ON CONTINUITY OF FUNCTIONS DEFINED ON UNRESTRICTED POINT SETS

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CHAPTER I

INTRODUCTION

Preliminary Remarks

1.1 This paper will be concerned with an investigation of the generalizations of continuous real functions of a real variable. In particular, the relationship between uniform continuity and ordinary continuity is considered. The concept of uniform continuity was first introduced by Heine about 1900.[1]

1.2 A point set will mean a set of real numbers.

1.3 The set of all real numbers will be spoken of as the points of the linear continuum, and is denoted by C.

1.4 The notation $x \in E$ will mean x is an element of E and will refer to elements of sets.

1.5 The notation $E \subset S$ will mean the set E is contained in the set S, or E is a subset of the set S.

1.6 A function (f, E, S) is a mathematical system consisting of a set E and a set S and a correspondence f which mates every element x of E with a unique element or set of elements y of S, the latter being denoted by f(x). The sets E and S are called, respectively, the

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[1]Mathematical Association, Journal für die reine und Angewandte Mathematik, Vol. LXXIV (1871), edited by A. L. Crelle (Berlin, 1826), p. 188.

the domain and the range of the function and the variables x and y of the sets E and S are respectively called the independent and dependent variables.

1.7 (f, E, S) is said to be single-valued if for each $x \in E$, f(x) is a unique element in S.

1.8 (f, E, S) is said to be a real function if S is a set of real numbers, and to be a function of a real variable if E is a set of real numbers.

1.9 $(f_1, E_1, S_1) = (f_2, E_2, S_2)$ will mean (1) $E_1 = E_2$ and (2) for every $x \in E_1 = E_2$, $f_1(x) = f_2(x)$. It is not necessary for S_1 to equal S_2.

1.10 By definition 1.9, a function (f, E, S) = (f, E, C) where C is the linear continuum. This fact justifies using the symbol (f, E) in place of (f, E, S). Hereafter, the notation (f, E) will be used, and will always denote a single-valued real function of a real variable. In other words, E will always denote a set of real numbers, and S will be understood to be C.

1.11 If (f, E) is a function and $E_1 \subset E$, then (f, E_1) will be used to denote a function (g, E_1) where if $x \in E_1$ then g(x) = f(x). Such a function (f, E_1) will be called a contraction of (f, E).

1.12 (F, E_1) is an extension of (f, E) means (1) $E \subset E_1$ and (2) whenever $x \in E$, then F(x) = f(x).

1.13 A closed interval [a, b] is the set of real numbers x, such that $a \leq x \leq b$.

1.14 An open interval (a, b) is the set of real numbers x, such that $a < x < b$.

1.15 ξ is a <u>limit point</u> of E, if every open interval I contains E contains points of E.

1.16 If E is a set, then the <u>complement</u> cE of E is the set of all real numbers not in E, that is, cE = C - E. If $E \subset S$, then S - E is called the <u>complement</u> of E <u>relative</u> to S.

1.17 ξ is an <u>interior point</u> of E, if ξ is not a limit point of cE. and if $\xi \in E$.

1.18 A set of points E is <u>closed</u>, if E contains all of its limit points.

1.19 A set of points E is <u>open</u>, if every point of E is an interior point of E.

1.20 U is an <u>upper bound</u> of (f, E) on E, if for every $x \in E$, $f(x) \leq U$.

1.21 L is a <u>lower bound</u> of (f, E) on E, if for every $x \in E$, $f(x) \geq L$.

1.22 U is the <u>least upper bound</u> of (f, E) on E, if it is an upper bound and for every $\in > O$, there exists an $x \in E$ such that $f(x) > U - \in$.

1.23 L is the <u>greatest lower bound</u> of (f, E) on E, if it is a lower bound and for every $\in > 0$, there exists an $x \in E$ such that $f(x) \angle L + \in$.

1.24 A set E is <u>dense-in-itself</u> if every point of E is a limit point of E.

1.25 A set E is <u>nowhere dense</u> if every interval contains a subinterval which contains no points of E.

1.26 A set is _everywhere dense_ if every interval contains points of E.

1.27 A _finite covering_ T of a set E means T is a finite set of open intervals $I_1, I_2, \ldots, I_n$ such that every point of E is in at least one of the intervals of T.

1.28 A _boundary point_ of E is a point of E which is not interior to E.

1.29 A _null sequence_ will mean any sequence of numbers that converges to zero.

1.30 A _neighborhood_ of a point ξ will mean an open interval containing ξ and will be denoted by N_ξ.

1.31 The _derivative_ of a set E of real numbers is the set of limit points of E, and will be denoted by E'.

1.32 The _union_ E + S of two sets E and S is the set consisting of all the elements either in E or in S.

1.33 The _intersection_ ES of two sets E and S is the set consisting of all the elements belonging to both E and S.

1.34 The _closure_ of E is the union of E and E', and will be denoted by $\overline{\overline{E}}$.

1.35 An _isolated point_ of E will mean any point $x \in E$ which is not a limit point of E.

1.36 E is an _isolated_ set means E is a set of isolated points of E.

1.37 E is a _δ - isolated_ set means for every $\xi \in E$, $EN_{\xi\delta} = (\xi)$, where $N_{\xi\delta} = (\xi - \delta, \xi + \delta)$.

1.38 If $E_1 \subset E$ is such that every point of E is a limit point of E_1, then E_1 is said to be <u>dense</u> on E.

1.39 A point ξ of a set E_1 is said to be an <u>interior point</u> of E_1 relative to E if a neighborhood N_ξ of ξ exists, for which all the points of E in N_ξ are also points of E_1.

1.40 A sequence $\{x_n\}$ is <u>convergent</u> if there is a number ξ such that for $\epsilon > 0$, there exists an integer N such that whenever $n > N$, $|x_n - \xi| < \delta$.

1.41 $E_1 \subset E$ is said to be <u>open relative</u> to E if every point of E_1 is an interior point of E_1 relative to E.

1.42 A set $E_1 \subset E$ is <u>closed relative</u> to E if every limit point of E_1 which is in E, is also in E_1.

1.43 <u>Definition</u> I (Cauchy). (f, E) is <u>continuous</u> at ξ if (1) $\xi \in E$ and (2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - \xi| < \delta$ and $x \in E$, then $|f(x) - f(\xi)| < \epsilon$.

1.44 <u>Definition</u> II (Heine). (f, E) is continuous at ξ if (1) $\xi \in E$ and (2) for every convergent sequence $\{x_n\} \in E$ whose limit is ξ, the sequence $\{f(x_n)\}$ converges and the $\lim_{n \to \infty} f(x_n) = f(\xi)$.

1.45 The Definitions I and II are equivalent.[2]

1.46 (f, E) is <u>continuous</u> on E_1 means (f, E) is continuous at every point of E_1.

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[2]For a proof of this equivalence, see H. C. Parrish, "The Analogues for T-Continuity of Certain Theorems of Ordinary Continuity," unpublished master's thesis, Department of Mathematics, North Texas State Teachers College, Denton, Texas, 1941, p. 7.

1.47 It should be noticed that if (f, E) is <u>continuous</u> on E_1, then necessarily $E_1 \subset E$.

1.48 (f, E) is <u>continuous</u> at ξ <u>relative</u> to $E_1 \subset E$ if (1) $\xi \in E$ and (2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in E_1$ and $|x - \xi| < \delta$, then $|f(x) - f(\xi)| < \epsilon$.

1.49 Examining definitions 1.43 and 1.48, it is seen that (f, E) <u>is continuous at</u> ξ <u>if and only if</u> (f, E) <u>is continuous at</u> ξ <u>relative to</u> E.

1.50 (f, E) is <u>continuous</u> on E_1 <u>relative</u> to E_2 where $E_1 \subset E$ and $E_2 \subset E$, means (f, E) is continuous at ξ relative to E_2 for every point ξ of E_1.

1.51 By definitions 1.46, 1.49 and 1.50, it can be seen that (f, E) <u>is continuous on</u> E_1 <u>if and only if</u> (f, E) <u>is continuous on</u> E_1 <u>relative to</u> E.

1.52 (f, E) is <u>lower semi-continuous</u> at ξ if (1) $\xi \in E$ and (2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - \xi| < \delta$ and $x \in E$, then $f(\xi) > f(x) - \epsilon$.

1.53 (f, E) is <u>upper semi-continuous</u> at ξ if (1) $\xi \in E$ and (2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - \xi| < \delta$ and $x \in E$, then $f(\xi) < f(x) + \epsilon$.

1.54 (f, E) is <u>lower semi-continuous</u> at ξ relative to $E_1 \subset E$ if (1) $\xi \in E$ and (2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in E_1$ and $|x - \xi| < \delta$ then $f(\xi) > f(x) - \epsilon$.

1.55 (f, E) is <u>upper semi-continuous</u> at ξ relative to $E_1 \subset E$ if (1) $\xi \in E$ and (2) for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in E_1$ and $|x - \xi| < \delta$ then $f(\xi) < f(x) + \epsilon$.

1.56 (f, E) is <u>lower</u> (<u>upper</u>) <u>semi-continuous</u> on E_1 means (f, E) is lower (upper) semi-continuous at every point of E_1.

1.57 (f, E) is <u>lower</u> (<u>upper</u>) <u>semi-continuous</u> on E_1 <u>relative</u> to E_2 if it is lower (upper) semi-continuous at ξ relative to E_2 for every point ξ of E_1.

1.58 (f, E) is <u>uniformly continuous</u> on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x, x' \in E$ and $|x-x'| < \delta$, then $|f(x)-f(x')| < \epsilon$.

1.59 (f, E) is <u>uniformly continuous</u> on $E_1 \subset E$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for $x \in E_1$ and $x' \in E_1$ with $|x-x'| < \delta$, then $|f(x)-f(x')| < \epsilon$.

1.60 (f, E) is <u>uniformly continuous</u> on E_1 relative to E where $E_1 \subset E$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in E_1$ and $x' \in E$ with $|x-x'| < \delta$, then $|f(x)-f(x')| < \epsilon$.

1.61 By definitions 1.58 and 1.60 it can be seen that (f, E) <u>is uniformly continuous on</u> E_1 <u>if</u> (f, E) <u>is uniformly continuous on</u> E_1 <u>relative to</u> E.

1.62 (f, E) is uniformly continuous on E_1 relative to E_2, where $E_1 \subset E$ and $E_2 \subset E$, means for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x \in E_1$, $x' \in E_2$, and $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$.

1.63 By definitions 1.59 and 1.61, it can be seen that if (f, E) is uniformly continuous on E, then (f, E) is uniformly continuous on each subset of E <u>relative</u> to E.

1.64 (f, E) is <u>absolutely continuous</u> on E, if for every $\epsilon > 0$ there exists a $\delta > 0$, such that whenever x_i, $x_i' \in E$ for $1 \leq i \leq n$ and $\sum |(x_i' - x_i)| < \delta$, then $\sum |f(x_i') - f(x_i)| < \epsilon$.

1.65 (f, E) is <u>absolutely continuous</u> on E_1 relative to E, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x_i \in E_1$ and $x_i' \in E$ for $1 \leq i \leq n$ and $\sum |(x_i' - x_i)| < \delta$, then $\sum |f(x_i') - f(x_i)| < \epsilon$.

1.66 (f, E) is <u>absolutely continuous</u> on E_1 relative to E_2 where $E_1 \subset E$ and $E_2 \subset E$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $x_i \in E_2$ and $x_i' \in E_1$ for $1 \leq i \leq n$ and $\sum |(x_i' - x_i)| < \delta$, then $\sum |f(x_i') - f(x_i)| < \epsilon$.

1.67 (f, E) is <u>bounded</u> from <u>above</u> if there exists an $M > 0$ such that for all $x \in E$, $f(x) \leq M$.

1.68 (f, E) is <u>bounded</u> from <u>below</u> if there exists an $m > 0$ such that for all $x \in E$, $f(x) \geq m$.

1.69 (f, E) is <u>bounded</u> if there exists an $M > 0$ such that for every $x \in E$, $|f(x)| < M$.

1.70 (f, E) is bounded at a <u>point</u> $\xi \in E$ if there exists a neighborhood N_ξ of ξ and an $M > 0$ such that for every $x \in EN_\xi$, $|f(x)| < M$.

Assumed Theorems

1.71 If a closed and bounded set E is covered by a class T of open intervals, then there is a finite number of intervals $J_1, J_2, \ldots, J_n$ of the class T which covers E.

1.72 Every bounded infinite set E has at least one limit point.

1.73 Every infinite bounded sequence of numbers $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$, where $n_1 < n_2 < \ldots$.

1.74 If A and B are real numbers, then $|A + B| \leq |A| + |B|$.

1.75 If A and B are real numbers, then

$$||A| - |B|| \leq |A - B| \leq |A| + |B| .$$

1.76 If A, B and C are real numbers, then $|A - C| \leq |A - B| + |B - C|$.

1.77 Every convergent sequence $\{x_n\}$ has a unique limit.

1.78 If $\xi \in E$ and for every sequence $\{x_n\}$ which converges to ξ, $\lim f(x_n)$ exists, then all the sequences have a common limit $f(\xi)$.

1.79 A sequence $\{x_n\}$ of real numbers converges if, and only if, it is a Cauchy sequence, that is, if $\epsilon > 0$ is chosen, there exists an N such that whenever $n > N$ and $m > N$, $|x_m - x_n| < \epsilon$.

Theorems

The following theorems, along with their related definitions will be required in this paper.

1.80 Let (f, E) be bounded and assume $EI \neq 0$. Then the following interval functions are associated with (f, E).

(i) $M\left[(f,E),\ I\right]$ will mean the l. u. b of $f(x)$ for $x \in EI$, and will be denoted by $M(I)$.

(ii) $m\left[(f,E),\ I\right]$ will mean the g. l. b of $f(x)$ for $x \in EI$, and will be denoted by $m(I)$.

(iii) $s\left[(f,E),\ I\right]$ will mean $M\left[(f,E),\ I\right] - m\left[(f,E),\ I\right]$ and will be called the <u>saltus</u> of $f(x)$ on I and denoted by $s(I)$.

<u>Remark</u>. <u>The saltus</u> $(s(I)$ = <u>the</u> l. u. b <u>of</u> $|f(x_1) - f(x_2)|$ <u>for all</u> $x_1, x_2 \in EI$.

1.81 Let (f,E) be bounded on E and $\xi \in E$. Then the following point properties are associated with (f,E).

(i) The <u>maximum</u> of (f,E) at ξ, denoted by $M\left[(f,E),\xi\right] = M(\xi)$, means the g. l. b of $M\left[(f,E),\ I\right]$ for all $I \supset \xi$.

The function thus defined will be denoted by (M,E).

(ii) The <u>minimum</u> of (f,E) at ξ, denoted by $m\left[(f,E),\xi\right] = m(\xi)$, means the l. u. b of $m\left[(f,E),\ I\right]$ for all $I \supset \xi$.

The function thus defined will be denoted by (m,E).

(iii) The <u>saltus</u> of (f,E) at ξ, denoted by $s\left[(f,E),\xi\right] = s(\xi)$, means the l. u. b of $s\left[(f,E),\ I\right]$ for $I \supset \xi$.

<u>Remark</u>. $s(\xi) = M(\xi) - m(\xi)$ for $\xi \in E$.

1.82 (f,E) <u>is continuous at</u> ξ <u>if and only if</u> $s(\xi) = 0$.

<u>Proof</u>. Assume that $s(\xi) = 0$. Choose $\epsilon > 0$, then there exists a neighborhood $I \supset \xi$ such that $s(I) < \epsilon$. Then for every $x \in EI$, $|f(x) - f(\xi)| < M(I) - m(I) = s(I) < \epsilon$. Hence, $f(x)$ is continuous at $x = \xi$.

Suppose now, (f, E) is continuous at ξ. Choose $\epsilon > 0$. There exists a neighborhood I of ξ, so that for $x \in EI$, $|f(x) - f(\xi)| < \frac{\epsilon}{2}$. Then, $s(\xi) \leq s(I) = M(I) - m(I) \leq \epsilon$. Hence, $s(\xi) = 0$.

1.83 <u>If E is a closed and bounded set and if (f, E) is bounded at every point of E, then (f, E) is bounded on E.</u>

= *Proof*. For every $\xi \in E$, there is a neighborhood N_ξ of ξ and an $M_\xi > 0$ such that for every $x \in EN_\xi$, $|f(x)| < M_\xi$. The open intervals N_ξ, where $\xi \in E$, cover the closed and bounded set E. By Theorem 1.71, a finite number of these open intervals, N_1 N_2, . . ., N_n covers E.

Let $M = \max [M_1, M_2, \ldots, M_n]$, where M_i is an upper bound of $|f(x)|$ on EN_i. Every $x \in E$ is also in N_i for some integer i, $1 \leq i \leq n$, so that $|f(x)| < M_i \leq M$. It follows that (f, E) is bounded on E.

1.84 <u>A set of real numbers E is closed and bounded if and only if every sequence $\{x_n\} \in E$, contains a subsequence $\{x_{n_i}\}$ converging to a point of E.</u>

Proof. Assume that every $\{x_n\} \in E$ contains a convergent subsequence $\{x_{n_i}\}$ whose limit is, say ξ, where $\xi \in E$. Now, it is to be shown that E is closed and bounded. If E were not bounded, there would exist a sequence $\{x_n\} \in E$ such that for every n, $|x_n| > n$ and thus no subsequence could converge to any point. If E were not closed, there would exist a sequence $\{x_n\} \in E$ converging to ξ, where ξ is not in E. Thus, every subsequence

$\{x_{n_i}\}$ would likewise converge to ξ, and no subsequence could converge to a point in E. This contradiction shows that E must be closed and bounded.

Assume E is closed and bounded. Choose a sequence $\{x_n\} \in E$. Since E is bounded, $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$, by Theorem 1.73. Since E is closed, $\xi \in E$.

CHAPTER II

CONTINUOUS FUNCTIONS

2.1 <u>Theorem</u>. <u>If</u> (f, E) <u>and</u> (g, E) <u>are continuous at</u> ξ, <u>then</u> (1) $(f + g, E)$, (2) $(f - g, E)$ <u>and</u> (3) $(f \cdot g, E)$ <u>are continuous at</u> ξ. <u>Also, if</u> $g(\xi) \neq 0$, <u>then</u> (4) $(\frac{f}{g}, E)$ <u>is continuous at</u> .

<u>Proof of</u> (1). Choose $\epsilon > 0$. There exists a neighborhood N_1 of ξ such that whenever $x \in EN_1$, then $|f(x) - f(\xi)| < \frac{\epsilon}{2}$. There exists a neighborhood N_2 of ξ such that whenever $x \in EN_2$, then $|g(x) - g(\xi)| < \frac{\epsilon}{2}$. Let $N = N_1N_2$. Then N is a neighborhood of E, and for $x \in EN$,

$$\left|[f(x) + g(x)] - [f(\xi) + g(\xi)]\right| \leq |f(x) - f(\xi)| + |g(x) - g(\xi)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

<u>Proof of</u> (3). Since (f, E) is continuous at ξ, there exists a N_1 of ξ such that whenever $x \in EN_1$, then $|f(x) - f(\xi)| < 1$. Hence, $|f(x)| < |f(\xi)| + 1$.

Let $M = \max(|g(\xi)|, |f(\xi)| + 1)$. Choose $\epsilon > 0$. There exists a neighborhood N_2 of ξ such that whenever $x \in EN_2$, then $|f(x) - f(\xi)| < \frac{\epsilon}{2M}$. Also, there exists a neighborhood N_3 of ξ such that whenever $x \in EN_3$, then $|g(x) - g(\xi)| < \frac{\epsilon}{2M}$. Let $N = N_1N_2N_3$, then for $x \in EN$,

$$\left|f(x)g(x) - f(\xi)g(\xi)\right| = \left|f(x)g(x) - f(x)g(\xi) + f(x)g(\xi) - f(\xi)g(\xi)\right|$$
$$\leq \left|f(x)g(x) - f(x)g(\xi)\right| + \left|f(x)g(\xi) - f(\xi)g(\xi)\right|$$
$$\leq \left|f(x)\right| \left|g(x) - g(\xi)\right| + \left|g(\xi)\right| \left|f(x) - f(\xi)\right|$$
$$< \left|f(x)\right| \frac{\epsilon}{2M} + \left|g(\xi)\right| \frac{\epsilon}{2M}$$
$$< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon.$$

The proofs of (2) and (4) are omitted.

2.2 Theorem. If (f, E) and (g, E) are continuous on E, then (1) (f + g, E), (2) (f - g, E) and (3) (f. g, E) are continuous on E. Also, if for every $x \in E$, $g(x) \neq 0$, then (4) $(\frac{f}{g}, E)$ is continuous on E.

This result follows immediately by Theorem 2.1 and Definition 1.45.

2.3 Theorem. If E is a closed and bounded set, and if (f, E) is continuous on E, then (f, E) is bounded on E.

Proof. Assume that (f, E) is not bounded. Then there exists a sequence of points $\{x_n\}$ and a sequence of positive numbers $k_1 < k_2 < k_2 < \ldots$ such that for every $x \in E$,

$\left|f(x_n)\right| > k_n$ and $\lim_{n\to\infty} k_n = \infty$.

Since E is closed and bounded, $\{x_n\}$ must be bounded. By Theorem 1.73, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging to a limit in E, say ξ. Hence, from the continuity of (f, E) at ξ, it is clear that $\{f(x_{n_i})\}$ must converge to $f(\xi)$. This cannot happen since $\left|f(x_{n_i})\right| > k_i$ and $\lim_{i\to\infty} \left|f(x_{n_i})\right| = \infty$. This contradicts the assumption. Thus, (f, E) must be bounded on E.

2.4 Theorem. *If E is a closed and bounded set, then (f, E) assumes its least upper bound and its greatest lower bound.*

Proof. By Theorem 2.3 (f, E) is bounded above, and for $x \in E$, $f(x)$ has a least upper bound, say M. Thus, $f(x) \leq M$.

Choose a sequence of positive numbers $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \ldots$ From the sequence $\{x_n\} \in E$, choose points $x_1, x_2, \ldots, x_n \ldots$ such that respectively,

$$f(x_1) > M - 1,\ f(x_2) > M - \frac{1}{2}, \ldots, f(x_n) > M - \frac{1}{n} \ldots$$

By Theorem 1.73 there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to some ξ in E. Now $f(x_{n_i}) > M - \frac{1}{n_i}$, so that the $\lim_{i \to \infty} f(x_{n_i}) \geq \lim_{i \to \infty} (M - \frac{1}{n_i}) = M$. By Definition 1.44, $\lim_{i \to \infty} f(x_{n_i}) = \lim_{n \to \xi} f(x_n) = f(\xi)$, so that $f(\xi) \geq M$. But $f(\xi) \leq M$. Combining $f(\xi) \leq M$ and $f(\xi) \geq M$, $f(\xi) = M$.

2.5 Corollary. *If E is a closed and bounded set and if (f, E) is continuous on E, then there exists a point ξ in E such that for every $x \in E$, $f(x) \leq f(\xi)$.*

This result follows immediately from Theorem 2.4.

2.6 Corollary. *If E is a closed and bounded set and if (f, E) is continuous on E, then there are points $x', x'' \in E$ where $f(x')$ = the g.l.b of $f(x)$ for $x \in E$, and $f(x'')$ = the l.u.b of $f(x)$ for $x \in E$.*

This result follows immediately from Theorem 2.4.

2.7 Theorem. *If (f, E) is continuous on E and ξ is a point in E with $f(\xi) > 0$, then there exists a number $p > 0$ and a N_ξ such that for every $x \in EN_\xi$, $f(x) > p$.*

Proof. Choose $p = \frac{f(\xi)}{2} > 0$. Since (f, E) is continuous at ξ, there exists a $\delta > 0$ such that whenever $x \in EN_\xi$ and $|x - \xi| < \delta$, then $|f(x) - f(\xi)| < p$. For every such x, it follows that,

$$\begin{aligned} f(x) &= f(x) - f(\xi) + f(\xi) \\ &\geq f(\xi) - |f(x) - f(\xi)| \\ &> f(\xi) - p \\ &> 2p - p = p. \end{aligned}$$

2.8 Theorem. *If (f, E) is continuous on $E = [a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, then there exists a point $\xi \in [a, b]$ for which $f(\xi) = 0$.*

Proof. Assume that $f(a) < 0$ and $f(b) > 0$. Let F be the set of x's for which $x \in [a, b]$ and $f(x) < 0$. Then $F \subset [a, b]$, $a \in F$, and F is bounded. Let ξ = the least upper bound of F.

Assume that $f(\xi) > 0$. By Theorem 2.6 there exists a N_ξ such that for $x \in N_\xi$, $f(x) > 0$. It follows that a number less than ξ is an upper bound of F. This contradicts the definition of ξ as the least upper bound of F.

Assume that $f(\xi) < 0$. Then there exists a N_ξ such that for $x \in EN_\xi$, $f(x) < 0$. Since $\xi < b$, then for some $x > \xi$, $f(x) < 0$. This contradicts the fact that ξ is an upper bound of F. Hence, $f(\xi)$ must equal zero.

2.9 Theorem. *If $[a, b]$ is an interval with $a, b \in E$ and there exists a point $\xi \in [a, b]$, ξ not in E, then there exists a* *function* (f, E) *continuous on* E *and such that for every* $x \in [a, b]$, $f(x) \neq 0$.

Consider the function $f(x) = x - \xi$, for $x \in E$. This function is continuous on E and, in fact, is uniformly continuous on E. Yet, for $x \in E$, $f(x) \neq 0$.

2.10 *Theorem. If* E *is an isolated set, then every function* (f, E) *is continuous on* E.

Proof. Let $\xi \in E$. Then ξ is an isolated point of E. By Definition 1.44, (f, E) is continuous at ξ. By Definitions 1.36 and 1.46 (f, E) is continuous on E.

2.11 *Theorem. If* (f, E) *is continuous on* E *and* $\bar{E}$ *is the closure of* E, *and if there exists a continuous extension* $(F, \bar{E})$ *of* (f, E), *then the values of* $(F, \bar{E})$ *are uniquely determined*.

Proof. Let $\xi \in \bar{E}$. By Definition 1.12, if $\xi \in E$, then $F(\xi) = f(\xi)$. If $\xi \notin E$, then ξ is a limit point of E. Hence, there exists a sequence $\{X_n\}$ of points of E converging to ξ. Since $(F, \bar{E})$ is continuous at ξ, $F(\xi) = \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} f(x_n)$.

Remark. As an example, let E *be the rational points in an interval* $I = [a, b]$. *Then* $I = \bar{E}$.

2.12 *Corollary. If the values of a continuous function* (f, E) *are known for a set* E_1 *of points everywhere dense in* E, *then the value of* (f, E) *is determined for all points* $x \in E$, *by* (f, E_1).

Proof. The proof follows immediately by Theorem 2.12 since $\bar{E}_1 \supset E$.

2.13 *If* (s, E) *is the saltus of* (f, E) *and if* k *is a real number, then* $E_1 = E_x \left[x \in E,\ s(x) \geq k \right]$ *is closed relative to* E.

2.14 *A set* $E_1 \subset E$ *is an* F_σ *relative to* E, *if* $E_1 = F_1 + F_2 + \ldots + F_n + \ldots$, *where* F_n *is closed relative to* E, *for every* n. *If* E *is the linear continuum, then* E_1 *is said to be an* F .

2.15 *Theorem*. *If* D *is the set of points* x *at which* (f, E) *is discontinuous, then* D *is an* F_σ *relative to* E.

Proof. Let $F_n = E_x \left[x \in E,\ s(x) \geq \frac{1}{n} \right]$, and $D = F_1 + F_2 + \ldots + F_n + \ldots$. By Theorem 2.13, the points where the saltus is $\geq \frac{1}{n}$, form a closed set relative to E. By Theorem 1.82, D is the set of points of discontinuity of (f, E). Since D is an F_σ relative to E, the theorem follows.

CHAPTER III

UNIFORM CONTINUITY

3.1 __Theorem__. __If__ (1) E __is a closed and bounded set__, (2) $E_1 \subset E$ __is closed relative to__ E, __and__ (3) (f, E) __is continuous on__ E_1, __then__ (f, E) __is uniformly continuous on__ E_1.

__Proof__. Choose $\epsilon > 0$. It is to be proved that there exists a $\delta > 0$ such that whenever $x' \in E_1$, $x \in E$ and $|x' - x| < \delta$, then $|f(x') - f(x)| < \epsilon$.

Assume that the conclusion does not follow, namely: there is some $\epsilon_1 > 0$ such that for every $\delta > 0$ there exists an $x' \in E_1$, $x \in E$, with $|x' - x| < \delta$ such that $|f(x') - f(x)| \geq \epsilon_1$.

Choose a null sequence $\{\delta_n\}$ of δ's. For every δ_n, there exists points $x'_n \in E_1$ and $x_n \in E$ with $|x'_n - x_n| < \delta_n$ such that $|f(x'_n) - f(x_n)| \geq \epsilon_1$. Since E_1 is closed and bounded, there exists a subsequence $\{x'_{n_i}\}$ of $\{x'_n\}$, $n_1 < n_2 < \ldots$, and a $\xi \in E_1$ with $\lim_{i \to \infty} x'_{n_i} = \xi$. Since $\xi \in \bar{E}_1$ and E_1 is closed, $\xi \in E_1$. The corresponding subsequence $\{x_{n_i}\}$ of $\{x_n\}$ also converges to ξ, due to the fact that $(x'_{n_i} - x_{n_i})$ approaches zero as i approaches ∞.

By hypothesis (3), (f, E) is continuous at ξ. Hence, there exists a $\delta > 0$ so that if $|x - \xi| < \delta$, $|f(x) - f(\xi)| < \frac{\epsilon_1}{2}$.

Since $\lim x'_{n_i} = \lim x_{n_i} = \xi$, there exists a positive integer N such that whenever $i \geq N$, $|x_{n_i} - \xi| < \delta$ and $|x'_{n_i} - \xi| < \delta$. Hence, for $i \geq N$, $|f(x'_{n_i}) - f(x_{n_i})| \leq$

$$|f(x'_{n_i}) - f(\xi)| + |f(x_{n_i}) - f(\xi)|$$

$$< \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1.$$

This contradicts the fact that for every i, $|f(x'_{n_i}) - f(x_{n_i})| \geq \epsilon_1$. The theorem follows.

3.2 __Theorem.__ __If__ E __is__ __a__ __closed__ __and__ __bounded__ __set__ __and__ __if__ (f, E) __is__ __continuous__ __on__ E, __then__ (f, E) __is__ __uniformly__ __continuous__ __on__ E.

__Proof.__ Choose $\epsilon > 0$. There is an open interval I_ξ having ξ as center and such that whenever $x \in EI_\xi$, then $|f(x) - f(\xi)| < \frac{\epsilon}{2}$. Let $x, x' \in EI_\xi$, then

$$|f(x) - f(x')| \leq |f(x) - f(\xi)| + |f(x') - f(\xi)|$$

$$\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

For $\xi \in E$, let J_ξ denote the interval of center ξ and of length half of I_ξ. The set of all intervals J_ξ thus defined is obviously a covering of E. Since E is closed and bounded, it follows by Theorem 1.71 that there exists a finite number of these intervals J_ξ that covers E, say $J_{\xi_1}, J_{\xi_2}, \ldots, J_{\xi_n}$.

Consider the set of open intervals $I_{\xi_1}, I_{\xi_2}, \ldots, I_{\xi_n}$. Let $\delta = \min(\delta_1, \delta_2, \ldots, \delta_n)$, where δ_i is half the length of J_{ξ_1}. Choose two points $x, x' \in E$ with $|x - x'| < \delta$. Then x is contained in some interval J_{ξ_1}, and it follows that $x' \in I_{\xi_i}$. Hence,

$$|f(x) - f(x')| < \epsilon.$$

3.3 Remark. In fact Theorem 3.1 and Theorem 3.2 are equivalent.

Proof. Theorem 3.2 can be obtained from Theorem 3.1 by letting $E_1 = E$. Conversely, by Definition 1.42, the set E_1 in Theorem 3.1 is a closed bounded set, satisfying condition (2). By Definition 1.46, (f, E) is continuous on E_1 satisfying condition (3), and $E = E_1$. Thus, (f, E) is uniformly continuous on E_1. The proof of Theorem 3.2 has been given merely to show a different method of the proof of Theorem 3.1, using the Borel Covering Theorem.

3.4 Corollary. If (f, E) is continuous on $[a, b]$, then (f, E) is uniformly continuous on $[a, b]$.

Proof. The proof follows immediately from Theorem 3.2.

3.5 Theorem. If (f, E) is uniformly continuous on E and if x_n and x'_n are sequences of points in E for which $\lim_{n\to\infty} |x_n - x'_n| = 0$, then $\lim_{n\to\infty} |f(x_n) - f(x'_n)| = 0$.

Proof. Choose $\epsilon > 0$, then there exists a $\delta > 0$ such that whenever $x, x' \in E$ and $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$.

For any two sequences of points x_n, x'_n in E with $\lim |x_n - x'_n| = 0$, there exists an $N > 0$ such that for $n > N$, $|x_n - x'_n| < \delta$. Hence, for $n > N$, $|f(x_n) - f(x'_n)| < \epsilon$, and the $\lim_{n\to\infty} |f(x_n) - f(x'_n)| = 0$.

3.6 Theorem. (f, E) is uniformly continuous on the closed and bounded set E if and only if (f, E) is continuous on E and the

$\lim_{n\to\infty} |f(x_n) - f(x'_n)| = 0$, <u>for every pair of sequences</u> $\{x_n\}$, $\{x'_n\}$ <u>of points in</u> E <u>such that the</u> $\lim_{n\to\infty} |x_n - x'_n| = 0$.

<u>Proof</u>. One part of the theorem follows by 3.5.

Suppose (f, E) is not uniformly continuous on E. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, there are two points $x, x' \in E$ with $|x - x'| < \delta$ such that $|f(x) - f(x')| \geq \epsilon$.

Now, choose a null sequence of δ's, $\delta_1 > \delta_2 > \ldots \to 0$. For every δ_n, there exists two points x_n, $x'_n \in E$ with $|x_n - x'_n| < \delta_n$ such that $|f(x_n) - f(x'_n)| \geq \epsilon$. By a proper choice of x_n and x'_n, it follows that $f(x_n) > f(x'_n)$ and hence, $f(x_n) \geq f(x'_n) + \epsilon$, n = 1, 2, ...

Since E is bounded, the sequence $\{x_n\}$ is bounded. By Theorem 1.73, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to some ξ. The corresponding subsequence $\{x'_{n_i}\}$ of $\{x'_n\}$ must also converge to ξ due to the fact that $(x_{n_i} - x'_{n_i}) \to 0$ as $i \to \infty$. Since E is closed, ξ belongs to E. Hence, (f, E) is continuous at ξ, and therefore $\lim_{x\to\xi} f(x) = f(\xi)$. Thus, $\lim_{i\to\infty} f(x_{n_i}) = f(\xi)$ and $\lim_{i\to\infty} f(x'_{n_i}) = f(\xi)$. Hence, the $\lim_{i\to\infty} |f(x_{n_i}) - f(x'_{n_i})| = 0$. But by the assumption, $\lim_{i\to\infty} f(x_{n_i}) \geq \lim_{i\to\infty} f(x'_{n_i}) + \epsilon$, which is a contradiction of the fact that $\lim_{n\to\infty} |f(x_n) - f(x'_n)| = 0$, and the Theorem is proved.

3.7 <u>Theorem.</u> <u>If</u> E <u>is a closed and bounded set and</u> $p > 0$ <u>and if at every point</u> ξ <u>of</u> E, $s(\xi) < p$, <u>then there exists a</u> $\delta > 0$ <u>such</u>

that whenever $x, x' \in E$ and $|x - x'| < \delta$, then $|f(x) - f(x')| < p$.[1]

Proof. Assume the contrary, that for every $\delta > 0$ there exists points x and $x' \in E$ with $|x - x'| < \delta$, such that $|f(x) - f(x')| \geq p$.

Choose a null sequence of positive δ's. There exists for each positive integer n two points x_n, $x'_n \in E$ such that whenever $|x_n - x'_n| < \delta_n$, then $|f(x_n) - f(x'_n)| \geq p$. By Theorem 1.71, the sequence $\{x_n\}$ has a limit point, say ξ. Let I be an interval with ξ as center. Then since ξ is a limit point of $\{x_n\}$ and since $|x_n - x'_n| \to 0$ as $n \to \infty$, it follows that there are two points x_n, $x'_n \in E$ which are interior to I and for which $|f(x_n) - f(x'_n)| \geq p$. Since there is no restriction on the length of I, it follows that at ξ, the saltus of (f, E) is not less than P. This is a contradiction.

3.8 Corollary. If E is a closed and bounded set and if $s(\xi) = 0$ for every point ξ in E, then (f, E) is uniformly continuous on E.

Proof. Choose $\epsilon > 0$. For every $\xi \in E$, $s(\xi) = 0$. Hence, $s(\xi) < \epsilon$ for every $\xi \in E$. By 3.7, there exists a $\delta > 0$ such that whenever $x, x' \in E$ and $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$, and the result follows.

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[1] R. L. Jeffery, The Theory of Functions of a Real Variable (Toronto, 1951), p. 34.

3.9 <u>Theorem</u>. <u>If</u> (f, E) <u>is uniformly continuous on</u> E_1 <u>relative to</u> E, <u>then</u> (f, E) <u>is continuous on</u> E_1 <u>relative to</u> E.

<u>Proof</u>. Choose $\epsilon > 0$. By Definition 1.60, there exists a $\delta > 0$ such that whenever $x \in E_1$, $x' \in E$ and $|x - x'| < \delta$ then $|f(x) - f(x')| < \epsilon$. Let ξ be any fixed point contained in E_1. Choose any point x contained in E such that $|x - \xi| < \delta$. Then $|f(x) - f(\xi)| < \epsilon$ and (f, E) is continuous at $x = \xi$. Since ξ was arbitrary, (f, E) is continuous on E_1.

3.10 <u>Example</u>. Consider the function (f, E), where $E = (0, 1)$ and $f(x) = \frac{1}{x}$.

Let $\xi \in (0, 1)$. Choose $\epsilon > 0$ and let $\delta = \min\left(\frac{\xi}{2}, \frac{\xi^2}{2} \cdot \epsilon\right)$. Let $x \in (0, 1)$ and $|x - \xi| < \delta$. Then $|f(x) - f(\xi)| < \epsilon$. Thus, (f, E) is continuous at each point $0 < x < 1$.

Let $\epsilon = 1$. For $\delta > 0$, there exists a positive number p such that $\frac{1}{p} < \delta$. Choose $N >$ the max $(1, p)$. Let $x = \frac{1}{N}$ and $x' = \frac{1}{2N}$. Then x and $x' \in (0, \delta) \subset (0, 1)$, and $|f(x) - f(x')| = |N - 2N| = N > 1$. Hence, (f, E) is not uniformly continuous on $(0, 1)$ and the converse of Theorem 3.9 is not true.

3.11 <u>Theorem</u>. <u>If</u> (f, E) <u>is absolutely continuous on</u> E, <u>then</u> (f, E) <u>is uniformly continuous on</u> E.

<u>Proof</u>. This result follows immediately by Definitions 1.64 and 1.58.

3.12 Theorem. If (f, E) is uniformly continuous on E and $E_1 \subset E$, then (f, E) is uniformly continuous on E_1.

Proof. Let x and x' be contained in E_1, then x and x' are contained in E. Choose $\epsilon > 0$, then there exists a $\delta > 0$ such that whenever $x, x' \in E$ and $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$.

Since E_1 is a subset of E, if x and x' are contained in E_1 and $|x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$. Hence, (f, E) is uniformly continuous on E_1.

3.13 Theorem. If E is δ-isolated for at least one positive δ, then (f, E) is uniformly continuous on E.

Proof. Choose $\epsilon > 0$, there exists a $\delta > 0$ such that E is δ-isolated. Let $x, x' \in E$, and $|x-x'| < \delta$. Then $x = x'$, and therefore $|f(x) - f(x')| = 0 < \epsilon$.

3.14 Corollary. If E is a finite set, then every function (f, E) is uniformly continuous on E.

Proof. This result follows immediately by Theorem 3.12, since every finite set is δ-isolated for some positive number δ.

3.15 Corollary. If $E' = 0$ and E is bounded, then every function (f, E) is uniformly continuous on E.

Proof. Since $E' = 0$ and E is bounded, it follows by Theorem 1.71 that E is finite. Thus the proof is immediate.

3.16 Theorem. If E is closed and bounded, and if (f, E) and (g, E) are both uniformly continuous on E then (1) $(f+g, E)$, (2) $(f - g, E)$ and (3) $(f. g, E)$ are uniformly continuous on E. Also, if

<u>for every</u> $x \in E$, $g(x) \neq 0$, <u>then</u> (4) $(\frac{f}{g}, E)$ <u>is uniformly continuous on</u> E.

<u>Proof</u>. The result follows immediately by Theorems 2.2, 3.1, 3.9 and definition 1.59.

3.17 <u>Theorem</u>. <u>If</u> (f, E) <u>and</u> (g, E) <u>are uniformly continuous on</u> E, then (1) (f + g, E) <u>and</u> (2) (f - g, E) <u>are uniformly continuous on</u> E.

<u>Proof of</u> (1). Choose $\epsilon > 0$. Then there exists a $\delta_1 > 0$ such that whenever $x, x' \in E$ and $|x - x'| < \delta_1$, then $|f(x) - f(x')| < \frac{\epsilon}{2}$. There exists a $\delta_2 > 0$ such that whenever $x, x' \in E$ and $|x-x'| < \delta_2$, then $|g(x) - g(x')| < \frac{\epsilon}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$. Then, for $x, x' \in E$,

$$|f(x) + g(x) - f(x') + g(x')| \leq |f(x) - f(x')| + |g(x) - g(x')|$$

$$\frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

3.18 <u>Theorem</u>. <u>If</u> (f, E) <u>and</u> (g, E) <u>are uniformly continuous</u> and bounded on E, then (1) (f. g, E) <u>is uniformly continuous and bounded on</u> E. <u>Further</u>, <u>if</u> (2) <u>for every</u> $x \in E$, $g(x) \neq 0$ <u>and</u> $\frac{1}{g}$ <u>is bounded on</u> E, <u>then</u> $(\frac{f}{g}, E)$ <u>is uniformly continuous and bounded on</u> E.

<u>Proof of</u> (1). If M_1, M_2 are bounds respectively of $|f(x)|$ and $|g(x)|$, then let $M = \max(M_1 M_2)$. Choose $\epsilon > 0$. There exists a $\delta_1 > 0$ such that if $x, x' \in E$ and $|x-x'| < \delta_1$, then $|f(x) - f(x')| < \frac{\epsilon}{2M}$. There exists a $\delta_2 > 0$ such that if $|x - x'| < \delta_2$, then $|g(x) - g(x')| < \frac{\epsilon}{2M}$. Let $\delta = \min(\delta_1 \delta_2)$.

Then for $x, x' \in E$ and $|x - x'| < \delta$,

$$\begin{aligned} |f(x)\, g(x) - f(x')\, g(x')| &= |f(x)\, g(x) - f(x)\, g(x') + f(x)\, g(x') - f(x')\, g(x')| \\ &\leq |f(x)\, g(x) - f(x)\, g(x')| + |f(x)\, g(x') - f(x')\, g(x')| \\ &\leq |f(x)|\; |g(x) - g(x')| + |g(x')|\; |f(x) - f(x')| \\ &< |f(x)|\; \frac{\epsilon}{2M} + |g(x')|\; \frac{\epsilon}{2M} \\ & \quad M\, \frac{\epsilon}{2M} + M\, \frac{\epsilon}{2M} = \epsilon . \end{aligned}$$

3.19 <u>Theorem</u>. <u>If</u> (f, E_1) <u>is uniformly continuous and if</u> E_1 E <u>is dense on</u> E, <u>then there exists a uniquely determined function</u> (F, E) <u>which</u> (1) <u>coincides with</u> (f, E_1) <u>on</u> E_1 <u>and</u> (2) <u>is uniformly continuous on</u> E.

<u>Proof</u>. For every contained in E, F( ) is defined to be the $\lim_{x} f(x)$ where x E_1. The uniqueness of (F, E) is assured by the uniqueness of the limit of a function at a point. To verify that this limit always exists and is unique, take any sequence x_n in E_1, converging to a point in E. $f(x_n)$ is a Cauchy sequence since f(x) is uniformly continuous on E_1. Any Cauchy sequence $f(x_n)$ is convergent and thus by Theorem 1.78, $\lim_{x} f(x)$ will exist. Therefore, if we define F( ) = $\lim_{x} f(x)$, the condition (1) is satisfied.

To prove the second part, namely, (F, E) is uniformly continuous on E, choose O. Then there exists a O such that whenever x - x' and x, x' are contained in E_1, then

f(x) - f(x') $\frac{}{3}$. Consider two points y and y' in E for which

y - y' $\frac{}{2}$. Since $\lim_{x\ \ y} f(x) = F(y)$ if x is restricted to be in E_1,

there exists an x E_1 with y - x $\frac{}{4}$ and F(y) - f(x) $\frac{}{3}$.

Likewise, there exists an $x' \in E_1$ with $|y' - x'| < \frac{\delta}{4}$ and $|F(y') - f(x')| < \frac{\epsilon}{3}$. Hence, $|x - x'| < |y - x| + |y - y'| + |y' - x'| < \delta$, and therefore

$$|F(y) - F(y')| \leq |F(y) - f(x)| + |f(x) - f(x')| + |F(y') - f(x')|$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

3.20 Corollary. If (f, E_1) is uniformly continuous on an everywhere dense set $E_1 \subset E$, then there exists a uniquely determined function (F, E) which (1) coincides with (f, E_1) on E_1 and (2) is continuous on E relative to E_1.

Proof. Since uniform continuity implies continuity, the proof follows immediately from Theorem 3.19.

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