A RELATION FOR POINT SETS IN A TOPOLOGICAL SPACE

THESIS

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CHAPTER I

INTRODUCTION

The purpose of this paper is the investigation of the relation $Z$ for point sets in a topological space. There were two original goals which caused the study. However, it later became evident that the goals were optimistic ones.

It was conjectured that the relation $Z$ would provide a useful tool for the study of dimension in a separable metric space. The last theorem in the third chapter of this thesis proves that the relation $Z$ is too general to be of use in any study concerning separable metric spaces.

A second assumption was that, based on the relation $Z$, a sequence of relations, varying in strength gradually from that of the relation $Z$ to that of the topological equivalence relation, could be defined. There is no logical way to exclude the possibility of such a sequence. However, the theorems in the fourth chapter of this paper clarify the difference between the two relations and imply that any transition from one to the other would be sudden rather than gradual.

Therefore the content of this paper and the order of its presentation have been chosen only for the purpose of investigating the relation $Z$. 
In this paper the words "point" and "region" are undefined, and a point set is a non-empty set of points.

**Axiom I.** There exists a region, and every region is a point set.

**Axiom II.** If each of \(Q\) and \(R\) is a region which contains the point \(p\), then there exists a region \(U\) such that \(p \in U\), and \(U \subseteq (Q \cap R)\).

**Axiom III.** If \(p\) and \(q\) are two points, then there exists a region which contains \(p\) but not \(q\).

**Definition 1.1.** If \(p\) is a point, then the statement that \(R\) is a region of \(p\) means that \(R\) is a region, and \(p \in R\).

**Definition 1.2.** If \(M\) is a point set, and \(p\) is a point, then the statement that \(p\) is a limit point of \(M\) means that every region of \(p\) contains a point of \(M\) distinct from \(p\).

**Definition 1.3.** If \(M\) is a point set, then the statement that \(G\) is the closure of \(M\) means that \(G\) is the set of all points \(p\) such that \(p \in M\), or \(p\) is a limit point of \(M\).

**Definition 1.4.** The statement that \(G\) is an open subset of the point set \(M\) means that \(G \subseteq M\), and if \(p \in G\), then there exists a region \(R\) of \(p\) such that \((R \cap M) \subseteq G\).

**Notation 1.1.** If \(M\) is a point set, then let \(\overline{M}\) denote the closure of \(M\).

**Definition 1.5.** If each of \(M\) and \(G\) is a point set, then the statement that \(H\) is the closure of \(G\) in \(M\) means that \(H = \overline{G} \cap M\).
Definition 1.6. If each of $M$ and $G$ is a point set, then the statement that $G$ is closed in $M$ means that $G$ is the closure of $G$ in $M$.

Definition 1.7. The statement that $G$ is a basis of the point set $M$ means that $G$ is a collection of open subsets of $M$, and if $p \in M$, then every region $R$ of $p$ contains a set $g$ in $G$ such that $p \in g$.

Theorem 1.1. If $G$ is a basis of the point set $M$, and each of $g'$ and $g''$ is a set in $G$ which contains the point $p$, then there exists a set $g$ in $G$ such that $p \in g$, and the set $g \subseteq (g' \cap g'')$.

Proof. There exists a region $R'$ of $p$ and a region $R''$ of $p$ such that $(R' \cap M) \subseteq g'$, and $(R'' \cap M) \subseteq g''$. Since the point $p \in (R' \cap R'')$, there is a region $V$ of $p$ such that $V \subseteq (R' \cap R'')$. Hence there exists a set $g$ in $G$ such that $p \in g$, and $g \subseteq V$. Therefore $g \subseteq M$, and $g \subseteq (R' \cap R'')$. The set $g \subseteq (g' \cap g'')$.

Theorem 1.2. If $G$ is a basis of the point set $M$, and $p$ and $q$ are two points of $M$, then there is a set in $G$ which contains $p$ but not $q$.

Proof. There exists a region $R$ of $p$ which does not contain $q$. Therefore there is a set $g$ in $G$ such that $p \in g$, and $g \subseteq R$. Hence $g$ does not contain $q$.

Notation 1.2. If $f$ is a reversible function, then let $f^{-1}$ denote the inverse function of $f$. 
**Notation 1.3.** If $f$ is a function, then let "d.f" denote the domain of $f$, and let "r.f" denote the range of $f$.

**Notation 1.4.** If $f$ is a function, and $G \subseteq d.f$, then let "$f(G)$" denote the set of all elements $y$ of r.f for which there is an element $x$ of $G$ such that $f(x) = y$.

**Notation 1.5.** If $G$ is a collection of sets, then let "G" denote the union of all sets of $G$.

**Definition 1.8.** The statement that $f$ is a z-function of $(A,B)$ means that each of $A$ and $B$ is a point set, $f$ is a function, d.f is a basis of $A$, r.f is a basis of $B$, and if each of $Q$ and $R$ is in d.f, then $Q \subseteq R$ if and only if $f(Q) \subseteq f(R)$.

**Definition 1.9.** The relation $Z$ is the set of all ordered pairs $(A,B)$ such that there exists a z-function of $(A,B)$.

**Notation 1.6.** Let "AzB" denote the statement that the ordered pair $(A,B) \subseteq Z$.

If $f$ is a z-function of $(A,B)$, then $f$ is reversible, and $f^{-1}$ is a z-function of $(B,A)$. Therefore if AzB, then BzA, and the relation $Z$ is symmetric.

If $M$ is a point set, then let $G$ denote the collection of all open subsets of $M$. Let $f$ denote the function $h$ such that d.h = G, and if $R \subseteq d.h$, then $h(R) = R$. The function $f$ is a z-function of $(M,M)$. Therefore MzM, and the relation $Z$ is reflexive.
**Conjecture.** If \( A \subseteq B \), and \( B \subseteq C \), then \( A \subseteq C \).

If this conjecture is correct, then the relation \( Z \) is transitive and is an equivalence relation. Whether or not the conjecture is correct is the major unsolved problem of this paper. It will be proven in the next chapter that the relation \( Z \) is transitive for point sets in a regular space.

**Theorem 1.3.** If \( f \) is a \( z \)-function of \((A, B)\), and each of \( Q \) and \( R \) is a set in \( d.f \), then \( Q \) intersects \( R \) if and only if \( f(Q) \) intersects \( f(R) \).

**Proof.** Suppose \( Q \) intersects \( R \). Let \( p \) denote a point of the intersection of \( Q \) and \( R \). There is a set \( g \) of \( d.f \) such that \( p \in g \), and \( g \) is a subset of each of \( Q \) and \( R \). Thus \( f(g) \) is a subset of each of \( f(Q) \) and \( f(R) \). Therefore \( f(Q) \) intersects \( f(R) \).

Suppose \( f(Q) \) intersects \( f(R) \). Let \( g \) denote a set in \( r.f \) which is contained in each of \( f(Q) \) and \( f(R) \). Then \( f^{-1}(g) \) is a subset of each of \( Q \) and \( R \), and \( Q \) intersects \( R \).

**Theorem 1.4.** If each of \( A \) and \( B \) is a point set, and \( A \) is topologically equivalent to \( B \), then \( A \subseteq B \).

**Proof.** Let \( T \) denote a topological transformation from \( A \) to \( B \). If \( R \) is an open subset of \( A \), then \( T(R) \) is an open subset of \( B \). If each of \( Q \) and \( R \) is an open subset of \( A \), then \( Q \subseteq R \) if and only if \( T(Q) \subseteq T(R) \). Let \( G \) denote the collection of all open subsets of \( A \). Then \( G \) is a basis of \( A \). Let \( f \) denote the function \( h \) such that \( d.h = G \), and if
$R \in d.h$, then $h(R) = T(R)$. Then $r.f$ is a basis of $B$. Hence the function $f$ is a z-function of $(A, B)$. Therefore the ordered pair $(A, B)$ is in $Z$.

**Theorem 1.5.** There exists a point set $A$ and a point set $B$ such that $AzB$, and $A$ is not topologically equivalent to $B$.

**Proof.** Two dimensional Euclidian space with region defined to be the interior of a circle satisfies axioms I, II, and III. Let $A$ denote the circle $x^2 + y^2 = 1$. Let $B$ denote the closed interval $[0,1]$ on the $x$-axis. The set $A$ is not topologically equivalent to $B$. Let $s$, $t$, and $u$ denote the points $(0,0)$, $(1,0)$, and $(0,1)$ respectively. Let $A'$ denote the set of all points of $A$ except $u$, and let $B'$ denote the set of all points of $B$ except $t$ and $s$. Let $M_1$ denote the set of all points $(x,y)$ such that $(x,y) \in A$, $y > 0$, and $x < 0$. Let $M_2$ denote the set of all points $(x,y)$ such that $(x,y) \in A$, $x > 0$, and $y > 0$. Let $T$ denote a topological transformation from $A'$ to $B'$ such that $s$ is a limit point of $T(M_1)$, and $t$ is a limit point of $T(M_2)$. Let $G$ denote the collection of all connected, open subsets $R$ of $A$ such that one of the following:

1. The point $u$ is not in $R$.
2. The point $u \in R$, $u$ is not in $R$, and $R \subset M_1$, or $R \subset M_2$. 

(3) The point \( u \in R \), and \( R \) is a subset of the sum of \( M_1 \), \( M_2 \), and the point \( u \).

The collection \( G \) is a basis of \( A \). Let \( f \) denote the function \( h \) such that \( d.h = G \), and if \( R \in d.h \), then one of the following:

(1) The point \( u \) is not in \( R \), and \( h(R) = T(R) \).

(2) The point \( u \in R \), \( u \) is not in \( R \), and if \( R \subseteq M_1 \), then \( h(R) \) is the sum of \( T(R) \) and the point \( s \), and if \( R \subseteq M_2 \), then \( h(R) \) is the sum of \( T(R) \) and the point \( t \).

(3) The point \( u \in R \), and \( h(R) \) is the sum of the points \( s \) and \( t \) and \( T[(M_1 \cup M_2) \cap R] \).

The function \( f \) is a z-function of \((A, B)\), and \( A \subseteq B \).

Notice that in the proof of Theorem 1.5 the sets \( A \) and \( B \) are closed and compact subsets of a Euclidian space. Thus these restrictions on \( A \) and \( B \) are not enough to imply that if \( A \subseteq B \), then \( A \) is topologically equivalent to \( B \).

**Theorem 1.6.** If \( f \) is a z-function of \((A, B)\), and \( R \) is a set in \( d.f \), then \( R \subseteq f(R) \).

**Proof.** Let \( g \) denote the collection of all ordered pairs \((x,y)\) in \( f \) such that \( x \subseteq R \). Then \( d.g \) is a basis of \( R \). If \( p \in f(R) \), and \( N \) is a region of \( p \), then there is a set \( N' \) in \( r.f \) such that \( p \in N' \), and \( N' \subseteq N \). There exists a set \( N'' \) in \( r.f \) such that \( p \in N'' \), and \( N'' \subseteq [N' \cap f(R)] \). Then the set \( f^{-1}(N'') \subseteq R \). Therefore \( N'' \subseteq r.g \), and \( r.g \) is a basis of
f(R). If U and V are sets in d.g, then U ⊆ V if and only if g(U) ⊆ g(V). Thus g is a z-function of (R, f(R)), and R z f(R).

**Theorem 1.7.** If A ≤ B, then there is a function f such that f is a z-function of (A, B), A is in d.f, and f(A) = B.

**Proof.** Let g denote a z-function of (A, B). If A is in d.g, then every set of r.g is a subset of g(A). Therefore r.g is a subset of g(A), and B ⊆ g(A). Since g(A) ⊆ B, g(A) = B. If A is not in d.g, then let f denote the set of all ordered pairs (x, y) such that (x, y) ∈ g, or x is A and y is B. Then f is a z-function of (A, B).

**Theorem 1.8.** If f is a z-function of (A, B), and R is a degenerate set in d.f, then f(R) is degenerate.

**Proof.** Suppose f(R) is not degenerate. Let p and q denote two points of f(R). There exists a set Q' of r.f which contains p but not q. The intersection of Q' and f(R) contains a set Q of r.f. Since q is not in Q, Q is not f(R). Therefore f⁻¹(Q) is not R. Since R is degenerate, f⁻¹(Q) is not a subset of R. However, Q is a subset of f(R). Therefore f(R) is degenerate.

Notice that Theorem 1.8 shows that if f is a z-function of (A, B), then the degenerate open subsets of A are mated with the degenerate open subsets of B by the function f. Therefore in the investigation of the relation Z, the major
concern is with point sets $A$ such that each point of $A$ is a limit point of $A$.

**Theorem 1.9.** It is not true that if $f$ is a z-function of $(A, B)$, and $A$ is compact, then $B$ is compact; and it is not true that if $f$ is a z-function of $(A, B)$, $G \subseteq d.f$, and there is a point common to every set in $G$, then there is a point common to every set in $f(G)$.

**Proof.** The set of all real numbers with region defined to be an open interval satisfies axioms I, II, and III. Let $A$ denote the closed interval $[0,1]$, and let $B$ denote the intersection of $A$ and the set of all rational numbers. Let $G$ denote the collection of all connected, open subsets of $A$. Let $f$ denote the function $h$ such that $d.h = G$, and if $R$ is in $d.h$, then $h(R) = R \cap B$. The function $f$ is a z-function of $(A, B)$. The set $A$ is compact, and the set $B$ is not compact. There exists a set $H$ such that $H$ is a subset of $d.f$, and the intersection of all sets in $H$ is an irrational number. The intersection of all sets in $f(H)$ does not exist.

**Definition 1.10.** The statement that the point set $A$ is completely separable means that there exists a countable basis of $A$.

**Definition 1.11.** The statement that the point set $A$ is regular means that if $p$ is a point of $A$, then for each open
subset R of A which contains p, there exists an open subset Q of A such that \( p \in Q \), and \( \overline{Q} \cap A \subseteq R \).

**Theorem 1.10.** It is not true that if \( A \subseteq B \), and A is completely separable, then B is completely separable, and it is not true that if \( A \subseteq B \), and A is regular, then B is regular.

**Proof.** Let S denote the sum of the set of all real numbers and the imaginary number i. A subset R of S will be called a region if and only if R is an open interval, or R contains i and all but a finite set of real numbers. With this definition of region the set S satisfies axioms I, II, and III. Let A denote the set of all real numbers, and let B denote S. If \( p \in A \), then every region of p contains an open interval R such that \( p \in R \). There is an open interval \( R' \) with rational end points such that \( p \in R' \), and \( R' \subseteq R \). Therefore A is regular, and since the set of all open intervals with rational end points is countable, A is completely separable. If R is a region of i, and p is a point not in R, then every region of p contains a point of R. Therefore p is a limit point of R, and \( \overline{R} = S \). Let Q denote a region of i different from S. Then there exists no region \( X \) of i such that \( Q \supseteq X \). Therefore B is not regular. If \( G \) is a countable collection of regions of i, let p denote a point in the intersection of the sets in G. Let U denote the set of all points of S except p. Then no set in G is a subset
of \( U \). Therefore every basis of \( B \) is uncountable, and \( B \) is not completely separable.

Let \( H \) denote the set of all regions. Let \( f \) denote the function \( h \) such that \( d.h = H \), and if \( R \in d.h \), then \( h(R) \) is the set of all points in \( R \) different from \( i \). The function \( f \) is a \( z \)-function of \((B,A)\), and \( BzA \). Hence \( AzB \).

**Theorem 1.11.** If \( AzB \), and there exists a basis of \( A \) with cardinal \( \alpha \), then there exists a subset \( M \) of \( B \) with cardinal less than or equal to \( \alpha \) such that \( \overline{M} \supset B \).

**Proof.** Let \( f \) denote a \( z \)-function of \((A,B)\), and let \( G \) denote a basis of \( A \) with cardinal \( \alpha \). For each set \( g \) of \( G \) there is a set \( g' \) of \( d.f \) such that \( g' \subseteq g \). Let \( P_g \) denote a point in \( f(g') \). Let \( M \) denote the set of all points \( p \) for which there is a set \( g \) of \( G \) such that \( P_g = p \). The set \( M \) has cardinal less than or equal to \( \alpha \). Suppose there is a point \( q \) of \( B \) such that \( q \) is not a point or a limit point of \( M \). Then there is a set \( Q \) in \( r.f \) such that \( q \in Q \), and \( Q \) does not intersect \( M \). There is a set \( h \) in \( G \) such that \( h \subseteq f^{-1}(Q) \). Therefore \( P_h \in Q \). However, \( P_h \in M \). Therefore every point of \( B \) is either a point or a limit point of \( M \), and \( \overline{M} \supset B \).

**Definition 1.12.** The statement that the point set \( M \) is separable means that there exists a countable subset \( M' \) of \( M \) such that \( \overline{M'} \supset M \).

**Corollary 1.11.** If \( AzB \), and \( A \) is completely separable, then \( B \) is separable.
Conjecture. If \( f \) is a \( z \)-function of \((A, B)\), each of \( A \) and \( B \) is completely separable, each point of \( A \) is a limit point of \( A \), and each point of \( B \) is a limit point of \( B \), then there exists a function \( g \) such that \( g \) is a \( z \)-function of \((A, B)\), and \( d.g \) is countable.

The statement of this conjecture is proven for more restricted point sets in the third chapter of this paper.

Theorem 1.12. It is not true that if \( f \) is a \( z \)-function of \((A, B)\), \( R \in d.f \), \( G \subseteq d.f \), and \( R \subseteq G^* \), then \( f(R) \subseteq f(G)^* \).

Proof. Let \( S \) denote the interior of the unit circle in two dimensional Euclidean space. If region means the interior of a circle whose interior is a subset of \( S \), then \( S \) with this definition of region satisfies axioms I, II, and III. Let \( A \) denote the set of all points in \( S \) except the origin, and let \( B \) denote \( S \). Let \( f \) denote the function \( h \) such that \( d.h \) is the collection of all point sets \( X \) such that \( X \) is the intersection of a region with \( A \), and if \( R \subseteq d.h \), then \( h(R) \) is the region \( N \) such that \( N \cap A = R \). Then \( f \) is a \( z \)-function of \((A, B)\). Let \( G \) denote the set of all sets \( X \) in \( d.f \) such that \( X \) is a region. Then \( G^* \supset A \). Since \( f(G)^* \) does not contain the origin, \( f(G)^* \) does not contain \( B \). By definition of \( f \), \( A \subseteq d.f \), and \( f(A) = B \).

Theorem 1.13. If \( f \) is a \( z \)-function of \((A, B)\), \( G \) is a subset of \( d.f \), and \( H \) is a subset of \( d.f \) such that \( H \supset G^* \), then \( f(H)^* \supset f(G)^* \).
Proof. Suppose there is a point $p$ of $f(G)^*$ such that $p$ is not in $\overline{f(H)^*}$. Then there is a set $R$ in $r.f$ such that $p \in R$, and $R$ does not intersect $\overline{f(H)^*}$. Then $f^{-1}(R)$ intersects no set in $H$. Since $R$ intersects a set in $f(G)$, $f^{-1}(R)$ intersects a set $g$ in $G$. Let $q$ denote a point of the intersection of $g$ and $f^{-1}(R)$. Then $q$ is a point of $G^*$, and $q$ is not a limit point or a point of $H^*$. Therefore $\overline{H^*}$ does not contain $G^*$. Hence each point of $f(G)^*$ is in $\overline{f(H)^*}$.

Corollary 1.13. If $f$ is a $z$-function of $(A,B)$, and $Q$ and $R$ are two sets in $d.f$ such that the closure of $Q$ in $A$ is the closure of $R$ in $A$, then the closure of $f(Q)$ in $B$ is the closure of $f(R)$ in $B$.

Corollary 1.13 is a direct consequence of Theorem 1.13.

The results of this chapter indicate the need for study of the relation $Z$ in a space which offers more information concerning the closures of regions. The basic difference between the relation $Z$ and the topological equivalence relation is suggested by the limitations of Theorem 1.13. This will be shown in the fourth chapter. Regular spaces will be investigated in the next chapter.
CHAPTER II

REGULAR SPACE

In this chapter the following axiom will be assumed.

Axiom IV. If p is a point, and R is a region of p, then there exists a region Q of p such that \( \overline{Q} \subseteq R \).

Theorem 2.1. If G is a basis of the point set M, and g is a set in G which contains a point p of M, then there is a set \( g' \) in G such that \( p \in g' \), and the closure of \( g' \) in M is a subset of g.

Proof. There exists a region R of p such that \( R \cap M \subseteq g \). Let \( R' \) denote a region X of p such that \( X \subseteq R \). Let \( g' \) denote a set X in G such that \( p \in X \), and \( X \subseteq R' \). Then \( \overline{g'} \subseteq R \), and \( \overline{g'} \cap M \subseteq R \cap M \). Therefore the closure of \( g' \) in M is a subset of g.

Definition 2.1. The statement that the point p is a boundary point of the point set M means that every region of p contains a point in M and a point not in M.

Definition 2.2. If M is a point set, and N is a subset of M, then the statement that the point p is a boundary point of N in M means that \( p \in M \), and every region of p contains a point of N and a point of M not in N.
**Definition 2.3.** The statement that \( G \) is the completion of the point set \( R \) means that \( G \) is the set of all points \( x \) in \( \overline{R} \) such that \( x \) is not a boundary point of \( \overline{R} \).

**Notation 2.1.** If the completion of the point set \( R \) exists, then let "\( R^+ \)" denote the completion of \( R \).

**Definition 2.4.** If \( M \) is a point set, then the statement that \( G \) is the completion of the point set \( R \) in \( M \) means that \( G \) is the set of all points \( x \) in the closure of \( R \) in \( M \) such that \( x \) is not a boundary point of the closure of \( R \) in \( M \) in \( M \).

**Notation 2.2.** If the completion of the point set \( R \) in the point set \( M \) exists, then let "\( R^+ \) in \( M \)" denote the completion of \( R \) in \( M \).

Notice that if \( M \) is an open point set, then \( M \subseteq M^+ \). Also, if \( M \) is a point set, and \( N \) is an open subset of \( M \), then \( N \subseteq (N^+ \text{ in } M) \).

**Definition 2.5.** The statement that the point set \( M \) is a completed point set means that \( M^+ = M \).

**Definition 2.6.** The statement that the point set \( G \) is completed in the point set \( M \) means that \( G^+ \text{ in } M \) is \( G \).

Notice that if \( Q \) is a completed point set, then \( Q \) is open, and if \( R \) is a completed point set different from \( Q \), then \( \overline{Q} \) is not \( \overline{R} \). Also, if the point set \( H \) is completed in the point set \( M \), then \( H \) is an open subset of \( M \), and if the point set \( K \) is completed in \( M \), and \( K \) is different from \( H \),
then the closure of \( H \) in \( M \) is different from the closure of \( K \) in \( M \).

**Definition 2.7.** The statement that \( G \) is a completed basis of the point set \( M \) means that \( G \) is a basis of \( M \), and if \( R \subseteq G \), then \( R \) is completed in \( M \).

**Definition 2.8.** The statement that \( f \) is a completed \( z \)-function of \((A,B)\) means that \( f \) is a \( z \)-function of \((A,B)\), \( d.f \) is a completed basis of \( A \), and \( r.f \) is a completed basis of \( B \).

**Definition 2.9.** If \( M \) is a point set, and \( G \) is a basis of \( M \), then the statement that \( H \) is the completion of \( G \) means that \( H \) is the collection of all sets \( X \) such that for a set \( g \) in \( G \), \( g^+ \) in \( M \) is \( X \).

**Theorem 2.2.** If \( M \) is a point set, and \( G \) is a basis of \( M \), then the completion of \( G \) is a basis of \( M \).

**Proof.** Suppose \( p \subseteq M \), and \( R \) is a region of \( p \). Let \( R' \) denote a region \( X \) of \( p \) such that \( \overline{X} \subseteq R \). There exists a set \( g \) in \( G \) such that \( p \subseteq g \), and \( g \subseteq R' \). Since \( g^+ \) in \( M \) is a subset of \( \overline{g} \), \( g^+ \) in \( M \) is a subset of \( R \). Therefore, since \( g^+ \) in \( M \) is in the completion of \( G \), the completion of \( G \) is a basis of \( M \).

**Theorem 2.3.** There exists a space \( S \) which does not satisfy Axiom IV for which Theorem 2.2 is not true.

**Proof.** Let \( S \) denote the set of all real numbers. Let "region" mean a set of points which contains all but a
countable infinite set of points. The set \( S \) with this definition of "region" satisfies axioms I, II, and III, but it does not satisfy Axiom IV. If \( R \) is a region, and \( p \) is a point not in \( R \), then every region of \( p \) contains a point in \( R \). Therefore \( R^+ \) in \( S \) is \( S \). Let \( H \) denote the completion of the collection of all open subsets of \( S \). Then \( H \) is the degenerate set whose only element is \( S \). If \( q \in S \), and \( Q \) is a region of \( q \), then no set in \( H \) is a subset of \( Q \).

**Conjecture.** A space \( S \) satisfies Axiom IV if and only if there exists a completed basis of \( S \).

If this conjecture is not true, and the existence of a completed basis is weaker than the condition of regularity, then the following question occurs: "Is the existence of a completed basis stronger than the statement that for each pair of points \( p \) and \( q \) there exists a pair of disjoint regions \( P \) and \( Q \) such that \( p \in P \), and \( q \in Q \)?"

**Theorem 2.4.** If \( f \) is a z-function of \((A, B)\), then there exists a function \( g \) such that \( g \) is a completed z-function of \((A, B)\), \( d, g \) is the completion of \( d, f \), and \( r, g \) is the completion of \( r, f \).

**Proof.** Let \( G \) denote the completion of \( d, f \), and let \( H \) denote the completion of \( r, f \). Suppose \( Q \) and \( R \) are two sets in \( d, f \), and \( Q^+ \) in \( A \) is \( R^+ \) in \( A \). Then \( Q \cap A = R \cap A \). Thus \( Q \supset R \), and \( R \supset Q \). Therefore by Theorem 1.13 \( f(Q) \supset f(R) \).
and $\overline{f(R)} \supseteq f(Q)$. Hence $\overline{f(Q)} \cap B = \overline{f(R)} \cap B$, and $f(Q)^+ \in B$ is $f(R)^+ \in B$.

For each set $R$ in $G$ let $U_R$ denote a set $X$ in $d.f$ such that $X^+$ in $A$ is $R$. Let $V_R$ denote $f(U_R)^+ \in B$. Let $H'$ denote the collection of all sets $X$ such that, for a set $R$ in $G$, $V_R = X$. Then $H' \subseteq H$. Suppose $Q \in H$. Let $N$ denote a set $X$ in $r.f$ such that $X^+$ in $B$ is $Q$. Let $N'$ denote $f^{-1}(N)^+ \in A$. Then $N' \subseteq G$, and $U_N$, and $f^{-1}(N)$ have the same completion in $A$. Therefore $V_{N'} = (N^+ \text{ in } B) = Q$. Thus $Q \in H'$, $H \subseteq H'$, and $H' = H$.

Suppose each of $Q$ and $R$ is in $G$, and $Q \subseteq R$. Then

$q \subseteq R$, and $U_Q \subseteq U_R$. Therefore $U_Q \subseteq U_R^\uparrow$ and $f(U_Q) \subseteq f(U_R^\uparrow)$. The set $\overline{f(Q)} \subseteq \overline{f(R)}$, and since $V_Q \subseteq \overline{f(Q)}$, $V_Q \subseteq \overline{f(U_R)}$.

Since $V_Q$ is an open subset of $B$, no point of $V_Q$ is a boundary point of $\overline{f(U_R)} \cap B$ in $B$. Therefore $V_Q \subseteq V_R$.

Suppose that each of $M$ and $N$ is in $H$, and $M \subseteq N$. There is a set $Q$ in $G$ and a set $R$ in $G$ such that $V_Q = M$, and $V_R = N$. Therefore $V_Q \subseteq V_R$. Then $V_Q \subseteq V_R$, and $\overline{f(U_Q)} \subseteq f(U_R)$. Therefore $f(U_Q) \subseteq f(U_R)$, and $U_Q \subseteq U_R$. Hence $U_Q \subseteq U_R$, and $Q \subseteq R$. Thus $Q \subseteq R$. Since $Q$ is an open subset of $A$, no point of $Q$ is a boundary point of $R \cap A$ in $A$. Therefore $Q \subseteq R$.

Let $g$ denote the function $h$ such that $d.h = G$, and if $R \in d.h$, then $h(R) = V_R$. Then $r.g = H$, and $g$ is a completed $z$-function of $(A,B)$. 
Theorem 2.5. If AzB, and BzC, then AzC.

Proof. Let f denote a completed z-function of (A, B), and let g denote a completed z-function of (B, C). For each set R in d.f let U_R denote the collection of all sets X in d.g such that X is a subset of f(R). Then U_R* = f(R). Let V_R denote g(U_R)* in C. Since g(U_R)* is an open subset of C, g(U_R)* is a subset of V_R.

Suppose each of Q and R is in d.f, and Q \subset R. Then f(Q) \subset f(R). Therefore every set x in d.g such that x is a subset of f(Q) is a subset of f(R). Hence U_Q \subset U_R, and g(U_Q) \subset g(U_R). Hence g(U_Q)* \subset g(U_R)*, and V_Q \subset V_R.

Suppose each of Q and R is in d.f, and V_Q \subset V_R. Then g(U_Q)* \subset V_R, and g(U_Q)* \subset V_R. Hence g(U_Q)* \subset g(U_R)*. Thus U_Q* \subset V_R*, and f(Q) \subset f(R). Hence Q \subset R. Since Q is an open subset of A, no point of Q is a boundary point of the closure of R in A. Therefore Q \subset R.

Let H denote the collection of all sets X such that, for a set R in d.f, X = V_R. If H is a basis of C, then let t denote the function h such that d.h = d.f, and if R \in d.h, then h(R) = V_R. Then t is a z-function of (A, C), and AzC.

Suppose H is not a basis of C. Let W denote the collection of all sets X in d.g such that for a point p in X no set in H which contains p is a subset of X. Then the sum of H and W is a basis of C.
For each set $R$ in $W$ let $U'_R$ denote the collection of all sets $X$ in $r.f$ such that $X \subseteq g^{-1}(R)$. Let $V'_R$ denote $f^{-1}(U'_R)^*$ in $A$. Then each of the following is true:

(1) If each of $Q$ and $R$ is in $W$, and $V'_Q$ is $V'_R$, then $Q = R$.

(2) If each of $Q$ and $R$ is in $W$, and $Q \subseteq R$, then $V'_Q \subseteq V'_R$.

(3) If each of $Q$ and $R$ is in $W$, and $V'_Q$ is a subset of $V'_R$, then $Q \subseteq R$.

Suppose $R \in d.f$, $Q \in W$, and $R \subseteq V'_Q$. Then $R$ is a subset of $\overline{V'}_Q$. Hence $R \subseteq f^{-1}(U'_Q)^*$, and $f(R) \subseteq \overline{U'}_Q^*$. Hence $\overline{U'}_Q^* \supseteq U_R^*$, and $g^{-1}(Q) \supseteq U_R^*$. Therefore $Q \supseteq g(U_R)^*$, and $\overline{Q} \supseteq V_R$. Since $V_R$ is an open subset of $C$, $Q \supseteq V_R$.

Suppose $R \in d.f$, $Q \in W$, and $V_R \subseteq Q$. Since $g(U_R)^* \subseteq V_R$, $g(U_R)^* \subseteq Q$. Then every set in $g(U_R)$ is a subset of $Q$, and every set in $U_R$ is a subset of $g^{-1}(Q)$. Hence $U_R^* \subseteq g^{-1}(Q)$, and $U_R^* \subseteq U'_Q^*$. Thus $f(R) \subseteq U'_Q^*$, and $f(R) \subseteq \overline{U'}_Q^*$. Hence $R \subseteq f^{-1}(U'_Q)^*$, and since $\overline{V'}_Q$ is the set $f^{-1}(U'_Q)^*$, $R \subseteq \overline{V'}_Q$.

Since $R$ is an open subset of $A$, and $V'_Q$ is completed in $A$, $R \subseteq V'_Q$.

For similar reasons the following are true:

(1) If $R \in d.f$, $Q \in W$, and $V'_Q \subseteq R$, then $Q \subseteq V_R$.

(2) If $R \in d.f$, $Q \in W$, and $Q \subseteq V_R$, then $V'_Q \subseteq R$.

By the definition of $H$ and the definition of $W$ no set in $H$ is identical to a set in $W$. Suppose there is a set $R$
in \( d.f \) and a set \( Q \) in \( W \) such that \( R = V'_Q \). Then \( R \subseteq V'_Q \) and \( V'_Q \subseteq R \). Therefore \( V_R \subseteq Q \), and \( Q \subseteq V_R \). Hence \( Q \) is \( V_R \), and a set in \( H \) is identical to a set in \( W \). Thus \( R \) is not \( V'_Q \).

Let \( K \) denote the collection of all sets \( X \) such that, for a set \( R \) in \( W \), \( X = V'_R \). Let \( E \) denote the sum of \( d.f \) and \( K \), and let \( F \) denote the sum of \( H \) and \( W \). Then \( E \) is a basis of \( A \), and \( F \) is a basis of \( C \).

Let \( t \) denote the function \( h \) such that \( d.h = E \), and if \( R \) is in \( d.h \), then one of the following:

1. The set \( R \in d.f \), and \( h(R) = V_R \).
2. The set \( R \in K \), and \( h(R) \) is the set \( Q \) in \( W \) such that \( R = V_Q' \).

The function \( t \) is a \( z \)-function of \((A, C)\), and \( AzC \).

**Theorem 2.6.** If \( M \) is a point set, and \( G \) is a completed basis of \( \overline{M} \), then there exists a function \( f \) such that \( f \) is a \( z \)-function of \((\overline{M}, M)\), and \( d.f = G \).

**Proof.** If \( g \in G \), then \( g \cap M \) exists, and \( \overline{g} = \overline{g} \cap M \).

Then \( (g \cap M)^{+} \) in \( \overline{M} \) is \( g^{+} \) in \( \overline{M} \). Therefore \( (g \cap M)^{+} \) in \( \overline{M} \) is \( g \).

Let \( f \) denote the function \( h \) such that \( d.h = G \), and if \( R \in d.h \), then \( h(R) = R \cap M \).

If each of \( Q \) and \( R \) is in \( d.f \), and \( f(Q) \subseteq f(R) \), then \( (Q \cap M) \subseteq (R \cap M) \). Therefore \( (Q \cap M)^{+} \) in \( \overline{M} \) is a subset of \( (R \cap M)^{+} \) in \( \overline{M} \), and \( Q \subseteq R \).
If each of $Q$ and $R$ is in $d.f$, and $Q \subseteq R$, then $Q \cap M$ is a subset of $R \cap M$. Therefore $f(Q) \subseteq f(R)$.

Therefore $f$ is a $z$-function of $(\overline{M}, M)$.

**Corollary 2.6.** If $M$ is a point set, then $M \sim \overline{M}$.

The last two theorems serve as tools for the investigation of metric spaces in the next chapter.
CHAPTER III

METRIC SPACE

In this chapter the assumption of Axiom IV is continued for convenience. The restrictions on the point sets discussed in this chapter are sufficiently strong to imply the fourth axiom.

Definition 3.1. The statement that \( t \) is a distance function for the point set \( M \) means that \( t \) is a function, and the following:

1. The domain of \( t \) is the collection of all ordered pairs \((x,y)\) such that \( x \in M \), and \( y \in M \).
2. The range of \( t \) is a subset of the set of all non-negative real numbers.
3. If each of \( x \) and \( y \) is a point of \( M \), then \( t(x,y) = 0 \) if and only if \( x = y \).
4. If each of \( x \) and \( y \) is a point of \( M \), then \( t(x,y) = t(y,x) \).
5. If each of \( w, x, \) and \( y \) is a point of \( M \), then \( t(w,x) + t(x,y) \geq t(w,y) \).
6. If \( N \subseteq M \), and \( p \in M \), then \( p \) is a limit point of \( N \) if and only if for each positive number \( j \), there is a point \( q \) in \( N \) different from \( p \) such that \( t(p,q) < j \).
**Definition 3.2.** The statement that the ordered pair \((M, t)\) is a metric space means that \(M\) is a point set, and \(t\) is a distance function for \(M\).

**Definition 3.3.** If \((M, t)\) is a metric space, then the statement that \(R\) is a neighborhood in \((M, t)\) means that, for a point \(p\) in \(M\) and a positive number \(k\), \(R\) is the set of all points \(q\) in \(M\) such that \(t(p, q) < k\).

**Definition 3.4.** If \((M, t)\) is a metric space, and \(R\) is a neighborhood in \((M, t)\), then the statement that \(p\) is a center of \(R\), and \(k\) is a radius of \(R\) for \(p\) means that \(p\) is a point in \(R\), \(k\) is a positive number, and \(R\) is the set of all points \(q\) in \(M\) such that \(t(p, q) < k\).

**Theorem 3.1.** If \((M, t)\) is a metric space, then the collection of all sets \(X\) such that \(X\) is a neighborhood in \((M, t)\) is a basis of \(M\).

**Proof.** Suppose \(R\) is a neighborhood in \((M, t)\), and \(R\) is not an open subset of \(M\). There exists a point \(p\) and a positive number \(k\) such that \(p\) is a center of \(R\), and \(k\) is a radius of \(R\) for \(p\). Let \(q\) denote a point \(x\) in \(R\) such that each region of \(x\) contains a point of \(M\) not in \(R\). Then \(q\) is a limit point of \(M - R\). Let \(r\) denote a point \(x\) of \(M - R\) such that \(t(q, x) < k - t(p, q)\). Then \(k > t(p, q) + t(q, r)\). Therefore \(k > t(p, r)\), and \(r\) is in \(R\). Since \(r\) is not in \(R\), every neighborhood in \((M, t)\) is an open subset of \(M\).
If \( p \) is a point in \( M \), and \( R \) is a region of \( p \), then the set \( R \cap M \) is \( M \), or \( M - R \) exists. If \( R \cap M \) is \( M \), then let \( K \) denote the set of all points \( x \) in \( M \) such that \( t(p,x) < 1 \). Then \( K \) is a neighborhood in \((M,t)\), \( K \subset R \), and \( p \in K \). If \( M - R \) exists, then \( p \) is not a limit point of \( M - R \). Let \( k \) denote a positive number \( x \) such that if \( q \in (M - R) \), then \( t(p,q) > x \). The neighborhood in \((M,t)\) with center \( p \) and radius \( k \) is a subset of \( R \). Therefore the collection of all neighborhoods in \((M,t)\) is a basis of \( M \).

**Theorem 3.2.** If \((M,t)\) is a metric space, \( M \) is a separable point set, and \( G \) is a basis of \( M \), then there exists a countable subset \( G' \) of \( G \) such that \( G' \) is a basis of \( M \).

**Proof.** There exists a countable subset \( N \) of \( M \) such that \( \overline{N} \supseteq M \). For each positive integer \( n \) let \( H_n \) denote the collection of all neighborhoods \( X \) in \((M,t)\) for which there is a point \( p \) in \( N \) and a positive rational number \( k \) which is less than \( \frac{1}{n} \) such that \( p \) is a center of \( X \), and \( k \) is a radius of \( X \) for \( p \). Then if \( i \) is a positive integer, \( H_i \) is a countable basis of \( M \). For each positive integer \( n \) let \( G_n \) denote the set of all sets \( X \) in \( G \) such that a set in \( H_n \) contains \( X \). Then if \( i \) is a positive integer, \( G_i \) is a basis of \( M \). For each positive integer \( n \), let \( K_n \) denote the set of all sets \( X \) in \( H_n \) such that a set in \( G_n \) contains \( X \). Hence, if \( i \) is a positive integer, then \( K_i \) is countable, and \( M \) is a subset of \( K_i \). If \( j \) is a positive integer, and \( R \in K_j \), then
let \( g_{jR} \) denote a set in \( G_j \) which contains \( R \). Let \( G' \) denote the set of all sets \( X \) such that for a positive integer \( j \) and a set \( Q \) in \( K_j, X = g_{jQ} \). Then \( G' \) is a countable subset of \( G \).

Suppose \( p \in M \), and \( R \) is a region of \( p \). Then there is a positive number \( r \) such that if \( q \in M \), and \( t(p, q) < r \), then \( q \in R \). Let \( n \) denote a positive integer \( x \) such that \( \frac{1}{x} \) is less than \( \frac{r}{2} \). There is a set \( Q \) in \( K_n \) such that \( p \in Q \). Then \( g_{nQ} \) contains \( p \), and a set \( H \) in \( H_n \) contains \( g_{nQ} \). Let \( u \) and \( v \) denote a point \( x \) of \( H \) and a positive number \( y \), respectively, such that \( y < \frac{r}{2} \), \( x \) is a center of \( H \), and \( y \) is a radius of \( H \) for \( x \). If \( s \in g_{nQ} \), then \( s \in H \), and \( t(u, s) < \frac{r}{2} \). Since \( p \) is in \( H \), \( t(u, p) < \frac{r}{2} \). Therefore \( t(p, s) < r \), and \( s \in R \). Thus the set \( g_{nQ} \subseteq R \), and \( G' \) is a basis of \( M \).

**Theorem 3.3.** If each of \((A, r)\) and \((B, t)\) is a metric space, each of \( A \) and \( B \) is separable, and \( A \cap B \), then there exists a function \( f \) such that \( f \) is a \( z \)-function of \((A, B)\), and \( d.f \) is countable.

**Proof.** Let \( g \) denote a \( z \)-function of \((A, B)\). Let \( H \) denote a countable subset \( X \) of \( d.g \) such that \( X \) is a basis of \( A \). Let \( K \) denote a countable subset \( X \) of \( r.g \) such that \( X \) is a basis of \( B \). Let \( f \) denote the function \( h \) such that \( d.h \) is the sum of \( H \) and \( g^{-1}(K) \), and if \( R \in d.h \), then \( h(R) \) is \( g(R) \). Then \( f \) is a \( z \)-function of \((A, B)\), and \( d.f \) is countable.
Theorem 3.4. If each of \((A,t)\) and \((B,r)\) is a metric space, each of \(A\) and \(B\) is separable, every point of \(A\) is a limit point of \(A\), and every point of \(B\) is a limit point of \(B\), then \(A \subset B\).

Proof. If \(A' \subset A\), \(B' \subset B\), \(A^r \supset A\), \(B^r \supset B\), and \(A' \subset B'\), then by Theorem 2.5 and Theorem 2.6 \(A \subset B\). Hence it is sufficient to prove that there is a subset \(A'\) of \(A\) and a subset \(B'\) of \(B\) such that \(A^r \supset A\), \(B^r \supset B\), and \(A' \subset B'\).

There exist two disjoint countably infinite subsets \(A'\) and \(A''\) of \(A\) such that \(A^r \supset A\), and \(A'' \supset A\). Let \(G''\) denote the collection of all neighborhoods \(X\) in \((A,t)\) for which there is a point \(p\) in \(A''\) such that \(p\) is a center of \(X\). Let \(G'\) denote the collection of all open subsets \(X\) of \(A'\) for which there is a set \(g\) in \(G''\) such that \(g \cap A' = X\). There are only countably many numbers \(x\) such that, for a point \(p\) in \(A''\) and a point \(q\) in \(A'\), \(x = t(p,q)\). Therefore, for each point \(p\) in \(A''\), there is a sequence \(R_{p_1}, R_{p_2}, R_{p_3}, \ldots\) of sets in \(G'\) such that, for each positive integer \(i\), \(R_{p_i}\) is the intersection of \(A'\) with a set \(Q\) in \(G''\), \(p\) is a center of \(Q\), and there is a radius \(k\) of \(Q\) for \(p\) such that \(\frac{1}{i+1} < k < \frac{1}{i}\), and if \(q \in A'\), then \(t(p,q)\) is not \(k\); and if \(i\) is a positive integer, then \(R_{p}(i+1)\) is a proper subset of \(R_{p_i}\). Then for each positive integer \(i\), \(R_{p_i}\) is closed in \(A'\), \(R_{p_i}\) is an open subset of \(A'\), and no point is common to every set in \(R_{p_1}, R_{p_2}, R_{p_3}, \ldots\).

Let \(G\) denote the collection of all sets \(g\) in \(G'\) such that,
for a point $p$ in $A''$ and a positive integer $k$, $g = R_{pk}$. Then $G$ is a basis of $A'$.

If $p \in A''$, and $i$ is a positive integer, then $R_{pi}$ is the union of the sets in $R_{pi} - R_{p(i+1)}$, $R_{p(i+1)} - R_{p(i+2)}$, $R_{p(i+2)} - R_{p(i+3)}$, ... . Therefore every set in $G$ is the union of a countably infinite collection of disjoint open subsets of $A'$.

For each positive integer $i$, let $H_i$ denote the set of all sets $X$ in $G$ such that, for a point $p$ in $A''$ and a positive integer $k$ such that $k \geq i$, $X = R_{pk}$. Then, for each positive integer $i$, $H_i$ is a countable basis of $A'$. Suppose $Q$ is an open subset of $A'$ and $k$ is a positive integer. If $Q$ is the union of a finite collection $\alpha$ of disjoint open subsets $X$ of $A'$ such that $X$ is a subset of a set in $H_k$, then let $a_1$ denote an element of $\alpha$. There is a proper subset $\beta$ of $a_1$ such that $\beta \subseteq G$. Then $a_1 - \beta$ is an open subset of $A'$, and $\beta$ is the union of a countably infinite collection of disjoint open subsets of $A'$. Therefore $Q$ is the union of a countably infinite collection of disjoint open subsets $X$ of $A'$ such that $X$ is a subset of a set in $H_k$. If $Q$ is not the union of a finite collection $\alpha$ of disjoint open subsets of $A'$ such that each set in $\alpha$ is a subset of a set in $H_k$, then let $h_1$, $h_2$, $h_3$, ... denote the sets $x$ in $H_k$ such that $x \subseteq Q$. Let $n_l$ denote 1. Let $n_2$ denote the smallest positive integer $x$ such that $h_x$ is not a subset of $h_{n_1}$. Suppose $n_1$, $n_2$, ...
$n_3, \ldots, n_j$ is a finite increasing sequence of positive integers, and for each positive integer $i$ such that $1 < i \leq j$, $i$ is the smallest positive integer $x$ such that $h_x$ is not a subset of $h_{n(i-1)} \cup h_{n(i-2)} \cup h_{n(i-3)} \cup \ldots \cup h_{n_1}$. Then the union of the sets in $h_{n_2} - h_{n_1}, h_{n_3} - (h_{n_1} \cup h_{n_2}), \ldots, h_{n_j} - (h_{n_1} \cup h_{n_2} \cup \ldots \cup h_{n(j-1)})$ does not contain $Q$. There is a positive integer $n(j+1)$ such that $n(j+1)$ is the smallest positive integer $x$ for which $h_{nx}$ is not a subset of $h_{n_1} \cup h_{n_2} \cup \ldots \cup h_{n_j}$. Thus there is a sequence $h_{n_1}, h_{n_2}, h_{n_3}, \ldots$ such that, for each positive integer $i$ such that $i > 1$, $n_i$ is the smallest positive integer $x$ such that $h_x$ is not a subset of the union of the sets in $h_{n_1}, h_{n_2}, h_{n_3}, \ldots, h_{n(i-1)}$. Then $Q$ is the union of all sets in the sequence $h_{n_2} - h_{n_1}, h_{n_3} - (h_{n_1} \cup h_{n_2}), \ldots$. Hence $Q$ is the union of a countably infinite collection of disjoint open subsets $x$ of $A'$ such that $x$ is a subset of a set in $H_k$.

Let $K_1, K_2, K_3, \ldots$ denote the elements of a countably infinite collection $W$ of disjoint open subsets of $A'$ such that $W^* = A'$, and every set in $W$ is a subset of a set in $H_1$. For each positive integer $n$, let $K_n, 1', K_n, 2', K_n, 3', \ldots$ denote the elements of a countably infinite collection $Y$ of open subsets of $A'$ such that $Y^* = K_n$, and every set in $Y$ is a subset of a set in $H_2$. Suppose $k$ is a positive integer; if $\alpha$ is a finite sequence of positive integers with no more
than k terms, then \( K_a \) is an open subset of \( A' \); and the following:

1. If \( i \) is a positive integer, and \( i \leq k \), and \( V \) is the collection of all sets \( x \) such that, for a finite sequence \( \beta \) of positive integers in which there are exactly \( i \) terms, \( x = K_\beta \), then \( V \) is a collection of disjoint sets, each set in \( V \) is a subset of a set in \( H_i \), and \( V^* = A' \).

2. If each of \( n_1, n_2, \ldots, n_i \) and \( m_1, m_2, \ldots, m_j \) is a finite sequence of positive integers, \( i \leq k \), and \( j \leq k \), then \( K_{n_1, n_2, \ldots, n_i} \subseteq K_{m_1, m_2, \ldots, m_j} \) if and only if \( j \leq i \), and for each positive integer \( u \) such that \( u \leq j \), \( mu = nu \).

For each finite sequence \( n_1, n_2, \ldots, n_k \) with \( k \) terms, let

\[ K_{n_1, n_2, \ldots, n_k, 1}, K_{n_1, n_2, \ldots, n_k, 2}, K_{n_1, n_2, \ldots, n_k, 3}, \ldots \]

denote the elements of a countably infinite collection \( X \) of disjoint open subsets of \( A' \) such that \( X^* = K_{n_1, n_2, \ldots, n_k} \) and each set in \( X \) is a subset of a set in \( H_{k+1} \).

Then there exists a reversible function \( T \) such that \( d.T \) is the collection of all finite sequences of positive integers, \( r.T \) is a collection of open subsets of \( A' \), and the following:

1. If \( i \) is a positive integer, and \( V \) is the collection of all sets \( X \) such that, for an element \( a \) of \( d.T \) with exactly \( i \) terms, \( X = T(a) \),
then each set in $V$ is a subset of a set in $H_t$, and $V\subseteq A'$.

(2) If each of $n_1, n_2, \ldots, n_i$ and $m_1, m_2, \ldots, m_j$ is in $d.T$, then $T(n_1, n_2, \ldots, n_i) \subseteq T(m_1, m_2, \ldots, m_j)$ if and only if $j \leq i$, and for each positive integer $u$ such that $u \leq j$, $mu = nu$.

If $q \in A'$, and $R$ is a region of $q$, then there is a set $g$ in $G$ such that $q \subseteq g$, and $g \subseteq R$. Then, for a point $p$ in $A''$ and a positive integer $k$, $g = R_{pk}$. Let $u$ denote a radius of $R_{pk}$ for $p$. There is a positive integer $j$ such that the number $\frac{u}{j} < \frac{1}{2}[u - t(p, q)]$. Then every set in $H_j$ which contains $q$ is a subset of $R$. Let $a$ denote a sequence of positive integers with $j$ terms such that $p \in T(a)$. Then $T(a) \subseteq R$, and $r.T$ is a basis of $A'$.

Similarly, there is a subset $B'$ of $B$ such that $\overline{B'} \supseteq B$. There is a function $T'$ such that $r.T'$ is a basis of $B'$, $d.T'$ is $d.T$, and if each of $n_1, n_2, \ldots, n_i$ and $m_1, m_2, \ldots, m_j$ is in $d.T'$, then $T'(n_1, n_2, \ldots, n_i) \subseteq T'(m_1, m_2, \ldots, m_j)$ if and only if $j \leq i$, and for each positive integer $u$ such that $u \leq j$, $nu = mu$.

Let $f$ denote the function $h$ such that $d.h = r.T$, and if $R \in d.h$, then $h(R) = T'[T^{-1}(R)]$. Then $f$ is a $z$-function of $(A', B')$, and $A' z B'$. Therefore $A \subseteq B$. 
CHAPTER IV

TOPOLOGICAL EQUIVALENCE

In this chapter Axiom IV is not assumed.

**Definition 4.1.** The statement that $f$ is a topological $z$-function of $(A, B)$ means that $f$ is a $z$-function of $(A, B)$, and if $R \in d.f$, and $G \subseteq d.f$, then $G \supseteq R$ if and only if $f(G) \supseteq f(R)$.

**Definition 4.2.** The statement that $f$ is an open $z$-function of $(A, B)$ means that $f$ is a $z$-function of $(A, B)$, $d.f$ is the collection of all open subsets of $A$, and $r.f$ is the collection of all open subsets of $B$.

**Theorem 4.1.** If $f$ is an open $z$-function of $(A, B)$, then $f$ is a topological $z$-function of $(A, B)$.

**Proof.** Suppose $R \in d.f$, $G \subseteq d.f$, and $G \supseteq R$. Let $H$ denote $f(G)^*$. Then $H$ is an open subset of $B$, and $H$ contains every set in $f(G)$. Therefore $f^{-1}(H)$ contains every set in $G$. Hence $f^{-1}(H) \supseteq R$, and $H \supseteq f(R)$.

Similarly, if $R \in r.f$, $G \subseteq r.f$, and $G \supseteq R$, then the set $f^{-1}(G)^* \supseteq f^{-1}(R)$. Hence $f$ is a topological $z$-function of $(A, B)$.

**Theorem 4.2.** If each of $A$ and $B$ is a point set, then there exists an open $z$-function of $(A, B)$ if and only if there exists a topological $z$-function of $(A, B)$.
Proof. If $f$ is an open $z$-function of $(A,B)$, then $f$ is a topological $z$-function of $(A,B)$. Suppose $f$ is a topological $z$-function of $(A,B)$. For each open subset $G$ of $A$, let $G$ denote the collection of all sets $X$ in $d.f$ such that $X \subseteq G$. Let $g$ denote the function $h$ such that $d.h$ is the collection of all open subsets of $A$, and if $R \subseteq d.h$, then $h(R)$ is the set $f(R)^*$.

If each of $Q$ and $R$ is in $d.g$, and $Q \subseteq R$, then $Q \subseteq R$. Therefore $f(Q) \subseteq f(R)$, and $g(Q) \subseteq g(R)$.

If each of $Q$ and $R$ is in $d.g$, and $g(Q) \subseteq g(R)$, then $f(Q)^* \subseteq f(R)^*$. Therefore each set in $f(Q)$ is a subset of $f(R)^*$, and each set in $Q$ is a subset of $R^*$. Hence the set $Q^*$ is a subset of $R^*$. Thus $Q \subseteq R$.

If $R$ is an open subset of $B$, then let $\bar{R}$ denote the set of all sets $X$ in $r.f$ such that $X \subseteq R$. Let $Q$ denote the set $f^{-1}(R)^*$. If $U$ is a set in $d.f$, and $U \subseteq Q$, then $f(U) \subseteq \bar{R}$. Hence $f(U) \subseteq \bar{R}$, and $Q = f^{-1}(R)$. Therefore $R = g(Q)$, and $r.g$ is the collection of all open subsets of $B$. Hence $g$ is an open $z$-function of $(A,B)$.

Theorem 4.3. If each of $A$ and $B$ is a point set, then $A$ is topologically equivalent to $B$ if and only if there exists a function $f$ such that $f$ is a topological $z$-function of $(A,B)$.

Proof. Suppose $A$ is topologically equivalent to $B$. Let $T$ denote a topological transformation from $A$ to $B$. Let
$f$ denote the function $h$ such that $d.h$ is the collection of all open subsets of $A$, and if $R \subseteq d.h$, then $f(R) = T(R)$.

Then $f$ is a topological $z$-function of $(A,B)$.

Suppose $f$ is a topological $z$-function of $(A,B)$. Then let $g$ denote an open $z$-function of $(A,B)$. If $A$ is a degenerate point set, then $B$ is degenerate, and $A$ and $B$ are topologically equivalent.

Suppose $A$ is not degenerate. For each point $p$ in $A$, let $G_p$ denote the set of all points $x$ in $A$ such that $x$ is not $p$. Then $G_p$ is an open subset of $A$, and $G_p$ is a proper subset of $A$. Therefore $g(G_p)$ is a proper subset of $B$. If $u$ and $v$ are two points in $B - g(G_p)$, then let $R$ denote the set of all points in $B$ except $v$. Then $R$ is a proper subset of $B$, and $g(G_p)$ is a proper subset of $R$. Thus $g^{-1}(R)$ is a proper subset of $A$, and $G_p$ is a proper subset of $g^{-1}(R)$. Since there is only one point of $A$ not in $G_p$, $G_p$ is not a proper subset of a proper subset of $A$. Therefore $B - g(G_p)$ is degenerate.

Let $T$ denote the function $h$ such that $d.h = A$, and if $p$ is in $d.h$, then $h(p)$ is the point in $B - g(G_p)$. If $q \subseteq B$, then let $H_q$ denote the set of all points in $B$ except $q$. The set $A - g^{-1}(H_q)$ contains only one point $s$, and $T(s) = q$. Thus $r.T$ is $B$.

If each of $p$ and $q$ is in $A$, and $T(q) = T(p)$, then $g(G_q)$ is $g(G_p)$, and $G_q = G_p$. Therefore $q = p$, and $T$ is reversible.
Suppose \( p \in A \), and \( R \) is a set in \( r.g \) which contains \( T(p) \). If \( q \in g^{-1}(R) \), then \( G_q \) does not contain \( g^{-1}(R) \). Thus \( g(G_q) \) does not contain \( R \), and \( T(q) \in R \). Therefore the set \( T[g^{-1}(R)] \subseteq R \). If \( p \) is not in \( g^{-1}(R) \), then \( G_p \supseteq g^{-1}(R) \), and \( g(G_p) \supseteq R \). However, since \( T(p) \in R \), \( g(G_p) \) does not contain \( R \). Therefore \( p \in g^{-1}(R) \), and \( T \) is continuous from \( A \) to \( B \).

Similarly, if \( p \in A \), and \( R \) is a set in \( d.g \) which contains \( p \), then \( g(R) \) contains \( T(p) \), and \( T^{-1}[g(R)] \subseteq R \). Therefore \( T^{-1} \) is continuous from \( B \) to \( A \). Thus \( T \) is a topological transformation from \( A \) to \( B \), and \( A \) is topologically equivalent to \( B \).
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Books

