FINITE DIMENSIONAL VECTOR SPACE

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FINITE DIMENSIONAL VECTOR SPACE

THESIS

Presented to the Graduate Council of the
North Texas State College in Partial
Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

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Denton, Texas
August, 1960
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CHAPTER I

INTRODUCTION

The object of this thesis is to examine properties of an abstract
vector space of finite dimension n. The properties of the set of
complex numbers are assumed, and the definition of a field and of an
abelian group are not stated, although reference to these systems is
made.

Definition 1.1. Let $[F; \cdot, \times]$ be a field and $[V; +]$ be an abelian
group. Also let there be a well defined operation, $\cdot$, such that if
$a \in F$ and $X \in V$, $a \cdot X \in V$. Then $V$ is a vector space over $F$ if and only if
for every $a, b \in F$ and $X, Y \in V$

1. $a \cdot (X + Y) = a \cdot X + a \cdot Y$
2. $(ab) \cdot X = a \cdot (b \cdot X)$
3. $(a \times b) \cdot X = a \cdot (b \cdot X)$
4. If $u$ is the multiplicative identity in $F$, $u \cdot X = X$.

Definition 1.2. A vector is an ordered $n$-tuple of complex numbers.
The vector $X$ has components $x_1, x_2, \ldots, x_n$ means $X = (x_1, x_2, \ldots, x_n)$.
The set of all vectors is $V$.

Definition 1.3. Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$.
$X \neq Y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$. 

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**Definition 1.4.** Let \( X = (x_1, x_2, \ldots, x_n) \) and \( Y = (y_1, y_2, \ldots, y_n) \). 
\( X = Y \) if and only if \( x_1 = y_1, x_2 = y_2, x_3 = y_3, \ldots, x_n = y_n \).

**Definition 1.5.** A scalar is a complex number.

**Definition 1.6.** Let \( a \) be a scalar and \( X = (x_1, x_2, \ldots, x_n) \).
\[ aX = (ax_1, ax_2, \ldots, ax_n). \]

**Theorem 1.1.** \( V \) is a vector space over the field of complex numbers.

**Proof:** Let \( a \) and \( b \) be complex numbers and let \( X, Y \in V \).

1. \( a(X + Y) = a(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \)
   \[ = (ax_1 + ay_1, ax_2 + ay_2, \ldots, ax_n + ay_n) \]
   \[ = aX + aY. \]

2. \( a + b \ X = (a + b \ x_1, a + b \ x_2, \ldots, a + b \ x_n) \)
   \[ = (ax_1 + bx_1, ax_2 + bx_2, \ldots, ax_n + bx_n) \]
   \[ = aX + bX. \]

3. \( (ab)X = (abx_1, abx_2, \ldots, abx_n) \)
   \[ = a(bX). \]

4. \( 1X = (1x_1, 1x_2, \ldots, 1x_n) \)
   \[ = (x_1, x_2, \ldots, x_n) \]
   \[ = X. \]
CHAPTER II

LINEAR VECTOR SPACE

Definition 2.1. The zero vector \( \theta = (0, 0, \ldots, 0) \).

Definition 2.2. The set of vectors \( X_1, X_2, \ldots, X_k \) is linearly independent if and only if \( a_1X_1 \neq a_2X_2 \neq \ldots \neq a_kX_k = \theta \) implies \( a_1 = a_2 = \ldots = a_k = 0 \).

Definition 2.3. A set of vectors is linearly dependent if and only if it is not linearly independent.

Definition 2.4. Let \( X_1, X_2, \ldots, X_k \) be vectors and \( a_1, a_2, \ldots, a_k \) be scalars. Then \( a_1X_1 + a_2X_2 + \ldots + a_kX_k \) is a linear combination of \( X_1, X_2, \ldots, X_k \).

Theorem 2.1. The vectors \( X_1, X_2, \ldots, X_k \) are linearly dependent if and only if one vector is a linear combination of the preceding vectors.

Proof: Let the vectors \( X_1, X_2, \ldots, X_k \) be linearly dependent. Then there exists a set of scalars, \( a_1, a_2, \ldots, a_k \) such that \( a_1X_1 \neq a_2X_2 \neq \ldots \neq a_kX_k = \theta \) and some \( a_i \neq 0 \). Let \( c \) be the largest number such that \( a_c \neq 0 \). Then \( a_1X_1 \neq a_2X_2 \neq \ldots \neq a_cX_c = \theta \). Hence, \( a_cX_c = -a_1X_1 -a_2X_2 -\ldots -a_{c-1}X_{c-1} \) or \( X_c = -(a_1/a_c)X_1 -(a_2/a_c)X_2 -\ldots -(a_{c-1}/a_c)X_{c-1} \) and \( X_c \) is a linear combination of the preceding vectors.
Let some vector $X_j$ be a linear combination of the preceding vectors $X_1, X_2, \ldots, X_{j-1}$. Then
\[
X_j = d_1X_1 \neq d_2X_2 \neq \ldots \neq d_{j-1}X_{j-1} \text{ and }
\]
\[
(-1)X_j \neq d_1X_1 \neq d_2X_2 \neq \ldots \neq d_{j-1}X_{j-1} = 0
\]
and at least one coefficient, $-1$, does not equal zero. Therefore $X_1, X_2, \ldots, X_k$ is linearly dependent.

**Definition 2.5.** Let $M$ be a subset of the vector space $V$. Then $M$ is a linear manifold if and only if for every pair $X, Y \in M$ and every pair of scalars $a, b$ ($aX \neq bY \in M$).

**Remark:** The intersection of a collection of linear manifolds is a linear manifold.

**Definition 2.6.** $X_1, X_2, \ldots, X_k$ is a linear basis for a linear manifold $M$ if and only if

1. $X_1, X_2, \ldots, X_k$ are linearly independent.

2. Every element of $M$ is a linear combination of $X_1, X_2, \ldots, X_k$ and conversely.

**Theorem 2.2.** If $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_j$ form linear bases for a linear manifold $M$, then $k = j$.

**Proof:** Suppose $j > k$. If $Z$ is a vector in $M$, $Z$ is a linear combination of $X_1, X_2, \ldots, X_k$. Further, all linear combinations of $Y_1, Y_2, \ldots, Y_j$ are in $M$. Then $Y$ is a vector in $M$ and consequently a linear combination of $X_1, X_2, \ldots, X_k$. Certainly then $Y_1, X_1, X_2, \ldots, X_k$ are dependent.
By Theorem 2.1, some element, $X_1'$, is a linear combination of the preceding elements.

Let $Y_1', X_{11}', X_{12}', \ldots, X_{1(k+1)}$ be the set $Y_1', X_{11}', X_{12}', \ldots, X_{1(k-1)}$ excluding $X_1'$. Every vector in $M$ is a linear combination of $Y_1', X_{11}', X_{12}', \ldots, X_{1(k-1)}$ span $M$. Then $Y_2'$ is a linear combination of $Y_1', X_{11}', X_{12}', \ldots, X_{1(k-1)}$, hence $Y_1', Y_{11}', Y_{12}', \ldots, Y_{1(k-1)}$ are linearly dependent. Now $Y_1, Y_2, \ldots, Y_j$ are linearly independent. Therefore, for no $i = 1, 2, \ldots, j$ is $Y_i$ a linear combination of the preceding vectors. Then some vector, $X_g'$, is a linear combination of the preceding vectors. Let $Y_1', Y_2', X_{21}', X_{22}', \ldots, X_{2(k-2)}$ be the set $Y_1', Y_2', X_{11}', X_{12}', \ldots, X_{1(k-1)}$ excluding $X_g'$. Since it is assumed the $j > k$, one may continue this process until all $X_1', i = 1, 2, \ldots, k$, are replaced. Then every element in $M$ is a linear combination of $Y_1', Y_2', \ldots, Y_k$ and $Y_{k+1}$ is a linear combination of $Y_1, Y_2, \ldots, Y_k$. This is a contradiction since $Y_1, Y_2, \ldots, Y_j$ are linearly independent. Therefore $j < k$.

In a like manner, $k < j$ and consequently $k = j$.

**Theorem 2.3.** Suppose $X_1, X_2, \ldots, X_k$ are linearly independent vectors of a linear manifold $M$ and $Y_1, Y_2, \ldots, Y_k$ form a linear basis for $M$. Then $X_1, X_2, \ldots, X_k$ form a linear basis for $M$.

**Proof:** Let $Y_1, Y_2, \ldots, Y_k$ be a linear basis for $M$. Then

\[ X_1 = a_{11}Y_1 + a_{12}Y_2 + \cdots + a_{1k}Y_k \]
\[ X_2 = a_{21}Y_1 + a_{22}Y_2 + \ldots + a_{2k}Y_k \\
\vdots \\
X_k = a_{k1}Y_1 + a_{k2}Y_2 + \ldots + a_{kk}Y_k \]

Since \( X_1, X_2, \ldots, X_k \) are linearly independent, clearly \( Y_1, Y_2, \ldots, Y_k \) can be determined as a linear combination of \( X_1, X_2, \ldots, X_k \) and consequently \( X_1, X_2, \ldots, X_k \) span \( M \). Then \( X_1, X_2, \ldots, X_k \) are a linear basis for \( M \).

**Theorem 2.4.** Suppose that \( X_1, X_2, \ldots, X_m \) are linearly independent vectors of the vector space \( V \) and \( m < n \). Then there exist vectors \( X_{m+1}, X_{m+2}, \ldots, X_n \) such that \( X_1, X_2, \ldots, X_n \) form a linear basis for \( V \).

Proof: Let \( Y_1, Y_2, \ldots, Y_n \) be a linear basis for \( V \). There exists an \( i \) such that \( 1 \leq i \leq n \) and \( Y_i \) is not a linear combination of \( X_1, X_2, \ldots, X_m \). If not \( X_1, X_2, \ldots, X_m \) would span \( V \) and \( m = n \). Let \( i_1 \) be the smallest integer such that \( Y_{i_1} \) is not a linear combination of \( X_1, X_2, \ldots, X_m \).

Let \( i_2 \) be the smallest integer such that \( Y_{i_2} \) is not a linear combination of \( X_1, X_2, \ldots, X_{m}, X_{m+1} \). Let \( Y_{i_2} = X_{m+2} \).

In general, let \( i_j \) be the smallest integer such that \( Y_{i_j} \) is not a linear combination of \( X_1, X_2, \ldots, X_m, X_{m+1}, \ldots, X_{m+j-1} \). Clearly, if this process is repeated, all elements of \( Y_1, Y_2, \ldots, Y_n \) are elements or a linear combination of \( X_1, X_2, \ldots, X_k \). Then \( X_1, X_2, \ldots, X_k \) span \( V \).

Further, since no vector in \( X_1, X_2, \ldots, X_k \) is a linear combination of the preceding vectors, \( X_1, X_2, \ldots, X_k \) are linearly independent.
Then $X_1, X_2, \ldots, X_k$ is a linear basis for $V$ and $k = n$.

**Definition 2.7.** The scalar product of two vectors $X, Y$ is

$$(X, Y) = x_1\bar{y}_1 + x_2\bar{y}_2 + \ldots + x_n\bar{y}_n$$

**Remark:** If $X = Y$, $(X, Z) = (Y, Z)$.

The following properties are direct consequences of the definition of scalar product.

(1) $(aX, Y) = a(X, Y)$ and $(X, aY) = a(X, Y)$.

(2) $(X \not\subset Y, U \not\subset V) = (X, U) \not\subset (X, V) \not\subset (Y, U) \not\subset (Y, V)$.

(3) $| (X, Y) |^2 = (X, Y) (Y, X)$.

**Definition 2.8.** $(X, X)$ is the square of the norm of $X$; that is,

$$(X, X) = \| X \|^2.$$

A direct consequence of definition 2.8 is

$$\| aX \| = a \| X \|.$$

**Theorem 2.5.** Cauchy-Schwarz Inequality: $(X, Y) \leq \| X \| \| Y \|$. 

**Proof:** If $Y = 0$, the theorem is trivially true. Suppose $Y \neq 0$. If $a$ is a scalar,

$$0 \leq (X \not\subset aY, X \not\subset aY) = (X, X) \not\subset (aY, aY) \not\subset (X, aY) \not\subset (aY, X)$$

$$= (X, X) \not\subset a\bar{a}(Y, Y) \not\subset a(X, Y) \not\subset a(X, Y).$$

Now let $a = -(X, Y)/\| Y \|^2$. Then

$$0 \leq (X, X) \not\subset [(X, Y)(Y, X)/\| Y \| ^4] (Y, Y) - (Y, X)(X, Y)/\| Y \| ^2 - (X, Y)(Y, X)/\| Y \|$$

$$0 \leq (X, X) \not\subset (X, Y)(Y, X)/\| Y \| ^2 - 2(X, Y)(Y, X)/\| Y \| ^2$$

$$0 \leq (X, X) - (X, Y)(Y, X)/\| Y \| ^2.$$

But this means
(X,Y)(Y,X) \leq \|Y\|^2 \leq (X,X) = \|X\|^2 \quad \text{and} \\
(X,Y)(X,Y) \leq \|X\|^2 \|Y\|^2 \quad \text{or} \\
(X,Y) \leq \|X\| \|Y\|.

**Theorem 2.6.** Triangle Inequality: \( \|X \neq Y\| \leq \|X\| + \|Y\| \).

**Proof:** \[ \|X \neq Y\|^2 = (X,Y) \neq (Y,X) \neq (X,Y) \]
\[ = (X,X) \neq (X,Y) \neq (Y,X) \neq (Y,Y) \]
\[ = (X,X) \neq 2\Re(X,Y) \neq (Y,Y) \]
\[ \leq (X,X) \neq 2\|X\| \neq (X,Y) \]
\[ \leq \|X\|^2 \neq 2\|X\| \neq \|Y\|^2 \]
\[ \leq (\|X\| \neq \|Y\|)^2 \text{ or} \]
\[ \|X \neq Y\| \leq \|X\| \neq \|Y\|. \]

**Definition 2.9.** X and Y are orthogonal if and only if \((X,Y) = 0\).

**Definition 2.10.** The set \(X_1, X_2, \ldots, X_k\) forms an orthogonal system if and only if \((X_r,X_s) = 0\) if \(r \neq s\).

**Definition 2.11.** The set \(X_1, X_2, \ldots, X_k\) forms an orthonormal system if and only if the set is orthogonal and \(\|X_1\|^2 = 1\). If \(k = n\) the set is a complete orthonormal system.

**Theorem 2.7.** The vectors in an orthonormal system are linearly independent.

**Proof:** Let \(X_1, X_2, \ldots, X_k\) be vectors in an orthonormal system and suppose
\[ a_1X_1 \neq a_2X_2 \neq \ldots \neq a_kX_k = 0. \] Then
\[(a_1x_1 \neq a_2x_2 \neq \ldots \neq a_kx_k, x_i) = (0, x_i) = 0 \text{ and} \]
\[(a_1x_1, x_i) \neq (a_2x_2, x_i) \neq \ldots \neq (a_kx_k, x_i) = 0. \text{ But} \]
\[a_1(x_1, x_i) \neq a_2(x_2, x_i) \neq \ldots \neq a_k(x_k, x_i) = a_i(x_i, x_i) = a = 0 \text{ and} \]
\[a_i = 0 \text{ for } i = 1, 2, \ldots, k. \]

**Theorem 2.8.** Given the set of vectors \(X_1, X_2, \ldots\), let \((X_1, X_2, \ldots)^\perp\) be the set of vectors orthogonal to \(X_1, X_2, \ldots\). Then \((X_1, X_2, \ldots)^\perp\) is a linear manifold.

**Proof:** Let \(a, b\) be any two scalars and let \(X, Y \in (X_1, X_2, \ldots)^\perp\). Then
\[(X, x_i) = (Y, x_i) = 0 \text{ for } i = 1, 2, \ldots, \text{ and} \]
\[(ax \neq bY, x_i) = (ax, x_i) \neq (bY, x_i) \]
\[= a(x, x_i) \neq b(y, x_i) \]
\[= a0 \neq b0 \]
\[= 0. \]

Then \((ax \neq bY)\) is orthogonal to \(X_1, X_2, \ldots\) and \((ax \neq bY) \in (X_1, X_2, \ldots)^\perp\). Therefore \((X_1, X_2, \ldots)^\perp\) is a linear manifold.

**Theorem 2.9.** Gram-Schmidt Orthogonalization Process:

Suppose \(Y_1, Y_2, \ldots, Y_k\) are linearly independent elements in \(V\). Then there exist elements \(X_1, X_2, \ldots, X_k\) which are orthonormal and if
\[
[X_1, X_2, \ldots, X_k] \text{ is the linear manifold spanned by } X_1, X_2, \ldots, X_k, \]
\[
[X_1] = [Y_1] \\
[X_1, X_2] = [Y_1, Y_2] \\
\vdots \ \\
[X_1, X_2, \ldots, X_k] = [Y_1, Y_2, \ldots, Y_k].
\]
Proof: Let $X_1 = Y_1 / \| Y_1 \|$. Then

$$(X_1, X_1) = (Y_1 / \| Y_1 \|, Y_1 / \| Y_1 \|) = (1/ \| Y_1 \|)^2 (Y_1, Y_1) = (1/ \| Y \|)^2 \| Y \|^2 = 1.$$ Clearly $[X_1] = [Y_1]$. 

To determine $X_2$, let $X_2^g = Y_2 \not\sim aX_1$ where $a$ is to be determined. Then

$$0 = (X_2^g, X_1) = (Y_2 \not\sim aX_1, X_1) (Y_2, X_1) = (Y_2, X_1) \not\sim a.$$ Then $a = -(Y_2, X_1)$ and $X_2^g = Y_2 - (Y_2, X_1) X_1$. Let $X_2 = X_2^g / \|X_2^g\|$.

In a like manner $X_3$ can be determined by letting $X_3^g = b_1 X_1 \not\sim b_2 X_2 \not\sim Y_3$ and solving for $b_1$ and $b_2$. Clearly the process may be completed until $V$ is spanned by a set of $n$ orthonormal vectors.

**Theorem 2.10.** If $M$ is a linear manifold, then $(M^\perp)^\perp = M$.

Proof: Let $X \in M$. Then $X$ is orthogonal to every vector in $M^\perp$ and therefore $X \in (M^\perp)^\perp$.

Let $Y_1, Y_2, \ldots, Y_k$ be an orthonormal basis for $M$ and let $Y_1, Y_2, \ldots, Y_n$ be a complete orthonormal system. Let $H \in (M^\perp)^\perp$. Then $H = a_1 Y_1 \not\sim a_2 Y_2 \not\sim \ldots \not\sim a_n Y_n$. Suppose some element in the set $a_{k+1}, a_{k+2}, \ldots, a_n$ is not zero. Let this element be $a_i$. Then

$$(H, Y_i) = (a_1 Y_1 \not\sim a_2 Y_2 \not\sim \ldots \not\sim a_n Y_n, Y_i)$$

$$= a_1 (Y_1, Y_i) \not\sim a_2 (Y_2, Y_i) \not\sim \ldots \not\sim a_n (Y_n, Y_i)$$

$$= a_1 0 \not\sim a_2 0 \not\sim \ldots \not\sim a_i 1 \not\sim \ldots \not\sim a_n 0$$

$$= a_i \not\sim 0.$$ Then $H$ is not orthogonal to $Y$ and $H \not\sim (M^\perp)^\perp$ contrary to assumption. Then $H$ is a linear combination of $Y_1, Y_2, \ldots, Y_k$ and therefore $H \in M$. 


Theorem 2.11. Parsevals Relation: If \( X_1, X_2, \ldots, X_n \) form a complete orthonormal system, then for any \( X, Y \in V \), 
\[
(X, Y) = \sum_{i=1}^{n} (X, X_i)(X_i, Y).
\]

Proof: Suppose \( X_1, X_2, \ldots, X_n \) form a complete orthonormal system.

Then if \( X, Y \in V \), there exist sets of constants \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) such that

\[
X = a_1X_1 \not\perp a_2X_2 \not\perp \ldots \not\perp a_nX_n \text{ and }
\]

\[
Y = b_1X_1 \not\perp b_2X_2 \not\perp \ldots \not\perp b_nX_n. \text{ Hence }
\]

\[
(X, Y) = (a_1X_1 \not\perp a_2X_2 \not\perp \ldots \not\perp a_nX_n, b_1X_1 \not\perp b_2X_2 \not\perp \ldots \not\perp b_nX_n)
\]

\[
= a_1b_1 \not\perp a_2b_2 \not\perp \ldots \not\perp a_nb_n = \sum_{i=1}^{n} a_i b_i. \text{ Since } (X_1, X_1) = 1
\]

and \( (X_i, X_j) = 0 \) if \( i \neq j \).

Now \( a_1 b_1 = \sum_{i=1}^{n} (a_1X_1 \not\perp a_2X_2 \not\perp \ldots \not\perp a_nX_n, X_i)(X_i, b_1X_1 \not\perp b_2X_2 \not\perp \ldots \not\perp b_nX_n)
\]

\[
= \sum_{i=1}^{n} (X, X_i)(X_i, Y).
\]
CHAPTER III

LINEAR OPERATORS

**Definition 3.1.** A mapping of \( V \) into \( V \) is a transformation on \( V \); that is, for \( Y \in V \), \( AY = X \in V \) is a transformation of \( V \). \( A \) is called an operator.

**Definition 3.2.** The operator \( A \) is linear if and only if \( A(aX \not\rightarrow bY) = aAX \not\rightarrow bAY \).

This paper deals with linear operators only. Some properties of linear operators are

1. \((aA)X = a(AX)\)
2. \((A \not\rightarrow B)X = AX \not\rightarrow BX\)
3. \((AB)X = A(BX)\).

As an example of a transformation, let \( A \) be an \( n \) by \( n \) matrix and \( AX \) be the product of the matrix with \( X \), that is

\[
AX = ((a_{ij}))(X_i) = ((\sum_{j=1}^{n} a_{ij}x_j)).
\]

To prove the operator \( A \) is a linear transformation let \( X = (x_1, x_2, \ldots, x_n) \) and \( Y = (y_1, y_2, \ldots, y_n) \). Now if \( d \) and \( b \) are scalars, \( dX \not\rightarrow bY = (dx \not\rightarrow by, dx \not\rightarrow by, \ldots, bx \not\rightarrow by) \). Then

\[
A(dX \not\rightarrow bY) = ((\sum_{j=1}^{n} a_{ij}dx_i \not\rightarrow bby_j)) = ((\sum_{j=1}^{n} [a_{ij}dx_i \not\rightarrow a_{ij}bY_j])) = ((d \sum_{j=1}^{n} a_{ij}x_j \not\rightarrow b \sum_{j=1}^{n} a_{ij}y_j))
\]
\[
= ((d \sum_{j=1}^{n} a_{ij}x_{j})) \neq ((b \sum_{j=1}^{n} a_{ij}y_{j}))
\]
\[
= d((\sum_{j=1}^{n} a_{ij}x_{j})) \neq b((\sum_{j=1}^{n} a_{ij}y_{j}))
\]
\[
= dAX \neq bAY.
\]

**Theorem 3.1.** Let M be a linear manifold and let AM be the set of elements AX where \(X \in M\). Then AM is a linear manifold.

**Proof:** Let \(a, b\) be any two scalars and let \(X, Y \in AM\). Then there exist elements \(X', Y' \in M\) such that \(AX' = X\) and \(AY' = Y\). Since \(X', Y' \in M\) and M is a linear manifold, \(ax' \neq bY' \in M\). Then \(A(ax' \neq bY') \in AM\) and \(A(ax' \neq bY') = a(AX') \neq b(AY') = aX \neq bY\). Hence, AM is a linear manifold.

**Definition 3.3.** An operator is the unit operator, I, if and only if \(IX = X\) for all \(X \in V\).

**Definition 3.4.** An operator is the zero operator, \(\mathcal{O}\), means \(\mathcal{O}X = 0\) for all \(X \in V\).

**Theorem 3.2.** A is the zero operator if and only if \((AX, Y) = 0\) for every \(X, Y \in V\).

**Proof:** Suppose \((AX, Y) = 0\) for every \(X, Y \in V\). Then \((AX, AX) = 0\) and \(AX = 0\) for all \(X \in V\). By definition, \(A = \mathcal{O}\).

Suppose \(A = \mathcal{O}\). Then \(AX = 0\) and certainly \((AX, Y) = 0\) for all \(X, Y \in V\).

**Theorem 3.3.** A is the zero operator if and only if \((AX, X) = 0\) for all \(X \in V\).

**Proof:** Suppose \((AX, X) = 0\) for all \(X \in V\). Then
\[
(A [ax \neq Y], ax \neq Y) = a\mathcal{O}(AX, A) \neq a(AX, Y) \neq \bar{a}(AY, X) \neq (AY, Y)
\]
\[
= 0 \neq a(AX, Y) \neq \bar{a}(AY, X) \neq 0 = 0 \text{ for any}
\]
scalar a and any vectors $X, Y \in V$. Let $a = 1$. Then

$$(AX, Y) \neq (AY, X) = 0.$$  If $a = 1$, $i(AX, Y) - i(AY, X) = 0$. Consequently

$$i^2(AX, Y) - i^2(AY, X) = 0 \text{ or}$$

$$(AY, X) - (AX, Y) = 0.$$  Then $(AX, Y) \neq (AY, X) \neq (AY, X) = (AX, Y) = 2(AY, X) = 0$ and $(AY, X) = 0$ for any $X, Y \in V$. Hence, $A = \emptyset$.

Suppose $A = \emptyset$. Then $AX = \emptyset$ for any $X \in V$ and $(AX, X) = 0$.

**Definition 3.5.** Two operators, $A$ and $B$, are equal if and only if $AX = BX$ for all $X \in V$.

**Theorem 3.4.** $A = B$ if and only if $(AX, X) = (BX, X)$ for all $X \in V$.

Proof: Suppose $(AX, X) = (BX, X)$ for all $X \in V$. Then

$$(AX, X) - (BX, X) = 0 = (AX - BX, X) = ([A-B]X, X), \text{ and } [A-B] = \emptyset.$$  Hence,

$$AX - BX = \emptyset \text{ or } AX = BX \text{ and by definition } A = B.$$

Suppose $A = B$. Then $AX = BX$ and $(AX, X) = (BX, X)$.

**Definition 3.6.** The null space of $A$, $N_A$, is the set of all vectors $X \in V$ such that $AX = \emptyset$.

**Theorem 3.5.** The null space of $A$, $N_A$, is a linear manifold.

Proof: Let $a, b$ be any two scalars and let $X, Y \in N_A$. Then

$$A(ax \neq by) = a(AX) \neq b(AY) = \emptyset.$$  Consequently $ax \neq by \in N_A$ and $N_A$ is a linear manifold.
Definition 3.7. \( A^* \) is the adjoint of \( A \) if and only if \( (AX, Y) = (X, A^*Y) \) for all \( X, Y \in V \).

Some properties of the adjoint are as follows:

1. \((A + B)^* = A^* + B^*\)


   Hence, \((A + B)^* = A^* + B^*\).

2. \((aA)^* = \overline{a} A^*\)

   **Proof:** \(( (aA)^*, Y) = (X, (aA)^*Y) = (X, aAY) = \overline{a}(X,AY) = \overline{a}(AY, X) = \overline{a}(A^*X, Y) = (\overline{a}^*X, Y)\).

   Hence, \((aA)^* = \overline{a} A^*\).

3. \((AB)^* = B^*A^*\)

   **Proof:** \(( (AB)^*, Y) = (X, (AB)^*Y) = (X, (AB)Y) = (X, A(BY)) = (A(BY), X) = (BY, A^*X) = \overline{(Y, B^*(A^*X))} = (B^*A^*X, Y)\). Then

   \((AB)^* = B^*A^*\).

4. \((I^*, Y) = (X, I^*Y) = (X, IX) = (X, Y) = (DX, Y) \) for \( X, Y \in V \).

   Hence, \( I^* = I \).

5. \( A^{**} = A \).

   **Proof:** \((AX, Y) = (X, A^*Y) = (A^*Y, X) = (Y, A^{**}X) = (A^{**}X, Y) \) for all \( X, Y \in V \). Therefore \( AX = A^{**}X \) for all \( X \in V \), and \( A = A^{**} \).

As an example of an adjoint, if \( A \) is the matrix \( (a_{ij}) \), then

\( A^* = (\overline{(a_{ji})}) \).
Theorem 3.6. There is no more than one adjoint for a given A.

Proof: Suppose $A_1^*$ and $A_2^*$ are adjoints of A. Then

$$(AX, Y) = (X, A_1^* Y) = (X, A_2^* Y)$$
or

$$(X, A_1^* Y) - (X, A_2^* Y) = 0$$
and

$$(X, A_1^* Y - A_2^* Y) = (X, [A_1^* - A_2^*] Y) = (\overline{[A_1^* - A_2^*] Y}, X) = 0.$$

Now $([A_1^* - A_2^*] Y, X) = 0$ if and only if $([A_1^* - A_2^*] Y, X) = 0$. Then

$A_1^* - A_2^* = 0$ and $A_1^* = A_2^*$. Hence, there is only one adjoint for A.

Definition 3.8. The range of A, $R_A^*$, is the set $AX$ for all $X \in V$.

Definition 3.9. The rank of A is the dimension of the range of A.

Theorem 3.7. $N_A = (R_A^*)^1$.

Proof: Suppose $X \in N_A$. Then $AX = \theta$ and $(AX, Y) = 0 = (X, A^* Y)$
for all $Y \in V$. But this means $X$ is orthogonal to any element in $R_A^*$,
therefore $X \in (R_A^*)^1$.

Suppose $X \in (R_A^*)^1$. Then if $Y \in R_A^*$, $(Y, X) = \theta$. But $Y = A^* Z$ for
some $Z \in V$. Then $(A^* Z, X) = 0 = (A^* Z, X) = (AX, Z) = 0$ or $AX = \theta$ and $X \in N_A$.

Theorem 3.8. Rank $A = \text{Rank } A^*$.

Proof: Let the rank A be $n - k$. Then the dimension of $(R_A^*)^1$ is k.

Now $(R_A^*)^1 = N_A^*$. Hence the dimension of $N_A^*$ is k. Let $X_1, X_2, \ldots, X_k$
be base vectors for $N_A^*$. Then a set of vectors $X_1, X_2, \ldots, X_k, X_{k+1}, \ldots,$
$X_n$ may be constructed which span V. Now $A^* X = \theta$ if $i = 1, 2, \ldots, k$
since $X_1, X_2, \ldots, X_k$ form a basis for $N_A^*$. Then the dimension of $R_A^*$
is $n - k$. 
Definition 3.10. A is nonsingular if and only if the rank of A is n. Otherwise A is singular.

Definition 3.11. If A is nonsingular, AX = Y has a unique solution X for every Y ∈ V. The solution is written X = A⁻¹Y where A⁻¹ is called the inverse of A.

Properties of the inverse are

1. (A⁻¹)⁻¹ = A
2. AA⁻¹ = A⁻¹A = I
3. (aA)⁻¹ = (1/a)(A⁻¹)
4. (AB)⁻¹ = B⁻¹A⁻¹.

Theorem 3.9. (A⁻¹)* = (A*)⁻¹.

Proof: (AA⁻¹)* = (A⁻¹A)*

(A⁻¹)*A* = (A⁻¹A)*

(A⁻¹) *A*A⁻¹ = (A⁻¹A)*A⁻¹

(A⁻¹)*I = I*(A*)⁻¹ = I(A*)⁻¹ and

(A⁻¹)* = (A*)⁻¹.

Theorem 3.10. A⁻¹ is a linear operator.

Proof: Let a, b be scalars and let X, Y ∈ V. There exist elements Y, W ∈ V such that AY = X and AW = Y. Then aAV = aX and bAW = bY. By definition of A⁻¹, Y = A⁻¹X and W = A⁻¹Y. Hence, aY = aA⁻¹X and bW = bA⁻¹Y. Now A(aY ≠ bW) = aAV ≠ bAW = aX ≠ bY or aY ≠ bW = A⁻¹(aX ≠ bY). But aY ≠ bW = aA⁻¹X ≠ bA⁻¹Y. Hence A⁻¹(aX ≠ bY) = aA⁻¹X ≠ bA⁻¹Y and A⁻¹ is a linear operator.
Theorem 3.11. Alternative Theorem:

Either (1) the homogeneous system $AX = \theta$ has no non-trivial solutions and consequently the system $AX = Y$ has exactly one solution for every $Y$ (which means the rank of $A$ is $n$ and $A$ is non-singular).

or (2) the homogeneous system $AX = \theta$ has $r$ linearly independent solutions and consequently $A^*X = \theta$ also has $r$ linearly independent solutions, say $X_1, X_2, \ldots, X_r$ and $AX = Y$ has solutions if and only if $(Y, X_1) = (Y, X_2) = \ldots = (Y, X_r) = 0$.

Proof: Consider the function $AX = \theta$. Either $AX = \theta$ has no non-trivial solutions or $AX = \theta$ has $r$ linearly independent solutions.

Case I: Suppose $AX = \theta$ has no non-trivial solutions. Then $AX = \theta$ if and only if $X = \theta$. Consequently, $N_A = \theta$. Now $N_A = (R_A^*)^\perp$ or $R_A^* = (N_A)^\perp$. Then the rank of $A^*$ is $n$ and therefore the rank of $A$ is $n$. Hence $A$ is non-singular and if $Y \in V$ there exists an $X \in V$ such that $AX = Y$.

Suppose $AZ = Y$. Then $AX = AZ$ or $A(X - Z) = \theta$ and $X - Z = \theta$. Then $X = Z$ and $AX = Y$ has exactly one solution.

Case II: Suppose $AX = \theta$ has $r$ linearly independent solutions. Then the dimension of $N_A$ is $r$. Now $N_A = (R_A^*)^\perp$. Then the rank of $A^*$ is the rank of $A$ which is $n - r$ and the rank of $(R_A^*)^\perp$ is $r$. But $(R_A^*)^\perp = N_A^*$ and the dimension of $N_A^*$ is the dimension of $N_A$. Then there exists $r$ linearly independent vectors $X_1, X_2, \ldots, X_r$ such that $A^*X_i = \theta$, $i = 1, 2, \ldots, r$ and $N_A^*$ is spanned by $X_1, X_2, \ldots, X_r$. Now $N_A^* = (R_A)^\perp$. Suppose $AX = Y$ has solutions. Then $Y \in R_A$ and $(Y, X_i) = 0$,
1 \leq i \leq n. Suppose \( (X,X) = 0 \), \( i = 1, 2, \ldots, r \). Then \( Y \in (N_A^*)^1 \) or \( Y \in N_A \) and there exists an \( X \) such that \( AX = Y \).

**Definition 3.12.** An operator \( A \) is Hermitian if and only if \( A = A^* \).

Properties of Hermitian operators are

1. \( aA \) is Hermitian if and only if \( a \) is real.

2. \( A \neq B \) is Hermitian if and only if \( A \) and \( B \) are Hermitian.

3. \( AB \) is Hermitian if and only if \( A \) and \( B \) are Hermitian.

**Theorem 3.12.** \( A \) is Hermitian if and only if \( (AX,X) \) is real for all \( X \in V \).

Proof: Suppose \( A \) is Hermitian. Then if \( X \in V \), \( (AX,X) = (X,A^*) = (X,AX) = (AX,X) \). But \( (AX,A) = (AX,X) \) if and only if \( (AX,X) \) is real.

Suppose \( (AX,X) \) is real for all \( X \in V \). Then \( (AX,X) = (AX,X) = (X,A^*X) = (A^*X,X) \). By Theorem 2.4, \( A = A^* \).

**Definition 3.13.** \( A \) is a positive operator if and only if \( (AX,X) > 0 \) for all \( X \neq 0 \).

**Definition 3.14.** \( A \) is a negative operator if and only if \( (AX,X) < 0 \) for all \( X \neq 0 \).

**Theorem 3.13.** If \( A \) is either positive or negative then \( A \) is non-singular.

Proof: Suppose \( A \) is either positive or negative. Then \( AX = 0 \) if and only if \( X = 0 \) or \( AX = 0 \) has no non-trivial solutions and by the alternative theorem, \( A \) is non-singular.
Theorem 3.14. If $A$ is a positive operator, $A$ is Hermitian.

Proof: Suppose $A$ is a positive operator. Then $(AX, X)$ is real since $(AX, X) > 0$. But this means $A$ is Hermitian by Theorem 3.1.

Theorem 3.15. Let $M$ be a linear manifold and $M^\perp$ be the orthogonal complement of $M$. If $X$ is any vector, then there exist vectors $W$ and $U$ such that $X = U \neq W$ where $U \in M$ and $W \in M^\perp$. Furthermore, the decomposition is unique.

Proof: Let $X_1, X_2, \ldots, X_s$ be a linear bases of $M$. Let $Y_1, Y_2, \ldots, Y_r$ be a linear bases of $M^\perp$. Now $M \cup M^\perp = V$ and $X_1, X_2, \ldots, X_s, Y_1, Y_2, \ldots, Y_r$ form a linear basis for $V$. Let $X \in V$. Then $X = a_1X_1 \neq a_2X_2 \neq \ldots \neq a_nY_r$. Let $U = a_1X_1 \neq a_2X_2 \neq \ldots \neq a_sX_s$ and $W = a_{s+1}Y_1 \neq a_{s+2}Y_2 \neq \ldots \neq a_nY_r$. Then $X = U \neq W$ and $U \in M, W \in M^\perp$.

Suppose $X = Q \neq Z$ where

\[ Q = b_1X_1 \neq b_2X_2 \neq \ldots \neq b_sX_s \]
\[ Z = b_{s+1}Y_1 \neq b_{s+2}Y_2 \neq \ldots \neq b_nY_r. \]

Then $a_1X_1 \neq a_2X_2 \neq \ldots \neq a_nY_r = b_1X_1 \neq b_2X_2 \neq \ldots \neq b_nY_r$, or

\[(a_1 - b_1)X_1 \neq (a_2 - b_2)X_2 \neq \ldots \neq (a_n - b_n)Y_r = 0. \]

Now if $a_i \neq b_i$ for $i = 1, 2, \ldots, n$ then the sets $X_1, X_2, \ldots, X_s$ and $Y_1, Y_2, \ldots, Y_r$ are not linearly independent contrary to assumption. Therefore $Q = W$ and $Z = U$ and the decomposition is unique.

Definition 3.15. Let $M$ be a linear manifold, $X \in V$ and $X = X_1 \neq X_2$ where $X_1 \in M$ and $X_2 \in M^\perp$. $P_M$ is a projection operator onto $M$ if and
**Theorem 3.16.** A projection operator, $P_M$, is linear.

**Proof:** Let $X, Y \in V$ and $a, b$ be complex numbers. If $M$ is any linear manifold there exists an element $X_1 \in M$ and an element $X_2 \in M_1$ such that $X_1 \not\perp X_2 = X$. Similarly, there exist elements $Y_1 \in M$, $Y_2 \in M_1$ such that $Y = Y_1 \not\perp Y_2$. Then $aX = aX_1 \not\perp aX_2$, $aX_1 \in M$ and $aX_2 \in M_1$ and $P_M(aX) = aX_1 = aP_MX$. Also, $bY = bY_1 \not\perp bY_2$, $bY_1 \in M$, $bY_2 \in M_1$ and $P_M(bY) = bY_1 = bP_MY$. Now $X \not\perp Y = aX_1 \not\perp bY_1 \not\perp aX_2 \not\perp bY_2 aX_1 \not\perp bY_1 \in M$, $aX_2 \not\perp bY_2 \in M_1$, and $P_M(aX \not\perp bY) = aP_MX \not\perp bP_MY$.

**Theorem 3.17.** $P$ is a projection operator with respect to $M$ if and only if for every $X \in M$, $Y \in M$, $PX = X$ and $PY = \emptyset$.

**Proof:** Suppose $P$ is a projection operator. Then, if $X \in M$, $X = X \not\perp \emptyset$, $X \in M$ and $\emptyset \in M_1$, and $PX = X$. Similarly, $Y \in M_1$, $Y = \emptyset \not\perp Y$, $\emptyset \in M$, $Y = M_1$ and $PY = \emptyset$.

Suppose for every $X \in M$, $Y \in M_1$, $PX = X$, $PY = \emptyset$. Let $Z \in V$. Then $Z = Z_1 \not\perp Z_2$ where $Z_1 \in M$, $Z_2 \in M_1$, and $PZ = P(Z_1 \not\perp Z_2)$.

$PZ = PZ_1 \not\perp PZ_2 = PZ_1 \not\perp \emptyset = Z_1$.

**Theorem 3.18.** The projection operator $P_M$ is the zero operator, $\emptyset$, if and only if $M$ is null.

**Proof:** Suppose $M$ is null. Then if $X \in V$, $X \in M_1$ and $P_MX = \emptyset$.

By definition, $P_M = \emptyset$.

Suppose $P_M = \emptyset$. Then if $X \in V$, $P_MX = \emptyset$ and the decomposition of $X$ is $\emptyset \not\perp X$, and the only vector in $M$ is $\emptyset$, hence $M$ is null.
Theorem 3.19. The projection operator $P_M$ is the identity operator, $I$, if and only if $M$ is $V$.

Proof: If $M = V$, certainly $P_M X = X$ for all $X \in V$. Suppose $P_M = I$. Then $P_M X = X$ for all $X \in V$ and the decomposition of $X$ is $X \neq 0$. Hence, $M^{-1}$ is null and $M = V$.

Theorem 3.20. If $P$ is a projection operator, then $P_M^2 = P_M$.

Proof: Let $X \in V$. Then $P_M X = X_1$, $X_1 \in M$ and $P_M X_1 = X_1$.

Hence, $P_M (P_M X) = P_M^2 X = P_M X_1 = X_1 = P_M X$ and $P_M^2 = P_M$.

Theorem 3.21. If $P_M$ is a projection operator, then $P_M$ is Hermitian and $P_M \geq 0$.

Proof: Let $X_1, X_2, \ldots, X_n$ be an orthonormal basis for $V$. Then if $X \in V$, $X = a_1 X_1 \neq a_2 X_2 \neq \ldots \neq a_4 X_4 \neq \ldots \neq a_n X_n$ where $X_1, X_2, \ldots, X_4$ is a basis for the linear manifold $M$. Then

$$(P_M X, X) = (a_1 X_1 \neq a_2 X_2 \neq \ldots \neq a_4 X_4, a_1 X_1 \neq a_2 X_2 \neq \ldots \neq a_4 X_4)$$

$$= a_1^2 + a_2^2 + \ldots + a_4^2 = |a_1|^2 + |a_2|^2 + \ldots + |a_4|^2 \geq 0.$$

By Theorem 3.1, $P_M$ is Hermitian.

Theorem 3.22. The rank of the projector $P_M$ is the dimension of $M$.

Proof: By the definition of $P_M$, the range of $P_M$ is $M$ and the theorem follows.

Theorem 3.23. If the projection operator $P_M$ is not the identity operator, $I$, then $P_M$ is singular.

Proof: Since $P_M \neq I$, $M$ is not $V$ and the dimension of $M$ is not $n$.

Hence the rank of $P$ is not $n$ and $P$ is singular.
Theorem 3.24. If $A$ is Hermitian and $A^2 = A$, then $A$ is a projection operator.

Proof: By Theorem 3.10, the range of $A$ is a linear manifold.

Case I: Suppose the range of $A$ is $V$. Then if $X \in V$, there exists an element $X'$ such that $AX' = X$. Since $A^2 = A$, $A(AX') = AX' = X$. But $A(AX') = A(X)$ and $AX = X$. Certainly if $Y \in V^1$, $AY = 0$.

Case II: Suppose the range of $A$ is $M$ where $M$ is a proper subset of $V$. If $M$ is null, $A = \emptyset$ and the theorem is true. If $M$ is not null, let $X \in M$. There exists an $X'$ such that $AX' = X$. Now $AAX' = AX' = X$ and $AAX' = AX$. Then $AX = X$.

Let $Y \in M^1$. $AY = Y_1 \in M$. Since $A$ is Hermitian, $(AY, Y_1) = (Y, AY_1) = (Y, Y_1) = 0$. Then $(AY, Y_1) = (Y_1, Y_1) = 0$ and $Y_1 = 0$. Therefore $A$ is a projection operator.

Theorem 3.25. Let $P_1 = P_{M_1}$ and $P_2 = P_{M_2}$ where $M_1$ and $M_2$ are linear manifolds. Then $P_1 \not\perp P_2$ is a projection operator if and only if $P_1P_2 = P_2P_1 = \emptyset$.

Proof: Since $P_1$ and $P_2$ are projection operators, $P_1^* = P_1$ and $P_2^* = P_2$. By the properties of adjoints, $(P_1 \not\perp P_2)^* = P_1^* \not\perp P_2^* = P_1 \not\perp P_2$. Hence $P_1 \not\perp P_2$ is Hermitian.

Suppose $P_1P_2 = P_2P_1 = \emptyset$. Then

$P_1^2 \not\perp P_1P_2 \not\perp P_2P_1 \not\perp P_2^2 = P_1 \not\perp P_2 \not\perp P_2$. By Theorem 3.12, $P_1 \not\perp P_2$ is a projection operator.
Suppose $P_1 \neq P_2$ is a projection operator. Then

$$P_1 \neq P_2 = (P_1 \neq P_2)^2$$

$$= P_1^2 \neq P_1 P_2 \neq P_2 P_1 \neq P_2^2$$

$$= P_1 \neq P_1 P_2 \neq P_2 P_1 \neq P_2 \text{ and } P_1 P_2 \neq P_2 P_1 = \emptyset.$$

Suppose $M_1$ and $M_2$ have a common vector, $X \neq \emptyset$. Then

$$P_1 P_2 X = P_1 X = X \text{ and } P_2 P_1 X = P_2 X = X.$$  Hence

$$(P_1 P_2 \neq P_2 P_1) X = 2X \text{ and not } \emptyset.$$  Consequently, it is not true that $M_1$ and $M_2$ have a vector in common. Then, for every $X \in M_1$, if $Y \in M_2$, $(X, Y) = 0$ and $M_1$ is orthogonal to $M_2$.

Now let $X$ be any vector. $P_2 X = X_2$ is a vector in $M_2$ and therefore in $M_1$. Then $P_2 X = P_1 P_2 X = \emptyset$, and $P_1 P_2 = \emptyset$. Likewise $P_1 X = X_1 \in M_1$ and therefore $X_1 \in M_2$. Then $P_2 X_1 = P_2 P_1 X = \emptyset$ and $P_1 P_2 = P_2 P_1 = \emptyset$.

**Theorem 3.26.** If $I - P$ is a projection operator, then $P$ is a projection operator. Furthermore $I - P_M = P_M^{-1}$.

**Proof:** Suppose $I - P$ is a projection operator. Then $I - P = (I - P)^* = I^* - P^*$ and since $I = I^*$, $I - P = I^* - P^*$ implies $P = P^*$, i. e., $P$ is Hermitian. Further $(I - P)^2 = I - P$ or

$$(I - P)^2 = I^2 - IP - PI \neq P^2 = I - P - P \neq P^2 = I - P \text{ and } P^2 = P.$$  Hence $P$ is a projection operator by Theorem 3.12.

Let $X \in V$. Then if $M$ is a linear manifold, $X = X_1 \neq X_2$ where $X_1 \in M$, $X_2 \in M^\perp$ and $(I - P_M) X = X - X_1 = X_1 \neq X_2 - X_1 = X_2 = P_M^{-1} X$. Hence

$I - P_M = P_M^{-1}$. 

Theorem 3.27. \( P_{M1} - P_{M2} \) is a projection operator if and only if
\[ P_{M1}P_{M2} = P_{M1}P_{M2} = P_{M2} \]
if and only if \( M_2 \subseteq M_1 \). Furthermore if \( M_2 \subseteq M_1 \), \( P_{M1} = P_{M2} = P_{M1-M2} \).

Proof: Suppose \( M_2 \subseteq M_1 \). Let \( X_1, X_2, \ldots, X_k \) form a linear basis for \( M_1 \) and let \( X_1, X_2, \ldots, X_1 \) form a linear basis for \( M_2 \). Obviously, \( 1 \leq k \).

Let \( X \in V \). Then \( P_{M1}X = a_1X_1 \not\in a_2X_2 \not\in \cdots \not\in a_kX_k \) and \( P_{M2}X = a_1X_1 \not\in a_2X_2 \not\in \cdots \not\in a_kX_k \). Hence, \( (P_{M1} - P_{M2})X = a_{1+1}X_{1+1} \not\in a_{1+2}X_{1+2} \not\in \cdots \not\in a_kX_k \). But, since \( M_1 - M_2 \) has the linear basis \( X_{1+1}, X_{1+2}, \ldots, X_k \), \( (P_{M1} - P_{M2})X = a_{1+1}X_{1+1} \not\in a_{1+2}X_{1+2} \not\in a_kX_k \) is a projection operator.

Let \( P_{M1} - P_{M2} \) be a projection operator and suppose it is not true that \( M_2 \subseteq M_1 \). Then there exists an element \( X \in M_2 \) such that \( X \not\in M_1 \).

Then \( (P_{M1} - P_{M2})X = \emptyset - X \) and
\[ (P_{M1} - P_{M2})^2X = (P_{M1} - P_{M2})(-X) = \emptyset - (-X) = X \]
and
\[ (P_{M1} - P_{M2})^2 \not\in P_{M1} - P_{M2} \] which is a contradiction. Then it must be true that \( M_2 \subseteq M_1 \).

Suppose \( P_{M1} - P_{M2} \) is a projection operator. Then \( I - (P_{M1} - P_{M2}) = (I - P_{M1}) + P_2 \) is a projection operator. Then \( (I - P_{M1})P_{M2} = P_{M2}(I - P_{M1}) = \emptyset \) and
\[ P_{M2} - P_{M1}P_{M2} = P_{M2} - P_{M2}P_{M1} = \varnothing \text{ or} \]

\[ P_{M2} = P_{M1}P_{M2} = P_{M2}P_{M1}. \]

Suppose \( P_{M1}P_{M2} = P_{M2}P_{M1} = P_{M2} \). Then

\[ -P_{M1}P_{M2} - P_{M2}P_{M1} = -2P_{M2} \text{ or} \]

\[ P_{M1} - P_{M1}P_{M2} - P_{M2}P_{M1} = P_{M1} - P_{M2}. \]

Hence \( (P_{M1} - P_{M2})^2 = P_{M1} - P_{M2} \). Furthermore

\[ (P_{M1} - P_{M2})^* = P_{M1} - P_{M2} \text{ and } P_{M1} - P_{M2} \text{ is Hermitian.} \]

Hence, \( P_{M1} - P_{M2} \) is a projection operator.

**Theorem 3.28.** \( P_{M1}P_{M2} \) is a projection operator if and only if

\[ P_{M1}P_{M2} = P_{M2}P_{M1} = P_{M1M2} \]

where \( M_1M_2 \) means the intersection of \( M_1 \) and \( M_2 \).

**Proof:** If \( P_{M1}P_{M2} = P_{M2}P_{M1} = P_{M1M2} \), certainly \( P_{M1M2} = P_{M1}P_{M2} \) is a projection operator.

Suppose \( P_{M1}P_{M2} \) is a projection operator. Let \( X \in M_1M_2 \).

Then \( P_{M1}P_{M2}X = P_{M1}X = X \). Let \( X \in (M_1M_2)^1 \). Then \( P_{M1M2}X = \varnothing \).

Further, \( P_{M1}P_{M2}X = P_{M1}Y \) where \( Y \in M_2 \) and \( P_{M1}Y = Z \in M_1M_2 \).

Suppose \( Z \neq \varnothing \). Then \( X = Y \neq Y' \) where \( Y' \in M_2^1 \) and \( X = Z \neq Z' \neq Y' \) where \( Z' \in M_2^1 \) or \( X = Z \neq (Z' \neq Y') \) and since \( Z \in M_1M_2 \), \( P_{M1M2}X \neq \varnothing \).

This is a contradiction, hence if \( X \in M_1M_2 \), \( P_{M1}P_{M2}X = X \) and if \( X \in (M_1M_2)^1 \), \( P_{M1}P_{M2} = \varnothing \). Then \( P_{M1}P_{M2} = P_{M1M2} \) and in like manner, \( P_{M2}P_{M1} = P_{M1M2} \).