

AN INVESTIGATION OF POINTS ABOUT
THE CIRCLE OF CONVERGENCE

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THESIS

Presented to the Graduate Council of the
North Texas State College in Partial
Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

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Denton, Texas

August, 1958

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CHAPTER I

INTRODUCTION

This paper will be concerned with the convergence of complex power series. Information concerning infinite and finite power series for real and complex numbers that is not listed below will be assumed. In the course of presenting these proofs several theorems and definitions will be used without proof.

Throughout this paper, the circle of convergence of a series will be denoted by $|z| = r'$; r' will denote the radius of convergence and z' will denote a point on the circle of convergence.

Definition 1.1. If to each positive integer, $n = 1, 2, 3, \dots$, there corresponds a definite number z_n , then the sum of these numbers $\sum_{n=0}^{\infty} z_n$ is said to be a series.

Definition 1.2. A series of the form $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$, where a_0, a_1, a_2, \dots are constants and z is a variable, is said to be a power series.

Definition 1.3. The power series $\sum_{n=0}^{\infty} a_n z^n$ converges to a number A at $z = z_0$ if, for $\epsilon > 0$ there is a positive integer N such that if $n > N$, $\left| \sum_{p=0}^{\infty} a_p z_0^p - A \right| < \epsilon$. A will be called the sum of the series.

Theorem 1.1. $\sum_{p=0}^{\infty} a_p z^p$ converges if and only if for each $\epsilon > 0$ there is an N such that if $m > n > N$, then

$$\left| \sum_{p=n}^m a_p z^p \right| < \epsilon.$$

Definition 1.4. The power series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n z^n|$ converges.

Theorem 1.2. If r is real and $-1 < r < 1$, then $\sum_{n=0}^{\infty} ar^n$ converges absolutely and $\sum_{n=0}^{\infty} |ar^n| = \frac{|a|}{1 - |r|}$.

Theorem 1.3. If $\sum_{p=0}^{\infty} a_p z^p$ converges then $\lim_{n \rightarrow \infty} |a_p z^p| = 0$.

Theorem 1.4. If $\sum_{n=0}^{\infty} a_n$ is a series of non negative terms which converge to A , and if $0 \leq b_n \leq a_n$ for every n , then $\sum_{n=0}^{\infty} b_n$ converges.

Theorem 1.5. If $\sum_{n=0}^{\infty} |a_n|$ converges then $\sum_{n=0}^{\infty} a_n$ converges.

Definition 1.5. The power series $\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent for a set M if, (1) $\sum_{n=0}^{\infty} a_n z^n$ converges for all z in M , (2) if $\epsilon > 0$, then there is a positive integer N such that if $n > N$, z is in M , and $\sum_{n=0}^{\infty} a_n z^n = A$ then $\left| \sum_{j=0}^n a_j z_0^j - A \right| < \epsilon$.

Definition 1.6. Any series that does not satisfy the conditions required for convergence is said to diverge.

Theorem 1.6. If $\sum_{n=0}^{\infty} b_n$ is a series of non negative terms which diverge and if $a_n \geq b_n$ for every n , then $\sum_{n=0}^{\infty} a_n$ diverges.

Definition 1.7. The circle of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is the greatest circle about the origin for which the power series will converge for any z interior to the circle.

Definition 1.8. The circle of convergence of the series $\sum_{n=1}^{\infty} \frac{b_n}{z^n}$ is the smallest circle about the origin for which the series $\sum_{n=1}^{\infty} \frac{b_n}{z^n}$ will converge for all points outside the circle.

Definition 1.9. The radius of convergence of the series $\sum_{p=0}^{\infty} a_p z^p$ or $\sum_{n=1}^{\infty} \frac{b_n}{z^n}$ is the radius of the circle of convergence.

Definition 1.10. If to each positive integer $n = 1, 2, 3, \dots$, there exists a definite number a_n , then these numbers are said to form a sequence.

Definition 1.11. The sequence $\{x_n\}$ converges to X if having chosen $\epsilon > 0$, there exists a positive integer N such that if $n > N$, $|x_n - X| < \epsilon$.

Theorem 1.7. If $0 \leq x < 1$, then the sequence x, x^2, x^3, \dots converges to zero.

Definition 1.12. A set M is bounded if and only if there is a real number k such that, for every element z in M , $|z| < k$.

Theorem 1.8. A convergent sequence is bounded. That is, if $\{a_n\} \rightarrow a$, there is a number k such that $|a_n| < k$ for all n .

Definition 1.13. A set M is not bounded if and only if for each real number k there is an element z in M such that $|z| > k$.

Definition 1.14. The set of points M is a neighborhood of z_0 if and only if there is a positive number ϵ such that z belongs to M if and only if $|z - z_0| < \epsilon$.

Definition 1.15. The point L is a limit point of the set M if every neighborhood of L contains at least one point of M other than L .

Theorem 1.9. If S is an infinite bounded set, then it has at least one limit point ξ .

Definition 1.16. The number k is an upper bound of the set of real numbers M if and only if, for every z in M , $z \leq k$.

Definition 1.17. The upper bound H is a least upper bound of the set L if, for every $\epsilon > 0$ and for every upper bound x of L , $x > H - \epsilon$.

Theorem 1.10. Every bounded set has a unique least upper bound.

Definition 1.18. If $\{s_n\}$ is a bounded sequence of real numbers, the limit superior, $\lim \sup s_n$, of $\{s_n\}$ is the largest limit of a subsequence of $\{s_n\}$.

Theorem 1.11. If $\{a_n\}$ is a bounded infinite sequence, then there is a subsequence of $\{a_n\}$ which has a sequential limit.

Theorem 1.12. If $\{a_n\} = a_1, a_2, a_3, \dots, a_n, \dots$ is a bounded sequence then $\lim \sup a_n$ exists.

Proof: Consider the set L consisting of points x if and only if x is the limit of a subsequence of $\{a_n\}$. There exists a number k such that every number in L is less than k . Let H be the least upper bound of L .

Case I. Suppose H is in L . H is a limit of a subsequence of $\{a_n\}$ since L consists only of limit points. H is also the largest limit of a subsequence of $\{a_n\}$ and is thus $\lim \sup a_n$.

Case II. Suppose H is not in L . Since H is the least upper bound of L , if $\epsilon > 0$, there is at least one point in L greater than $H - \epsilon$. Since H is not in L , and since H is the least upper bound of L , there are no points of L greater than H . Since for every $\epsilon > 0$, there exists a point of L contained in $(H - \epsilon, H)$, there is a point of L , say x_1 such that $|x_1 - H| < \frac{1}{2}$. Since x_1 is a limit of a subsequence of $\{a_n\}$, there is an integer n_1 such that $x_1 - \frac{1}{2} < a_{n_1} < x_1 + \frac{1}{2}$;

thus $|a_{n_1} - H| < 1$. There is an x_2 in L such that $|x_2 - H| < \frac{1}{4}$. Since x_2 is a limit of a subsequence of $\{a_n\}$, then there is an infinite number of elements in the interval $(x_2 - \frac{1}{4}, x_2 + \frac{1}{4})$ and there is an $n_2 > n_1$ such that $x_2 - \frac{1}{4} < a_{n_2} < x_2 + \frac{1}{4}$; thus $|a_{n_2} - H| < \frac{1}{2}$. There is an x_3 in L such that $|x_3 - H| < \frac{1}{6}$. Since x_3 is a limit of a subsequence of $\{a_n\}$, then there is an infinite number of elements in the interval $(x_3 - \frac{1}{6}, x_3 + \frac{1}{6})$ and there is an $n_3 > n_2$ such that $x_3 - \frac{1}{6} < a_{n_3} < x_3 + \frac{1}{6}$; thus $|a_{n_3} - H| < \frac{1}{3}$. In general there exists numbers n_1, n_2, n_3, \dots such that, if p is a positive integer greater than 1, there is an x_p in L such that $|x_p - H| < \frac{1}{2^p}$ and an integer $n_p > n_{p-1}$ such that $|a_{n_p} - x_p| < \frac{1}{2^p}$ and hence $|a_{n_p} - H| < \frac{1}{p}$. Hence H is a limit of a subsequence of $\{a_n\}$ and therefore belongs to L. H is the largest limit of a subsequence of $\{a_n\}$ and is thus the $\limsup a_n$. In either case, there is a $\limsup a_n = H \leq k$.

Definition 1.19. $f(z) = w$ is a single-valued function if, for every value of z, there is one and only one value for w.

Definition 1.20. A single-valued function $f(z)$ is continuous at a point z_0 if, for each positive number ϵ , there exists a number δ , such that $|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| < \delta$.

Theorem 1.13. (Cauchy-Hadamard): The power series $\sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence $r' = \frac{1}{\limsup \sqrt[n]{|a_n|}}$

if $\limsup \sqrt[n]{|a_n|} \neq 0$. If $\limsup \sqrt[n]{|a_n|}$ does not exist, then the power series diverges for any $z \neq 0$. If $\limsup \sqrt[n]{|a_n|} = 0$, then the power series converges for any z . The power series diverges for all z such that $|z| > r^*$.

Proof: Consider the sequence $|a_1|, \sqrt{|a_2|}, \sqrt[3]{|a_3|}, \dots$. Call this set of numbers M . Suppose M is not bounded, that is, if k is a number, then there is a g_0 such that $\sqrt[g_0]{|a_{g_0}|} > k$ and $\limsup \sqrt[n]{|a_n|}$ does not exist. Let $|z| = h \neq 0$. There is a g_1 such that $\sqrt[g_1]{|a_{g_1}|} > \frac{1}{h}$, $\sqrt[g_1]{|a_{g_1}|} \cdot h > 1$, $\sqrt[g_1]{|a_{g_1}|} \cdot |z| > 1$, or $\left[\sqrt[g_1]{|a_{g_1}|} \cdot |z| \right]^{g_1} = \left| a_{g_1} z^{g_1} \right| > 1$. There is a positive integer g_2 such that

$$\sqrt[g_2]{|a_{g_2}|} > |a_1| + \sqrt{|a_2|} + \sqrt[3]{|a_3|} + \sqrt[4]{|a_4|} + \dots + \sqrt[g_2]{|a_{g_2}|} + \frac{1}{h},$$

$g_2 > g_1$; thus $\left| a_{g_2} z^{g_2} \right| > 1$. There is a positive integer $g_3 > g_2$ such that

$$\sqrt[g_3]{|a_{g_3}|} > |a_1| + \sqrt{|a_2|} + \sqrt[3]{|a_3|} + \sqrt[4]{|a_4|} + \dots + \sqrt[g_3]{|a_{g_3}|} + \frac{1}{h}.$$

Thus $\left| a_{g_3} z^{g_3} \right| > 1$. For each positive integer n , there is a g_n such that $g_n \leq g_{n+1}$ and $\left| a_{g_n} z^{g_n} \right| > 1$. Hence $\lim_{n \rightarrow \infty} |a_n z^n| \neq 0$ and, by theorem 1.3, the series cannot converge for any $z \neq 0$.

Suppose M is bounded, that is, there is a k such that $\sqrt[n]{|a_n|} \leq k$ for every positive integer n . By theorem 1.12, if k is the least upper bound of M , $\limsup \sqrt[n]{|a_n|} = k \leq k$.

If $\limsup \sqrt[n]{|a_n|} = 0$, then let $|z| = r \neq 0$. There is at most a finite number of the terms of $\sqrt[n]{|a_n|} > \frac{1}{2r}$.

There is a positive integer N such that if $n > N$, then

$\sqrt[n]{|a_n|} < \frac{1}{2r}$ and $\sqrt[n]{|a_n|} \cdot r < \frac{1}{2}$. Hence $|a_n z^n| \leq \frac{1}{2^n}$. Since $\sum_{n=N}^{\infty} \frac{1}{2^n}$ converges by theorem 1.2, $\sum_{n=N}^{\infty} |a_n z^n|$ converges and $\sum_{n=0}^{\infty} |a_n z^n|$ converges.

If $R \neq 0$, let $r' = \frac{1}{\limsup n a_n}$ and let $0 < |z| = r < r'$.

Then $\frac{1}{r} > \frac{1}{r'}$ and $\limsup \sqrt[n]{|a_n|} = \frac{1}{r'} < \frac{1}{r}$. There are at most a finite number of terms of M greater than $\frac{r' + 1}{2}$. Therefore

there exists an N so that for $n > N$, $\sqrt[n]{|a_n|} \cdot |z| = \sqrt[n]{|a_n|} \cdot r \leq \frac{r' + 1}{2} \cdot r < \frac{1}{r} \cdot r = 1$. Since $r < r'$, $|a_n z^n| \leq \left(\frac{r' + 1}{2} \cdot r\right)^n = \left(\frac{r' + 1}{2}\right)^n < 1$. Thus the series $\sum_{n=0}^{\infty} \left(\frac{r' + 1}{2}\right)^n$ converges by theorem 1.2. Now $\sum_{n=0}^{\infty} |a_n z^n| \leq \sum_{n=0}^{\infty} \left(\frac{r' + 1}{2}\right)^n$ converges by theorem 1.4.

If $|z| = r > r'$, then $\frac{1}{r'} < \frac{1}{r}$, and $\limsup \sqrt[n]{|a_n|} = \frac{1}{r'} > \frac{1}{r}$. Since $\limsup \sqrt[n]{|a_n|} = \frac{1}{r'} > \frac{1}{r}$, $\frac{1}{r}$ is a limit point; thus there are an infinite number of terms of M greater than $\frac{1}{r}$. Therefore for an infinite number of integers n , $\sqrt[n]{|a_n|} \cdot |z| = \sqrt[n]{|a_n|} \cdot r > \frac{1}{r} \cdot r = 1$. Now $|a_n z^n| > 1$. Hence $\lim_{n \rightarrow \infty} |a_n z^n| \neq 0$ and by theorem 1.3 $\sum_{n=0}^{\infty} a_n z^n$ diverges for all z such that $|z| > r'$.

CHAPTER II

CONVERGENCE OF POWER SERIES WITH POSITIVE EXPONENTS

Theorem 2.1. If $\sum_{p=0}^{\infty} a_p z^p$ converges for all $|z| < h$ and $0 < g < h$, then $\sum_{p=0}^{\infty} |a_p z^p|$ converges uniformly for $|z| \leq g$.

To prove:

(1) $\sum_{p=0}^{\infty} |a_p z^p|$ converges.

(2) If $\epsilon > 0$ there is a positive integer N such that if $|z| \leq g$, and $m > n > N$, then $\sum_{p=n}^m |a_p z^p| < \epsilon$.

Proof: Since $\frac{g+h}{2} < h$, then $\sum_{p=0}^{\infty} a_p \left(\frac{g+h}{2}\right)^p$ converges and there is an M such that $|a_p \left(\frac{g+h}{2}\right)^p| < M$ for all p . Now $g < \frac{g+h}{2}$ and $0 < \frac{g}{g+h} = \frac{2g}{g+h} < 1$. Now the geometric series $\sum_{p=0}^{\infty} \left(\frac{2g}{g+h}\right)^p$ converges.

If $\epsilon > 0$, there is a positive integer N such that if $m > n > N$, then $\sum_{p=n}^m \left(\frac{2g}{g+h}\right)^p < \frac{\epsilon}{M}$.

$$\begin{aligned} \text{Now if } |z_0| < g, \text{ and } m > n > N, \text{ then } \sum_{p=n}^m |a_p z_0^p| \\ = \sum_{p=n}^m \left| \frac{2z_0}{g+h} \right|^p \cdot \left| a_p \left(\frac{g+h}{2}\right)^p \right| < \sum_{p=n}^m \left| \frac{2z_0}{g+h} \right|^p \cdot M = M \sum_{p=n}^m \left| \frac{2z_0}{g+h} \right|^p \\ < M \cdot \frac{\epsilon}{M} = \epsilon. \end{aligned}$$

Corollary 2.1. The power series $\sum_{p=0}^{\infty} a_p z^p$ is uniformly and absolutely convergent over any closed set of points S interior to its circle of convergence.

Proof: Since S is closed, let the shortest distance from any point in S to the circle of convergence be denoted by α . If r' is the radius of convergence, then all the points of S are within a circle of radius $r' - \frac{\alpha}{2}$. Now by theorem 2.1, where $h = r'$ and $g = r' - \frac{\alpha}{2}$, the proof is immediate.

Theorem 2.2. If the power series $\sum_{p=0}^{\infty} a_p z^p$ converges at a point $z' = r'(\cos p\theta + i \sin p\theta)$ on its circle of convergence, then $\sum_{p=0}^{\infty} a_p r'^p (\cos p\theta + i \sin p\theta)$ converges uniformly for $0 < r \leq r'$ and if the $\arg \alpha = 0$, and $|\alpha| < r'$ then $\lim_{\alpha \rightarrow z'} \sum_{p=0}^{\infty} a_p \alpha^p = \sum_{p=0}^{\infty} a_p z'^p$.

To prove:

(1) If $\epsilon > 0$, there is an N such that if $m > n > N$ and $0 < r \leq r'$, then $|\sum_{p=n}^m a_p r'^p (\cos p\theta + i \sin p\theta)| < \epsilon$.

(2) If $\epsilon > 0$, there is a $\delta > 0$ such that if $|\alpha - z'| < \delta$, $\arg \alpha = 0$, and $|\alpha| < r'$ then $|\sum_{p=0}^{\infty} a_p \alpha^p - \sum_{p=0}^{\infty} a_p z'^p| < \epsilon$.

Proof: Let $a'_p = a_p r'^p$. Now $\sum_{p=0}^{\infty} a'_p (\cos p\theta + i \sin p\theta) z'^p$ is a power series whose radius of convergence is 1. Let $d_p = a'_p (\cos p\theta + i \sin p\theta) = a_p r'^p (\cos p\theta + i \sin p\theta)$. Now $\sum_{p=0}^{\infty} d_p q^p$ converges for $0 < q \leq 1$. If $\epsilon > 0$, then since $\sum_{p=0}^{\infty} d_p$ converges, there is a positive integer N such that if $n > N$, then $|\sum_{p=n}^{n+p} d_p| < \epsilon$ for $p = 1, 2, 3, \dots$. Let $s_{p+1} = |\sum_{j=n}^{n+p} d_j| < \epsilon$ for $p = 0, 1, 2, 3, \dots$. If $m > n > N$ and $0 < q \leq 1$, then $|\sum_{p=n}^m d_p q^p|$
 $= |s_1 q^n + (s_2 - s_1) q^{n+1} + (s_3 - s_2) q^{n+2} + \dots + (s_{m-n} - s_{m-n-1}) q^{m-1} + (s_{m-n+1} - s_{m-n}) q^m|$

$$\begin{aligned}
&= |s_1(q^n - q^{n+1}) + s_2(q^{n+1} - q^{n+2}) + s_3(q^{n+2} - q^{n+3}) + \dots \\
&\quad + s_{m-n}(q^{m-1} - q^m) + s_{m-n+1}q^m| \\
&\leq |s_1|(q^n - q^{n+1}) + |s_2|(q^{n+1} - q^{n+2}) + |s_3|(q^{n+2} - q^{n+3}) + \dots \\
&\quad + |s_{m-n}|(q^{m-1} - q^m) + |s_{m-n+1}|q^m \\
&\leq \epsilon [q^n - q^{n+1} + q^{n+1} - q^{n+2} + q^{n+2} - \dots + q^{m-n} - q^m + q^m] \\
&= \epsilon q^n \leq \epsilon.
\end{aligned}$$

To prove that $\lim_{r \rightarrow 1^-} \sum_{p=0}^{\infty} a_p r^p (\cos p\theta + i \sin p\theta)$
 $= \sum_{p=0}^{\infty} r^p a_p (\cos p\theta + i \sin p\theta)$, let $\epsilon > 0$. If $0 < r \leq r'$,
then let $q = \frac{r}{r'}$. Now $0 < q \leq 1$. Since $\sum_{p=0}^{\infty} d_p q^p$ converges
uniformly for $0 < q \leq 1$, then there is a positive integer N
such that

$$\begin{aligned}
&\left| \sum_{p=0}^N a_p r^p (\cos p\theta + i \sin p\theta) - \sum_{p=0}^{\infty} a_p r^p (\cos p\theta + i \sin p\theta) \right| \\
&= \left| \sum_{p=N+1}^{\infty} a_p r^p - \sum_{p=0}^{\infty} d_p q^p \right| = \left| \sum_{p=N+1}^{\infty} a_p r^p \right| < \frac{\epsilon}{3} \text{ for all } 0 < r \leq r'.
\end{aligned}$$

$$\text{If } m > n > N \text{ then } \left| \sum_{p=n}^m a_p r^p (\cos p\theta + i \sin p\theta) \right| = \left| \sum_{p=n}^m d_p q^p \right| < \frac{\epsilon}{3}$$

If $\sum_{p=0}^N |a_p| pr^{p+1} = 0$, let $\delta = r'$. Otherwise let δ
be the minimum of r' and $\frac{\epsilon}{3 \sum_{p=0}^N |a_p| pr^{p+1}}$.

Hence if $|\alpha - z'| < \delta$, $\arg \alpha = 0$ and $|\alpha| = r < r'$, then

$$\begin{aligned}
0 < r' - r < \delta. \text{ Now } &\left| \sum_{p=0}^{\infty} a_p \alpha^p - \sum_{p=0}^{\infty} a_p z'^p \right| = \\
&\left| \sum_{p=0}^{\infty} a_p \alpha^p - \sum_{p=0}^N a_p \alpha^p + \sum_{p=0}^N a_p \alpha^p - \sum_{p=0}^N a_p z'^p + \sum_{p=0}^N a_p z'^p \right. \\
&\quad \left. - \sum_{p=0}^{\infty} a_p z'^p \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \left| \sum_{p=0}^{\infty} a_p z^p - \sum_{p=0}^N a_p z^p \right| + \left| \sum_{p=0}^N a_p z^p - \sum_{p=0}^N a_p z^{*p} \right| \\
&\quad + \left| \sum_{p=0}^N a_p z^{*p} - \sum_{p=0}^{\infty} a_p z^{*p} \right| \\
&< \frac{2\epsilon}{3} + \left| \sum_{p=0}^N a_p r^p (\cos p\theta + i \sin p\theta) \right. \\
&\quad \left. - \sum_{p=0}^N r^{*p} a_p (\cos p\theta + i \sin p\theta) \right| \\
&= \frac{2\epsilon}{3} + \left| \sum_{p=0}^N a_p (r^p - r^{*p}) (\cos p\theta + i \sin p\theta) \right| \leq \frac{2\epsilon}{3} \\
&+ \sum_{p=0}^N |a_p| (r^{*p} - r^p) = \frac{2\epsilon}{3} \\
&+ \sum_{p=0}^N |a_p| (r^{*p} - r^p) (r^{*p-1} + r^{*p-2} + r^2 r^{*p-3} + \dots + r^{p-1}) \\
&\leq |r^{*p} - r^p| \sum_{p=0}^N |a_p| pr^{*p-1}. \text{ If } \sum_{p=0}^N |a_p| pr^{*p-1} = 0 \text{ then } \frac{2\epsilon}{3} \\
&+ |r^{*p} - r^p| \sum_{p=0}^N |a_p| pr^{*p-1} = \frac{2\epsilon}{3} < \epsilon. \text{ Otherwise } \frac{2\epsilon}{3} \\
&+ 8 \sum_{p=0}^N |a_p| pr^{*p-1} < \frac{2\epsilon}{3} + \frac{\epsilon}{3 \sum_{p=0}^N |a_p| pr^{*p-1}} \left(\sum_{p=0}^N |a_p| pr^{*p-1} \right) = \epsilon.
\end{aligned}$$

Theorem 2.3. If $0 < \alpha < \frac{\pi}{2}$, $\phi = \frac{\alpha + \frac{\pi}{2}}{2}$, $a = \sin \alpha + i \cos \alpha$

then there is a w such that $w + \bar{w} > 2 \sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha}$,

$\bar{aw} + \bar{aw} - 2 \sin \alpha \leq 0$, and $aw + \bar{aw} - 2 \sin \alpha \leq 0$. Furthermore,

if w satisfies the above inequalities, and $w \neq 1$ then

$$\frac{|1-w|}{|1-w|} < \sec \phi.$$

Proof: The region $w + \bar{w} > 2 \sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha}$

is a half plane whose boundary is perpendicular to the real axis. The distance from the origin to this boundary is

$$\sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha}. \text{ Since } 0 < \alpha < \phi < \frac{\pi}{2}, \text{ then}$$

$$\sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi}} = \sqrt[3]{\frac{\cos^2 \alpha \cos^4 \alpha \tan^4 \alpha}{\tan^4 \phi}} = \cos^2 \alpha \sqrt[3]{\frac{\tan^4 \alpha}{\tan^4 \phi}}$$

$< \cos^2 \alpha$. Therefore $\sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha}$

$< \sqrt{\cos^2 \alpha + \sin^2 \alpha} = 1$ and the boundary of

$w + \bar{w} > 2 \sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha}$ intersects the real axis between zero and one.

Since $\sin \alpha < 1$ the distance from the origin to the boundary of the half plane $\bar{w} + w - 2 \sin \alpha \leq 0$ is less than 1 and the boundary intersects the unit circle at two places. Since $a(1) + \bar{a}(1) - 2 \sin \alpha = 2R(a) - 2 \sin \alpha = 2 \sin \alpha - 2 \sin \alpha = 0$, the boundary intersects the circle at 1.

If w is real and $\sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha} < w \leq 1$, then $w + \bar{w} = 2R(w) > 2 \sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha}$ also

$$\bar{w} + w - 2 \sin \alpha = w(a + \bar{a}) - 2 \sin \alpha = 2w \sin \alpha - 2 \sin \alpha \leq 2 \sin \alpha - 2 \sin \alpha = 0. \text{ Also } aw + \bar{aw} - 2 \sin \alpha = w(a + \bar{a}) - 2 \sin \alpha = 2w \sin \alpha - 2 \sin \alpha \leq 2 \sin \alpha - 2 \sin \alpha = 0.$$

These conditions give an isosceles triangle such that the boundaries are described by the three lines

$w + \bar{w} = 2 \sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha}$, $\bar{w} + w - 2 \sin \alpha = 0$ and $aw + \bar{aw} - 2 \sin \alpha = 0$. This triangle lies in the unit circle with one vertex at 1 and the opposite side perpendicular to the real axis.

If w satisfies the conditions of the triangle, let

$$|z_1| = \cos \alpha \sqrt{|w|^2 - \sin^2 \alpha} + \sin^2 \alpha + i(\cos \alpha \sin \alpha - \sin \alpha \sqrt{|w|^2 - \sin^2 \alpha}), \text{ Now } |z_1| = |w| \text{ and } \bar{a}z_1 + az_1 - 2 \sin \alpha = 0.$$

Let $z_2 = |w| + i(1 - |w|) \tan \phi$.

Consider the function

$$f(|w|) = 2 \left\{ \sqrt{|w|^2 - \sin^2 \alpha} \cos \alpha + \sin^2 \alpha - |w| \right\} + (1 - |w|)^2 \tan^2 \phi,$$

$$f(1) = 0,$$

$$f'(|w|) = 2 \left\{ \frac{|w| \cos \alpha}{\sqrt{|w|^2 - \sin^2 \alpha}} - 1 \right\} - 2(1 - |w|) \tan^2 \phi,$$

$$f'(1) = 0,$$

$$f''(|w|) = - \frac{\frac{2 \cos \alpha \sin^2 \alpha}{(|w|^2 - \sin^2 \alpha)^{3/2}} + 2 \tan^2 \phi}{},$$

$$f''(1) = - 2 \tan^2 \alpha + 2 \tan^2 \phi > 0.$$

Furthermore since $\sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha} < |w| \leq 1$ then

$$|w|^2 > \sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha},$$

$$|w|^2 - \sin^2 \alpha > \sqrt[3]{\frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi}},$$

$$(|w|^2 - \sin^2 \alpha)^3 > \frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi},$$

$$(|w|^2 - \sin^2 \alpha)^{3/2} > \frac{\cos \alpha \sin^2 \alpha}{\tan^2 \phi},$$

$$\tan^2 \phi > \frac{\cos \alpha \sin^2 \alpha}{(|w|^2 - \sin^2 \alpha)^{3/2}},$$

$$\text{and } f''(|w|) = 2 \left\{ \tan^2 \phi - \frac{\cos \alpha \sin^2 \alpha}{(|w|^2 - \sin^2 \alpha)^{3/2}} \right\} > 0,$$

Hence $f(|w|) \geq 0$ for $\sqrt{3 \frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha} < |w| \leq 1$ and

$$f(|w|) = 2 \left\{ \sqrt{|w|^2 - \sin^2 \alpha} \cos \alpha + \sin^2 \alpha - |w| \right\} + (1 - |w|)^2 \tan^2 \phi$$

$$= -2|w| + 2|w| \left\{ \cos \alpha \sqrt{1 - \frac{\sin^2 \alpha}{|w|^2}} + \frac{\sin^2 \alpha}{|w|} \right\} + (1 - |w|)^2 \tan^2 \phi$$

$$= 1 - 2|w| + |w|^2 - |w|^2 + 2|w| \left[\cos \alpha \sqrt{1 - \frac{\sin^2 \alpha}{|w|^2}} + \frac{\sin^2 \alpha}{|w|} \right]$$

$$-1 + (1 - |w|)^2 \tan^2 \phi$$

$$= (1 - |w|)^2 + (1 - |w|)^2 \tan^2 \phi - \left\{ 1 - 2|w| \left[\cos \alpha \sqrt{1 - \frac{\sin^2 \alpha}{|w|^2}} + \frac{\sin^2 \alpha}{|w|} \right] + |w|^2 \right\}$$

$$= |(1 - |w|) + (1 - |w|) \tan \phi|^2 - \left\{ 1 - 2|z_1| \left[\cos \alpha \sqrt{1 - \frac{\sin^2 \alpha}{|z_1|^2}} + \frac{\sin^2 \alpha}{|z_1|} \right] + |z_1|^2 \right\}$$

$$= |1 - z_2|^2 - \left\{ 1 - 2R(z_1) + |z_1|^2 \right\} = |1 - z_2|^2 - |1 - z_1|^2 > 0.$$

Hence $|1 - z_2| > |1 - z_1|$.

Consider $R(w) = \cos \alpha \sqrt{|w|^2 - \sin^2 \alpha} - \sin^2 \alpha$.

Case I. If $\ell(w) \geq 0$ then $\bar{w}w + \bar{w}w - 2 \sin \alpha \leq 0$.

$$2R(\bar{w}w) \leq 2 \sin \alpha,$$

$$R(\bar{w}w) \leq \sin \alpha,$$

$$\sin \alpha R(w) + \cos \alpha \ell(w) \leq \sin \alpha,$$

$$0 = \ell(w) = \frac{\sin \alpha}{\cos \alpha} [1 - R(w)],$$

$$[\ell(w)]^2 = \frac{\sin^2 \alpha}{\cos^2 \alpha} [1 - R(w)]^2.$$

Case II. If $\ell(w) < 0$ then $aw + \overline{aw} - 2 \sin \alpha \leq 0$.

$$2 R(aw) \leq 2 \sin \alpha,$$

$$R(aw) \leq \sin \alpha,$$

$$\sin \alpha R(w) - \cos \alpha \ell(w) \leq \sin \alpha,$$

$$-\cos \alpha \ell(w) \leq \sin \alpha [1 - R(w)],$$

$$0 < -\ell(w) \leq \frac{\sin \alpha}{\cos \alpha} [1 - R(w)],$$

$$[-\ell(w)]^2 = [-\ell(w)]^2 \leq \frac{\sin^2 \alpha}{\cos^2 \alpha} [1 - R(w)]^2.$$

$$\text{In either case, } [\ell(w)]^2 \leq \frac{\sin^2 \alpha}{\cos^2 \alpha} [1 - R(w)]^2.$$

$$\begin{aligned} & \text{Now } R(w) = \cos \alpha \sqrt{|w|^2 - \sin^2 \alpha} = \sin^2 \alpha \\ &= R(w) = \cos \alpha \sqrt{[R(w)]^2 + [\ell(w)]^2 - \sin^2 \alpha} = \sin^2 \alpha \\ &\geq R(w) = \cos \alpha \sqrt{[R(w)]^2 + \frac{\sin^2 \alpha}{\cos^2 \alpha} [1 - R(w)]^2 - \sin^2 \alpha} = \sin^2 \alpha \\ &= R(w) = \sqrt{\cos^2 \alpha \left\{ [R(w)]^2 + \frac{\sin^2 \alpha}{\cos^2 \alpha} [1 - R(w)]^2 - \sin^2 \alpha \right.} \\ &\quad \left. - \sin^2 \alpha \right\}} \\ &= R(w) = \sqrt{\cos^2 \alpha [R(w)]^2 + \sin^2 \alpha [1 - R(w)]^2 - \sin^2 \alpha \cos^2 \alpha} \\ &\quad - \sin^2 \alpha \\ &= R(w) = \sqrt{\cos^2 \alpha [R(w)]^2 + \sin^2 \alpha \left\{ 1 - 2R(w) + [R(w)]^2 - \sin^2 \alpha \cos^2 \alpha \right.} \\ &\quad \left. - \sin^2 \alpha \right\}} \\ &= R(w) = \sqrt{\cos^2 \alpha [R(w)]^2 + \sin^2 \alpha - 2 \sin^2 \alpha R(w) + \sin^2 \alpha [R(w)]^2 - \sin^2 \alpha \cos^2 \alpha} \\ &\quad - \sin^2 \alpha \\ &= R(w) = \sqrt{\sin^2 \alpha} \\ &= \sqrt{(\cos^2 \alpha + \sin^2 \alpha) [R(w)]^2 + \sin^2 \alpha - 2R(w)\sin^2 \alpha - \sin^2 \alpha(1 - \sin^2 \alpha)} \\ &= R(w) = \sqrt{[R(w)]^2 - 2R(w)\sin^2 \alpha + \sin^2 \alpha - \sin^2 \alpha + \sin^4 \alpha} \\ &\quad - \sin^2 \alpha \end{aligned}$$

$$\begin{aligned}
 R(w) &= \sqrt{[R(w)]^2 - 2R(w)\sin^2\alpha + \sin^4\alpha} - \sin^2\alpha \\
 &= R(w) - \sqrt{[R(w) - \sin^2\alpha]^2} - \sin^2\alpha \\
 &= R(w) - R(w) + \sin^2\alpha - \sin^2\alpha = 0.
 \end{aligned}$$

Therefore $R(w) = \cos\alpha\sqrt{|w|^2 - \sin^2\alpha} - \sin^2\alpha \geq 0$. Since

$$R(w) = R(z_1) = R(w) - \cos\alpha\sqrt{|w|^2 - \sin^2\alpha} - \sin^2\alpha \geq 0,$$

then $2R(w) - 2R(z_1) \geq 0$. But $2R(w) - 2R(z_1)$

$$= -1 + 2R(w) = |w|^2 + 1 - 2R(z_1) + |z_1|^2$$

$$= -(1 - 2R(w) + w^2) + 1 - 2R(z_1) + |z_1|^2$$

$$= -|1-w|^2 + |1-z_1|^2 \geq 0. \text{ Hence } |1-w|^2 \leq |1-z_1|^2$$

$$< |1-z_2|^2,$$

$$\frac{|1-z_2|}{1-|w|} > \frac{|1-z_1|}{1-|w|} \geq \frac{|1-w|}{1-|w|}, \text{ and}$$

$$\sec\phi = \frac{|1-z_2|}{1-|w|} > \frac{|1-w|}{1-|w|}.$$

Theorem 2.4. If $0 < \alpha < \frac{\pi}{2}$, $a = \sin\alpha + i\cos\alpha$, and if the series $\sum_{p=0}^{\infty} a_p z^p$ is convergent at the point $z = 1$ of its circle of convergence, then it is uniformly convergent in the triangle bounded by the lines $\overline{zw} + \overline{aw} - 2\sin\alpha = 0$,

$$aw + \overline{zw} - 2\sin\alpha = 0, \text{ and } w + \overline{w} > 2\sqrt{3 \frac{\cos^2\alpha \sin^4\alpha}{\tan^4\phi} + \sin^2\alpha}.$$

Proof: Let $\phi = \frac{\alpha + \frac{\pi}{2}}{2}$. Since the series $\sum_{p=0}^{\infty} a_p z^p$ converges at $z = 1$, $\sum_{p=0}^{\infty} a_p$ converges, that is, for every $\epsilon > 0$, there is a positive integer N such that if $m > n > N$,

$$\left| \sum_{p=0}^{\infty} a_p \right| < \frac{\epsilon}{1+\sec\phi}. \text{ Let } s_{p+1} = \left| \sum_{j=n}^{n+p} a_j \right|. p = 0, 1, 2, \dots.$$

Now $s_{p+1} < \frac{\epsilon}{1+\sec\phi}$ for $p = 0, 1, 2, 3, \dots$. If z is

any point in the triangle, $z \neq 1$, $\sum_{p=n}^{\infty} a_p z^p$
 $= s_1 z^n + (s_2 - s_1) z^{n+1} + (s_3 - s_2) z^{n+2} + \dots$
 $\quad \quad \quad + (s_{m-n} - s_{m-n-1}) z^{m-1} + (s_{m-n+1} - s_{m-n}) z^m$
 $= s_1 z^n + s_2 z^{n+1} - s_1 z^{n+1} + s_3 z^{n+2} + \dots$
 $\quad \quad \quad + s_{m-n} z^{m-1} - s_{m-n-1} z^{m-1} + s_{m-n+1} z^m - s_{m-n} z^m$
 $= s_1 (1 - z) z^n + s_2 (1 - z) z^{n+1} + s_3 (1 - z) z^{n+2} + \dots$
 $\quad \quad \quad + s_{m-n-1} (1 - z) z^{m-2} + s_{m-n} (1 - z) z^{m-1} + s_{m-n+1} z^m.$

Now $\left| \sum_{p=n}^{\infty} a_p z^p \right| \leq |1 - z| |z^n| \left| \sum_{p=1}^{m-n} s_p z^{p-1} \right| + |s_{m-n+1} z^m|$
 $< |1 - z| |z^n| \frac{\epsilon}{1 + \sec \phi} \left| \sum_{p=1}^{m-n} z^{p-1} \right| + |s_{m-n+1}|$
 $< |1 - z| |z^n| \frac{\epsilon}{1 + \sec \phi} \left[\frac{1 - |z|^{m-n}}{1 - |z|} \right] + \frac{\epsilon}{1 + \sec \phi}$
 $< \frac{|1 - z|}{1 - |z|} \frac{\epsilon}{1 + \sec \phi} + \frac{\epsilon}{1 + \sec \phi}. \text{ By theorem 2.3,}$

$$\frac{|1 - z|}{1 - |z|} < \sec \phi. \text{ Now } \frac{|1 - z|}{1 - |z|} \frac{\epsilon}{1 + \sec \phi} + \frac{\epsilon}{1 + \sec \phi}$$

 $< \sec \phi \frac{\epsilon}{1 + \sec \phi} + \frac{\epsilon}{1 + \sec \phi} \text{ and}$

$\left| \sum_{p=n}^{\infty} a_p z^p \right| < \frac{\epsilon}{1 + \sec \phi} (1 + \sec \phi) = \epsilon. \text{ Hence } \sum_{p=0}^{\infty} a_p z^p$ is uniformly convergent for all z within the triangle. If

$$z = 1 \text{ then } \left| \sum_{p=n}^{\infty} a_p z^p \right| = \left| \sum_{p=n}^{\infty} a_p \right| < \frac{\epsilon}{1 + \sec \phi}.$$

Theorem 2.5. If $\sum_{p=0}^{\infty} a_p z^p$ converges at the point $z = 1$ on its circle of convergence, $0 < \alpha < \frac{\pi}{2}$, and $\sin \alpha + i \cos \alpha = \sin \alpha + i \cos \alpha$, then $\lim_{\sigma \rightarrow 1} \sum_{p=0}^{\infty} a_p z^p = \sum_{p=0}^{\infty} a_p$ where σ is any point on a curve in the triangle bounded by the lines $aw + bw - 2 \sin \alpha = 0$, $aw + bw - 2 \sin \alpha = 0$ and

$$w + \bar{w} > 2 \sqrt{3 \frac{\cos^2 \alpha \sin^4 \alpha}{\tan^4 \phi} + \sin^2 \alpha}.$$

Proof: By theorem 2.3, $\sum_{p=0}^{\infty} a_p z^p$ is uniformly convergent at all points in the triangle. Since σ is restricted to be in the triangle, $\sum_{p=0}^{\infty} a_p z^p$ is uniformly convergent over any value σ may take on.

If $\epsilon > 0$, then since $\sum_{p=0}^{\infty} a_p z^p$ is uniformly convergent, there is an N such that $\left| \sum_{p=0}^N a_p z^p - \sum_{p=0}^{\infty} a_p z^p \right| < \frac{\epsilon}{3}$ for z on the curve.

Case I. If $\epsilon \geq 3N \sum_{p=0}^N |a_p|$, let $\delta = 1$.

Case II. If $\epsilon < 3N \sum_{p=0}^N |a_p|$, then $1 > \frac{\epsilon}{3N \sum_{p=0}^N |a_p|}$. Let

$$\delta = \frac{\epsilon}{3N \sum_{p=0}^N |a_p|}. \quad \text{In either case } 0 < \delta \leq 1.$$

Hence if $|\sigma - 1| < \delta$, $|\sigma| < 1$, and σ on the curve, then

$$\begin{aligned} 0 < 1 - |\sigma| < \delta. \quad &\text{Now } \left| \sum_{p=0}^{\infty} a_p \sigma^p - \sum_{p=0}^{\infty} a_p \right| \\ &= \left| \sum_{p=0}^{\infty} a_p \sigma^p - \sum_{p=0}^N a_p \sigma^p + \sum_{p=0}^N a_p \sigma^p - \sum_{p=0}^N a_p + \sum_{p=0}^N a_p - \sum_{p=0}^{\infty} a_p \right| \\ &\leq \left| \sum_{p=0}^{\infty} a_p \sigma^p - \sum_{p=0}^N a_p \sigma^p \right| + \left| \sum_{p=0}^N a_p \sigma^p - \sum_{p=0}^N a_p \right| + \left| \sum_{p=0}^N a_p - \sum_{p=0}^{\infty} a_p \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \left| \sum_{p=0}^N a_p (\sigma^p - 1) \right| \\ &= \frac{2\epsilon}{3} + \left| \sum_{p=0}^N a_p (\sigma - 1)(\sigma^{p-1} + \sigma^{p-2} + \sigma^{p-3} + \dots + 1) \right| \\ &\leq \frac{2\epsilon}{3} + \sum_{p=0}^N |a_p| |\sigma - 1| (|\sigma^{p-1}| + |\sigma^{p-2}| + |\sigma^{p-3}| + \dots + 1) \\ &\leq \frac{2\epsilon}{3} + |\sigma - 1| \sum_{p=0}^N |a_p| N < \frac{2\epsilon}{3} + 8N \sum_{p=0}^N |a_p|. \quad \text{In case I,} \\ \frac{2\epsilon}{3} + 8N \sum_{p=0}^N |a_p| &= \frac{2\epsilon}{3} + N \sum_{p=0}^N |a_p| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \quad \text{In case II,} \\ \frac{2\epsilon}{3} + 8N \sum_{p=0}^N |a_p| &= \frac{2\epsilon}{3} + \frac{\epsilon}{3N \sum_{p=0}^N |a_p|} N \sum_{p=0}^N |a_p| = \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

CHAPTER III

CONVERGENCE OF POWER SERIES WITH NEGATIVE EXPONENTS

Theorem 3.1. If the series $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ converges for all $|z| > h$ and $0 < h < g$, then $\sum_{p=1}^{\infty} \left| \frac{b_p}{z^p} \right|$ converges uniformly for $|z| \geq g$. That is:

$$(1) \sum_{p=1}^{\infty} \left| \frac{b_p}{z^p} \right| \text{ converges.}$$

(2) If $\epsilon > 0$ there is an N such that if $|z| \geq g$, and $m > n > N$, then $\left| \sum_{p=n}^m \frac{b_p}{z^p} \right| < \epsilon$.

Proof: Since $\frac{g+h}{2} > h$, then $\sum_{p=1}^{\infty} \frac{b_p}{(\frac{g+h}{2})^p}$ converges and there is an M such that $\left| \frac{b_p}{(\frac{g+h}{2})^p} \right| < M$ for all p . Now

$g > \frac{g+h}{2}$ and $\frac{g}{\frac{g+h}{2}} > 1$. Thus $\frac{g+h}{2g} < 1$ and the geometric series

$$\sum_{p=1}^{\infty} \left(\frac{g+h}{2g} \right)^p \text{ converges.}$$

If $\epsilon > 0$, there is a positive integer N such that if

$$m > n > N, \text{ then } \sum_{p=n}^m \left(\frac{g+h}{2g} \right)^p < \frac{\epsilon}{M}$$

Now if $|z_0| \geq g$ and $m > n > N$, then $\sum_{p=n}^m \left| \frac{b_p}{z_0^p} \right|$

$$\begin{aligned}
 &= \sum_{p=n}^m \left| \frac{g+h}{2} \right|^p \left| \frac{b_p}{(\frac{g+h}{2})^p} \right| = \sum_{p=n}^m \left| \frac{g+h}{2z_0} \right|^p \left| b_p \left(\frac{2}{g+h} \right)^p \right| \\
 &< \sum_{p=n}^m \left| \frac{g+h}{2z_0} \right|^p M = M \sum_{p=n}^m \left| \frac{g+h}{2z_0} \right|^p \leq M \sum_{p=n}^m \left| \frac{g+h}{2g} \right|^p < M \frac{\epsilon}{M} = \epsilon.
 \end{aligned}$$

Theorem 3.2. If the series $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ converges at a point

$z' = r'(\cos \theta + i \sin \theta)$ on its circle of convergence, then

$\sum_{p=1}^{\infty} \frac{b_p}{r^p(\cos p\theta + i \sin p\theta)}$ converges uniformly for $r \geq r'$

and if $\arg \alpha = \theta$, and $|\alpha| > r'$ then $\lim_{\alpha \rightarrow z'} \sum_{p=1}^{\infty} \frac{b_p}{\alpha^p} = \sum_{p=1}^{\infty} \frac{b_p}{z'^p}$.

To prove:

(1) If $\epsilon > 0$, there is an N such that if $m > n > N$ and $r \geq r'$, then $\left| \sum_{p=n}^m \frac{b_p}{r^p(\cos p\theta + i \sin p\theta)} \right| < \epsilon$.

(2) If $\epsilon > 0$, there is a $S > 0$ such that if $|\alpha - z'| < S$, $\arg \alpha = \theta$, and $|\alpha| > r'$ then $\left| \sum_{p=1}^{\infty} \frac{b_p}{\alpha^p} - \sum_{p=1}^{\infty} \frac{b_p}{z'^p} \right| < \epsilon$.

Proof: Let $b'_p = \frac{b_p}{r'^p}$. Now $\sum_{p=1}^{\infty} \frac{b'_p}{z'^p(\cos p\theta + i \sin p\theta)}$

is a series whose radius of convergence is 1. Let

$c_p = \frac{b'_p}{\cos p\theta + i \sin p\theta} = \frac{b_p}{r'^p(\cos p\theta + i \sin p\theta)}$. Now $\sum_{p=1}^{\infty} \frac{c_p}{t^p}$

converges for $t \geq 1$. If $\epsilon > 0$, then since $\sum_{p=1}^{\infty} c_p$ converges, there is a positive integer N such that if $n > N$,

then $\left| \sum_{j=n}^{n+p} c_j \right| < \epsilon$ for $p = 1, 2, 3, \dots$. Let

$s_{p+1} = \left| \sum_{j=n}^{n+p} c_j \right| < \epsilon$ for $p = 0, 1, 2, 3, \dots$. If $m > n > N$

and $t \geq 1$, then $\left| \sum_{p=n}^m \frac{b_p}{r'^p(\cos p\theta + i \sin p\theta)} \right| = \left| \sum_{p=n}^m \frac{c_p}{t^p} \right|$

$$\begin{aligned}
&= \left| \frac{s_1}{t^n} + \frac{(s_2 - s_1)}{t^{n+1}} + \frac{(s_3 - s_2)}{t^{n+2}} + \dots \right. \\
&\quad \left. + \frac{(s_{m-n} - s_{m-n-1})}{t^{m-1}} + \frac{(s_{m-n+1} - s_{m-n})}{t^m} \right| \\
&= \left| \frac{s_1(t^{n+1} - t^n)}{t^n t^{n+1}} + \frac{s_2(t^{n+2} - t^{n+1})}{t^{n+1} t^{n+2}} + \frac{s_3(t^{n+3} - t^{n+2})}{t^{n+2} t^{n+3}} + \dots \right. \\
&\quad \left. + \frac{s_{m-n}(t^m - t^{m-1})}{t^{m-1} t^m} + \frac{s_{m-n+1}}{t^m} \right| \\
&\leq |s_1| \left| \frac{(t^{n+1} - t^n)}{t^n t^{n+1}} \right| + |s_2| \left| \frac{(t^{n+2} - t^{n+1})}{t^{n+1} t^{n+2}} \right| + |s_3| \left| \frac{(t^{n+3} - t^{n+2})}{t^{n+2} t^{n+3}} \right| + \dots \\
&\quad + |s_{m-n}| \left| \frac{(t^m - t^{m-1})}{t^{m-1} t^m} \right| + |s_{m-n+1}| \frac{1}{t^m} \\
&< \epsilon \left[\frac{t^{n+1} - t^n}{t^n t^{n+1}} + \frac{t^{n+2} - t^{n+1}}{t^{n+1} t^{n+2}} + \frac{t^{n+3} - t^{n+2}}{t^{n+2} t^{n+3}} + \dots \right. \\
&\quad \left. + \frac{t^m - t^{m-1}}{t^{m-1} t^m} + \frac{1}{t^m} \right] \\
&= \epsilon \left[\frac{1}{t^n} - \frac{1}{t^{n+1}} + \frac{1}{t^{n+1}} - \frac{1}{t^{n+2}} + \frac{1}{t^{n+2}} - \frac{1}{t^{n+3}} + \dots \right. \\
&\quad \left. + \frac{1}{t^{m-1}} - \frac{1}{t^m} + \frac{1}{t^m} \right] \\
&= \frac{\epsilon}{t^n} \leq \epsilon
\end{aligned}$$

To prove that $\lim_{r \rightarrow \infty} \sum_{p=1}^{\infty} \frac{b_p}{r^p(\cos p\theta + i \sin p\theta)}$

$= \sum_{p=1}^{\infty} \frac{b_p}{r^p(\cos p\theta + i \sin p\theta)}$, let $\epsilon > 0$. If $r > r'$, then let $t = \frac{r}{r'}$. Now $t \geq 1$. Since $\sum_{p=1}^{\infty} \frac{c_p}{t^p}$ converges uniformly for $t \geq 1$, then there is a positive integer N such that $\left| \sum_{p=1}^N \frac{b_p}{r^p(\cos p\theta + i \sin p\theta)} - \sum_{p=1}^{\infty} \frac{b_p}{r^p(\cos p\theta + i \sin p\theta)} \right|$

$\bullet \left| \sum_{p=1}^N \frac{c_p}{r^p} - \sum_{p=1}^{\infty} \frac{c_p}{r^p} \right| = \left| \sum_{p=N+1}^{\infty} \frac{c_p}{r^p} \right| < \frac{\epsilon}{3}$ for all $r \geq 1$. If $n > N$ then $\left| \sum_{p=n}^{\infty} \frac{b_p}{r^p(\cos p\theta + i \sin p\theta)} \right| = \left| \sum_{p=n}^{\infty} \frac{b_p}{r^p} \right| < \frac{\epsilon}{3}$. If $\left| \sum_{p=1}^N b_p \right| (p-1) \frac{1}{r^{p+1}} = 0$, let $s = r'$. Otherwise let s be the minimum of r' and $\frac{\epsilon}{3 \sum_{p=1}^N |b_p| (p-1) \frac{1}{r^{p+1}}}$. Hence if $|\alpha - z| < s$, $\arg \alpha = 0$, and $|z| = r > r'$, then

$$\begin{aligned}
 0 < r - r' < s. \quad & \text{Now } \left| \sum_{p=1}^{\infty} \frac{b_p}{\alpha^p} - \sum_{p=1}^{\infty} \frac{b_p}{z^p} \right| \\
 &= \left| \sum_{p=1}^{\infty} \frac{b_p}{\alpha^p} - \sum_{p=1}^N \frac{b_p}{\alpha^p} + \sum_{p=1}^N \frac{b_p}{\alpha^p} - \sum_{p=1}^N \frac{b_p}{z^p} + \sum_{p=1}^N \frac{b_p}{z^p} - \sum_{p=1}^{\infty} \frac{b_p}{z^p} \right| \\
 &\leq \left| \sum_{p=1}^{\infty} \frac{b_p}{\alpha^p} - \sum_{p=1}^N \frac{b_p}{\alpha^p} \right| + \left| \sum_{p=1}^N \frac{b_p}{\alpha^p} - \sum_{p=1}^N \frac{b_p}{z^p} \right| + \left| \sum_{p=1}^N \frac{b_p}{z^p} - \sum_{p=1}^{\infty} \frac{b_p}{z^p} \right| \\
 &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \left| \sum_{p=1}^N \frac{b_p}{r^p(\cos p\theta + i \sin p\theta)} \right| \\
 &\quad + \sum_{p=1}^N \left| \frac{b_p}{r'^p(\cos p\theta + i \sin p\theta)} \right| \\
 &= \frac{2\epsilon}{3} + \left| \sum_{p=1}^N \frac{b_p r'^p - b_p r^p}{r'^p r^p (\cos p\theta + i \sin p\theta)} \right| \\
 &\leq \frac{2\epsilon}{3} + \sum_{p=1}^N \left| \frac{b_p (r'^p - r^p)}{r'^p r^p} \right| = \frac{2\epsilon}{3} + \sum_{p=1}^N \left| b_p \frac{(r^p - r'^p)}{r'^p r^p} \right| \\
 &= \frac{2\epsilon}{3} + \sum_{p=1}^N \left| b_p \frac{(r - r')}{r'^p r^p} (r^{p-1} + r^{p-2} r^1 + r^{p-3} r^2 + \dots + r^{1-p}) \right| \\
 &\leq \frac{2\epsilon}{3} + \sum_{p=1}^N \left| b_p \frac{(r - r')}{r'^p r^p} (r^{p-1} + r^{p-1} + r^{p-1} + \dots + r^{p-1}) \right| \\
 &= \frac{2\epsilon}{3} + \sum_{p=1}^N \left| b_p \frac{(r - r')}{r'^p r^p} (p-1) r^{p-1} \right| = \frac{2\epsilon}{3} + (r - r') \sum_{p=1}^N \left| b_p \frac{(p-1)}{r'^p r^p} \right|
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2\epsilon}{3} + 8 \sum_{p=1}^N |b_p| \frac{(p-1)}{r^p p^p}. \quad \text{If } \sum_{p=1}^N |b_p| \frac{(p-1)}{r^p p^p} = 0 \text{ then } \frac{2\epsilon}{3} \\
 &+ \sum_{p=1}^N |b_p| \frac{(p-1)}{r^p p^p} = \frac{2\epsilon}{3} < \epsilon. \quad \text{Otherwise } \frac{2\epsilon}{3} + \sum_{p=1}^N |b_p| \frac{(p-1)}{r^p p^p} \\
 &= \frac{2\epsilon}{3} + \frac{\epsilon}{3 \sum_{p=1}^N |b_p| \frac{(p-1)}{r^p p^p}} \sum_{p=1}^N |b_p| \frac{(p-1)}{r^p p^p} = \epsilon.
 \end{aligned}$$

Theorem 3.3 If $0 < \alpha < \frac{\pi}{2}$, $\phi = \frac{\alpha + \frac{\pi}{2}}{2}$, $a = \sin \alpha + i \cos \alpha$

then there is a w such that $\bar{w}w + aw - 2 \sin \alpha \geq 0$ and $aw + \bar{w}w - 2 \sin \alpha \geq 0$. Furthermore if w satisfies the

above inequalities and $w \neq 1$ then $\frac{|w-1|}{|w|-1} < \sec \phi$.

$$\begin{aligned}
 \text{Proof: Let } z_1 &= |w|(\cos \alpha \sqrt{1 - \frac{\sin^2 \alpha}{|w|^2} + \frac{\sin^2 \alpha}{|w|}}) \\
 &\quad + i(\frac{\cos \alpha \sin \alpha}{|w|} - \sin \alpha \sqrt{1 - \frac{\sin^2 \alpha}{|w|^2}})
 \end{aligned}$$

$$\text{and } z_2 = |w| + i(|w| - 1) \tan \phi. \quad \text{Now } |z_1| = |w|.$$

Consider the function

$$f(|w|) = 2 \left\{ \cos \alpha \sqrt{|w|^2 - \sin^2 \alpha} + \sin^2 \alpha - |w| \right\} + (|w| - 1)^2 \tan^2 \phi.$$

$$\begin{aligned}
 f'(|w|) &= 2 \left\{ \frac{|w| \cos \alpha}{\sqrt{|w|^2 - \sin^2 \alpha}} - 1 \right\} + 2(|w| - 1) \tan^2 \phi, \\
 f'(1) &= 0,
 \end{aligned}$$

$$f''(w) = 2 \left\{ - \frac{\cos \alpha \sin^2 \alpha}{(|w|^2 - \sin^2 \alpha)^{3/2}} + \tan^2 \phi \right\}$$

$$\geq 2 \left\{ - \frac{\cos \alpha \sin^2 \alpha}{(1 - \sin^2 \alpha)^{3/2}} + \tan^2 \phi \right\}$$

$$= 2 \left\{ - \frac{\cos \alpha \sin^2 \alpha}{\cos^3 \alpha} + \tan^2 \phi \right\} = 2 \left\{ - \tan^2 \alpha + \tan^2 \phi \right\} > 0 \text{ for}$$

$$\text{all } |w| \geq 1.$$

Hence $f(|w|) \geq 0$ for all $|w| \geq 1$ and

$$\begin{aligned}
 f(|w|) &= 2\left\{\cos\alpha\sqrt{|w|^2 - \sin^2\alpha} + \sin^2\alpha - |w|\right\} + (|w| - 1)^2\tan^2\phi \\
 &= -2|w| + 2|w|\left\{\cos\alpha\sqrt{1 - \frac{\sin^2\alpha}{|w|^2}} + \frac{\sin^2\alpha}{|w|}\right\} + (|w| - 1)^2\tan^2\phi \\
 &= 1 - 2|w| + |w|^2 - |w|^2 + 2|w|\cos\alpha\sqrt{1 - \frac{\sin^2\alpha}{|w|^2}} + \frac{\sin^2\alpha}{|w|} - 1 \\
 &\quad + (|w| - 1)^2\tan^2\phi \\
 &= (|w| - 1)^2 + (|w| - 1)^2\tan^2\phi \\
 &\quad - \left\{1 - 2|w|\left[\cos\alpha\sqrt{1 - \frac{\sin^2\alpha}{|w|^2}} + \frac{\sin^2\alpha}{|w|}\right] + |w|^2\right\} \\
 &= \left|(|w| - 1) + i\left((|w| - 1)\tan\phi\right)\right|^2 - |z_1 - 1|^2 \\
 &= |z_2 - 1|^2 - |z_1 - 1|^2 > 0.
 \end{aligned}$$

Thus $|z_2 - 1| > |z_1 - 1|$.

Consider $\cos\alpha\sqrt{|w|^2 - \sin^2\alpha} + \sin^2\alpha = R(w)$.

Case I. If $\ell(w) \geq 0$ then $aw + \bar{a}w - 2\sin\alpha \geq 0$.

$$2R(aw) \geq 2\sin\alpha,$$

$$R(aw) \geq \sin\alpha,$$

$$\sin\alpha R(w) - \cos\alpha \ell(w) \geq \sin\alpha,$$

$$-\cos\alpha \ell(w) \geq \sin\alpha [1 - R(w)],$$

$$[\ell(w)]^2 \leq \frac{\sin^2\alpha}{\cos^2\alpha} [R(w) - 1]^2$$

Case II. If $\ell(w) < 0$ then $\bar{a}w + aw - 2\sin\alpha \geq 0$,

$$2R(\bar{a}w) \geq 2\sin\alpha,$$

$$R(\bar{a}w) \geq \sin\alpha,$$

$$\sin\alpha R(w) + \cos\alpha \ell(w) \geq \sin\alpha,$$

$$-\sin\alpha R(w) - \cos\alpha \ell(w) \leq -\sin\alpha,$$

$$-\cos\alpha \ell(w) \leq \sin\alpha [R(w) - 1],$$

$$0 < -\ell(w) = \frac{\sin \alpha}{\cos \alpha} [R(w) - 1],$$

$$[\ell(w)]^2 = [-\ell(w)]^2 \leq \frac{\sin^2 \alpha}{\cos^2 \alpha} [R(w) - 1]^2.$$

$$\text{In either case, } [\ell(w)]^2 \leq \frac{\sin^2 \alpha}{\cos^2 \alpha} [R(w) - 1]^2.$$

$$\text{Now } \cos \alpha \sqrt{w^2 - \sin^2 \alpha} + \sin^2 \alpha = R(w)$$

$$= \cos \alpha \sqrt{[R(w)]^2 + [\ell(w)]^2 - \sin^2 \alpha + \sin^2 \alpha} = R(w)$$

$$\leq \cos \alpha \sqrt{[R(w)]^2 + \frac{\sin^2 \alpha}{\cos^2 \alpha} [R(w) - 1]^2 - \sin^2 \alpha} = R(w)$$

$$= \sqrt{\cos^2 \alpha \left\{ [R(w)]^2 + \frac{\sin^2 \alpha}{\cos^2 \alpha} [R(w) - 1]^2 - \sin^2 \alpha \right\} + \sin^2 \alpha} = R(w)$$

$$= \sqrt{\cos^2 \alpha [R(w)]^2 + \sin^2 \alpha [R(w) - 1]^2 - \sin^2 \alpha \cos^2 \alpha + \sin^2 \alpha} = R(w)$$

$$= \sqrt{\cos^2 \alpha [R(w)]^2 + \sin^2 \alpha \left\{ [R(w)]^2 - 2R(w) + 1 \right\} - \sin^2 \alpha \cos^2 \alpha + \sin^2 \alpha} = R(w)$$

$$= \sqrt{\cos^2 \alpha [R(w)]^2 + \sin^2 \alpha - 2R(w)\sin^2 \alpha + \sin^2 \alpha [R(w)]^2 - \sin^2 \alpha \cos^2 \alpha + \sin^2 \alpha} = R(w)$$

$$= \sqrt{(\cos^2 \alpha + \sin^2 \alpha) [R(w)]^2 + \sin^2 \alpha - 2R(w)\sin^2 \alpha - \sin^2 \alpha (1 - \sin^2 \alpha) + \sin^2 \alpha} = R(w)$$

$$= \sqrt{[R(w)]^2 + \sin^2 \alpha - 2R(w)\sin^2 \alpha - \sin^2 \alpha + \sin^4 \alpha + \sin^2 \alpha} = R(w)$$

$$= \sqrt{[R(w) - \sin^2 \alpha]^2 + \sin^2 \alpha} = R(w)$$

$$= R(w) - \sin^2 \alpha + \sin^2 \alpha - R(w) = 0. \text{ Hence}$$

$$\cos \sqrt{|w|^2 - \sin^2 \alpha} + \sin^2 \alpha - R(w) \leq 0. \text{ Since } R(z_1) = R(w)$$

$$= \cos \sqrt{|w|^2 - \sin^2 \alpha} + \sin^2 \alpha - R(w) \leq 0, \text{ then } 2R(z_1)$$

$$- 2R(w) \leq 0.$$

$$\text{But } 2R(z_1) - 2R(w) = -|w|^2 + 2R(z_1) - 1 + |w|^2 - 2R(w) + 1$$

$$= -|z_1|^2 + 2R(z_1) - 1 + |w - 1|^2 = -|z_1 - 1|^2$$

$$+ |w - 1|^2 \leq 0,$$

$$|w - 1|^2 \leq |z_1 - 1|^2,$$

$$|z_1 - 1| \geq |w - 1|.$$

$$\frac{|z_2 - 1|}{|w - 1|} > \frac{|z_1 - 1|}{|w - 1|} = \frac{|w - 1|}{|w - 1|}.$$

$$\frac{|z_2 - 1|}{|w - 1|} > \frac{|w - 1|}{|w - 1|},$$

$$\sec \phi > \frac{|w - 1|}{|w - 1|}.$$

Theorem 3.4. If $0 < \alpha < \frac{\pi}{2}$, $a = \sin \alpha + i \cos \alpha$, and if the series $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ is convergent at the point $z = 1$ of its circle of convergence, then it is uniformly convergent for all values of z which lie in both of the half planes $\operatorname{Re} z + \operatorname{Im} z - 2 \sin \alpha \geq 0$ and $\operatorname{Re} z + \operatorname{Im} z - 2 \sin \alpha \leq 0$.

Proof: Let $\frac{\alpha + \frac{\pi}{2}}{2} = \phi$. Since the series $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ is convergent at $z = 1$, $\sum_{p=1}^{\infty} b_p$ converges. That is, for every $\epsilon > 0$, there is a positive integer N such that if $m > n > N$, $\left| \sum_{p=n}^m b_p \right| < \frac{\epsilon}{1 + \sec \phi}$.

Let $s_{p+1} = \left| \sum_{j=n}^{n+p} b_j \right|$ for $p = 0, 1, 2, 3, \dots$. If z is any point in the regions, $z \neq 1$,

$$\begin{aligned}
\sum_{p=n}^m \frac{b_p}{z^p} &= \frac{s_1}{z^n} + \frac{(s_2 - s_1)}{z^{n+1}} + \frac{(s_3 - s_2)}{z^{n+2}} + \dots \\
&\quad + \frac{(s_{m-n} - s_{m-n-1})}{z^{m-1}} + \frac{(s_{m-n+1} - s_{m-n})}{z^m} \\
&= \frac{s_1}{z^n} + \frac{s_2}{z^{n+1}} - \frac{s_1}{z^{n+1}} + \frac{s_3}{z^{n+2}} - \frac{s_2}{z^{n+2}} + \dots \\
&\quad + \frac{s_{m-n}}{z^{m-1}} - \frac{s_{m-n-1}}{z^{m-1}} - \frac{s_{m-n+1}}{z^m} - \frac{s_{m-n}}{z^m} \\
&= \frac{s_1(z-1)}{z^{n+1}} + \frac{s_2(z-1)}{z^{n+2}} + \frac{s_3(z-1)}{z^{n+3}} + \dots + \frac{s_{m-n}(z-1)}{z^m} \\
&\quad + \frac{s_{m-n+1}}{z^m} \\
&= \frac{(z-1)}{z^{n+1}} \left[s_1 + \frac{s_2}{z} + \frac{s_3}{z^2} + \dots + \frac{s_{m-n}}{z^{m-n-1}} \right] + \frac{s_{m-n+1}}{z^m}. \quad \text{Now} \\
\sum_{p=n}^m \frac{b_p}{z^p} &\leq \left| \frac{z-1}{z^{n+1}} \left| \sum_{p=1}^{m-n} \frac{s_p}{z^{p-1}} \right| \right| + \left| \frac{s_{m-n+1}}{z^m} \right| \leq \frac{|z-1|}{|z|^{n+1}} \sum_{p=1}^{m-n} \left| \frac{s_p}{z^{p-1}} \right| + \left| \frac{s_{m-n+1}}{z^m} \right| \\
&< \frac{\epsilon}{1 + \sec \phi} \left[\frac{|z-1|}{|z|^{n+1}} \sum_{p=1}^{m-n} \left| \frac{1}{z^{p-1}} \right| + \frac{1}{|z|^m} \right] \\
&= \frac{\epsilon}{1 + \sec \phi} \left\{ \frac{|z-1|}{|z|^{n+1}} \left[\frac{|z|^{m-n-1} - 1}{|z|^{m-n}(|z| - 1)} \right] + \frac{1}{|z|^m} \right\} \\
&= \frac{\epsilon}{1 + \sec \phi} \left\{ \frac{|z-1|}{|z| - 1} \left[\frac{|z|^{m-n-1} - 1}{|z|^{m+1}} \right] + \frac{1}{|z|^m} \right\} \\
&< \frac{\epsilon}{1 + \sec \phi} \left\{ \frac{|z-1|}{|z| - 1} \left[\frac{|z|^{m-n-1}}{|z|^{m+1}} \right] + \frac{1}{|z|^m} \right\}
\end{aligned}$$

$$= \frac{\epsilon}{1 + \sec \phi} \left\{ \frac{|z - 1|}{|z| - 1} \left[\frac{1}{r^n} \right] + \frac{1}{r^m} \right\} = \frac{\epsilon}{1 + \sec \phi} \left\{ \frac{|z - 1|}{|z| - 1} \frac{1}{r^n} + \frac{1}{r^m} \right\}$$

$$\leq \frac{\epsilon}{1 + \sec \phi} \left\{ \frac{|z - 1|}{|z| - 1} \frac{1}{r^n} + \frac{1}{r^m} \right\}. \quad \text{By theorem 3.3,}$$

$$\frac{|z - 1|}{|z| - 1} < \sec \phi. \quad \text{Now } \frac{\epsilon}{1 + \sec \phi} \left\{ \frac{|z - 1|}{|z| - 1} \frac{1}{r^n} + \frac{1}{r^m} \right\}$$

$$< \frac{\epsilon}{1 + \sec \phi} \left\{ \frac{\sec \phi}{r^n} + \frac{1}{r^m} \right\} < \frac{\epsilon}{1 + \sec \phi} (\sec \phi + 1) = \epsilon.$$

Hence $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ is uniformly convergent for all z within the regions.

Theorem 3.5. If $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ converges at the point $z = 1$ on its circle of convergence, $0 < \alpha < \frac{\pi}{2}$, and $a = \sin \alpha - i \cos \alpha$, then $\lim_{\sigma \rightarrow 1} \sum_{p=1}^{\infty} \frac{b_p}{z^p} = \sum_{p=1}^{\infty} b_p$ where σ is any point on a curve in the half planes $\bar{w} + a\bar{v} - 2 \sin \alpha \geq 0$ and $aw + \bar{av} - 2 \sin \alpha \geq 0$.

Proof: By theorem 3.4, $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ is uniformly convergent at all points in the triangle. Since σ is restricted to be in the triangle, $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ is uniformly convergent over the set of values σ may take on.

If $\epsilon > 0$, then since $\sum_{p=1}^{\infty} \frac{b_p}{z^p}$ is uniformly convergent, there is an N such that $\left| \sum_{p=1}^N \frac{b_p}{z^p} - \sum_{p=1}^{\infty} \frac{b_p}{z^p} \right| < \frac{\epsilon}{3}$ for all z on the curve.

Case I. If $\epsilon \geq 3N \sum_{p=1}^N |b_p|$ let $S = 1$.

Case II. If $\epsilon < 3N \sum_{p=1}^N |b_p|$ then $1 > \frac{\epsilon}{3N \sum_{p=1}^N |b_p|}$. Let

$S = \frac{\epsilon}{3N \sum_{p=1}^N |b_p|}$. In either case $0 < S \leq 1$.

Hence if $|\sigma - 1| < \delta$, $|\sigma| > 1$, and σ on the curve, then

$0 < |\sigma| - 1 < \delta$. Now

$$\begin{aligned}
 \left| \sum_{P=1}^{\infty} \frac{b_p}{\sigma^p} - \sum_{P=1}^{\infty} b_p \right| &= \left| \sum_{P=1}^{\infty} \frac{b_p}{\sigma^p} - \sum_{P=1}^N \frac{b_p}{\sigma^p} + \sum_{P=1}^N \frac{b_p}{\sigma^p} - \sum_{P=1}^{\infty} b_p \right| \\
 &\leq \left| \sum_{P=1}^{\infty} \frac{b_p}{\sigma^p} - \sum_{P=1}^N \frac{b_p}{\sigma^p} \right| + \left| \sum_{P=1}^N \frac{b_p}{\sigma^p} - \sum_{P=1}^N b_p \right| + \left| \sum_{P=1}^N b_p - \sum_{P=1}^{\infty} b_p \right| \\
 &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \left| \sum_{P=1}^N b_p \left(\frac{1}{\sigma^p} - 1 \right) \right| = \frac{2\epsilon}{3} + \left| \sum_{P=1}^N b_p \left(\frac{1 - \sigma^{-p}}{\sigma^p} \right) \right| \\
 &= \frac{2\epsilon}{3} + \left| \sum_{P=1}^N \frac{b_p}{\sigma^p} (\sigma^{-p} - 1) \right| \\
 &= \frac{2\epsilon}{3} + \left| \sum_{P=1}^N \frac{b_p}{\sigma^p} (\sigma - 1) (\sigma^{p-1} + \sigma^{p-2} + \sigma^{p-3} + \dots + 1) \right| \\
 &\leq \frac{2\epsilon}{3} + \sum_{P=1}^N \left| \frac{b_p}{\sigma^p} \right| |\sigma - 1| (|\sigma^{p-1}| + |\sigma^{p-2}| + |\sigma^{p-3}| + \dots + 1) \\
 &\leq \frac{2\epsilon}{3} + \sum_{P=1}^N \left| \frac{b_p}{\sigma^p} \right| |\sigma - 1| (|\sigma^p| + |\sigma^p| + |\sigma^p| + \dots + |\sigma^p|) \\
 &\leq \frac{2\epsilon}{3} + |\sigma - 1| \sum_{P=1}^N |b_p| \cdot p \leq \frac{2\epsilon}{3} + \delta N \sum_{P=1}^N |b_p|. \quad \text{In case I,} \\
 \frac{2\epsilon}{3} + \delta N \sum_{P=1}^N |b_p| &= \frac{2\epsilon}{3} + N \sum_{P=1}^N |b_p| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \quad \text{In case II,} \\
 \frac{2\epsilon}{3} + \delta N \sum_{P=1}^N |b_p| &= \frac{2\epsilon}{3} + \frac{\epsilon}{3N} \sum_{P=1}^N |b_p| = \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
 \end{aligned}$$

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