A SET OF AXIOMS FOR A TOPOLOGICAL SPACE

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TABLE OF CONTENTS

CHAPTER I

Z SPACES

Axioms for a topological space are generally based on neighborhoods where "neighborhood" is an undefined term. Then, limit points are defined in terms of neighborhoods. However, limit points seem to be the basic concept of a topological space, rather than neighborhoods. For this reason, it will be attempted to state a set of axioms for a topological space, using limit point as the undefined concept, and to delete the idea of neighborhoods from the theory.

If H and K are two point sets, H+K will mean the union of H and K, H-K will mean the points of H which are not in K, and H·K will mean the set of points in both H and K.

A Z space is a set E in which taere is associated with each element a class Q of nonempty subsets of E. The elements of E will be called points.

Definition 1.01: If P is a point and M is an element of Q_P, then P is said to be a limit point of M.

Definition 1.02: The statement that P is a boundary point of M means that either (1) P is a point of M, and P is a limit point of a subset of E-M, or (2) P is not a point of M, and P is a limit point of a subset of M.

Definition 1.03: The statement that the point set M is open means that no point of M is a boundary point of M.

Definition 1.04: The statement that the point set M is closed means that if P is a limit point of M, then P is a point of M.

Definition 1.05: If M is a point set, the statement that M' is the derived set of M means that M' is the set of all points P such that P is a limit point of M.

Theorem 1.01: The point set M is closed if, and only if, the derived set of M is a subset of M.

Proof that the condition is necessary: Since M is closed, every limit point of M is a point of M. Hence, every point of M', the derived set of M, is a point of M. Therefore, M' is a subset of M.

Proof that the condition is sufficient: Since M' is a subset of M, every limit point of M is a point of M. Hence, M is closed.

Theorem 1.02: The point set M is open if, and only if, for each point P in M and each set H in Q_P, M contains a point of H.

Proof that the condition is necessary: Since M is open, no point of M is a boundary point of M. Let P denote a point of M and let H denote a set in Q_P. Since P is not a boundary point of M, H is not a subset of E-M. Hence, M contains a point of H. Therefore, if P is a point of M and H is a set in Q_P, M contains a point of H.

Proof that the condition is sufficient: Let P denote a point of M. Since M contains a point of every set in Q_P, P is not a limit point of a subset of E-M. Therefore, P is not a boundary point of M. Hence, M is open.

Definition 1.06: If T is a collection of sets, then T^* will denote the union of all the sets in T.

Theorem 1.03: If T is a collection of open sets, then T^* is open.

Proof: Let P denote a point of T^*. There exists a set M in T such that P is a point of M. Since M is open, M contains a point of every set in Q_P by Theorem 1.02. Then T^* contains a point of every set in Q_P, since M is a subset of T^*. Hence, by Theorem 1.02, T^* is open.

Definition 1.07: The statement that the point set K is closed in M means that if P is a point of M, and P is a limit point of K, then P is in K.

Theorem 1.04: If M is a point set, H is a subset of M, and K is closed in M, then K is closed in H.

Proof: Let P denote a point of H such that P is a limit point of K. Since H is a subset of M, P is a point of M. By Definition 1.07, P is in K. Hence, K is closed in H by Definition 1.07.

Definition 1.08: The statement that the point sets H and K are mutually exclusive means that H and K are nonempty, and no point of one is a point of the other.

<u>Definition 1.09</u>: The statement that the point sets H and K are mutually separated means that H and K are nonempty, and no point of one is a point or limit point of the other.

<u>Definition 1.10</u>: The statement that the point set M is connected means that M is not the sum of two mutually separated point sets.

<u>Theorem 1.05</u>: The point set M is connected if, and only if, it is not the sum of two mutually exclusive sets, each closed in M.

<u>Proof that the condition is necessary</u>: Suppose M is the sum of two mutually exclusive sets, each closed in M. By Definition 1.07, no point of one is a limit point of the other. Then, M is the sum of two mutually separated sets. By Definition 1.10, M is not connected. A contradiction. Hence M is not the sum of two mutually exclusive sets, each closed in M.

<u>Proof that the condition is sufficient</u>: Suppose M is not connected. Then M is the sum of two mutually separated sets H and K. Since no point of K is a limit point of H, H is closed in M by Definition 1.07. Likewise, K is closed in M. Also, H and K are mutually exclusive. Hence M is the sum of two mutually exclusive sets, each closed in M. A contradiction. Therefore, M is connected.

<u>Definition 1.11</u>: The statement that the point set H is open in M means that no point of H is a limit point of M-H.

Theorem 1.06: The point set M is connected if, and only if, no nonempty proper subset of M is both open in M and closed in M.

Proof that the condition is necessary: Suppose there exists a nonempty proper subset H of M such that H is both open in M and closed in M. Now H and M-H have no point in common. Since H is open in M, no point of H is a limit point of M-H. Also, no point of M-H is a limit point of H by Definition 1.07. Hence H and M-H are mutually separated by Definition 1.09, and so M is not connected. A contradiction. Therefore, no nonempty proper subset of M is both open in M and closed in M.

Proof that the condition is sufficient: Suppose M is not connected. Then M is the sum of two mutually separated sets H and K. Now H is a nonempty proper subset of M. No point of H is a limit point of K by Definition 1.09. Hence H is open in M by Definition 1.11. Also, no point of K is a limit point of H, hence H is closed in M by Definition 1.07. Hence there exists a nonempty proper subset of M which is both open in M and closed in M. A contradiction. Therefore, M is connected.

Theorem 1.07: If M is a point set and H is a connected set which contains a point of M and a point not in M, then H contains a boundary point of M.

Proof: Suppose no point of H is a boundary point of M. Consider the point sets H-M and H·M. Since no point of H is a boundary point of M, no point of H-M is a limit point of H·M and no point of H·M is a limit point of H-M by Definition 1.02. Then H-M and H·M are mutually separated, since no point of one is a point or limit point of the other. Hence H is not connected by Definition 1.10. A contradiction. Hence H contains a boundary point of M.

Axiom A_Q: If P is a point, M is in Q_P and N contains M, then N is in Q_P.

This axiom will also be referred to as the inclusion property. The inclusion property is a condition which seems to be intuitively true in one's normal concept of a limit point. Z spaces which satisfy the inclusion property will now be considered.

Theorem 1.08: If T is a collection of closed point sets which have a point in common, then the set of all points common to the elements of T is closed.

Proof: Let H denote the set of all points common to the elements of T. Let P denote a limit point of H. Now H is a subset of each set in T. Hence, by the inclusion property, P is a limit point of each set in T. Therefore, since each set in T is closed, P is a point of each set in T. So P is a point of H. It then follows that H is closed.

Example 1.01: This example will illustrate a Z space for which Theorem 1.08 does not hold. Let E consist of three elements: A, B, and C. The assignment of classes of subsets of E for each point will be as follows: Q_A is empty, Q_B contains one set which contains only the point C, and Q_C is also empty. It is seen that the inclusion property does not hold for this Z space. Let M_1 denote the set which consists of the points A and C. Let M_2 denote the set which consists of the points B and C. Let T denote the collection which consists of the sets M_1 and M_2. Now M_1 and M_2 are closed, since no point is a limit point of M_1 or M_2. Let M denote the set of points common to M_1 and M_2. Now M consists of the point C. Therefore, B is a limit point of M. Hence M is not closed.

Definition 1.12: The statement that the point set M is dense in itself means that if P is a point of M, then P is a limit point of M.

Definition 1.13: The statement that K is the nucleus of M means that K is the sum of all the subsets of M which are dense in themselves.

Theorem 1.09: If M is a point set and K is the nucleus of M, then K is closed in M.

Proof: Let R denote a point of K. Since K is the nucleus of M, there exists a set H containing R such that H is contained in M, H is contained in K, and H is dense in itself. Then R is a limit point of H, since H is dense in itself. By A_0, R is a limit point of K. Since every point of K is a limit point of K, K is dense in itself. Let P denote a point of M such that P is a limit point of K. Since P is a limit point of K, $K+\{P\}$ is dense in itself. Hence K contains $K+\{P\}$. Therefore, K is closed in M.

Theorem 1.10: If M is a closed point set and H is a subset of M which contains M', then H is closed.

Proof: Let P denote a limit point of H. Then P is a limit point of M by the inclusion property. Hence by Definition 1.05, P is a point of M'. Therefore, P is a point of H, and thus H is closed.

Theorem 1.11: The point set M is open if, and only if, E-M is closed.

Proof that the condition is necessary: Let P denote a limit point of E-M. Suppose that P is a point of M. By Theorem 1.02, M contains a point of every set in Q_P. But E-M is a set in Q_P and contains no point of M. A contradiction. Hence P is in E-M, and by Definition 1.04, E-M is closed.

Proof that the condition is sufficient: Let R denote a point of M. Now E-M is not in Q_R by Definition 1.04. By the inclusion property, no subset of E-M is in Q_R. Hence M contains a point of every set in Q_R. Therefore, M is open by Theorem 1.02.

Theorem 1.12: If T is a collection of point sets such that each set in T is dense in itself, then T^* is dense in itself.

Proof: Let P denote a point of T^*. There exists a set M in T such that P is a point of M. Since M is dense in itself, P is a limit point of M by Definition 1.12. Hence P is a limit point of T^* by the inclusion property. Therefore, T^* is dense in itself.

Theorem 1.13: If M is a point set and T is a collection of point sets which have a point in common such that each set in T is closed in M, then the set of all points common to the elements of T is closed in M.

Proof: Let H denote the set of all points common to the elements of T. Let P denote a point of M which is a limit point of H. Since H is a subset of each set in T, P is a limit point of each set in T by the inclusion property. Then, since each set in T is closed in M, P is a point of each set in T by Definition 1.07. So P is in H. Therefore, H is closed in M.

Theorem 1.14: A closed point set M is connected if, and only if, it is not the sum of two mutually exclusive, closed point sets.

Proof that the condition is necessary: Suppose M is the sum of two mutually exclusive, closed point sets. Then no point of one is a point of the other, and since each set is closed, no point of one is a limit point of the other. Hence M is the sum of two mutually separated sets by Definition 1.09. Therefore, M is not connected. A contradiction.

So M is not the sum of two mutually exclusive, closed point sets.

Proof that the condition is sufficient: Suppose M is not connected. Then M is the sum of two mutually separated point sets H and K. No point of one is a point of the other by Definition 1.09. Hence H and K are mutually exclusive. Also, no point of one is a limit point of the other. Let P denote a limit point of H. By the inclusion property, P is a limit point of M. Then P is in M, since M is closed. Therefore, P is in H, since P cannot be in K. Hence H is closed. Likewise, K is closed. So M is the sum of two mutually exclusive, closed point sets. A contradiction. Hence M is connected.

Definition 1.14: If M is a point set, the statement that $\overline{M}$ is the closure of M means that $\overline{M}$ is the sum of M and M'.

Theorem 1.15: If M is a connected point set, then $\overline{M}$ is connected; and if H is a subset of $\overline{M}$ which contains M, then H is connected.

Proof: Suppose $\overline{M}$ is not connected. Then $\overline{M}$ is the sum of two mutually separated sets K and J. Now by Definition 1.14, M is a subset of $\overline{M}$. Suppose M·K and M·J both are non-empty. Now K and J have no point in common, and so M·K and M·J have no point in common. Suppose there exists a point P of M·K which is a limit point of M·J. Then P is a limit point of J by the inclusion property. But no point of H is a limit point of J. Hence no point of $M \cdot K$ is a limit point of $M \cdot J$. Likewise, no point of $M \cdot J$ is a limit point of $M \cdot K$. Hence $M \cdot K$ and $M \cdot J$ are mutually separated. However, $M \cdot K + M \cdot J$ is M. Therefore, M is not connected. A contradiction. So either K or J contains M. Suppose K contains M. By Definition 1.14, J is a subset of M'. Let P denote a point of J. Then P is a limit point of K by the inclusion property. Therefore, K and J are not mutually separated. A contradiction. Hence $\overline{M}$ is connected. A similar argument follows for the proof that H is connected.

<u>Theorem 1.16</u>: If H and K are two mutually separated point sets and M is a connected subset of H+K, then M is a subset of H or a subset of K.

<u>Proof</u>: Suppose $H \cdot M$ and $K \cdot M$ are both nonempty. Since H and K have no point in common by Definition 1.09, $H \cdot M$ and $K \cdot M$ have no point in common. Suppose there exists a point P of $H \cdot M$ which is a limit point of $K \cdot M$. Then P is a limit point of K by the inclusion property. But no point of H is a limit point of K. Hence no point of $H \cdot M$ is a limit point of $K \cdot M$. Likewise, no point of $K \cdot M$ is a limit point of $H \cdot M$. Now $H \cdot M + K \cdot M$ is M. Therefore, M is the sum of two mutually separated sets. Hence M is not connected. A contradiction. So M is a subset of H or a subset of K.

Theorem 1.17: If M is a point set such that if P and R are two points of M, then there exists a connected subset of M containing P and R; then M is connected.

Proof: Suppose M is not a connected point set. Then M is the sum of two mutually separated point sets H and K. Let P denote a point of H, and let R denote a point of K. Let N denote a subset of M which contains P and R. Now H·N and K·N are both nonempty, and no point of one is a point of the other. Also, no point of H·N or K·N is a limit point of the other by Definition 1.09 and the inclusion property. Hence H·N and K·N are two mutually separated sets. Now H·N + K·N is N, and so N is not connected. A contradiction. Therefore, M is connected.

Lemma: If H and K are two connected point sets such that H and K have a point in common, then H+K is connected.

Proof: Suppose H+K is not connected. Then H+K is the sum of two mutually separated point sets I and J. Now suppose H·I and H·J are both nonempty. Since I and J are mutually exclusive, H·I and H·J are mutually exclusive. Since no point of I or J is a limit point of the other, no point of H·I or H·J is a limit point of the other by the inclusion property. Hence H·I and H·J are two mutually separated sets by Definition 1.09. Then, since H·J + H·I is H, H is not connected. A contradiction. Therefore, H is a subset of I or J. Suppose I contains H. Since H and K have a point in common, K·I and K·J are both nonempty. This leads, in a similar manner, to the contradiction that K is not connected. Hence H+K is connected.

Theorem 1.18: If T is a collection of connected point sets such that if H and K are two elements of T, then H and K have a point in common; then T^* is connected.

Proof: Let P and R denote two points of T^*. By Definition 1.06, there exists a set in T which contains P, and there exists a set in T which contains R. Let H denote a set of T which contains P, and let K denote a set of T which contains R. Now H and K have a point in common. Hence by the lemma, H+K is connected. Now H+K is a subset of T^* by Definition 1.06. Therefore, by Theorem 1.17, T^* is connected.

CHAPTER II

TOPOLOGICAL SPACES

In Chapter I, Z spaces which were not subjected to any conditions, and Z spaces which satisfied the inclusion property, were investigated. In this chapter, Z spaces which satisfy the following conditions will be considered.

Axiom A_1: If P is a point and M is a point set in Q_P, then M is nondegenerate.

Axiom A_2: If P is a point, M is a point set in Q_P, and N is a subset of M such that N is not in Q_P, then M N is in Q_P.

Axiom A_3: If P is a point and M is a point set such that M is not in Q_P, then there exists an open set H which contains P and contains no point of M distinct from P.

A Z space which satisfies axioms A_1, A_2, and A_3 is called a topological space.

Definition 2.01: The statement that the point set M is compact means that, if K is an infinite subset of M, then there exists a point P which is a limit point of K.

Theorem 2.01: If M_1, M_2, M_3,... is a sequence of non-empty, closed and compact point sets such that for each n, M_n contains M_{n+1}, then there is a point common to all the sets of the sequence, and the set of all points common to the sets of the sequence is closed.

<u>Proof</u>: If there exists a positive integer N such that if $n>N$, then M_n is M_N; then M_N is the set of points common to the elements of the sequence, and M_N is closed.

If for each positive integer N, there exists a positive integer $n>N$ such that M_n is not M_N; then there exists a monotone increasing sequence $n_1, n_2, n_3, \ldots$ of positive integers such that, for each i, $M_{n_{i+1}}$ is a proper subset of M_n . Let p_1 denote a point of $M_{n_1}-M_{n_2}$. Let p_2 denote a point of $M_{n_2}-M_{n_3}$. Let p_3 denote a point of $M_{n_3}-M_{n_4}$. Continuing this process, a sequence $p_1, p_2, p_3, \ldots$ of points is obtained such that if i and j are two positive integers, then p_i is not p_j. Now each point of the sequence $p_1, p_2, p_3, \ldots$ is a point of M_1. Since M_1 is compact, there exists a point P such that P is a limit point of $\{p_1, p_2, p_3, \ldots\}$. Then P is in M_1 by the inclusion property and the fact that M_1 is closed. Now P is not a limit point of $\{p_1\}$ by A_1. Therefore, P is a limit point of $\{p_2, p_3, p_4, \ldots\}$. Then P is in M_2 by the inclusion property and the fact that M_2 is closed. In a similar manner, it can be shown that P is a point of each set in the sequence $M_1, M_2, M_3, \ldots$. Let K denote the set of all points common to the elements of $M_1, M_2, M_3, \ldots$. By Theorem 1.08, K is closed.

<u>Theorem 2.02</u>: If H and K are two point sets and P is a limit point of H+K, then P is a limit point of H or K.

Proof: If P is not a limit point of H, then P is a limit point of (H+K)-H by A_2. Now K contains (H+K)-H. Hence, by the inclusion property, P is a limit point of K. Therefore, P is a limit point of H or K.

Theorem 2.03: If P is a limit point of the point set M, then M is infinite.

Proof: Let p_1 denote a point of M. Now P is not a limit point of $\{p_1\}$ by A_1. Then P is a limit point of M-$\{p_1\}$ by A_2. Let p_2 denote a point of M-$\{p_1\}$. Now P is not a limit point of $\{p_2\}$. Therefore, P is a limit point of M-($\{p_1\}$+$\{p_2\}$) by A_2. This process may be continued indefinitely, and hence M is infinite.

Theorem 2.04: Every finite point set is closed.

Proof: If P is a point and M is a finite point set, then P is not a limit point of M by Theorem 2.03. Hence every finite set is closed by Definition 1.04.

Theorem 2.05: If M is a nondegenerate, connected point set, then M is dense in itself.

Proof: Let P denote a point of M. Consider the sets M-$\{P\}$ and $\{P\}$. Now no point of one is a point of the other, but since M is connected, M-$\{P\}$ and $\{P\}$ are not mutually separated. No point of M-$\{P\}$ is a limit point of $\{P\}$ by A_1. Hence P is a limit point of M-$\{P\}$. Therefore, every point of M is a limit point of a subset of M and, thus, of M. Hence M is dense in itself by Definition 1.12.

<u>Theorem 2.06</u>: If M is a point set, then M' is closed.

<u>Proof</u>: Suppose M' is not closed. Then there exists a point P such that P is a limit point of M', and P is not a point of M'. Since P is not a point of the derived set of M, P is not a limit point of M by Definition 1.05. Hence there exists an open set H such that H contains P, and H contains no point of M distinct from P. Let R denote a point of H. Since H contains no point of M-$\{P\}$, R is not a limit point of M-$\{P\}$ by Theorem 1.02. Also, R is not a limit point of $\{P\}$ by A_1. Hence R is not a limit point of M by Theorem 2.02. No point of H is a limit point of M, and so no point of H is a point of M'. Hence P is not a limit point of M' by Theorem 1.02. A contradiction. Therefore, M' is closed.

<u>Theorem 2.07</u>: If T is a finite collection of closed point sets, then T^* is closed.

<u>Proof</u>: Let P denote a limit point of T^* and let $M_1, M_2, M_3, \ldots, M_N$ denote the elements of T. Now P is a limit point of M_1 or $M_2+M_3+\ldots+M_N$ by Theorem 2.02. If P is a limit point of M_1, then P is a point of M_1, and hence P is in T^*. If P is not a limit point of M_1, then P is a limit point of M_2 or $M_3+M_4+\ldots+M_N$. In this manner, it can be shown that P is a limit point of one of the sets of T and, hence, in that set. Then by Definition 1.06, P is in T^*. Therefore, T^* is closed.

<u>Lemma</u>: If M and N are two open sets, then $M \cdot N$ is open.

Proof: Suppose $M \cdot N$ is not open. Then, by Theorem 1.02, there exists a point P in $M \cdot N$ and a set H in Q_P such that $M \cdot N$ contains no point of H. Now M contains no point of $H \cdot N$. Hence $H \cdot N$ is not an element of Q_P by the Theorem 1.02. Then $H - H \cdot N$ is in Q_P by A_2. But N contains no point of $H - H \cdot N$. Therefore, N is not open. A contradiction. Hence $M \cdot N$ is open.

Theorem 2.08: If T is a finite collection of open sets, then the set of all points common to the elements of T is open.

Proof: Let M_1, M_2, $M_3, \ldots,$ M_N denote the elements of T. Now $M_1 \cdot M_2$ is open by the lemma. Also, $M_1 \cdot M_2 \cdot M_3$ is open by the lemma. Continuing this process, $M_1 \cdot M_2 \cdot M_3 \ldots \cdot M_N$ is open. Hence the set of all points common to the elements of T is open.

CHAPTER III

INVESTIGATION OF THE AXIOMS

Independence

The independence of the axioms for a topological space will now be investigated. Also, the connection of A_0 and the axioms for a topological space will be considered.

It will now be shown that the axioms imply A_0. Let E denote a set such that the associated Z space is a topological space. Let P denote a point of E; let M denote a point set such that M is in Q_P; and let N denote a point set which contains M. Suppose N is not in Q_P. Then there exists an open set H which contains P and such that H contains no point of N distinct from P by A_3. Also, P is not a limit point of $\{P\}$ by A_1. Therefore, by A_2, P is a limit point of $M-\{P\}$. Since H is open, H contains a point R of $M-\{P\}$ by Theorem 1.02. And, since R is a point of N, H contains a point of N distinct from P. A contradiction. Therefore, if P is a point, M is a point set such that M is in Q_P, and N is a point set which contains M, then N is in Q_P. Hence the axioms for a topological space imply A_0.

It will now be shown that A_1 is independent of A_0, A_2, and A_3. Let E denote the set which consists of two points, P and R. The sets $\{P\}$ and E will be the elements of Q_P, and $\{R\}$ and E are the elements of Q_R. It is seen from the description of Q_P and Q_R that A_0 is satisfied, and likewise A_2 is satisfied. The point set $\{P\}$ is open, since P is not a limit point of $\{R\}$ and hence not a boundary point of $\{P\}$. Similarly, $\{R\}$ is open. Therefore, A_3 is satisfied. However, A_1 is not satisfied. So A_1 is independent of A_0, A_2, and A_3.

It will now be shown that A_2 is independent of A_0, A_1, and A_3. Let E denote the set of positive integers. If P is a point and M is a point set, then M is in Q_P if, and only if, there exists a positive integer N such that if $n>N$, then n is in M.

Let P denote a point, let H denote a point set such that H is in Q_P, and let K denote a point set which contains H. There exists a positive integer N such that if $n>N$, then n is in H. Since K contains H, if $n>N$, n is in K. Therefore, P is a limit point of K, and so A_0 is satisfied. If M is in Q_P, then M is infinite. Hence A_1 is satisfied. Let R denote a point and let J denote a point set which is not in Q_P. Then there exists a positive integer N_1 such that $N_1>R+1$, and N_1 is not in J. There exists a positive integer N_2 such that $N_2>N_1+1$, and N_2 is not in J. Continuing this process, a monotone increasing sequence N_1, N_2, N_3,... of positive integers is obtained. Let I denote $\{P, N_1, N_2, N_3,...\}$. Now no point of $I-\{P\}$ is a point of J. Also, no point of I is a limit point of a subset of E-I, and so no point of I is a boundary point of I. Therefore, I is open. Hence A_3 is satisfied. The point 1 is a limit point of the set of positive integers. The point 1 is not a limit point of the set of even, positive integers, and 1 is not a limit point of the set of odd, positive integers. Therefore, A_2 is not satisfied. Hence A_2 is independent of A_0, A_1, and A_3.

It will now be shown that A_3 is independent of A_0, A_1, and A_2. Let M_1 denote $\{1/2, 1/4, 1/8,\ldots\}$. Let M_2 denote $\{3/4, 5/8, 9/16,\ldots\}$; in other words, M_2 is the set of points relative to $[1/2, 1]$ as the points of M_1 were relative to $[0, 1]$. Let M_3 denote $\{3/8, 5/16, 9/32,\ldots\}$; in other words, M_3 is the set of points relative to $[1/4, 1/2]$ as the points of M_1 were relative $[0, 1]$. Continuing this process, a sequence M_1, M_2, $M_3,\ldots$ of point sets is obtained. Let E denote $\{0\}+M_1+M_2+M_3+\ldots$. The point set M is in Q_0 if, and only if, M contains infinitely many points of M_1. The point set M is in Q_P, $P=1/2^n$, if and only if M contains infinitely many points of M_{n+1}. If P is a point, P is not 0, and P is not in M_1; then Q_P is empty. It is seen from the classes Q that A_0 is satisfied and that A_1 is satisfied. Let H denote a point set in Q_0, and let K denote a subset of H which is not in Q_0. Then H contains infinitely many points of M_1, and K contains only a finite number of points of M_1. So H-K contains infinitely many points of M_1, and hence H-K is in Q_0. Let J denote a point set in Q_p, $P = 1/2^n$, and let I denote a subset of J which is not in Q_p. Then J contains infinitely many points of M_{n+1}, and I contains only a finite number of points of M_{n+1}. So J-I contains infinitely many points of M_{n+1}, and hence J-I is in Q_p. Therefore, A_2 is satisfied. No point of M_1 is a point of $M_2+M_3+M_4+\ldots$. Hence $M_2+M_3+M_4+\ldots$ is not in Q_0. Let U denote an open set which contains O. M_1 is in Q_0. So there exists a point P, $P = 1/2^n$, in U by Theorem 1.02. Since P is in U, there exists a point of M_{n+1} in U by Theorem 1.02. Hence every open set which contains O contains a point of $M_2+M_3+M_4+\ldots$. So A_3 is not satisfied. Therefore, A_3 is independent of A_0, A_1, and A_2.

Hence the axioms for a topological space are independent; and, moreover, no two axioms along with the inclusion property imply the third axiom.

Equivalence to Sierpinski's Axioms

An S space consists of a set E and a class of subsets of E called "neighborhoods" which satisfy the following four conditions:

I. Every element of E possesses at least one neighborhood. Every element is contained in all its neighborhoods.

II. If V_1 and V_2 are two neighborhoods of an element P, there exists a neighborhood V of P such that V is contained in $V_1 \cdot V_2$.

III. If R is an element of E different from P, there exists a neighborhood V of P which does not contain R.

IV. If V is a neighborhood of P, and R is in V, there exists a neighborhood W of R such that V contains W.[1]

These are Sierpinski's axioms for a topological space. It will now be shown that an S space and a Z space which is a topological are equivalent.

Definition 3.01: In an S space, if M is a point set, the statement that the point P is a limit point of M means that every neighborhood of P contains a point of M distinct from P.

Consider a Z space which is a topological space. It will be shown that a class of neighborhoods can be associated with E such that an S space is obtained such that, if M is in Q_P, then P is a limit point of M by Definition 3.01, and conversely. Let "neighborhood" of a point mean open set which contains that point. The set E is open, and hence E is a neighborhood of each point of E. By the definition of neighborhood, each point is contained in all its neighborhoods. So condition I is satisfied.

Let V_1 and V_2 denote two neighborhoods of the point P. By Theorem 2.08, $V_1 \cdot V_2$ is open. Since P is a point of $V_1 \cdot V_2$, $V_1 \cdot V_2$ is a neighborhood of P, and $V_1 \cdot V_2$ is contained in $V_1 \cdot V_2$. Hence condition II is satisfied.

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[1]Waclaw Sierpinski, General Topology (Toronto, 1956), p. 38.

Let P and R denote two points. By A_1, $\{R\}$ is not in Q_P. Hence there exists an open set V which contains P and does not contain R by A_3. Since V is a neighborhood of P, condition III is satisfied.

Let V denote a neighborhood of P, and let R denote a point of V. Since V is a neighborhood of R, there exists a neighborhood of R contained in V. So, condition IV is satisfied.

Let P denote a point and let M denote a set in Q_P. By A_1, $\{P\}$ is not Q_P. Hence $M-\{P\}$ is in Q_P by A_2. Therefore, every open set which contains P contains a point of $M-\{P\}$ by Theorem 1.02. So, P is a limit point of M by Definition 3.01.

Let P denote a point and M denote a point set such that P is a limit point of M by Definition 3.01. Hence every open set which contains P contains a point of M distinct from P. Suppose M is not in Q_P. Then there exists an open set which contains P and contains no point of M distinct from P. A contradiction. Hence M is in Q_P.

Consider now an S space. It will be shown that with each point of E there can be associated a class Q such that the Z space thus obtained is a topological space. If P is a point and M is a point set, then M is in Q_P if, and only if, P is a limit point of M by Definition 3.01. Let P denote a point and let M denote a set in Q_P. By Definition 3.01, every neighborhood of P contains a point of M distinct from P, and so M is nondegenerate. Hence A_1 is satisfied.

Let P denote a point; let M denote a set in Q_P; and let N denote a subset of M which is not in Q_P. Suppose M-N is not in Q_P. By Definition 3.01, there exists a neighborhood V_1 of P which contains no point of N distinct from P, and there exists a neighborhood V_2 of P which contains no point of M-N distinct from P. Then there exists a neighborhood V of P which is a subset of $V_1 \cdot V_2$ by condition II. Since V is a subset of $V_1 \cdot V_2$, V contains no point of M distinct from P. Then P is not a limit point of M by Definition 3.01. Hence M is not in Q_P. A contradiction. So M-N is in Q_P, and A_2 is satisfied.

Let R denote a point and let H denote a set which is not in Q_R. By Definition 3.01, there exists a neighborhood V of R which contains no point of H distinct from R. If S is a point of V, then V contains a point of every set in Q_S. Therefore, V is open by Theorem 1.02. So A_3 is satisfied.

Therefore, an S space and a topological space are equivalent.

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