reprints from Harvard Business Review (May-June, 1954), p. 117.

and interpreting a large volume of data and facts; using approximations in mathematical processing of the data and facts; evaluating the benefits of profits that should come from apportioning or allocating the available resources to different opportunities and/or providing a better way of determining how to assign or use limited resources among different opportunities. Thus, linear programming provides a better way of handling the information used in making decisions.

CHAPTER III

METHODS OF LINEAR PROGRAMMING

The methodology of linear programming ranges in complexity from the use of paper and pencil, as in the graphical method, to the use of high-speed digital computers sometimes required of the simplex transportation methods.

A simplex method problem of a given complexity of a 20 by 40 matrix can be solved with the aid of a desk calculator in about three hours per iteration. If an automatic multiplier, such as the IBM 604 is used, the time per iteration will be decreased to thirty minutes. If, on the other hand, a linear program on an IBM 1620 computer is used, each iteration would be carried out in less than twenty seconds.

The simplest method available to work certain types of problems of limited complexity is the graphical method. [All linear programming problems can be handled by the simplex method or by the distribution method, also called the transportation method, which is a derivative of the simplex method.] *

Graphical method.--This method is limited to two-dimensional framework problems, those problems of complexity limited to two variables only. The optimum solution will be found at one of the vertexes of the convex polygon so constructed as proven by Dresher.¹ (See Fig. 2) The accuracy

¹Melvin Dresher, Games of Strategy: Theory and Applications, (Englewood Cliffs, N. J., 1961), pp. 43-44.

of these solutions obtained by this method depends upon the ability of the draftsman and the width of the lines drawn by the pencil. Algebraic solutions are recommended as a check to graphical solution.

As an example of the graphical method there is a small leather goods manufacturer who produces only two products: toy saddles and handbags, and has a limited stock of hide. (See Table I) The manufacturer wishes to maximize his profit and optimize his resources. Observe that to maximize profits he will not have to exhaust all his resources: time and hide.

The maximum use of time or hide will not necessarily maximize profit. As will be shown, profit will be reduced if his resources are exhausted.

TABLE I

LEATHER GOODS PROBLEM

Hide in stock	104 pieces
Time available	68 hours
Hide required to make one saddle . . .	8 pieces
Hide required to make one handbag . . .	2 pieces
Time required to make one saddle . . .	2 hours
Time required to make one handbag . . .	4 hours

Table II shows some possible combinations based on the available time and hide. Plotting the number of handbags versus the number of saddles in rectangular Cartesian

TABLE II
POSSIBLE COMBINATIONS OF TIME AND HIDE

Hide: 104 pieces		Time: 68 hours	
Number of		Number of	
Handbags (2 pieces each)	Saddles (8 pieces each)	Handbags (4 hours each)	Saddles (2 hours each)
0	13	17	0
4	12	16	2
8	11	15	4
12	10	14	6
52	0	0	34

coordinates shows two straight lines representing time and hide. (See Fig. 2) The lines show the limits. Any point on the lines makes maximum use of either time or hide. The area of the polygon, ABCD, defines the limits of the solution, hence the solution must, when plotted, be within polygonal area.

Any point on line BC makes optimum use of production time; any point on line AB makes optimum use of materials. Point B, where optimum lines AB and BC meet best utilizes the resources available and thus produces the highest profit.

Since this enterprise is in business for a profit, assume that each unit of production brings a profit of five dollars. At point A, the profit would be $\$5 \times (13 + 0) = \65 . At point C, the profit would be $\$5 \times (17 + 0) = \85 . Any

Fig. 2--Polygon ABCD showing the limits of the resources available for production.

and all lines which cross points E or A which are perpendicular to the bisectrix of the 90° angle are lines of equal profit. These lines are called isoprofit lines. The further away from the bisectrix within the realm of the area ABCD, the higher the total number of production units. Mathematically the highest profit in this case is:

$$\$5 \times (12 + 10) = \$110 \text{ (at point B).}$$

See Figure 3.

If, however, the profit is \$5 for each handbag and \$40 for each saddle, it is obvious that the best combination would be to produce all saddles and no handbags. The point of optimum production, profit-wise, would be at A. Hence,

$$5x + 40y = 520; \text{ since } x = 0 \text{ and } y = 13.$$

See Figure 4.

In this sample problem it can be seen that not all resources were used to maximize the profit. This means that utilization of resources has been optimized though not maximized. If additional resources were employed this would result in a lower profit potential.

The Simplex Method

George B. Dantzig² formulated the simplex method in 1947. So far it is the best-known method for solving linear programming problems. The only drawback it may have is that of lengthy arithmetics. This is a limiting factor to both the

²Charnes, Cooper and Henderson, op. cit.

Fig. 3--Polygon ABCD showing isoprofit line and the point of optimum production,

Fig. 4--Polygon ABCD showing point of maximum profit if $x = \$5$, $y = \$40$.

application of the simplex method and to the acceptance of linear programming as a manager's tool. Managers, upon investigation of the simplex method, may conclude that they need an electronic computer to be able to utilize linear programming. Many problems can be solved without resort to the simplex method and many complex problems can be simplified so that the simplex method may be used without computers.

Management should not be discouraged if it needs to use linear programming to solve complex problems when the companies cannot afford to rent, let alone purchase, an electronic computer. There are many service bureaus that are equipped with up-to-date equipment and have a staff of linear programming experts. These service bureaus charge nominal fees for solving linear programming problems.

The approach to the simplex method is to begin with a feasible solution to the problem and then to proceed to calculate a number of feasible solutions to the problem through a series of iterations until an optimal solution is found.

The Transportation Method

This method is rather short and simple for solving linear programming problems of a certain category characterized by:

1. One unit of input produces one unit of output. The ratio of output to input is 1:1;
2. The cost per unit remains fixed throughout different production levels; and

3. The quantity of production is known in advance. If a problem does not satisfy these conditions it cannot be solved by the transportation method.

In some cases when the ratio of input to output is not 1:1, it is possible to set up extra dummy outputs or inputs so as to make the solution feasible, i. e., if the inputs exceed the outputs, set up a dummy output with requirements set at the difference between the total input capacity and the total output capacity. The cost of this difference is set at zero and the units which are assigned to this dummy station in the final program become idle capacity, thus not affecting the final solution in any way.

Also when the total production is unknown, it is possible to set up dummy inputs and outputs so as to take care of this unknown portion. At any rate the problem can be solved through the transportation method and the solutions remain the same.³

The Transportation Problem

A manufacturing company has ten warehouses scattered over the country. As a result of an advertising campaign, in five of the warehouses demand exceeds supply and in the other five, the supply exceeds the demand. The Operations Research staff is instructed to arrange the best shipping plan that would take care of this discrepancy. Table III shows the supply of merchandise at the respective warehouses.

³A. Henderson and R. Schlaifer, "Mathematical Programming: Better Information for Better Decision-Making," Harvard Business Review, XXXII (May-June, 1954), 139.

TABLE III
SURPLUS MERCHANDISE

Warehouse	Surplus Units of Merchandise
SW ₁	12
SW ₂	9
SW ₃	6
SW ₄	15
SW ₅	24

Table IV shows the demand of merchandise at the respective warehouse.

TABLE IV
MERCHANDISE DEMANDED

Warehouse	Demanded Units of Merchandise
DW ₁	15
DW ₂	6
DW ₃	6
DW ₄	12
DW ₅	9

It should be noted that there are enough units of merchandise in surplus at warehouses SW₁, SW₂, SW₃, SW₄, and SW₅ to satisfy the demands of DW₁, DW₂, DW₃, DW₄, and DW₅ warehouses.

Operations Research men have estimated that it costs \$1.00 to move one unit of merchandise per mile.

The following table shows the distances from the SW warehouses to the DW warehouses.

TABLE V
MILEAGE

Demand Warehouse	Supply Warehouse				
	SW ₁	SW ₂	SW ₃	SW ₄	SW ₅
DW ₁	690	690	835	300	1000
DW ₂	240	290	290	510	390
DW ₃	190	90	270	320	540
DW ₄	80	50	190	400	500
DW ₅	130	30	220	320	510

In Figure 5 negative values are given to the distances because they represent subtractions from gains. The objective is to minimize cost. If negative values were not assigned, the program would develop a maximum solution.

The values on the last row of the matrix of Figure 5 represent the supply at the SW warehouses, respectively, and similarly, the values on the rightmost column represent the demand requirements at the DW warehouses, respectively. Entry of 6.6 of the matrix is the difference between the total supply and the total demand; and it is included in the matrix to satisfy the requirement number 1 of the general transportation problem which states that total demand must equal total supply.

Steps in Solving Problem

To simplify solution, the problem must be framed. (See Figure 5) The construction of a matrix, Table V establishes a computational framework in which the slack requirements have been stated.

The next step is finding a feasible solution, and from it develop the optimal solution utilizing one of the following distribution methods:

- a. Trial and error method
- b. The Northwest Corner method
- c. The Mutually Preferred method
- d. Vogel's approximation method.

If a transportation program for a computer is available, then it is not necessary to obtain the first feasible solution. The computer can obtain the optimal solution in a matter of seconds.

Trial and Error Method

If the problem is not complex and it does not involve many variables, it is possible to use this method. It is clear that the longer the total distance travelled in this particular transportation problem the higher the cost of transportation.

See Figure 6 for a feasible solution by the trial and error method.

Source Desti- nation	SW ₁	SW ₂	SW ₃	SW ₄	SW ₅	Demand
DW ₁	-690	-690	-835	-300	-1000	15
DW ₂	-240	-290	-290	-510	-390	6
DW ₃	-190	- 90	-210	-320	-540	6
DW ₄	- 80	- 50	-190	-400	-500	12
DW ₅	-130	- 30	-220	-320	-510	9
Slack	0	0	0	0	0	18
Supply	12	9	6	15	24	66

Fig. 5--Distance - Demand - Supply

Source Desti- nation	SW ₁	SW ₂	SW ₃	SW ₄	SW ₅	Demand
DW ₁	-690	-690	-835	-300	-1000	15
			3	12		
DW ₂	-240	-290	-290	-510	-390	6
		6				
DW ₃	-190	-90	-270	-320	-540	6
		3	3			
DW ₄	-80	-50	-190	-400	-500	12
	9				3 (3) <i>not needed</i>	
DW ₅	-130	-50	-220	-320	-510	9
	3			3	3	
	0	0	0	0	0	
Slack						18
Supply	12	9	6	15	24	66

Fig. 6--Trial and Error Solution.

The Stepping Stone Method

The Northwest Corner Rule: This method assigns orders without regard to cost. Assignments are started at the warehouse at the upper left-hand corner of the matrix. See Figure 7.

If the demand at the warehouse DW_1 is greater than the supply at SW_1 , the total supply at SW_1 is assigned to DW_1 . In this case the supply at warehouse SW_2 is considered to satisfy the remaining demand at warehouse DW_1 ; if after allocating all of the supply from SW_2 there still is demand at DW_1 , warehouse SW_3 is considered in a similar manner and so on until all the demand at DW_1 is satisfied.

In the case in which the supply at a warehouse is greater than the demand at the one with which it is being considered, enough of the supply at that warehouse is assigned to the demand-warehouse to satisfy its demand completely, and the remaining supply is assigned to the next demand-warehouse. The same procedure is followed with this demand-warehouse as with the preceding one, and so on, until a stairway pattern is developed, and all demands are satisfied.

The circled numbers represent the number of units assigned to each warehouse.

The Mutually Preferred Method

This method assigns orders to the square at the lowest cost in each row and column. Once the "mutually preferred" plan has been found, a maximum assignment is made. As all

Source Desti- nation	SW ₁	SW ₂	SW ₃	SW ₄	SW ₅	Demand
DW ₁	-690 12	-690 3	-835	-300	-1000	15
DW ₂	-240	-290 6	-290	-510	-390	6
DW ₃	-190	-90	-270 6	-320	-540	6
DW ₄	-80	-50	-190	-400 12	-500	12
DW ₅	-130	-50	-220	-320 3	-510 6	9
Slack	0	0	0	0	0 18	18
Supply	12	9	6	15	24	66

Fig. 7--The Northwest Corner Method Solution.

requirements are satisfied in a cell, the rest of the cells are dropped from the solution. The procedure is repeated until all assignments are made. (See Figure 8.)

Source Desti- nation	SW ₁	SW ₂	SW ₃	SW ₄	SW ₅	Demand
DW ₁	-690	-690	-835	-500 15	-1000	15
DW ₂	-240	-290	-290	-510	- 390 6	6
DW ₃	-190	- 90	-210 6	-320	- 540	6
DW ₄	- 80 3	- 50 9	-190	-400	- 500	12
DW ₅	-130 9	- 50	-220	-320	- 510	9
Slack	0	0	0	0	0	18
Supply	12	9	6	15	24	66

Fig. 8--The Mutually Preferred Method Solution.

VAM - Vogel's Approximation Method

Starting with the first row, the difference between the two lowest cost values is recorded in annexed columns and

rows to the matrix. This process is continued for all rows and all columns until the difference row and the difference column are completed.

The next step is to choose the highest number in the difference columns and the difference rows and allocate to the lowest cost combined square in that row or column, and drop it from the matrix.

As cells are dropped from the solution, the differences are again computed and the procedure is repeated until all requirements are met. This method does not give the optimal solution always. That is why it is called an approximation method. The solution is always close to the optimal solution.

Transportation Program Solution

1. Construct an initial matrix.
2. Set up data for computer in necessary format for computer input.
3. Feed data along with program.

There is a unique optimal solution to demonstrate this point; the problem was solved three times starting at different rows in the matrix: the first, the third, and lastly the fifth row. The optimal solution was found to be in all three cases 6,390 miles total distance travelled, at a total cost of \$6,390.

Solution I: Beginning with the first row, the computer uses the following pattern of assignment.

1. Twelve units from supply warehouse SW_1 to demand warehouse DW_4 and no units to DW_5 at a cost of \$960; total distance travelled is 960 miles.

2. Nine units from SW_2 to DW_5 ; total distance travelled is 270 miles (cost \$270).

3. Six units from SW_3 to DW_2 and no units to DW_3 ; total distance travelled 1,740 miles (cost \$1,740).

4. Six units to DW_3 , no units to DW_5 and nine units to DW_6 , all from SW_4 . Total distance travelled 1,920 miles (cost \$1,920). Total number of units distributed: 15.

5. Fifteen units to DW_1 and nine units to DW_6 from SW_5 . Total cost = \$1,500. Total number of units assigned = 24.

6. Total cost, objective functions: \$6,390; that is a total distance travelled of 6,390 miles.

The optimal solution was obtained after six iterations. See Table VI which is a typed copy of the actual computer output.

Solution II: If the program is started from the third row, the following pattern is developed:

1. From SW_1 - no units are assigned to DW_2

12 units are assigned to DW_4

Total cost is \$960 for 960 miles travelled.

2. From SW_2 - no units are assigned to DW_3

9 units are assigned to DW_5

Total cost \$270 for 270 miles travelled.

TABLE VI

COMPUTER OUTPUT

Transportation Problem - Optimal Solution
Beginning from the 1st Row of the Matrix

2600011000129RS

TRANSPORTATION PROBLEM FOR IBM 1620 WITH INDIRECT
ADDRESSING

PHASE 1 NOW LOADING

TURN SS 2 ON IF NO DEVIATIONS

TYPE IN DEVIATIONS FOR COST AND UNITS

M=05 N=006 SUPPLY-DEMAND=0000

LAST MEMORY POSITION USED 06589

PHASE 2 NOW LOADING

FIRST ROW FOR IBFS IS

PHASE 3 NOW LOADING

00012 004 00000 005	01 000000000960 00012
00009 005	02 000000000270 00009
00006 002 00000 003	03 000000001740 00006
00006 003 00000 005 00009 006	04 000000001920 00015
00015 001 00009 006	05 000000001500 00024

OBJECTIVE FUNCTION 000000006390

SOLUTION OPTIMUM 0006

3. From SW_3 - 6 units are assigned to DW_2
Total cost of \$1,740, distance travelled 1,740 miles.
4. From SW_4 - 6 units are assigned to DW_3
9 units are assigned to DW_6
Total units from SW_4 = 15 at a cost of \$1,920 per
1920 miles travelled.
5. From SW_5 - 15 units are assigned to DW_1
no units are assigned to DW_2
9 units are assigned to DW_6
Total units assigned: 24.
6. Objective function minimized to a cost of \$6,390 in
the fifth iteration.

See Table VII.

Solution III: If the program is started from the fifth row, the solution is again the same. Thus:

1. From SW_1 - 12 units are assigned to DW_4
no units are assigned to DW_2
Total cost: \$960.
2. From SW_2 - no units are assigned to DW_3
9 units are assigned to DW_5
Total cost \$270.
3. From SW_3 - 6 units are assigned to DW_2
no units are assigned to DW_3
Total cost is \$1,740.
4. From SW_4 - 9 units are assigned to DW_6
6 units are assigned to DW_3
Total cost \$1,920.

TABLE VII

COMPUTER OUTPUT

Transportation Problem - Optimal Solution
Beginning from the 3rd Row of the Matrix

2600011000129RS

TRANSPORTATION PROBLEM FOR IBM 1620 WITH INDIRECT
ADDRESSING

PHASE 1 NOW LOADING

TURN SW 2 ON IF NO DEVIATIONS

TYPE IN DEVIATIONS FOR COSTS AND UNITS

M=05 N=006 SUPPLY-DEMAND=0000

LAST MEMORY POSITION USED 06589

PHASE 2 NOW LOADING

FIRST ROW FOR IBFS IS 03RS

PHASE 3 NOW LOADING

00000 002 00012 004	01 000000000960 00012
00009 005 00000 003	02 000000000270 00009
00006 002	03 0000000001740 00006
00006 003 00009 006	04 0000000001920 00015
00015 001 00000 002 00009 006	05 0000000001500 00024
OBJECTIVE FUNCTION 000000006390	

SOLUTION OPTIMUM 0005

5. From SW_5 - 15 units are assigned to DW_1

9 units are assigned to DW_6

Total cost is \$1,500.

See Table VIII.

Since all three solutions were alike, it can be said with a high degree of confidence that there is only one optimal solution to this transportation problem.

Degeneracy

Degeneracy occurs in transportation problems when a program of movements can be divided into two or more groups where the combined capacities satisfy the requirements of the destination.

To correct this degeneracy, an additional amount is added to the program of movements by increasing the surplus available at the origin and the shortage at the destinations.

Procedure:

1. If the number of origins is smaller than the number of destinations divide one unit of the item by twice the number of origins.
2. Select a number smaller than the quotient (of step 1) and add it to the capacity of the origins.
3. Add the total amount of the additions of the capacity of the origins to the requirements of any one destination.
4. Solve as usual.

TABLE VIII

COMPUTER OUTPUT

Transportation Problem - Optimal Solution
Beginning from the 5th Row of the Matrix

2600011000129RS

TRANSPORTATION PROBLEM FOR IBM 1620 WITH INDIRECT
ADDRESSING

PHASE 1 NOW LOADING

TURN SW 2 ON IF NO DEVIATIONS

TYPE IN DEVIATIONS FOR COST AND UNITS

M=05 N=006 SUPPLY-DEMAND=0000

LAST MEMORY POSITION USED 06589

PHASE 2 NOW LOADING

FIRST ROW FOR IBFS IS 05RS

PHASE 3 NOW LOADING

00012 004	00000 002	01	000000000960	00012
00000 003	00009 005	02	000000000270	00009
00006 002	00000 003	03	000000001740	00006
00009 006	00006 003	04	000000001920	00015
00015 001	00009 006	05	000000001500	00024

OBJECTIVE FUNCTION 000000006390

SOLUTION OPTIMUM 0007

Input-Output Method

The input-output method formulated by W. W. Leontief⁴ in 1956 proposes that the study of an economy can be and should be based upon an analysis of the interrelationships of the industries which comprise that economy. The output of one industry becomes ultimately a fragment of the input of several other industries, and the outputs of these industries become also the inputs of more industries through this system. The impact of a change in the output of one industry can be traced throughout the economy.

Leontief's analysis of inter-industry economics has been incorporated to linear programming. The importance of the Leontief model lies in its empirical aspect, the way in which statistical data of the industry are used for the purpose of economic analysis; and in turn, economic analysis is made applicable for practical adaptations.

The Leontief system has linear equations but it has neither maxima nor minima functions or objectives. However, it can be written as a special case for linear programming by relaxing the rigid assumption of fixed coefficients of production, as the maximizing function is introduced.

⁴W. W. Leontief, The Structure of the American Economy, 1919-1929 (New York, 1951).

CHAPTER IV

SOME APPLICATIONS OF LINEAR PROGRAMMING

Early applications of linear programming fell into three major categories: military applications, inter-industry problems, and game theory. [Since the development of high-speed digital computers, linear programming has become one of management's most important tools.] As a mathematical device for solving management and operational problems, it has few rivals. Its application is limited only by human ingenuity in relating it to actual conditions and specific problems. It has been applied successfully in solving problems in most fields and practically all industries; such as military, mathematical, structural design, engineering and managerial problems, like which products to make so as to maximize profits; where to make these products so as to minimize costs; optimum machine allocation; inventory control; personnel assignment; economic analysis; from what plant to ship to which warehouse so as to minimize freight cost; where to sell; and how much to make; optimum price-setting; optimum blend of raw materials, and many other problems of allocation of resources.¹

¹Saul I. Gass, Linear Programming Methods and Applications (New York, 1960), pp. 180-187; Henderson and Schlaifer, pp. 121-134.

The following are the most important applications which will be presented:

A. Military Applications

- 1) Optimum aircraft scheduling
- 2) Deployment of aircraft
- 3) Optimum defense system
- 4) Contract awards

B. Industrial Applications

- 1) Chemical industry
- 2) Petroleum industry
 - a. gasoline blending
 - b. transportation of crude oil to refineries
- 3) Communications
- 4) Iron and steel industry
- 5) Railroad industry

C. Agriculture

- 1) Farm economics
- 2) Farm management

D. The Paper Industry

- 1) The trim problem
- 2) Optimum transportation of materials and finished products

E. Samples of Industrial Management Problems

- 1) Machine loading problem
- 2) Production scheduling problems
- 3) Personnel assignment problems
- 4) Transportation problems
- 5) The problem of where to produce.

F. Economic Analysis

G. Additional Applications of Linear Programming.

Military Applications²

One of the earliest applications of linear programming was in the field of the military. The problem here involved was to find an optimum schedule of aircraft transportation of weapons and men to Berlin during the war. The objective was to minimize cost for a given tonnage or to maximize tonnage for a given amount of money and aircraft. The resources to be optimized were the aircraft, crew, runways, money and time.

The Air Force also has applied linear programming to maximize the use of deployed aircraft. In this case, a given number of airplanes were being produced each month, some aircraft were left for training, depending upon the number of crews to be trained, and the rest were deployed (sent to combat). Other military problems have been the selection of an optimal defense weapons system, a community defense system to give the adequate protection at the lowest cost possible.

The U. S. Air Force Procurement Office has utilized linear programming to solve their purchasing problems, minimizing the combined cost of purchase and freight. Whenever the U. S. Air Force desires to purchase from civilian sources, the producers are invited to bid for contracts.

²Ibid., p. 184.

If, for example, manufacturers submit bids for delivering their products to ten quartermaster depots, and the procurement office works out a schedule of freight cost and unit cost combined, they will end up with one hundred different prices. The difficulty may be seen clearly in the selection of the lowest possible cost contract. Linear programming helps the procurement office to award the lowest possible cost contracts.³

Industrial Applications

Chemical industry.--Most of the work done with linear programming in the chemical industry has been in the areas of production scheduling and inventory control and in the transportation of raw materials to the plants.

Many studies have been made to schedule electrochemical machinery of different capacities to produce aluminum oxide films on etched aluminum foils of different combinations of voltage capacity and width (each width requires a different current input).

The constraints are the capacity of the machinery and the available power at the plant.

Quite frequently, in the chemical industries, it is encountered that some of the constraints are non-linear; however, in most of such cases these constraints are simple

³The Application of Linear Programming Techniques to Air Force Problems: A Non-Technical Discussion, Computation Headquarters, U. S. Air Force (Washington, D. C., 1954), pp. 22-26.

curves which can be approximated by several tangent lines, thus making it possible to employ linear programming techniques.⁴

Petroleum industry.---Extensive work has been done with linear programming in the petroleum industry. Typical of the many kinds of problems where it has been applied are:

a. Gasoline blending: an oil company needs to blend a number of stock gasolines to produce marketable gasolines. Each marketable gasoline must meet certain requirements such as octanage, vapor pressure, molecular weight, color, etc. For each product, the price per barrel or price per gallon (unit cost of raw material) and the volume of available raw material are known. The problem is to find the optimum blends or what volume each type of fuel should be blended in order to yield the maximum possible profit meeting the specifications. Symonds,⁵ in a study of the application of linear programming to the solution of the refinery problems in blending, gave a very clear and simple presentation of the simplex method.

The problem concerns the optimum allocation of materials which are used in two procedures for making fuel oil from pitch and flux stock, the latter being available in any desired quantity. One method is to blend the pitch directly

⁴Rafael B. Cruz, Informal Lectures in Linear Programming, The University of Texas, Austin (Spring, 1961).

⁵Gifford H. Symonds, Linear Programming: The Solution of Refinery Problems, Esso Standard Oil Company (New York, 1955).

with flux stock; the other, is to use the pitch to make a tar and then to blend the tar with flux stock to make the fuel oil. The problem is illustrated in Figure 9.

Fig. 9--Block diagram

Suppose that P gallons per hour of pitch are mixed directly with the flux stock, while V gallons per hour of pitch are used to make tar which, in turn, is mixed with flux stock. Each gallon of pitch used to make tar will yield 0.8 gallons of tar and by-products of negligible value. If a total of F gallons per hour of flux are used, then the output of fuel oil will be $P + 0.8V + F$ gallons per hour.

The fuel oil is worth five cents per gallon and the flux stock is worth eight cents per gallon, and the producer has reasons for neglecting, in this particular calculation, the

cost of pitch, of pitch to tar conversion, and of tar. The net value of the output will be, therefore: $5(P + 0.8 + F) - 8F = 5P + 4V - 3F$ cents per hour, and it is this figure that the producer desires to maximize.

The fuel oil must meet certain specifications as to viscosity and specific gravity. These technical requirements turn out to be expressed respectively by the inequalities, $2P + V - 2F \leq 0$ and $P + V - 3F \leq 0$. The producer has a source of pitch delivering 1000 gallons per hour, and clearly cannot use pitch at a rate faster than this, whence $P + V = 1000$. From its practical source the problem accordingly is this: maximize the profit function: Profit = $5P + 4V - 3F$ subject to: $P \geq 0$, $V \geq 0$, $F \geq 0$, $2P + V - 2F \leq 0$, $P + V - 3F \leq 0$, $P + V \leq 1000$.⁶

The above case represents the maximum possible profit in this operation since there is no potential profit . . . to be obtained by using any other variable in the basis. This case includes vis-breaking all of the pitch (1000 gph) with 500 gph of flux. The minimum viscosity specification is exactly met, but there is a give-away of about 1.5 points in gravity. The maximum profit is 1500 cents per hour (\$600 per day).⁷

Charnes, Cooper and Mellon⁸ made a similar study in the blending of aviation gasolines. This was one of the first studies made in the field of linear programming.

⁶Adaptation from Symonds, Ibid., pp. 4-11.

⁷Ibid.

⁸Charnes, Cooper and Mellon, Blending Aviation Gasolines: A Study in Programming Interdependent Activities in an Integrated Oil Company (New York, 1952).

b. Transportation of crude oil to refineries: The application of linear programming to long-range investment planning and to operations scheduling for an integrated oil company is represented in the following example. An oil company extracts and processes crude oil from different fields and transports the oil to sales terminals as needed. At the refineries gasoline and diesel fuel are processed from two types of crude oils of light and heavy density. The company is integrated as follows: the extraction of raw materials; transportation to refineries; and refining the crude oil and transporting it to market. These functions are inter-related closely. The output of one function is the input of another consecutive function and each function consists of parallel elements. There are many oil fields and more than one refinery.

This system offers a wide choice of alternatives for the performance of each function with corresponding constraints at each level of operation. This very complex type of problem can be broken up into several smaller linear programming problems to facilitate working out a solution by making use of Dantzig's principles. Many modern industrial enterprises are made up of more than one plant. The management of each plant enjoys a certain amount of autonomy in the planning of operations. The decisions of one local manager may affect the operation of the other plants which shared capital, raw materials and markets. It is for this reason that large

enterprises employ a staff to coordinate and plan the activities centrally--to optimize the whole enterprise.

This central planning staff gives each plant manager a set of restrictions which reflect the interaction of the whole business. Each plant manager goes about optimizing his plant ignoring the effects his decision may have on other plants.

If the plant's economic functions are linear, then in principle the whole organization could be optimized by linear programming techniques.

As linear programming problems increase in complexity, the time and costs increase threefold; optimization of a large enterprise is, then, not very feasible. Dantzig has developed a practical way of solving complex problems of this nature by making use of a computational device called the Decomposition Principle.⁹

Example:¹⁰

Allocation of Scarce Raw Materials Between Two Plants

A company has two plants, A and B, which manufacture the same product but uses different raw materials. The only specification on the product is that the vapor pressure at 68° F. cannot exceed 11 Psia. Plant A has a stock of components 1, 2, and 3, while Plant B has available components

⁹Rafael B. Cruz, Informal Lectures in Linear Programming, University of Texas, Austin (Spring, 1961).

¹⁰Ibid.

3, 4, and 5 as shown in Table IX. The prospective markets demand 1000 ton-mols each of the product. Together they have unlimited stock of components 1, 2, 4, and 5, but only 1400

TABLE IX
VAPOR PRESSURE AND UNIT COST OF COMPONENTS

Component	Vapor Pressure at 68° F., Psia	Cost in Dollars per Lb.-Mol
1	10	6
2	70	2
3	0	3
4	20	6
5	30	1

ton-mols of component 3 is available. The problem is to find the amount of this component that should be allocated to each plant so as to optimize production of both plants combined.

Plant A:

$$\text{Vapor pressure} - 10x_1 + 70x_2 + 0x_{3a} \leq 11 \quad (1)$$

$$\text{or } 10x_1 + 70x_2 + V_a = 11 \quad (2)$$

Here the variable V_a has been introduced to turn this inequality into an equation, (2).

$$\text{*M. B.: } x_1 + x_2 + x_{3a} = 1 \quad (3)$$

*Material balance--the sum of the components must be equal to the whole.

Plant B:

$$\text{Vapor pressure: } 0x_{3b} + 20x_4 + 30x_5 \leq 11 \quad (4)$$

$$\text{or } 20x_4 + 30x_5 + V_b = 11 \quad (5)$$

$$\text{M.B.: } x_{3b} + x_4 + x_5 = 1 \quad (6)$$

Interconnecting constraint:

$$1000x_{3a} + 1000x_{3b} \leq 1400 \quad (7)$$

$$\text{or } x_{3a} + x_{3b} + V_c = 1.4 \quad (8)$$

Minimize cost function: (per pound-mol.)

$$C = 6x_1 + 2x_2 + 3x_{3a} + 3x_{3b} + 6x_4 + 1x_5 \quad (9)$$

The problem once set up can be solved by the simplex method.

Communications.--This is another area in which only a few applications have been used. The main work has been in the optimal design and utilization of communication networks. Problems involving facilities for transmission, switching, relaying, etc., have been solved by linear programming methods. These methods provide a general system approach for solving the complex interactions among system capacities of users' demands, and economic factors.

Iron and steel industry.--This industry has made use of linear programming to optimize the transportation of raw materials to the steel mills and the scheduling of machinery to meet the demand. The factors considered are the capacity of the machinery sales forecasts, shipping rates and the quality of the product.

Railroad industry.--The applications of linear programming to the railroad industry have been confined largely to the optimal distribution of freight cars; the distribution and classification of cars between railroad yards; and the scheduling problems where the objective was to minimize the total crew and engine expense.

Agriculture

Agricultural applications can be classified into two general categories: farm economics and farm management.

Farm economics.--In a case in which the prices of livestock feed at different shipping points and at destinations are related directly to the quantities shipped or received and the quantities of livestock feed available from local producers, it is desired to minimize the average cost over a whole region. This type of livestock feed economy model originally was formulated in terms of a ten-region subdivision of the United States. The problem was solved through a modification of the transportation method.

Farm management.--Managing an individual farm is the allocation of limited resources such as feed, acreage, water, ranchhands, and money for the general purpose of maximizing profits. A problem of this sort may involve deciding which crop or crops to plant from a wide choice of alternatives in a given acreage at a given time and the particular method of cultivation and periodic irrigation. Another problem may be the selection of a crop rotation plan.

Another case of farm management is the hog raiser desiring to determine the most economical mixture of ingredients for a hog feed that will supply his hogs with the known minimum requirements of proteins, vitamins, minerals, calories, and other nutritional factors to complement a healthy and fattening diet. This same type of problem is faced by cattle raisers, chicken raisers, and all others who grow livestock.

The Paper Industry

Paper mills produce their product in rolls of standard lengths and widths; in order to fill a customers order the rolls need to be cut in various widths. Here the manufacturer is faced with the problem of how to cut the paper rolls from the standard raw rolls so as to waste as little paper as possible. Maximum profit is derived from the system of slicing the standard rolls so that all orders are filled with a minimum of trim loss due to add-end pieces.

The trim problem is faced by textile manufacturers, pipe rollers, metallic foil producers, cellophane paper manufacturers as well as pulp paper manufacturers and any other industry where trim losses are incurred in the cutting of materials.

Linear programming helps the management of the industries with the trim problem not only in optimizing the use of their materials, but also in machine scheduling which includes the selection of the proper machinery, the number of blades and

their widths, the number of cutters available, their capacity, and time as well as money limitations.

Where an optimum program has been formed and it is already in operation when new orders come in, linear programming permits modifications of the original program, superimposing on the first optimum solution to yield the new optimum solution.

Another area where linear programming has been of help to the paper industry is in the transportation of materials to plants and of finished products to markets and warehouses. Transportation scheduling deals in this case with companies having several mills. The problem is how to assign the various orders to the mills so as to minimize freight expenses.

Samples of Industrial Management Problems

Machine loading problem.--In a particular paper form factory, business forms are printed by different types of machines. The machines have different operating costs for each type of business form. The plant manager must deliver to the customer the ordered forms within a given time period. If the machines cannot exceed their rate capacity, what allocation of product to machines would result in minimum costs?

Assuming that the capacity of machine A is 12 hours, that one unit of output (in this case 1000 business forms) = 3 hours. The program is $x = 4$ (the solution) since: $3x = 12$. If utilization to capacity is allowed but not required: $3x \leq 12$. If a minimum of 12 hours is set, then: $3x \geq 12$. If

a second type of business form is to be run through this machine, and one unit of output equals 2 hours, let x_1 = business forms with 3 hours production run, where $x_1 \geq 0$, x_2 = business forms with 2 hours production run, where $x_2 \geq 0$.

$$\text{Then: } 3x_1 + 2x_2 = 12$$

$$x_1 = 4, 2, 0$$

$$x_2 = 0, 6, 3.$$

Through linear programming the optimum machine scheduling program may be found.

Production scheduling problems.--Typical of this type of problem is the manufacturer faced with the problem of warehousing a seasonal product. The seasonal fluctuations, cost of raw materials, and the selling price of the finished product are known. Storage facilities have a limited capacity and the storage of a given quantity for a certain period of time has known costs. To meet a given sales potential during the season, the plant manager would like to know the optimal pattern of production and storage. A sales forecast chart is needed to set the production level to meet demand requirements.

If a plant manager decides to make exactly the demand requirements for each month, the result would be idle time and overtime costs during high production months; on the other hand, the plant manager may decide to overproduce in months of low demand and keep stocks for the high demand months. This type of plan may result in very high storage cost. The

optimum production program lies somewhere between the two extremes. The following is an example of a production scheduling problem. The production manager of a manufacturing firm has been instructed to produce p_i units in n months.

$$i = 1, \dots, n$$

CL = minimum production at regular time

CO = maximum production at overtime

CT_i = unit cost per month on regular time

CTO_i = unit cost per month on overtime

ST_c = storage cost per unit per month

X_i = total production in month i th at regular time

XO_i = total production in month i th at overtime

ST_i = total number of units stored during month i th.

The production manager would like to know whether it would be more profitable to produce in advance of scheduled delivery.¹¹

Constraints: $X_i \leq CL$, $XO_i \leq CO$, for $i = 1, \dots, n$.

$$X_1 + X_i - P_i - \dots - P_1 = ST_i + 1, i = 1, \dots, n-1.$$

$$X_i \geq 0, ST_i \geq 0$$

Minimizing function:

$$C_1X_1 + \dots + C_nX_n + CO_1XO_1 + \dots + CO_nXO_n + ST_c (ST_2 + \dots + ST_n).$$

Procedure: Transform into a transportation type problem. Introduce "unused capacity" so that numbers in sources add up

¹¹The above problem was taken from S. Vagda, Readings in Linear Programming (New York, 1960), pp. 13-15.

to CL or CO while those for requirements (destinations) add up to P_i . Total unused capacity is, therefore:

$$\sum_i CL + \sum_i CO - \sum_i P_i$$

Costs: 1) Unused capacity cost = 0.

2) Cost of one unit produced in i th month and used in the month J th ($J \geq i$) is either $CT_i + (J - i)STC$ or $CTO_i + (J - i)STC$.

3) Cost of any item with $J < i$ is very large, so they do not show up in the final solution.

The connection between X_{ij} and XO_{ij} of transportation type and original problem are:

$$\sum_{j=1}^n X_{ij} = X_i, \sum_{j=1}^n XO_{ij} = XO_i \text{ for } i = 1, \dots, n;$$

$$\sum_{i=1}^T, \sum_{j=t+1}^n X_{ij} + XO_{ij} = ST_{t+1} \text{ for } t = 1, \dots, n - 1;$$

Where: $X_{ij} \geq 0$ and $XO_{ij} \geq 0$

Subject to: $\sum_{i=1}^j X_i = P_j, \sum_{j=1}^n X_{ij} + Y_i = CL$

Y_i and YO_i refer to unused capacities.

For pedagogical purposes numerical values are assigned below. Let: $n = 4$

$$\begin{aligned} d &= 1 \\ i &= 1, 2, 3, 4 \\ P_i &= 5, 6, 8, 6 \\ CL &= 9, 9, 9, 9 \\ CO &= 3, 3, 3, 3 \\ CT_i &= 1, 4, 2, 4 \\ CTO_i &= 2, 6, 4, 6 \end{aligned}$$

Thus the following cost matrix is constructed.

	5	6	8	6	23 (unused capacity)
9	1	2	3	4	0
3	2	3	4	5	0
9		4	5	6	0
3		6	6	8	0
9			2	3	0
3			4	5	0
9				4	0
3				6	0

The following scheme is produced:

	5	6	8	6	23	*f.c.
9	5	4	.	.	.	0
3	.	2	.	.	1	1
9	.	.	.	.	9	1
3	.	.	.	.	3	1
9	.	.	8	1	.	0
3	.	.	.	.	3	1
9	.	.	.	5	4	1
3	.	.	.	.	3	1
*f.c.	1	2	2	3	-1	

*f.c. = final cost.

This scheme implies a schedule of production given as:

$$X_1 = 9, X_{O1} = 2, X_3 = 9, X_4 = 5, St_2 = 6, ST_4 = 1.$$

As overtime costs are higher than regular time costs, the scheme calls for overtime only when the regular capacity of the same month has been used.

Another type of production scheduling problems is that of assembly line balancing. Basically the problem is as

follows. The item consists of different components and it is necessary to assemble them in a given sequence. As it takes more time for some components to be assembled, a waiting line is formed and idle time is accumulated elsewhere. The problem is to minimize assembly line idle time.

Personnel assignment problems.---The assignment problem may be stated as follows: There are M men available to fill n jobs. C_{ij} is the cost for assigning man i to job j . The objective is to minimize the cost of assigning all men to all jobs.

The problem may be stated mathematically in the following manner: Given a cost matrix (C_{ij}) , find a matrix (X_{ij})

where $\sum_{i=1}^n \sum_{j=1}^n C_{ij}X_{ij}$ is at a minimum and $X_{ij} = 0$ or 1 , and

$$\sum_{j=1}^n X_{ij} = 1 \text{ and } \sum_{i=1}^n X_{ij} = 1.$$

Here is another form of the preceding problem: Corporation X has ten jobs open and ten candidates for the jobs. No one job is more important than another; and the corporation desires over-all optimization. It is possible to assign ratings to each candidate for each job. It is desired, then, to obtain the highest possible total from the summation of the candidates' ratings.¹²

Let i = candidate #1, 2, ..., 10

j = job # 1, 2, ..., 10

¹²Ibid., pp. 38-40.

r_{ij} = rating for candidate i in job j

$X_{ij} = 1$, if candidate i is allocated to job j , and
 0, if otherwise.

$X_{ij} \geq 0$ and $\sum_j X_{ij} = 1$ for all i ;

$\sum_i X_{ij} = 1$ for all j .

Find maximum of $\sum_{i=1}^{10} \sum_{j=1}^{10} r_{ij} \cdot x_{ij}$

The assignment problem may be viewed as a special case of a zero-sum two-person game, and solved by linear programming techniques after some modification.¹³

Transportation problems.--Managers of industrial enterprises are sometimes faced with questions such as the need of a new warehouse but where should it be built in order to minimize freight expenses. Is the present shipping pattern sufficiently economical? Production is being expanded on the eastern coast, how should the present shipping pattern be altered? What is the effect of seasonal fluctuations and of the business cycle on freight expenses to market?

The transportation program is more than an aid for reducing transportation costs. It helps management by providing important data essential for long-range decision-making.

As seasonal variations change supply and demand, shipment plans can be altered to keep costs always at a minimum. If

¹³Jerome M. Kurtzberg, "On Approximation Methods for the Assignment Problem," Journal of the Association for Computing Machinery, IX, No. 4 (October, 1962), 420.

freight rates change a new plan can be provided. The transportation program can tell management which of the production plants would be most profitable to expand, or where a new plant should be built. In the latter case, various locations can be analyzed to determine which location produces the greatest reduction in total transportation costs, thus indicating the rate of return on investment. Management may wish to know whether it would be more profitable to build a new plant or to expand an old plant. By analyzing various combinations of increased supplies at existing plants and new supplies at new hypothetical plants, the best alternative can be chosen.

Factors involved in the transportation problem are: 1) item to be transported, 2) origins, 3) destinations, and 4) costs per item.

The item to be transported from origins to destinations may be a manufactured product, a raw material, a carload of equipment, a tanker full of oil, or any unit of merchandise. The origins are the localities where the product or items are available for distribution. The supply is the amount of the product or items available at the origins. Origins may be production plants, warehouses, refineries or depots. The number of origins must remain unchanged, the supply of the item must be limited, known, and remain constant.

The demand is the quantity of items wanted at the destination points. The number of destination points must remain

constant, the demand must be limited, known, and remain fixed. Demand must equal supply.

The cost per item will vary according to the distances the item travels and the transportation media since truck, airplane, ship, vessel, barge, and railroad freight rates differ widely. The unit cost between origins and destinations must be known and remain the same.

It is possible in some cases when the actual transportation problems do not meet all requirements to transform the problem so as to meet them. The supply may not equal demand; the supply at the origins or the demand at the destinations may not be fixed. To meet the necessary requirements, dummy stations are set up to take care of the needed extra demand or the needed extra supply. The answer thus obtained does not change the value of the problem.

In case that more than one type of item needs to be included in the problem, the program may be modified so that the minimum cost solution will indicate the shipping plan for each item. A typical example is the refinery with varied products such as kerosene, oil, gasoline, and benzene.

The General Transportation Problem

The general transportation problem can be solved by a variation of the Simplex Method called the distribution or transportation method.

A company produces a homogeneous product in specific amounts at different plants scattered around the country.

The plants must ship the product to distribution points in large cities. Each of these points of distribution has a given demand for the product. The shipping costs from any plant to any distribution point are known. The ideal distribution would be for each plant to supply the nearest distribution points. The problem is not so simple, for the demands are different and if a particular plant were to ship to a nearby distribution point, that plant might not have enough to ship to another closeby distribution point.

Let A_{od} = product units to be shipped from origin o to destination d

B_o = product units available at origin o

N_o = product units required at origin o

C_{od} = product units cost of shipping from o to d .

So the total amount of units of product x to be shipped from any plant (or origin in case of several plants) to any number of destinations cannot be greater than the total availability at that origin: $\sum_d A_{od} \leq B_o$.

Above is a sample of functional constraint. The shipments from any number of origins to any destination must be greater than or equal to the total demand of product x at that point: $\sum_d A_{od} \geq N_o$. The problem is to minimize the total cost of shipping: $\sum_d, \sum_o C_{od} A_{od} = \text{minimum}$. All shipments must be equal to or greater than zero. $A_{od} \geq 0$. This is a sample of a negative constraint.

The J. J. Heinz Company was one of the first industrial enterprises to optimize their shipping schedules. In 1953 this company had six ketchup manufacturing plants in various parts of the country. These six plants supplied seventy warehouses from which distribution to market was made.

Heinz could sell all that it could produce. The management wanted to make a distribution plan that would keep distribution costs to a minimum. There was one problem, the Eastern demand exceeded its capacity and Western capacity exceeded its demand.

The problem was solved through the use of linear programming.¹⁴ The form used was that of the regular transportation problem:

Let A_{od} = product (ketchup) units to be shipped from o to d

B_o = product units available at o

N_o = product units required at o

C_{od} = product carload costs of shipping from o to d .

Thus

$$\sum d \cdot A_{od} \leq B_o \quad (1)$$

$$\sum d \cdot A_{od} \geq N_o \quad (2)$$

$$\sum_o d, \sum_o c \cdot C_{od} \cdot A_{od} = \text{minimum} \quad (3)$$

$$A_{od} \geq 0. \quad (4)$$

The problem of where to produce.--It is possible to minimize the cost of production by limiting production to the low cost plants where shipments would be made at the lowest costs. As an example here is another problem the Heinz

¹⁴Henderson and Schlaifer, op. cit., pp. 121-123.

Company encountered. It produced an item at several plants for distribution to a number of warehouses. The capacities of the plants exceeded the needs of the warehouses and the cost of production was different at each plant. This is the case of a double problem, how much to produce and where to produce it. The assumption was made that the cost of production at any plant is the sum of a fixed cost independent of volume, and a variable cost proportional to the volume in total, but fixed per unit. The variable cost of production per unit is added to the freight rates between the particular plant and the warehouses. This, when done for all plants, provides a combined unit cost of production and freight for every combination of plant and warehouse. The problem was solved then by the transportation procedure.¹⁵

Economic Analysis

The use of linear programming in the field of economics may be said to have started with Leontief's interindustry input-output model.

As an example of Leontief's model, assume the existence of an economy consisting of three types of industries:

1. Extractive industries (raw materials)
2. Railroad
3. Manufacturers

¹⁵Ibid., pp. 123-125.

These three types of industries are interrelated. The objective is to optimize the whole economy.¹⁶

The following are samples of economic analysis by linear programming techniques. The selection of an optimum investment portfolio, optimum plant site considering competitors, and an optimum marketing plan.

The selection of the optimum investment plans where there are alternatives and constraints limiting the choice and the quality of investment alternatives, is rather an easy task if linear programming techniques are applied.

Here is the case of an investment company manager who receives from a customer \$1,000,000 (one million dollars) and he is told to maximize the total return on investment but that the following regulations must be followed. Forty per cent of the whole amount or more must be invested in securities yielding 4 per cent.

Not more than 20 per cent can be invested in any one security yielding 6, 7, 8, and 8.5 per cent. To solve, let securities yielding up to 4 per cent = x

those yielding up to 6 per cent = y

those yielding up to 7 per cent = z

those yielding up to 8 per cent = t

those yielding up to 8.5 per cent = k

So: $x + y + z + t + k \leq 100$. (1)

¹⁶ For an excellent detailed example of Leontief's inter-industry economic model, refer to Gass, op. cit., pp. 164-170.

The restrictions on the system are: $x, y, z, t, k \geq 0$.
 $x \geq 40$; $y \leq 20$; $z \leq 20$; $t \leq 20$; and $k \leq 20$.

Non-negative slack variables m and V_1 are introduced to transform these inequalities into equations, thus yielding the system: $x + y + z + t + k + m = 100$ (2)

$$x - V_1 = 40 \quad (3)$$

$$y + V_2 = 20 \quad (4)$$

$$z + V_3 = 20 \quad (5)$$

$$t + V_4 = 20 \quad (6)$$

$$k + V_5 = 20 \quad (7)$$

The problem is to maximize the following yield function:

$$F = 4x + 6y + 7z + 8t + 8.5k. \quad (8)$$

The function F represents the per cent return and is independent of the amount invested.

Most businesses derive the greatest portion of their sales from a small section of their market, and mislocate their sales efforts.

Which markets should receive the greatest attention and promotion and which should receive the least promotion or none at all? This question usually arises when marketers strive to reduce their marketing costs and increase efficiency.

Linear programming can help to allocate best limited distribution time, effort and costs. The objective of the problem is to make optimum use of salesman efforts, advertising budgets, and warehouse space, which is the optimum

combination of sales activities that will make optimum use of the company's resources, and thus maximize gains.

Additional Applications of Linear Programming

Additional applications of linear programming include the design of optical filters and the operation of a system of dams so as to maximize the electrical energy obtained from the system. Linear programming has been of great help in solving some mathematical problems, both theoretical and applied. The theory of statistics has been applied to problems in which variables were subject to random errors. Other mathematical applications are combinatorial analysis, network and graph theory.

The following is a sample of a linear program applied to mathematics. In a solution of a set of simultaneous linear equations, a problem could be transformed into a linear programming problem by adding a slack variable to each one of the equations and minimizing a dummy cost function where a cost of zero is associated to each of the variables and a large cost associated to each of the slack variables. This forces all of the slack variables out of the basis and gives the unique solution to the system. This method becomes advantageous in the case where a computer program for solving linear programming problems is available to the mathematician.¹⁷

¹⁷Rafael B. Cruz, Informal Lectures on Linear Programming, University of Texas, Austin, Texas (Spring, 1961).

Most structural design problems are of a simple nature. The regular problem is as follows: The highway department is constructing a bridge which must be able to support a given distribution of weight. What is the minimum amount of structural steel that must be used to build a bridge of the required strength? The critical value of the loading parameter at which collapse of the bridge occurs can be reduced in various ways to the problem of maximizing a linear function subject to linear constraints.

CHAPTER V

OPTIMUM BLEND PROBLEM

The Kreck Packing Company Case

In a system of non-negative variables constrained by a set of inequalities, it might be desirable to optimize a function of these variables. In this case the variables are the per cent fraction of a number of meat products that possibly could be used in different and varied amounts in the blending of an all-meat frankfurter by the Kreck Packing Company of Dallas, Texas. The different meat products have different costs and proportions of water, protein and fat.

There are two controlling factors in quality: taste and content. A number of ingredients, if used disproportionately, might make the product hard and very chewy. Others might impair the flavor by excessive saltiness or some other peculiarity. Content quality of the product means the minimum required percentages of protein and the maximum allowed percentages of fat and water.

The constraints are the maximum amounts of certain meat products that can be used in the blend and the desired quality of the product.

Possible ingredients for an all-meat frankfurter and the limitations in composition.---The Kreck Packing Company can use

a blend of thirteen possible meat products in its manufacture of frankfurters.

These components are: bull meat, regular beef trimmings, cow meat, boneless beef navel, beef flanks, mutton meat, lean beef trimmings, pork jowls, bacon, ox lips, tripe, regular pork trimmings, and lean pork trimmings. Each one of these meat products contains different proportions of water, protein and fat and have different costs. See Table X for specific compositions and costs of the products.

It is desired to produce a pre-processed final blend of the following quality. (See Table XI on constraints.)

Fat should not make up over	40.3 per cent
Water should not make up over	44.6 per cent
Protein should not make up less than	13.8 per cent
Tripe should not be over	5.0 per cent
Lips should not be over	9.0 per cent
Mutton should not be over	10.0 per cent
Bacon should not be over	13.5 per cent
Pork should not be over	35.0 per cent

Translating data into mathematical equations.---Translation of data into mathematical equations is necessary so that they may be translated into computer language.

Constraint No. 1: Tripe should not make up over 5 per cent of the blend. $T \leq .05$.

Add slack variable V_1 to turn this inequation into an equation No. 1. $T + V_1 = .05$.

TABLE X
RAW MATERIALS FOR ALL-MEAT FRANKFURTER

Product	Code	Cost per Pound (Cents)	Composition (Percentage)
Bull meat	BULL	41	70 water 22 protein 7 fat
Cow meat	COW	38.5	70 water 19 protein 10 fat
Beef navel	BN	22	43 water 13 protein 45 fat
Beef flanks	BF	22	37 water 12 protein 50 fat
Mutton	M	29	70 water 19 protein 10 fat
Regular beef trimmings	RBT	22	37 water 12 protein 50 fat
Lean beef trimmings	LBT	33	61 water 18 protein 20 fat
Pork jowls	J	17.5	26 water 7 protein 65 fat
Bacon	BAC	12	19 water 8 protein 73 fat
Ox-lips	L	14.5	68 water 17 protein 13 fat
Tripe	T	7	80 water 17 protein 2 fat

TABLE X--Continued

Product	Code	Cost Per Pound (Cents)	Composition (Percentage)
Regular pork trimmings	RPT	19	37 water 7 protein 2 fat
Lean pork trimmings	LPT	21	42 water 11 protein 46 fat

The following table shows the taste and content constraints for the all-meat frankfurter.

TABLE XI

TASTE AND CONTENT CONSTRAINTS FOR
ALL-MEAT FRANKFURTER

Taste Constraint	Product	Per Cent of Blend Composition
Chewiness	Tripe	5.0
Peculiarity	Lips	9.0
Peculiarity	Mutton	10.0
Saltiness	Bacon	13.5
Peculiarity	Pork	35.0
	Fat	40.3
Content Constraints	Water	40.6
	Protein	13.8

Constraint No. 2: Lips should not make up over 9 per cent of the blend. $L \leq .09$.

Add slack variable V_2 to turn this inequation into equation No. 2. $L + V_2 = .09$.

Constraint No. 3: Mutton meat should not make up over 10 per cent of the blend. $M \leq .10$.

Add slack variable V_3 to turn this inequation into equation No. 3. $M + V_3 = .10$.

Constraint No. 4: Bacon should not make up over 13.5 per cent of the blend. $BAC \leq .135$.

Add slack variable V_4 to turn this inequation into equation No. 4. $BAC + V_4 = .35$.

Constraint No. 5: Pork should not make up more than 35 per cent of the blend. $J + RPT + LPT \leq .35$.

Add slack variable V_5 to turn this inequation into equation No. 5. $RPT + J + LPT + V_5 = .35$.

Constraint No. 6: The sum of the components should be equal to 100 per cent of the blend. $BULL + COW + BF + M + RBT + LBT + J + BAC + L + RPT = 1.00$.

Constraint No. 7: Protein should at least make up 13.8 per cent of the blend: $.22 BULL + .19 COW + .13 BN + .12 BF + .19 M + .12 RBT + .18 LBT + .07 J + .08 BAC + .17 T + .07 RPT + .17 L + .11 LPT = .138$.

Subtract variable V_7 since a minimum percentage of 13.8 protein is desired to turn this inequation into equation No. 7. $.22 BULL + .19 COW + .13 BN + .12 BF + .19 M + .12 RBT + .18 LBT + .07 J + .08 BAC + .17 L + .17 T + .07 RPT + .11 LPT - V_7 = .138$.

Constraint No. 8: Fat should not make up more than 40.3 per cent of the blend. $.07 \text{ BULL} + .10 \text{ COW} + .45 \text{ BN} + .50 \text{ RBT} + .10 \text{ M} + .20 \text{ LBT} + .65 \text{ J} + .73 \text{ BAC} + .02 \text{ T} + .55 \text{ RPT} + .46 \text{ LPT} \leq .403$. Add variable V_8 to turn this inequation into equation No. 8: $\text{BULL} + .10 \text{ COW} + .45 \text{ BN} + .50 \text{ BF} + .10 \text{ M} + .50 \text{ RBT} + .20 \text{ LBT} + .65 \text{ J} + .73 \text{ BAC} + .13 \text{ L} + .02 \text{ T} + .55 \text{ RPT} + .46 \text{ LPT} + V_8 = .403$.

Constraint No. 9: Water should not make up more than 44.6 per cent of the blend: $.70 \text{ BULL} + .70 \text{ COW} + .43 \text{ BN} + .37 \text{ BF} + .70 \text{ M} + .37 \text{ RBT} + .61 \text{ LBT} + .26 \text{ J} + .19 \text{ BAC} + .68 \text{ L} + .80 \text{ T} + .37 \text{ RPT} + .42 \text{ LPT} + V_9 \leq .446$.

Introduce a non-negative artificial variable V_1^i to each constraint not having a slack variable or having a negative one, that is, equations Nos. 6 and 7. $\text{BULL} + \text{COW} + \text{BF} + \text{M} + \text{RBT} + \text{LBT} + \text{J} + \text{BAC} + \text{L} + \text{RPT} + \text{LPT} + U_1^i = 1.00$.

$.22 \text{ BULL} + .19 \text{ COW} + .13 \text{ BN} + .12 \text{ BF} + .19 \text{ M} + .12 \text{ RBT} + .18 \text{ LBT} + .07 \text{ J} + .08 \text{ BAC} + .17 \text{ L} + .17 \text{ T} + .07 \text{ RPT} + .11 \text{ LPT} + V_1^i \geq .138$.

The artificial variables V_6^i and V_7^i were introduced arbitrarily.. They must be equal to zero.

. . ., $V_6^i = 0$ and $V_1^i = 0$.

As all $V^i = 0$ by definition, in the present problem, $V_6^i + V_7^i = 0$.

In the problem, $V_6^i = 1$ and $V_7^i = .138$, therefore $V_6^i + V_7^i = 1.138$. But $V_6^i + V_7^i$ must be equal to zero. In order to make $V_6^i + V_7^i = 0$, or $V_6^i = 0$ and $V_7^i = 0$, a very high cost will

be assigned to each of these variables and by so doing it will be ascertained that in obtaining a solution that would minimize the cost, these variables will be driven out of the basis.

The intermediate solution obtained upon driving out the artificial variables, V_6 and V_7 , of the basis will be a basic feasible solution. The proof of this not so obvious statement is one of Dantzig's contributions to the field.¹

From this basic feasible solution the simplex method will find the optimal solution by successive iterations.

The cost function to be minimized written in a mathematical equation is as follows: $C = .41 \text{ BULL} + .385 \text{ COW} + .22 \text{ BN} + .22 \text{ BF} + .29 \text{ M} + .22 \text{ RBT} + .33 \text{ LBT} + .175 \text{ J} + .12 \text{ BAC} + .145 \text{ L} + .07 \text{ T} + .19 \text{ RPT} + .21 \text{ LPT}$.

The next step consists of constructing a table or matrix. A matrix is a form in which problem information is put up for computation. "A matrix is a convenient representation of the coefficients of a system of linear equations."² See Table XII.

Note that in the starting matrix the basis is formed by the seven slack variables $V_1, V_2, V_3, V_4, V_5, V_8, V_9$, and by the two artificial variables, V_6 and V_7 . This basis defines a nine-dimensional space which the basis vector spans.

¹G. B. Dantzig, Maximization of a Linear Function of Variables Subject to Linear Inequalities (New York, 1951), p. 175.

²Walter W. Garving, Introduction to Linear Programming, (New York, 1960), p. 51.

TABLE XII
SIMPLEX MATRIX OF FRANKFURTER MIXTURE FOR THE KRECK PACKING
COMPANY OF DALLAS, TEXAS

1	2	3	4	5	6	7	8	9	10	11	12	13
BULL	COW	BN	BF	M	RBT	LBT	J	BAC	L	T	RPT	LPT
..	..	..	..	..	..	..	..	..	..	1	..	..
..	..	..	..	..	..	..	..	..	1.	..	..	..
..	..	..	..	1	..	..	..	..	..	..	..	..
..	..	..	..	..	..	..	..	1	..	..	..	..
..	..	..	..	..	..	..	..	1	..	..	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
.22	.19	.13	.12	.19	.12	.18	.07	.08	.17	.17	.07	.11
.07	.10	.45	.50	.10	.50	.20	.65	.73	.13	.02	.55	.46
.70	.70	.43	.37	.70	.37	.61	.26	.19	.68	.80	.37	.42
-.41	-.385	-.22	-.22	-.29	-.22	-.37	-.175	-.12	-.145	-.07	-.19	-.21
8811	8822	8833	8844	8855	8866	8877	8888	8899	8910	8920	8930	8940

TABLE XII--Continued

14	15	16	17	18	19	20	21	22	23	00
U ₇	U ₇	U ₁	U ₂	U ₃	U ₄	U ₅	U ₆	U ₇	U ₈	U ₉
..	1	..	..	..	..	..	..	..	..	.050
..	..	1	..	..	..	..	..	..	..	.090
..	..	..	1	..	..	..	..	..	..	.100
..	..	..	..	1	..	..	..	..	..	.135
..	..	..	..	..	1	..	..	..	..	.350
..	..	..	..	..	..	1	..	..	..	1.000
-1	..	..	..	..	..	..	1	..	..	.138
..	..	..	..	..	..	..	..	1	..	.403
..	..	..	..	..	..	..	..	..	1	.446
..	0	0	0	0	0	-999.999	-999.999	0	0	0
9001	9015	9025	9035	9045	9055	9065	9075	9085	9095	0

A set of vectors $x_1, x_2, \dots, x_k$ in space M is to span or to determine M if every vector y in M can be represented as a linear combination of $x_1, x_2, \dots, x_k$; that is, for any vector y in M there exists scalars $a_1, a_2, a_3, \dots, a_k$ depending on y , such that $y = a_1(x_1) + \dots + a_k(x_k)$.

The vectors $x_1, x_2, \dots, x_k$ form a basis for M if they span M and also are linearly independent.³

Computer Solutions to Blending Problems

After input data were written in computer language by means of card-input, it was presented to the computer. The output optimal blend was obtained after the ninth iteration.

Table XIII is the actual output printed by the machine. The optimum composition as given by the 1620 IBM Computer is as follows:

No. 8899 - Bacon		13.50 per cent blend
No. 8844 - Beef flank		55.63 per cent blend
No. 8811 - Bull meat		23.07 per cent blend
No. 8910 - Ox lips		07.79 per cent blend
Total		<u>99.99</u>

The quality of the product as calculated from the optimal blend is as follows:

Protein = .08 (13.50) Bacon + .12 (55.63) Beef flank +
 .22 (23.07) Bull meat + .17 (07.79) Ox lips =
 1.02 + 6.67 + 5.22 + 1.31.

Protein = 15.26 per cent blend.

³Bernard Friedman, Principles and Techniques of Applied Mathematics (New York, 1960), p. 11.

Fat = .73 (13.5) Bacon + .50 (55.63) Beef flank + .07
 (23.07) Bull meat + .13 (7.79) Ox lips = 9.85 +
 27.79 + 1.69 + 0.99.

Fat = 40.27.

Water = .19 (13.5) Bacon + .37 (55.63) Beef flank + .70 (23.07)
 Bull meat + .68 (7.79) Ox lips = 2.54 + 20.58 +
 16.58 + 5.28.

Water = 44.98.

The specifications under variable cost in the computer output sheet Table XIII are as follows:

9015 (V_1) = .05 (no tripe used).

9025 (V_2) = .012 (.012 less than the maximum amount of
 ox lips used).

9035 (V_3) = .10 (no mutton used).

9055 (V_5) = .35 (no pork meat was used).

9001 (V_7) = .0035 (.0035 more than the minimum amount
 of required protein was obtained).

Cost of blend: 24.45 cents per pound.

Shadow Price--Sensitivity Analysis

By a few calculations upon the shadow price (Table XIII), a sensitivity analysis can be performed. By means of this analysis, the penalty of extra cost to the system, if some amount of the unused component is introduced into the mixture, can be computed. The theory of sensitivity analysis was

TABLE XIII
COMPUTER OUTPUT

CASE KRECK

ITER NO	FUNCTIONAL	VAR OUT	VAR IN
001	3337298412	9075999999	8811000410
002	3336311803	9095000000	9001000000
003	3326476217	9045000000	8899000120
004	5244770350	9055000000	8888000175
005	5027070400	9065999999	8844000220
006	5027056677	9085000000	9095000000
007	5026916642	9095000000	8920000070
008	5026838459	8920000070	8910000145
009	5024453126	8888000175	9055000000
FUNCTIONAL	5024453126		

VAR/COST	ACTIVITY	LIM VAR	LOWER LIM	LIM VAR	UPPER LIM
9015000000	.0500	8920	.2021-	8930	9999.9000
9025000000	.0120	9065	34.3035-	8920	.0854
9035000000	.1000	8855	.1040-	0000	9999.9000
8899000120	.1350	99045	.1502-	0000	9999.9000
9055000000	.3500	8888	.0681-	0000	9999.9000
8844000220	.5563	8866	.2200-	9045	.2006-
8811000410	.2307	8866	.4100-	8888	.2784-
8910000145	.0779	8920	.2304-	9065	34.1585
9001000000	.0035	9045	3.4964-	9085	5.7678

VAR/COST	SHAD PRICE	LIM VAR	LOWER LIM	LIM VAR	UPPER LIM
9075999999	999.9990	9001	.0035-	0000	9999.9000
8822000385	.1990	9025	.0136-	8910	.0881
8833000220	.1758	9025	.0145-	8910	.0938
9065999999	992.6582	8910	.0026-	9025	.0004
8855000290	.1040	9025	.0136-	8910	.0881
8866000220	.0000	0000	9999.9000-	8844	.5563
8877000330	.0668	9025	.0322-	8910	.2077
8888000175	.0681	9025	.0615-	9055	.3500
9045000000	.0302	9025	.0903-	8899	.1350
8920000070	.2021	9025	.0051-	8910	.0329
9095000000	9.1544	9025	.0003-	8910	.0020
8930000190	.3434	9025	.0082-	8910	.0528
8940000210	.1490	9025	.0163-	8910	.1051
9085000000	7.4673	9025	.0004-	8910	.0026

developed by Kuhn and Tucker⁴ in 1951. The method employed by them made use of LaGrange Multipliers. Most people are not familiar with it unless they have had an extensive background of advanced mathematics.

As long as the price of the considered meat products remains the same, the obtained solution represents the optimum mixture under the prescribed specification. Nevertheless, in case the price of any one of the components should change or the specifications vary, a new matrix may be constructed and a new optimal blend be sought by the simplex method of linear programming. If the changes are minor, it is possible in some cases to predict the new optimum solution without going through the simplex method.

⁴H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (Berkeley, California, 1951).

CHAPTER VI

SUMMARY

Linear programming is an objective technique designed to help management decide how to use limited resources to obtain a particular objective, such as highest profit, least cost, or least time, when these resources have alternate demands.

The growing application of linear programming and other mathematical techniques has meant that more and better information has been available to guide management action.

Management decision-making depends on the quality and quantity of information available at the executive level. For there to be adequate action, management must be able to see the interactions between flows of information.

Linear programming helps management to understand the operations of a business in simple terms. The construction of a linear programming model makes management evaluate the essential elements of operations and the factors affecting results. After such evaluation management is conscious of making decisions that may improve a segment of a business and hamper another or the work of the whole organization.

Another way in which linear programming helps management is that it provides the optimal solution as well as alternate feasible solutions. Each alternative is evaluated, all factors considered and given, such as the margin of error, the risks

involved, the resources and efforts necessary, and the time consumed, so that management can insert all factors not previously considered and result the real optimum solution.

The Requirements of Linear Programming

The two basic requirements of linear programming problems are certainty and linearity. The objectives or goals must be precisely and concisely stated, that is, to minimize distance, costs, materials and time, or to maximize profits. The constraints must be clearly stated. For example, warehouse A can supply up to a maximum of 1000 bushels a day or the capacity of machine X is eight hours a day of continuous service.

The program must be instructed to produce positive answers only. If for a certain program a given warehouse is unprofitable, the program, unless instructed otherwise, would probably turn out negative shipments as an answer. For example a chain of warehouses supplies a given market. The maximum number of units that all the warehouses can supply to the market is 1000. There are three warehouses. The program could very well come out with answers such as: Warehouse I supplies 3000 units; Warehouse II, 1000 units; and Warehouse III, -3000 units. This occurs because Warehouse III was particularly unprofitable. The program was instructed to meet the maximum requirements of the market, 1000 units thus:

$$3000 + 1000 - 3000 = 1000.$$

One of the basic problems encountered in setting up a linear program is the validity of the data. Many variables which must be evaluated could hardly be measured and must be approximated or, to put it more bluntly, the value of the variables has to be guessed. When data of questionable validity are fed into a program, the result is a solution of questionable validity and limited usefulness; however, it helps management evaluate the course of action.

Operations research men, in setting up a program, usually discover that management wants all possible variables to be included, thus making a program difficult to solve because of its complexity, cost and time requirements. In most cases the simpler the program, the more valuable it is for it can be obtained quicker and at less cost. The basic variables are the ones that critically affect the solution and should be the only ones considered.

In many cases the optimal solution is not the best solution because some variables are actually neither linear nor realistic. Management should consider the near-optimal solutions as well as the optimal solution provided by the linear program.

Linear Programming - A Management Tool and Its Applications

Management can, with the aid of linear programming, maintain the operations of a business with a maximum efficiency.

For the sake of simplicity the operations of an industrial enterprise will be compounded into three major areas:

1. Management
2. Manufacturing and storage
3. Marketing. See Figure 6.

These three areas have been optimized successfully in many industries by the use of linear programming.

Management.---Linear programming can help management directly in many internal and external management problems.

1. In the selection of the best, most profitable investment portfolio, a new product line, foreign bonds or plant expansion.

2. In the selection of the most adequate personnel for available jobs.

3. In the selection of optimal plant sites considering the market competitors and the needs of the company.

4. In the selection of an optimal warehouse site.

5. In the research and development department it may help to decide where to put the greatest research efforts in order to maximize future income.

6. In prevention of bottleneck situations in internal office problems: the number of workers, the amount spent in office supplies and miscellaneous items.

7. To decide which departments of an operating enterprise can be allowed to work overtime so that the most

Fig.10--Operations Research Center through linear programming can help optimize the three M's of industry: Management, Manufacturing and Marketing.

important and expensively operated departments can save more valuable time.

8. To evaluate objectives; it makes management more conscious of the operations of business as well as the real goals.

9. Finally, to coordinate the whole business and orient it toward the common goals.

Manufacturing.--Linear programming can help:

1. In the selection of the optimal manufacturing plan. If a number of products can be manufactured, linear programming can tell which product should be selected (and in what proportion) for maximum profits, taking into consideration all factors such as marketing costs, efforts, and market absorption.

2. In the optimal allocation of the product to machine usage.

3. In production scheduling and inventory control problems. Linear programming can help in this case to decide whether to produce in advance of demand or to make exactly the demand requirement needed. All factors such as storage costs, overtime costs, and chances of finished product to become obsolete or spoiled are taken into consideration.

4. In assembly line balancing. This is another type of production scheduling problem where linear programming can play an important part. Different components of a product take different amounts of time to assemble. The problem is

to minimize assembly line idle time so as to minimize total assembly time.

5. Transportation problems: the transportation program points out the need for a new warehouse and the place where it should be constructed. It optimizes shipping by allocating the most economical shipping schedules and routes. If freight rates change, linear programming can rearrange the shipping pattern. It can tell the effect of seasonal fluctuations on freight expenses to market.

6. In deciding which processes to use in the manufacture of a product in order to optimize production.

Marketing.---Linear programming can help:

1. To analyze markets so as to decide where to sell.
2. To reduce marketing costs and increase efficiency.

In the selection of an optimum marketing plan, the program can tell how much effort to put in the right places and by what media.

3. Marketers in the selection of the proper channels of distribution.

4. To determine the price for the product that would bring the highest profits.

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