A NUMERICAL METHOD FOR THE CALCULATION OF THE INERTIAL LOADS ON AN AIRPLANE

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THESIS

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This paper is an extension of various projects that the writer has been associated with at Chance Vought Aircraft, Incorporated. The writer expresses thanks to the many persons at Chance Vought Aircraft whose advice and criticisms helped formulate the ideas upon which this paper is based.

The writer developed equations equivalent to those developed in the first two sections of Chapter VI while on a special job assignment at Chance Vought Aircraft. Dr. H. A. Wood, Supervisor of the Machine Computing Group at Chance Vought Aircraft, then developed the same equations using vector-matrix notation. Both Dr. Wood's development and results proved to be more compact than those of the writer. As a result, Chapter VI is written in the same vector-matrix notation as that employed by Dr. Wood.

INTRODUCTION

In order to successfully design an airplane, the stress analyst must have the shear and moment distributions for each component of the airplane. Also, the dynamics engineer needs the distribution of inertial properties of each component so that he may determine the aero-elastic properties of the airplane.

In the past, the above data have been computed manually. This task is a long and tedious one. In addition, the calculations have not been completed until well into the design phase.

The purpose of this thesis is to provide a numerical method for obtaining the above data that is applicable to digital computers. It then will be possible to obtain the data soon after the beginning of design with a small expenditure of manpower.

CHAPTER I

DEFINITION OF BASIC TERMS

Whenever the expression, "$\int f\,dv$" is used in this paper, it will mean the triple, Riemann integral of the function, f, over the volume under consideration. It will be assumed that for any function, f, used in this paper, $\int f\,dv$ is meaningful and is equal to the iterated integral over the same volume; that is,

$$\int_v f\,dv \qquad \int_a^b \int_c^d \int_e^g f\,dx\,dy\,dz,$$

where a, b, c, d, e, and g are the applicable limits of integration of the volume V.

Weight

Weight, W, can be expressed

$$W = \int \rho\,dv,$$

where ρ = density.

Centers of Gravity

The co-ordinates of the center of gravity, $\overline{x}$, $\overline{y}$, and $\overline{z}$, can be expressed

$$\overline{x} = \frac{\int \rho\,x\,dv}{\int \rho\,dv},$$

$$\overline{y} = \frac{\int \rho\,y\,dv}{\int \rho\,dv}, \text{ and}$$

$$\bar{z} = \frac{\int \rho\, z\, dv}{\int \rho\, dv}.$$

Define the $\bar{x}$, $\bar{y}$, $\bar{z}$ axis system to be respectively parallel to the x, y, z axis system and have its origin at $(\bar{x},\bar{y},\bar{z})$.

Moments of Inertia

The moments of inertia, $I\overline{xx}$, $I\overline{yy}$, and $I\overline{zz}$, about the $\bar{x}$-$\bar{x}$, $\bar{y}$-$\bar{y}$, and $\bar{z}$-$\bar{z}$ axes respectively, can be expressed

$$I\overline{xx} = \int \rho\, r_{\bar{x}}^{2}\, dv,$$

$$I\overline{yy} = \int \rho\, r_{\bar{y}}^{2}\, dv, \text{ and}$$

$$I\overline{zz} = \int \rho\, r_{\bar{z}}^{2}\, dv,$$

where $r_{\bar{x}}$, $r_{\bar{y}}$, and $r_{\bar{z}}$ are the distances of dv from the $\bar{x}$-$\bar{x}$, $\bar{y}$-$\bar{y}$, and $\bar{z}$-$\bar{z}$ axes respectively.

Products of Inertia

The products of inertia, $P\overline{xy}$, $P\overline{xz}$, and $P\overline{yz}$ with respect to the $\overline{xy}$, $\overline{xz}$, and $\overline{yz}$ planes respectively, can be expressed

$$P\overline{xy} = \int \rho\, r_{\overline{yz}}\, r_{\overline{xz}}\, dv,$$

$$P\overline{xz} = \int \rho\, r_{\overline{yz}}\, r_{\overline{xy}}\, dv, \text{ and}$$

$$P\overline{yz} = \int \rho\, r_{\overline{xz}}\, r_{\overline{xy}}\, dv,$$

where $r_{\overline{xy}}$, $r_{\overline{xz}}$, and $r_{\overline{yz}}$ are the signed distances of dv from the $\overline{xy}$, $\overline{xz}$, and $\overline{yz}$ planes respectively.

CHAPTER II

STATEMENT OF BASIC ASSUMPTIONS

The following assumptions will be made in this paper:

1. The airplane is a rigid body composed of a finite number of rigid parts.
2. The following data are known about each part:
   a. W_p = weight,
   b. $\overline{x}_p$, $\overline{y}_p$, $\overline{z}_p$ = co-ordinates of the center of gravity,
   c. L_x, L_y, L_z = dimensions of the part as measured parallel to the x, y, z axes respectively,
   d. Type of part:
      (1) Fuselage skin panel,
      (2) Fuselage frame
      (3) Fuselage bulkhead,
      (4) Fuselage part other than one of above,
      (5) Horizontal surface trapezohedron,
      (6) Horizontal surface beam,
      (7) Vertical surface trapezohedron,
3. Fuselage parts will be defined as follows:
   a. Skin panels - hollow right circular cylinders of diameter $\frac{L_y + L_z}{2}$ with the symmetrical axis parallel to the x-x axis. This circular

cylinder will replace the actual right elliptical cylinder with axes Ly and Lz for all computations.

b. Frames and bulkheads-- solid right elliptical cylinders with the symmetrical axis parallel to the x-x axis.

c. All other parts-- rectangular parallelepipeds with sides parallel to the reference planes.

4. Horizontal surface parts will be defined to be one of the following:

a. Trapezohedron-- six sided solid with the following specifications:

(1) Two faces are parallel to the xz plane and are rectangles,

(2) The projections of two other faces on the yz plane are identical parallegrams,

(3) The projections of the two remaining surfaces on the xy plane are identical trapezoids,

(4) $A/B = a/b$, where A and B are the lengths of the root and tip chords respectively of the surface and a and b are the lengths of the root and tip chords respectively of the trapezohedron,

(5) $C/S = c/Ly$,

$D/S = d/Ly$, and

$t = \frac{c + d}{2}$, where

C and D = the maximum depth of the surface at root and tip respectively,

c and d = the depths of the part at the root and tip respectively,

t = average thickness, and

s = the distance, measured parallel to the y-y axis, from the root to the tip of the surface,

(6) The trapezohedron has dimensions lx and ly as measured parallel to the x-x and y-y axes respectively,

(7) In order to force the slope of the detail part to have the same sign as the slope of the surface, let $\frac{M}{|M|} = \frac{m}{|m|}$ and $\frac{M_1}{|M_1|} = \frac{m_1}{|m_1|}$, where

M and M_1 = the slopes of the leading and trailing edges respectively of the surface, and

m and m_1 = the slopes of the leading and trailing edges respectively of the trapezohedron,

(8) $A > B$

(9) If $\frac{M}{|M|} \neq \frac{M_1}{|M_1|}$ then define $m = M$.

b. Beam-vertical plane, perpendicular to the xy plane, with the following specifications:

(1) The projection on the xy plane is one of the diagonal of the rectangle with sides lx and ly,

(2) The projection on the yz plane will be a trapezoid. The non-parallel sides of this trapezoid will coincide with the non-parallel sides of the trapezoid formed by the projection of the surface onto the yz plane. The parallel sides of the trapezoid will be parallel to the xz plane and perpendicular to the xy plane.

5. Vertical surface parts are defined in the same manner as horizontal surface parts if the y-y and z-z axes are interchanged.

6. Each part in the airplane is of uniform density.

CHAPTER III

BASIC CONCEPTS

Matrix-Vector Notation

The matrix-vector notation developed by Wood (1) will be used in Chapter VI. Using this system of notation, a vector, $\vec{a}$, with components a_1, a_2, a_3 has the following matrix representations:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{1}$$

$$a' = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \text{ and} \tag{2}$$

$$a^{\vee} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{3}$$

The following theorems are proven by Wood (1,2) for any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$:

$$\vec{a} \cdot \vec{b} = a' b = b' a, \tag{4}$$

$$\vec{a} \times \vec{b} = a^{\vee} b = -b^{\vee} a, \tag{5}$$

$$a^{\vee} b^{\vee} b^{\vee} c = -b^{\vee} a\, c' b - b' b a^{\vee} c, \tag{6}$$

$$a^{\vee} b^{\vee} b^{\vee} a = -b^{\vee} a^{\vee} a^{\vee} b, \text{ and} \tag{7}$$

$$a^{\vee} b^{\vee} c = -a^{\vee} c^{\vee} b. \tag{8}$$

Integration of a Matrix

The integration of a matrix is defined as

$$\int A = B, \qquad (9)$$

where A and B are matrices and for any row, i, and column, j,

$$b_{ij} = \int a_{ij}\, dv.$$

Chapter Bibliography

1. Wood, H. A., Matrices, Vectors, and Tensors, Sism-1009/ AWN-EI, Chance Vought Aircraft Publication, 1948
2. Wood, H. A., Unpublished notes, Chance Vought Aircraft, Incorporated, Dallas, Texas

CHAPTER IV

DISTRIBUTION EQUATIONS

General

Assume the following definitions:

R_1 = the distance from the forward most point of any part to the yz plane,

W_p = weight of any part,

$\overline{X}_p$, $\overline{Y}_p$, $\overline{Z}_p$ = co-ordinates of the center of gravity of any part,

lx, ly, lz = dimensions of the part as measured parallel to the x-x, y-y, z-z axes respectively,

W_{pi} = weight of that portion of any item in I_i (See definition of I_i under each subheading of this chapter),

$\overline{X}_{pi}$, $\overline{Y}_{pi}$, $\overline{Z}_{pi}$ = co-ordinates of the center of gravity of that portion of any part in I_i.

l_{pi} = length, measured parallel to interval length, of that portion of any item contained in I_i,

$I_{\overline{xx}pi}$, $I_{\overline{yy}pi}$, $I_{\overline{zz}pi}$ = Moments of inertia of that portion of any part in I_i about axes parallel to x-x, y-y, z-z with origin at ($\overline{X}_{pi}$, $\overline{Y}_{pi}$, $\overline{Z}_{pi}$),

$P_{\overline{xy}pi}$, $P_{\overline{xz}pi}$, $P_{\overline{yz}pi}$ = Products of inertia of that portion of any part in I_i with respect to orthogonal axes parallel to x-x, y-y, z-z, with origin at ($\overline{x}_{pi}$, $\overline{y}_{pi}$, $\overline{z}_{pi}$),

R_2 = the distance from the most inboard point of any part to the xz plane,

R_3 = the distance from the lowest point at $Y = R_2$ of any part to the xy plane.

Fuselage

Pass a finite set of planes, $x = x_0$, $x = x_1$, $x = x_2$, $\cdots$, $x = x_n$ through the fuselage. Now the fuselage may be thought of as being partitioned into a set of intervals, I_1, I_2, $\cdots$, I_n, where $I_i = (X_{i-1}, X_i]$.

Now consider any interval I_i. If a part lies completely within I_i (that is, $X_{i-1} < R_1 < (R_1 + l_x) \leqslant X_i$), then

$$l_{pi} = l_x. \tag{10}$$

But, if $X_{i-1} < R_1 < X_i < (R_1 + l_x)$,

$$l_{pi} = (X_i - R_1). \tag{11}$$

Now if $R_1 \leqslant X_{i-1} < (R_1 + l_x) \leqslant X_i$,

$$l_{pi} = (R_1 + l_x - X_{x-1}). \tag{12}$$

If $R_1 \leqslant X_{i-1} < X_i < (R_i + l_x)$,

$$l_{pi} = (x_i - x_{i-1}). \tag{13}$$

Finally, if $R_1 \geqslant X_i$ or $X_{i-1} \geqslant (R_1 + l_x)$,

$$l_{pi} = 0. \tag{14}$$

Obviously in equations (10)-(14), R_1 can be expressed as

$$R_1 = \overline{X}_p - \tfrac{1}{2} lx. \tag{15}$$

Now,

$$W_{pi} = \frac{l_{pi}}{l_x} W_p \tag{16}$$

$$\overline{X}_{pi} = \max \left[R_1, X_{i-1} \right] + \tfrac{1}{2} l_{pi} \tag{17}$$

$$\overline{Y}_{pi} = \overline{Y}_p, \text{ and} \tag{18}$$

$$Z_{pi} = Z_p. \tag{19}$$

It can be seen that a value $\neq 0$ may be obtained for equations (17)-(19) even when a part does not extend into I_i. However, this is not important, since each of the forementioned terms will always be multiplied by $W_{pi} = o$ before being used.

Now consider the following:

Theorem I: If a body is symmetrical about a plane, Q, then the product of inertia of the body with respect to an axis, $\alpha - \alpha$, in Q and an axis, $\beta - \beta$, perpendicular to both $\alpha - \alpha$ and Q is 0.

Proof:

By definition,

$$P_{\alpha\beta} = \int \rho \alpha \beta \, dv.$$

also by definition, ρ is a constant, so

$$P_{\alpha\beta} = \rho \int \alpha \beta \, dv.$$

Now define the surface of the body above Q as,

$$\beta_A = f(\alpha, \gamma),$$

where $\gamma - \gamma$ is the third orthogonal axis. Then, by hypotheses, the surface of the body below Q is

$$\beta_B = -f(\alpha, \gamma).$$

Therefore,

$$P_{\alpha\beta} = \rho \int_{g_1}^{g_2} \int_{g_3}^{g_4} \int_{-f(\alpha,\gamma)}^{f(\alpha,\gamma)} \alpha \beta \, d\beta \, d\gamma \, d\alpha,$$

where g_1 and g_2 are the limits of integration on α, and g_3 and g_4 are the limits of integration on γ.

Then $P_{\alpha\beta} = \frac{\rho}{2} \int_{g_1}^{g_2} \int_{g_3}^{g_4} \alpha \beta^2 \Big]_{-f(\alpha,\gamma)}^{f(\alpha,\gamma)} d\gamma \, d\alpha,$

So

$$P_{\alpha\beta} = 0.$$

Now since each fuselage part is symmetrical about the $\overline{X}_p\overline{Y}_p$, $\overline{X}_p\overline{Z}_p$, and $\overline{Y}_p\overline{Z}_p$ planes, and since the planes $X = X_{i-1}$ and $X = X_i$ are perpendicular to the x-x axis, the portion of any part in I_i will be symmetrical about each of the $\overline{X}_{pi}\ \overline{Y}_{pi}$, $\overline{X}_{pi}\ \overline{Z}_{pi}$, and $\overline{Y}_{pi}\ \overline{Z}_{pi}$ planes. So by theorem I

$$P_{\overline{xy}_{pi}} = P_{\overline{xz}_{pi}} = P_{\overline{yz}_{pi}} = 0. \qquad (20)$$

From the SAWE Weight Handbook (2):

(1) for frames and bulkheads

$$I_{\overline{xx}_{pi}} = \frac{W_{pi}}{4}\left(l_y^{\,2} + l_z^{\,2}\right), \qquad (21)$$

$$I_{\overline{yy}_{pi}} = \frac{W_{pi}}{12}\left(3/4\ l_z^{\,2} + l_{pi}^{\,2}\right), \text{ and} \qquad (22)$$

$$I_{\overline{zz}_{pi}} = \frac{W_{pi}}{12}\left(3/4\ l_y 2 + l_x 2\right); \qquad (23)$$

(2) for skin panels,

$$I_{\overline{xx}_{pi}} = \frac{W_{pi}}{4}\left(l_z + l_y\right)^2 \qquad (24)$$

$$I_{\overline{yy}_{pi}} = \frac{W_{pi}}{2}\left[\frac{(l_z + l_y)^2}{4} + \frac{l_{pi}^{\,2}}{6}\right], \text{ and} \qquad (25)$$

$$I_{\overline{zz}_{pi}} = I_{\overline{yy}_{pi}}; \qquad (26)$$

(3) for all other fuselage parts,

$$I_{\overline{xx}_{pi}} = \frac{W_{pi}}{12}\left(l_y^{\,2} + l_z^{\,2}\right), \qquad (27)$$

$$I_{yy_{pi}} = \frac{W_{pi}}{12}\left(l_{pi}^{\,2} + l_z^{\,2}\right), \qquad (28)$$

$$I_{\overline{zz}_{pi}} = \frac{W_{pi}}{12}\left(l_{pi}^{\,2} + l_y^{\,2}\right). \qquad (29)$$

Horizontal Surfaces

Pass a finite set of planes $Y=Y_0$, $Y=Y_1$, $\cdots$, $Y=Y_n$ through the surface. The surface now is partitioned into a set of intervals I_1, I_2, $\cdots$, I_i, $\cdots$, I_i, $\cdots$, I_n, where any $I_i = (Y_{i-1}, Y_i]$.

Consider a part that is a trapezohedron. It is obvious that

$$\overline{Y}_p = R_2 + \frac{\int \rho y_1 \, dv}{\int \rho \, dv},$$

where $Y_1 = (Y - R_2)$. Now

$$\int \rho Y_1 \, dv = \rho \int_0^{l_x} \int_{mY_1}^{a+m_1Y_1} Y_1 \, t \, dx \, dY,$$

$$\int \rho Y_1 dv = \left[t(m_1 - m) \frac{l_y^3}{3} + a t \frac{l_y^2}{2} \right] \rho .$$

But

$$\int \rho dv = \rho t \int_0^{l_y} \int_{mY_1}^{a+m_1Y_1} dx \, dz,$$

$$\int \rho dv = \rho \left[t \, (m_1 - m) \frac{l_y^2}{2} + t \cdot a \cdot l \, y \right).$$

So,

$$\overline{Y}_p = R_2 + \frac{t(m_1 - m) \dfrac{l_y^3}{2} + a \cdot t \cdot \dfrac{l_y^2}{2}}{t(m_1 - m) \dfrac{l_y^2}{2} + t \cdot a \cdot l_y},$$

and hence

$$R_2 = \overline{Y}_p - \frac{2(m_1 - m) \; l_y^2 + 3 \cdot a \cdot l_y}{3(m_1 - m) \; l_y + 6a} . \qquad (30)$$

Now, if $\frac{M}{|M|} \neq \frac{M}{|M_1|}$,

$$a = l_x, \qquad (31)$$

$$b = \frac{B}{A} \cdot a, \qquad (32)$$

$$m = M, \qquad (33)$$

$$m_1 = \frac{b + m \, l_y - l_x}{l_y}, \text{ and} \qquad (34)$$

$$\overline{m} = m + \frac{b - a}{2 l_y}. \qquad (35)$$

However, if $\frac{M}{|M|} = \frac{M_1}{|M_1|}$,

$$a = 2A \frac{l_x - 2 \, l_y \, |\overline{m}|}{A + B}, \qquad (36)$$

$$b = B/A \; a, \qquad (37)$$

$$m = \max \left[\frac{l_x - b}{l_y} \cdot \frac{M}{|M|}, \; \frac{l_x - a}{l_y} - \frac{M}{|M|} \right], \qquad (38)$$

$$m_1 = \min \left[\frac{l_x - b}{l_y} \cdot \frac{M}{|M|}, \; \frac{l_x - z}{l_y} \cdot \frac{M}{\overline{M}} \right], \text{ and} \qquad (39)$$

$$\overline{m} = \frac{2 \, l_x - a - b}{2 \, l_y} \cdot \frac{M}{|M|}. \qquad (40)$$

Now, by definition

$$t = \frac{c + d}{2} = \frac{l_y}{2S} (C + D). \qquad (41)$$

Also,

$$R_1 = \overline{X}_p - \frac{a}{2} - (\overline{Y}_p - R_2)\,\bar{m}. \tag{42}$$

Now consider any interval I_i. If the trapezohedron lies completely within I_i (that is, $Y_{i-1} < R_2 < (R_2 + L_y) \leq Y_i$), then

$$l_{pi} = l_y. \tag{43}$$

But, if $R_2 \leq Y_{i-1} < (R_2 + l_y) \leq Y_i$,

$$l_{pi} = (R_2 + l_y - Y_{i-1}). \tag{44}$$

Now, if $Y_{i-1} < R_2 < Y_i < (R_2 + l_y)$,

$$l_{pi} = (Y_i - R_2). \tag{45}$$

If $R_2 \leq Y_{i-1} < Y_i < (R_2 + l_y)$,

$$l_{pi} = (Y_i - Y_{i-1}). \tag{46}$$

Finally, if $R_2 \geq Y_i$ or $Y_{i-1} \geq (R_2 + ly)$,

$$l_{pi} = 0. \tag{47}$$

Now define a_0, a_1, a_2, $\cdots$, A_h as the chord length of the part at $Y = R_2$, $Y = Y_1$, $\cdots$, $Y = (R_2 + ly)$ respectively. Now if $Y_{i-1} \leq R_2 < (R_2 + ly) \leq Y_i$,

$$a_{i-1} = a, \text{ and}$$
$$a_i = b. \tag{48}$$

If $R_2 < Y_{i-1} < (R_2 + ly) \leq Y_i$,

$$a_{i-1} = a + (m_1 - m)(y_{i-1} - R_2) \text{ and} \tag{49}$$
$$a_i = b.$$

Or if $Y_{i-1} \leq R_2 < Y_i \leq (R_2 + Ly)$,

$$a_{i-1} = a, \text{ and}$$
$$a_i = a + (m_i - m)(Y_i - R_2). \tag{50}$$

But if $R_2 \leq Y_{i-1} < Y_i \leq (R_2 + ly)$,

$$A_{i-1} = a + (m_1 - m)(Y_{i-1} - R_2), \text{ and}$$

$$a_i = a + (m_1 - m)\ (Y_i - R_2). \qquad (51)$$

Finally, if $R_2 \ngtr Y_i$ or $Y_{i-1} \ngtr (R_2 + Ly)$, assume

$$a_{i-1} = a, \text{ and}$$

$$a_i = b.$$

Define V_{pi} to be the volumn of that portion of any part in I_i and V_p to be the total volumn of the same part. Then

$$W_{pi} = \frac{V_{pi}}{V_p} W_p. \qquad (53)$$

Define M_2 as the slope of the plane of symmetry of the surface with respect to the xy plane. Then from the SAWE Weight Handbook (2),

$$V_{pi} = \frac{l_{pi}}{2} (a_{i-1} + a_i)\ t \sec \tan^{-1} M_2, \text{ and}$$

$$V_p = \frac{ly}{2} (a + b)\, t \sec \tan^{-1} M_2. \qquad (54)$$

By substitution of (54) in (53),

$$W_{pi} = \frac{l_{pi}\ (a_{i-1} + a_i)}{ly(a + b)}\ W_p. \qquad (55)$$

In a manner similiar to that used in proving (30), it can be proven that

$$Y_{pi} = \max\left[R_2,\ Y_{i-1}\right] + \frac{2(m_1 - m)\ l_{pi}^{2} + 3\ a_{i-1}\ l_{pi}}{3(m_1 - m)\ l_{pi} + a_{i-1}}. \qquad (56)$$

Now, according to the SAWE Weight Handbook (2), the center of gravity of a trapezoid lies on the line connecting the mid-points of the parallel sides, so

$$\overline{X}_{pi} = R_i + \frac{a_{i-1}}{2} + (\overline{Y}_{pi} - \max\left[Y_{i-1},\ R_2\right]\ \overline{m}$$

$$+ Y_{i-1} - \min\left[R_2,\ Y_{i-1}\right] m. \qquad (57)$$

Assuming the part to be symmetrical about the surface plane of symmetry,

$$\overline{Z}_{pi} = \overline{Z}_p + M_2\ (\overline{Y}_{pi} - \overline{Y}_p). \qquad (58)$$

The following relationships are defined for later use:

$$R_4 = R_1 + (Y_{i-1} - \min\left[R_2,\ Y_{i-1}\right]),\ \text{and} \qquad (59)$$

$$R_5 = \max\left[R_2 - Y_{i-1}\right]. \qquad (60)$$

Now define a set of orthogonal axes such that the $\beta - \beta$ axis lies in the plane of symmetry of the surface, the $\beta\gamma$ plane is parallel to the YZ plane and passes through $X = R_4$, and the $\alpha\gamma$ plane passes through $Y = R_5$. Also define $l_{pi\beta}$ as the length, measured parallel to the $\beta - \beta$ axis, of that portion of the part being in I_i. Then,

$$l_{pi\beta} = l_{pi} \sec \tan^{-1} M_2. \qquad (61)$$

Assume that negligible error will result in the moments and products of inertia if the projection of the part onto the XZ plane is assumed to be a rectangle with sides t and l_{pi}. Then the moment of inertia of the portion of the part in I_i about $\gamma - \gamma$ is

$$I_{\gamma\gamma_{pi}} = \rho \int_{-t/2}^{t/2} \int_{0}^{l_{pi\beta}} \int_{m\beta}^{m_1\beta + a_{i-1}} (\alpha^2 + \beta^2)\, d\alpha\, d\beta\, d\gamma,$$

$$I_{\gamma\gamma_{pi}} = \rho t \Big[(l_{pi})^4 \left(\frac{m_1^3}{12} + \frac{m_1}{4} - \frac{m^3}{12} - \frac{m}{4}\right) + (l_{pi\beta})^3 \left(\frac{m_1^2\, a_{i-1}}{3} + \frac{a_{i-1}}{3}\right) + \frac{m_1\, a_{i-1}^2\, (l_{pi\beta})^2}{2} + \frac{(a_{i-1})^3\, (l_{pi\beta})}{3} \Big]. \qquad (62)$$

Using the moment of inertia transfer formula from Haggard (1),

$$I_{\bar{\gamma}\bar{\gamma}_{pi}} = I_{\gamma\gamma_{pi}} - W_{pi}\, (\bar{\alpha}_{pi}^2 + \bar{\beta}_{pi}^2), \qquad (63)$$

where $\bar{\alpha}_{pi}$, $\bar{\beta}_{pi}$, and $\bar{\gamma}_{pi}$ are the co-ordinates of the center of gravity of the portion of the part in I_i. $I_{\bar{\gamma}\bar{\gamma}_{pi}}$ is then the moment of inertia of the part in I_i about the $\bar{\gamma}_{pi} - \bar{\gamma}_{pi}$ axis. Now obviously

$$\bar{\alpha}_{pi} = X_{pi} - R_4,$$

$$\bar{\beta}_{pi} = (\bar{Y}_{pi} - R_5) \sec \tan^{-1} M_2, \text{ and} \tag{64}$$
$$\bar{\gamma}_{pi} = 0 .$$

Now the moment of inertia of that portion of the part in I_i about the $\alpha - \alpha$ axis is

$$I_{\alpha\alpha_{pi}} = \rho \int_0^{l_{pi\beta}} \int_{m\beta}^{m_1\beta + a_{i-1}} \int_{-t/2}^{t/2} (\beta^2 + \gamma^2)\, d\gamma\, d\alpha\, d\beta,$$

$$I_{\alpha\alpha_{pi}} = \frac{W_{pi}}{12(a_{i-1} + a_i)} \left\{ \left[6(l_{pi\beta})^3 + (l_{pi\beta})\, t^2 \right] (m_1 - m) + 2a_{i-1} \left[4\, (l_{pi\beta})^2 + t^2 \right] \right\}. \tag{65}$$

Again applying the moment of inertia transfer formula from Haggard (1), the moment of inertia about the $\bar{\alpha}_{pi} - \bar{\alpha}_{pi}$ axis is

$$I_{\bar{\alpha}\bar{\alpha}_{pi}} = I_{\alpha\alpha_{pi}} - W_{pi}\, (\bar{\gamma}_{pi}{}^2 + \bar{\beta}_{pi}{}^2). \tag{66}$$

Now consider the moment of inertia of the part in I_i about the $\beta - \beta$ axis,

$$I_{\beta\beta_{pi}} = \rho \int_0^{l_{pi\beta}} \int_{m\beta}^{m_1\beta + a_{i-1}} \int_{-t/2}^{t/2} (\gamma^2 + \alpha^2)\, d\gamma\, d\alpha\, d\beta,$$

$$I_{\beta\beta_{pi}} = \frac{W_{pi}}{12(a_{i-1} + a_i)} \Big[t^2\, (l_{pi\beta})(m_1 - m) + 2(l_{pi\beta})^3\, (m_1{}^3 - m^3) + 2\, a_{i-1}\, t^2 + 8\, a_{i-1}\, (m_1{}^2\, l_{pi\beta}{}^2 + a_{i-1}{}^2) + 12\, m_1\, (l_{pi\beta})\, a_{i-1}{}^2 \Big]. \tag{67}$$

Once again applying the moment of inertia transfer formula, the moment of inertia about the $\bar{\beta}_{pi} - \bar{\beta}_{pi}$ axis is

$$I_{\bar{\beta}\bar{\beta}_{pi}} = I_{\beta\beta_{pi}} - W_{pi}\, (\bar{\alpha}_{pi}{}^2 + \bar{\gamma}_{pi}{}^2). \tag{68}$$

Now by theorem I, the products of inertia, $P_{\bar{\gamma}\bar{\beta}_{pi}}$ and $P_{\bar{\alpha}\bar{\gamma}_{pi}}$ with respect to the $\bar{\alpha}_{pi}$ - $\bar{\alpha}_{pi}$, $\bar{\gamma}_{pi}$ - $\bar{\gamma}_{pi}$ and $\bar{\beta}_{pi}$ - $\bar{\beta}_{pi}$, $\bar{\gamma}_{pi}$ - $\bar{\gamma}_{pi}$ axes respectively are

$$P_{\bar{\gamma}\bar{\beta}_{pi}} = P_{\bar{\alpha}\bar{\gamma}_{pi}} = 0. \tag{69}$$

Now the product of inertia with respect to the α - α, β - β axes can be expressed as

$$P_{\alpha\beta_{pi}} = \rho t \int_{m\beta}^{m_1\beta + a_{i-1}} \int_0^{l_{pi\beta}} \alpha\beta \, d\beta \, d\alpha,$$

$$= \frac{W_{pi} \, l_{pi\beta}}{12(a_{i-1} + a_i)} \left[3(l_{pi\beta})^2 (m_1^2 - m^2) + A_{i-1} (8 \, m_1 \, l_{pi\beta} + 6 \, a_{i-1}) \right]. \tag{70}$$

Applying the transfer formula from Haggard (1) the product of inertia with respect to the $\bar{\alpha}_{pi}$ - $\bar{\alpha}_{pi}$, $\bar{\beta}_{pi}$ - $\bar{\beta}_{pi}$ axes is

$$P_{\bar{\alpha}\bar{\beta}_{pi}} = P_{\alpha\beta_{pi}} - W_{pi} \, (\bar{\alpha}_{pi} \, \bar{\beta}_{pi}). \tag{71}$$

Obviously

$$I_{\overline{XX}_{pi}} = I_{\overline{\alpha\alpha}_{pi}}. \tag{72}$$

Now define

$$X_1 = X - \bar{X}_{pi},$$

$$Y_1 = Y - \bar{Y}_{pi},$$

$$Z_1 = Z - \bar{Z}_{pi},$$

$$\alpha_1 = \alpha - \bar{\alpha}_{pi},$$

$$\beta_1 = \beta - \bar{\beta}_{pi},$$

$$\gamma_1 = \gamma - \bar{\gamma}_{pi}, \text{ and}$$

$$\phi = \tan^{-1} M_2 .$$

Then, by examination of the relationship between the two axis systems,

$$Z_1 = \beta_1 \sin\phi + \gamma_1 \cos\phi,$$
$$Y_1 = \beta_1 \cos\phi - \gamma_1 \sin\phi,$$
$$X_1 = \alpha_1 \qquad (73)$$
$$\gamma_1 = Z_1 \cos\phi - Y_1 \sin\phi,$$
$$\beta_1 = Y_1 \cos\phi + Z_1 \sin\phi.$$

So,

$$I_{\overline{yy}_{pi}} = \rho \int X^2\, dv + \rho \int Z^2\, dv,$$

$$I_{\overline{yy}_{pi}} = I_{\overline{\gamma\gamma}_{pi}} \sin^2\phi + I_{\overline{\beta\beta}_{pi}} \cos^2\phi$$
$$+2 P_{\overline{\gamma\beta}_{pi}} \sin\phi \cos\phi, \qquad (74)$$

$$I_{\overline{yy}_{pi}} = I_{\overline{\gamma\gamma}_{pi}} \sin^2\phi + I_{\overline{\beta\beta}_{pi}} \cos^2\phi. \qquad (74a)$$

Also,

$$I_{zz_{pi}} = \rho \int X_1^2\, dv + \rho \int Y_1^2\, dv,$$

$$I_{\overline{zz}_{pi}} = I_{\overline{\gamma\gamma}_{pi}} \cos^2\phi + I_{\overline{\beta\beta}_{pi}} \sin^2\phi$$
$$-2 P_{\overline{\beta\gamma}_{pi}} \sin\phi \cos\phi, \qquad (75)$$

$$I_{zz_{pi}} = I_{\bar\gamma\bar\gamma_{pi}} \cos^2\phi + I_{\bar\beta\bar\beta_{pi}} \sin^2\phi. \qquad (75a)$$

Then,

$$P_{\overline{xy}_{pi}} = \rho \int X_1\, Y_1\, dv,$$

$$P_{\overline{xy}_{pi}} = P_{\overline{\alpha\beta}_{pi}} \cos\phi - P_{\overline{\alpha\gamma}_{pi}} \sin\phi, \qquad (76)$$

$$P_{\overline{xy}_{pi}} = P_{\overline{\alpha\beta}_{pi}} \cos\phi. \qquad (76a)$$

Also,

$$P_{\overline{xz}_{pi}} = \rho \int X_1 \ Z_1 \ dv,$$

$$P_{\overline{xz}_{pi}} = P_{\overline{\alpha\beta}_{pi}} \sin \phi + P_{\overline{\alpha\gamma}_{pi}} \cos\phi, \qquad (77)$$

$$P_{\overline{xz}_{pi}} = P_{\overline{\alpha\beta}_{pi}} \sin\phi. \qquad (77a)$$

Finally,

$$P_{\overline{YZ}_{pi}} = \rho \int Y_1 \ Z_1 \ dv,$$

$$P_{\overline{yz}_{pi}} = (I_{\overline{\gamma\gamma}_{pi}} - I_{\overline{\beta\beta}_{pi}}) \sin \phi \cos \phi$$
$$+ P_{\overline{\beta\gamma}_{pi}} (\cos^2 \phi - \sin^2 \phi),$$

$$P_{\overline{yz}_{pi}} = (I_{\overline{\gamma\gamma}_{pi}} - I_{\overline{\beta\beta}_{pi}}) \sin \phi \cos\phi. \qquad (78a)$$

Next consider a part that is considered to be a beam. Define the following:

ψ = angle that the beam makes with the yz plane,

c = depth of the beam at the inboard end,

d = depth of the beam at the outboard end,

m_4 = slope of the upper edge of the beam with respect to the xy plane, and

m_3 = slope of the lower edge of the beam with respect to the xy plane.

By a proof similar to that used in (30),

$$R_2 = \overline{Y}_p - \frac{2 (m_4 - m_3) \ ly^2 + 3 c \ ly}{3 (m_4 - m_3) \ ly + 6 c} \quad . \qquad (79)$$

Now,

$$m_3 = M_2 + (\frac{C - D}{2\ S})$$

$$m_4 = M \quad - (\frac{C - D}{2\ S}) \text{ and} \tag{80}$$

$$c = C + R_2\ (m_4 - m_3).$$

Simultaneous solution of (79) and (80) will now yield R_2, m_3, m_4, and c. Obviously now

$$d = c + ly\ (m_4 - m_3), \tag{81}$$

$$\Psi = \tan^{-1} \frac{lx}{ly}\ ,$$

$$R_1 = \overline{X}_p - (\overline{Y}_p - R_2)\ \tan 4, \text{ and} \tag{83}$$

$$R_3 = \overline{Z}_p - (\overline{Y}_p - R_2)\ M_2 - \frac{C}{2} \tag{84}$$

Now consider that portion of a beam contained in I_i. Examination quickly shows that equations (43) through (47) also are valid for the beam. Now define C_0, C_1, C_2, $\cdots$, C_n as the depth of the beam at Y R_2, Y Y_1, $\cdots$, Y (R_2 ly) respectively. Then:

(1) if $Y_{i-1} \leq R_2 < (R_2 + ly) \leq Y_i$,

$C_{i-1} = C$, and

$C_i = d$; (85)

(2) if $R_2 < Y_{i-1} < (R_2 + ly) \leq Y_i$,

$C_{i-1} = C + (m_4 - m_3)\ (Y_{i-1} - R_2)$, and (86)

$C_i = d$;

(3) If $Y_{i-1} \leq R_2 < Y_i \leq (R_2 + ly)$,

$$C_{i-1} = C \text{ and}$$
$$C_i = C + (m_4 - m_3)(Y_i - R_2); \tag{87}$$

(4) if $R_2 \leq Y_{i-1} < Y_i \leq (R_2 + ly)$,

$$C_{i-1} = C + (m_4 - m_3)(Y_{i-1} - R_2), \text{ and}$$
$$C_i = C + (m_4 - m_3)(Y_i - R_2); \tag{88}$$

(5) finally, if $R_2 \geqslant Y_i$ or $Y_{i-1} \geqslant (R_2 + ly)$, assume

$$C_{i-1} = C, \text{ and}$$
$$C_i = b. \tag{89}$$

Using the same type of proof as used in (55),

$$W_{pi} = \frac{l_{pi}\ (C_{i-1} + C_i)}{ly\ (c+d)}\ W_p. \tag{90}$$

In a manner similar to that used in (30),

$$\overline{Y}_{pi} = R_2 + \frac{2\ (m_4 - m_3)\ (l_{pi})^2 + 3\ C_{i-1}\ (l_{pi})}{3\ (m_4 - m_3)\ (l_{pi}) + 6\ C_{i-1}}. \tag{91}$$

Also,

$$\overline{X}_{pi} = R_1 + \overline{Y}_{pi} \tan \Psi, \text{ and} \tag{92}$$

$$\overline{Z}_{pi} = R3 + C/2 + (\overline{Y}pi - R2)\ M_2. \tag{93}$$

Define,

$$l_{pi\psi} = l_{pi} \sec \psi. \tag{94}$$

Now define a set of orthogonal axes such that the beam lies in the $\beta_2\gamma_2$ plane, with C_{i-1} also lying in the $\alpha_2\gamma_2$ plane and $(Z = R_3 + C/2 + Y_{i-1} M_2)$ lying in the $\alpha_2\beta_2$ plane. So, the co-ordinates of the center of gravity, $\overline{\alpha}_{2pi}$, $\overline{\beta}_{2pi}$, $\overline{\gamma}_{2pi}$, are defined as follows:

$$\bar{\alpha}_{2pi} = 0,$$

$$\bar{\beta}_{2pi} = (\bar{Y}_{pi} - \max \left[Y_{i-1}, R_2 \right]) \sec \psi, \text{ and}$$

$$\bar{\gamma}_{2pi} = (\bar{Y}_{pi} - \max \left[Y_{i-1}, R_2 \right]) m_2.$$

Let $I_{\bar{\alpha}\bar{\alpha}\,2pi}$, $I_{\bar{\beta}\bar{\beta}\,2pi}$, ..., $I_{\bar{\beta}\bar{\gamma}\,2pi}$, be the moments and products of inertia of that portion of the part in I_i with respect to the indicated centroidal axes.

Then, by a proof similar to that used in (62) - (78),

$$I_{\bar{\alpha}\bar{\alpha}_{2pi}} = \frac{2\, W_{pi}}{C_{i-1} + C_i} \left[(l_{pi\psi})^3 \left(\frac{m_4^{\,3}}{12} + \frac{m_4}{4} - \frac{m_3^{\,3}}{12} - \frac{m_3}{4} \right) + (l_{pi\psi})^2 \cdot \left(\frac{m_4^{\,2}\, C_{i-1}}{3} + \frac{C_{i-1}}{3} \right) \right.$$

$$\left. \frac{m_4\, C_{i-1}^{\,2}\, l_{pi\psi}}{2} + \frac{(C_{i-1})^3}{3} \right]$$

$$- W_{pi} \left(\bar{\beta}_{pi}^{\,2} + \bar{\gamma}_{pi}^{\,2} \right), \tag{95}$$

$$I_{\bar{\gamma}\bar{\gamma}_{2pi}} = \frac{W_{pi}}{12(C_{i-1} + C_i)} \left\{ \left[6 (l_{pi\psi})^3 \right] (m_4 - m_3) + 2\, C_{i-1} \left[4 (l_{pi\psi})^2 \right] \right\}$$

$$- W_{pi} \left[(\bar{\alpha}_{2pi})^2 + (\bar{\beta}_{2pi})^2 \right], \tag{96}$$

$$I_{\beta\beta\,2pi} = \frac{W_{pi}}{12(C_{i-1} + C_i)} \left[2\, (l_{pi\psi})^3 (m_4^{\,3} - m_3^{\,3}) \right.$$

$$+ 8\, C_{i-1} (m_4^{\,2}\, l_{pi\psi}^{\,2} + C_{i-1}^{\,2})$$

$$+ 12\, m_4\, (l_{pi\psi})\, (C_{i-1})^2 \Big]$$
$$- W_{pi} \left[(\bar{\alpha}_{2pi})^2 + (\bar{\gamma}_{2pi})^2 \right], \qquad (97)$$

$$P_{\overline{\alpha\beta}_{2pi}} = P_{\overline{\alpha\gamma}_{2pi}} = 0, \qquad (98)$$

$$P_{\overline{\gamma\beta}_{2pi}} = \frac{W_{pi}\, (l_{pi\psi})}{12\, (C_{i-1} + C_i)} \Big[3\, (l_{pi\psi})^2\, (m_4^{\;2} - m_3^{\;2})$$
$$+ C_{i-1}\, (8 m_4\, l_{pi\psi} + 6\, C_{i-1}) \Big]$$
$$- W_{pi}\, (\bar{\gamma}_{2pi})\, (\bar{\beta}_{2pi}), \qquad (99)$$

$$I_{\overline{zz}_{pi}} = I_{\overline{\gamma\gamma}_{2pi}}, \qquad (100)$$

$$I_{\overline{yy}_{pi}} = I_{\overline{\alpha\alpha}_{2pi}} \sin^2 \psi + I_{\overline{\beta\beta}_{2pi}} \cos^2 \psi, \qquad (101)$$

$$I_{\overline{xx}_{pi}} = I_{\overline{\alpha\alpha}_{2pi}} \cos^2 \psi + I_{\overline{\beta\beta}_{2pi}} \sin^2 \psi, \qquad (102)$$

$$P_{\overline{yz}_{pi}} = P_{\overline{\gamma\beta}_{2pi}} \cos \psi, \qquad (103)$$

$$P_{\overline{xz}_{pi}} = P_{\overline{\gamma\beta}_{2pi}} \sin \psi, \text{ and} \qquad (104)$$

$$P_{\overline{xy}_{pi}} = (I_{\overline{\alpha\alpha}_{2pi}} - I_{\overline{\beta\beta}_{2pi}}) \sin \psi \cos \psi. \qquad (105)$$

Vertical Surfaces

Consider a mapping, δ, of a set, H, onto a set, V, as defined below.

$\underline{H}$		$\underline{V = \delta H}$
R_1	$\longrightarrow$	R_1
R_2	$\longrightarrow$	R_3

R_3	$\longrightarrow$	R_2
W_p	$\longrightarrow$	W_p
$\overline{X}_p$	$\longrightarrow$	$\overline{X}_p$
$\overline{Y}_p$	$\longrightarrow$	$\overline{Z}_p$
$\overline{Z}_p$	$\longrightarrow$	$\overline{Y}_p$
lx	$\longrightarrow$	lx
ly	$\longrightarrow$	lz
lz	$\longrightarrow$	ly
$W_{p(i)}$	$\longrightarrow$	$W_{p(i)}$
$\overline{X}_{p(i)}$	$\longrightarrow$	$\overline{X}_{p(i)}$
$\overline{Y}_{p(i)}$	$\longrightarrow$	$Z_{p(i)}$
$\overline{Z}_{p(i)}$	$\longrightarrow$	$Y_{p(i)}$
$l_{p(i)}$	$\longrightarrow$	$l_{p(i)}$
$I_{\overline{xx}p(i)}$	$\longrightarrow$	$I_{\overline{xx}p(i)}$
$I_{\overline{yy}p(i)}$	$\longrightarrow$	$I_{\overline{zz}p(i)}$
$I_{\overline{zz}p(i)}$	$\longrightarrow$	$I_{\overline{yy}p(i)}$
$P_{xyp(i)}$	$\longrightarrow$	$P_{xzp(i)}$
$P_{\overline{xz}p(i)}$	$\longrightarrow$	$P_{\overline{xy}p}$
$P_{\overline{yz}p}$	$\longrightarrow$	$P_{\overline{yz}p}$

$(Y_0, Y_1, \cdots Y_{i-1}, Y_i, \cdots Y_n) \longrightarrow (Z_0, Z_1, \cdots Z_{i-1}, Z_i, \cdots Z_n)$

$$X \longrightarrow X$$
$$Y \longrightarrow Y$$
$$Z \longrightarrow Y$$
$$K_x \longrightarrow K_x$$
$$K_y \longrightarrow K_z$$

Think of removing all of the terms in equations (30) - (105) that are elements of H and replacing them with their respective mates, $\mathcal{J}$H, from V. It can be proven that the resulting equations are valid for vertical surfaces.

Chapter Bibliography

1. Haggard, Chester, The Calculation of the Mass Moments of Inertia of Aircraft, Society of Aeronautical Weight Engineers Paper Number 148, 1956

2. Society of Aeronautical Weight Engineers, SAWE Weight Handbook, Volume I, 1944

CHAPTER V

CALCULATION OF INERTIA CHARACTERISTICS

General

Define the following basic symbols:

W = weight,

$\overline{X}$, $\overline{Y}$, $\overline{Z}$ = co-ordinates of the center of gravity,

$I_{\overline{xx}}$, $I_{\overline{yy}}$, $I_{\overline{zz}}$ = moments of inertia about the subscript axis, and

$P_{\overline{xy}}$, $P_{\overline{xz}}$, $P_{\overline{yz}}$ = products of inertia about the subscript axes.

Following are definitions of subscripts to be used in conjunction with the above symbols:

I = the portion of all parts contained in I_i,

C = a complete component, and

A = the complete airplane.

Intervals of a Component

Consider any interval, I_i, on any component of the airplane. A finite number of parts extend through this interval. The quantities W_{pi}, $\overline{X}_{pi}$, $\overline{Y}_{pi}$, $\overline{Z}_{pi}$, $I_{\overline{xx}pi}$, $I_{\overline{yy}pi}$, $I_{\overline{zz}pi}$, $P_{\overline{xy}pi}$, $P_{\overline{xz}pi}$, and $P_{\overline{yz}pi}$ can be computed for the portion of each part in I_i by the equations of the previous chapter. The inertia characteristics of the portion of the total component contained in I_i can then be written almost directly from definition:

$$W_i = \int_{I_i} \rho \, dv = \sum W_{pi}, \tag{106}$$

$$\bar{X}_i = \frac{\int_{I_i} X \rho \, dv}{\int_{I_i} \rho \, dv} = \frac{\sum (W_{pi} \bar{X}_{pi})}{W_i}, \tag{107}$$

$$\bar{Y}_i = \frac{\int_{I_i} Y \rho \, dv}{\int_{I_i} \rho \, dv} = \frac{\sum (W_{pi} Y_{pi})}{W_i}, \tag{108}$$

$$\bar{Z}_i = \frac{\int_{I_i} Z \rho \, dv}{\int_{I_i} \rho \, dv} = \frac{\sum (W_{pi} Z_{pi})}{W_i}, \tag{109}$$

$$I_{\overline{xx}_i} = \int_{I_i} \rho \, (r_{\bar{x}})^2 \, dv = \int_{I_i} \left[(Y - Y_i)^2 + (Z - Z_i)^2 \right] \rho \, dv$$

$$= \sum I_{\overline{xx}_{pi}} + \sum (W_{pi} Y_{pi}{}^2) + \sum (W_{pi} \bar{Z}^2{}_{pi})$$

$$- (\sum W_{pi}) (\bar{Z}_i{}^2 + \bar{Y}_i{}^2), \tag{110}$$

$$I_{\overline{yy}_i} = \int_{I_i} \rho \, (r_{\overline{y}})^2 \, dv,$$

$$= \sum I_{\overline{yy}_{pi}} + \sum (W_{pi} \, \overline{X}_{pi}{}^2) + \sum (W_{pi} \, \overline{Z}_{pi}{}^2)$$

$$- (\sum W_{pi}) \, (\overline{X}_i{}^2 + \overline{Z}_i{}^2), \qquad (111)$$

$$I_{\overline{zz}_i} = \int_{I_i} \rho \, (r_{\overline{z}})^2 \, dv,$$

$$= \sum I_{\overline{zz}_{pi}} + \sum (W_{pi} \, \overline{X}_{pi}{}^2) + \sum (W_{pi} \, \overline{Y}_{pi}{}^2)$$

$$- (\sum W_{pi}) \, (\overline{X}_i{}^2 + \overline{Y}_i{}^2), \qquad (112)$$

$$P_{\overline{xy}_i} = \int_{I_i} \rho \, r_{\overline{xy}} \, dv,$$

$$= \sum (W_{pi} \, \overline{X}_{pi} \, \overline{Y}_{pi}) - \overline{X}_i \, \overline{Y}_i \, W_i + \sum P_{\overline{xy}_{pi}}, \qquad (113)$$

$$P_{\overline{xz}_i} = \int_{I_i} \rho \, r_{\overline{xz}} \, dv,$$

$$= \sum (W_{pi} \, \overline{X}_{pi} \, \overline{Z}_{pi}) - \overline{X}_i \, \overline{Z}_i \, W_i + \sum P_{\overline{xz}_{pi}}, \text{ and} \qquad (114)$$

$$P_{\overline{yz}_i} = \int_{I_i} \rho \, r_{\overline{yz}} \, dv,$$

$$P_{\overline{yz}_i} = (W_{pi}\ \overline{Y}_{pi}\ \overline{Z}_{pi}) - \overline{Y}_i \overline{Z}_i W_i \qquad P_{\overline{yz}_{pi}}. \tag{115}$$

Complete Component

Each component may be thought of as being composed of a finite number of segments, a segment being that portion of a component within a given interval. The inertia characteristics of any component then can be proven, by methods similar to those used above, to be

$$W_c = \Sigma_c\ W_i, \tag{116}$$

$$\overline{X}_c = \frac{\Sigma_c\ (W_i\ \overline{X}_i)}{W_c} \tag{117}$$

$$\overline{Y}_c = \frac{\Sigma_c (W_i\ \overline{Y}_i)}{W_c} \tag{118}$$

$$\overline{Z}_c = \frac{\Sigma_c\ (W_i\ \overline{Z}_i)}{W_c} \tag{119}$$

$$I_{\overline{xx}_c} = \Sigma_c\ (W_i\ \overline{Y}_i^2) + \Sigma_c\ (W_i\ \overline{Z}_i^2) + \Sigma_c\ I_{\overline{xx}_i}$$

$$-(\overline{Y}_c^{\,2} + \overline{Z}_c^{\,2})\ W_c, \tag{120}$$

$$I_{\overline{yy}_c} = \Sigma_c\ (W_i\ X_i^2) + \Sigma_c\ (W_i\ Z_i^2) + \Sigma_c\ I_{\overline{yy}_i}$$

$$-\ (\overline{Z}_c^{\,2} + \overline{X}_c^{\,2})\ W_c, \tag{121}$$

$$P_{\overline{xy}_c} = \Sigma_c\ (W_i \overline{Y}_i \overline{Z}_i) - \overline{X}_c\ \overline{Y}_c\ W_c + \Sigma_c\ P_{\overline{yz}_i}, \tag{122}$$

$$I_{\overline{zz}_c} = \Sigma_c\ (W_i\ X_i^2) + \Sigma_c\ (W_i\ Y_i^2) + \Sigma_c\ I_{zz_i}$$

$$- (\overline{X}_c{}^2 + \overline{Y}_c{}^2)\ W_c, \tag{123}$$

$$P_{\overline{xz}_c} = \Sigma_c\ (W_i\ \overline{X}_i\ \overline{Z}_i) - \overline{X}_c \overline{Z}_c \overline{W}_c + \Sigma_c\ P_{\overline{xz}_i}, \text{ and} \tag{124}$$

$$P_{\overline{yz}_c} = \Sigma_c\ (W_i\ \overline{Y}_i\ \overline{Z}_i) - \overline{Y}_c \overline{Z}_c \overline{W}_c + \Sigma_c\ P_{\overline{yz}_i}. \tag{125}$$

Complete Airplane

In the same manner as above, the inertia characteristics of the total airplane may be expressed as

$$W_A = \Sigma\ W_c, \tag{126}$$

$$\overline{X}_A = \frac{\Sigma\ (W_c\ \overline{X}_c)}{W_A}, \tag{127}$$

$$\overline{Y}_A = \frac{\Sigma\ (W_c\ \overline{Y}_c)}{W_A}, \tag{128}$$

$$\overline{Z}_A = \frac{\Sigma\ (W_c\ Y_c)}{W_A}, \tag{129}$$

$$I_{\overline{xx}_A} = \Sigma\ (W_c\ \overline{Y}_c{}^2) + \Sigma\ (W_c\ \overline{Z}_c{}^2) + \Sigma\ I_{\overline{xx}_c}$$
$$-(Y_A{}^2 + \overline{Z}_A{}^2)\ W_A, \tag{130}$$

$$I_{\overline{yy}_A} = \Sigma\ (W_c\ \overline{X}_c{}^2) + \Sigma\ (W_c\ \overline{Z}_c{}^2) + \Sigma\ I_{\overline{yy}_c}$$
$$-(Z_A{}^2 + X_A{}^2)\ W_A, \tag{131}$$

$$I_{\overline{zz}_A} = \Sigma\ (W_c\ \overline{X}_c{}^2) + \Sigma\ (W_c\ \overline{Y}_c{}^2) + \Sigma\ I_{zz_c}$$

$$- (\bar{X}_A{}^2 + \bar{Y}_A{}^2)\ W_A, \tag{132}$$

$$P_{\overline{xy}_A} = \Sigma (W_c\ \bar{X}_c\ \bar{Y}_c) - \bar{X}_A\ \bar{Y}_A\ W_A + \Sigma\ P_{\overline{xy}c} \tag{133}$$

$$P_{\overline{xz}_A} = \Sigma (W_c\ \bar{X}_c\ \bar{Z}_c) - \bar{X}_A\ \bar{Z}_A\ W_A + \Sigma\ P_{\overline{xz}_c}, \text{ and} \tag{134}$$

$$P_{\overline{yz}_A} = \Sigma (W_c\ \bar{Y}_c\ \bar{Z}_c) - \bar{Y}_A\ \bar{Z}_A\ W_A + \Sigma\ P_{\overline{yz}_c}. \tag{135}$$

CHAPTER VI

CALCULATION OF UNIT INERTIAL LOADS ON AN AIRPLANE

Inertial Loads Within A Rigid Body

Consider a body that has been partitioned in the manner specified in Chapter IV. Define the body reference axes to be fixed relative to the body and have its origin at O. The body and hence the body reference axes are moving with respect to a set of fixed ground axes. Define the motion as follows:

$\vec{\gamma}$ = linear acceleration of O,

$\vec{W}$ = angular velocity of the body, and

$\vec{\alpha}$ = angular acceleration of the body.

Now consider any interval, I_p, of the body. Select finite points (n, n-1, ···, (p + 2), (P + 1), p, (p - 1), ···, 2, 1) in the intervals (I_n, I_{n-1}, ···, I_{p+1}, I_p, I_{p-1}, ···, I_2, I_1). The portion of the body in interval I_{p+1} exerts a force $\vec{X}_{p+1}$ and a moment $\vec{L}_{p+1}$ on interval I_p at the point (p+1). In like manner, the portion of the body in interval I_{p-1} exerts a force $-\vec{X}_p$ and a moment $-\vec{L}_p$ on the interval I_p at the point (P).

The portion of the body contained in interval I_p exerts $-\vec{F}_p$, a force, and $-\vec{M}_p$, a moment, at some reference point (R_p). Now define the following.

$\vec{x}_p$ = the vector from O to point (P),

$\vec{r}_p$ = the vector from 0 to point (R_p),

$\vec{x}$ = the vector from 0 to any point,

$\vec{z}$ = the vector from point (R_p) to any point, and

$\vec{x}_{p+1}$ = the vector from 0 to point $(P+1)$.

Summing the forces and moment on the interval I_p, the following is obtained:

$$\underline{\vec{X}}_p = \underline{\vec{X}}_{p+1} - \vec{F}_p \text{, and} \tag{131}$$

$$\vec{L}_p = \vec{L}_{p+1} - \vec{M}_p + (\vec{x}_{p+1} - \vec{x}_p) \times \underline{\vec{X}}_{p+1}$$
$$- (\vec{r}_p - \vec{x}_p) \times \vec{F}_p. \tag{132}$$

The relationship between the Body Axes and Ground Axes are defined as follows:

$\vec{r}_0$ = the vector from the origin of the ground axes to the origin of the body axes

$\vec{r}$ = the vector from the origin of the ground axes to any point.

A time derivative with respect to the ground axes and body axes will be indicated by a small circle $[\,\circ\,]$ and a dot $[\,\cdot\,]$ respectively.

For the remainder of the paper, define $dm = \rho\, dv$. Then $\vec{V}$ and $\vec{a}$, the true velocity and acceleration respectively of dm may be expressed

$\vec{V}$ $\dot{\vec{r}}$, and

$$\vec{a} = \ddot{\vec{r}}.$$

Also

$$\vec{\gamma} = \ddot{\vec{r}}_0, \text{ and}$$
$$\vec{\alpha} = \dot{\vec{W}}.$$

Wood (4) proves the following relationship for any vector, $\vec{h}$:

$$\dot{\vec{h}} = \overset{\circ}{\vec{h}} + \vec{W} \times \vec{h}.$$

For $\vec{W}$ itself,

$$\dot{\vec{W}} = \overset{\circ}{\vec{W}} + \vec{W} \times \vec{W} = \overset{\circ}{\vec{W}}.$$

Now consider,

$$\vec{r} = \vec{r}_0 + \vec{x},$$
$$\dot{\vec{r}} = \dot{\vec{r}}_0 + \overset{\circ}{\vec{x}} + \vec{W} \times \vec{x},$$
$$\ddot{\vec{r}} = \ddot{\vec{r}}_0 + \overset{\circ}{\overline{\left[\overset{\circ}{\vec{x}} + \vec{W} \times \vec{x}\right]}} + \vec{W} \times \left[\overset{\circ}{\vec{x}} + \vec{W} \times \vec{x}\right]$$
$$= \ddot{\vec{r}}_0 + \overset{\circ\circ}{\vec{x}} + \vec{W} \times \overset{\circ}{\vec{x}} + \overset{\circ}{\vec{W}} \times \vec{x} + \vec{W} \times \overset{\circ}{\vec{x}}$$
$$+ \vec{W} \times (\vec{W} \times \vec{x}).$$

Then

$$\vec{a} = \vec{\gamma} + \overset{\circ\circ}{\vec{x}} + 2\,\vec{W} \times \overset{\circ}{\vec{x}} + \vec{\alpha} \times \vec{x} + \vec{W} \times (\vec{W} \times \vec{x}).$$

But since the body is fixed relative to the body axes,

$$\overset{\circ\circ}{\vec{x}} = \overset{\circ}{\vec{x}} = 0.$$

Therefore, $\vec{a} = \vec{\gamma} + \vec{\alpha} \times \vec{x} + \vec{W} \times (\vec{W} \times \vec{x})$,

or using the notation of Wood (3)

$$a = \gamma + \alpha^{\vee} x + W^{\vee}(W^{\vee} x). \qquad (133)$$

Furthermore,

$$a = \gamma + (\alpha^{\vee} + W^{\vee} W^{\vee})\, x$$
$$= \gamma + (\alpha^{\vee} + W^{\vee} W^{\vee})\,(r_p + Z). \qquad (134)$$

Let $\int_p$ indicate the integral over the interval I_p.

Define

$$m_p = \int_p dm. \tag{135}$$

It follows that

$$m_p\ z_p = \int_p Z\, dm,$$

where $\vec{Z}_p$ is the vector from (R_p) to the center of gravity of the portion of the body contained in I_p.

Now define,

$$J_p = -\int_p Z^\vee Z^\vee\, dm. \tag{136}$$

From Newton's law,

$$F_p = \int_p a\ dm$$

$$= \int_p \left[\gamma + (\alpha^\vee + w^\vee w^\vee)(r_p + z) \right] dm, \tag{137}$$

and

$$M_p = \int_p Z^\vee a\ dm$$

$$= \int_p \left[Z^\vee \gamma + Z^\vee (\alpha^\vee + w^\vee w^\vee)(r_p + z) \right] dm.$$

By application of (6) and (8),

$$M_p = \int_p \left[Z^\vee \gamma - Z^\vee (r_p + Z)^\vee \alpha - w^\vee Z (r_p + z)' w \right.$$

$$\left. - w' w Z^\vee (r_p + Z) \right] dm. \tag{138}$$

By examination of (131), (132), (137), and (138), equations (131) and (132) may be expressed in the forms

$$\underline{\bar{X}}_p = s_p\,(-\gamma) + (-\alpha^{\vee} - W^{\vee} W^{\vee})\; C_p, \text{ and} \qquad (139)$$

$$L_p = b_p^{\vee}\,(-\gamma) + Q_p\,\alpha + W^{\vee} H_p\, W + W^2 d_p,$$

where:

s_p is a scalar,

b_p, c_p, d_p are column matrices, and

H_p, Q_p are 3X3 matrices.

Now define $\Delta_p s$, $\Delta_p C$, $\Delta_p b$, $\Delta_p d$, $\Delta_p H$, and $\Delta_p Q$ as the differences in s, c, b, d, H, and Q due to consideration of the loads imposed by the body in I_p and the transfer of loads to point (P).

From equation (139),

$$\underline{\bar{X}}_{p+1} = s_{p+1}\,(-\gamma) + (-\alpha^{\vee} - W^{\vee} W^{\vee})\; C_{p+1},$$

$$\begin{aligned}
(x_{p+1} - x_p)^{\vee}\, \underline{\bar{X}}_{p+1} &= \left[s_{p+1}\,(x_{p+1} - x_p)^{\vee} \right] (-\gamma) \\
&\quad + (x_{p+1} - x_p)^{\vee} (-\alpha^{\vee} - W^{\vee} W^{\vee})\; C_{p+1} \\
&= \left[s_{p+1}\,(x_{p+1} - x_p)^{\vee} \right] (-\gamma) \\
&\quad + W^{\vee} \left[(x_{p+1} - x_p)\,(C_{p+1}') \right] W \\
&\quad + W^2 \left[(x_{p+1} - x_p)^{\vee}\, C_{p+1} \right] \\
&\quad + \left[(x_{p+1} - x_p)^{\vee} C_{p+1}^{\vee} \right] \alpha.
\end{aligned}$$

Therefore, due to $(x_{p+1} - x_p)^{\vee} \underline{\bar{X}}_{p+1}$,

$$\Delta_p b = s_{p+1} (x_{p+1} - x_p)$$

$$\Delta_p Q = (x_{p+1} - x_p)^{\vee} (C_{p+1})^{\vee}, \qquad (141)$$

$$\Delta_p H = (x_{p+1} - x_p)(C_{p+1})', \text{ and}$$

$$\Delta_p d = (x_{p+1} - x_p)^{\vee} C_{p+1}.$$

From equation (139),

$$\underline{\bar{X}}_p = s_p (-\gamma) + (-\alpha^{\vee} - W^{\vee}W^{\vee}) \; C_p,$$

and

$$\underline{\bar{X}}_{p+1} = s_{p+1} (-\gamma) + (-\alpha^{\vee} - W^{\vee}W^{\vee}) \; C_{p+1},$$

but

$$-F_p = \underline{\bar{X}}_p - \underline{\bar{X}}_{p+1}$$

$$= \Delta_p s(-\gamma) + (-\alpha^{\vee} - W^{\vee}W^{\vee}) \Delta_p C. \qquad (142)$$

Now,

$$(r_p - x_p)^{\vee} (-F_p) = \Delta_p s(r_p - x_p)^{\vee} (-\gamma)$$

$$+ (r_p - x_p)^{\vee} (-\alpha^{\vee} - W^{\vee}W^{\vee}) \Delta_p C$$

$$= \left[\Delta_p s(r_p - x_p)^{\vee} \right] (-\gamma)$$

$$+ W^{\vee} \left[(r_p - x_p)(\Delta_p C') \right] W$$

$$+ W^2 \left[(r_p - x_p)^{\vee} \Delta_p C \right]$$

$$+ \left[(r_p - x_p)^{\vee} \Delta_p C^{\vee} \right] \alpha.$$

Hence due to $(r_p - x_p)^{\vee}(-F_p)$,

$$\Delta_p b = \Delta_p s(r_p - x_p),$$

$$\Delta_p Q = (r_p - x_p)^{\vee} (\Delta_p C)^{\vee}, \qquad (143)$$

$$\Delta_p H = (r_p - x_p) (\Delta_p C)', \text{ and}$$

$$\Delta_p d = (r_p - s_p)^{\vee} (\Delta_p C).$$

From equation (137),

$$(-F_p) = \int_p dm \, (-\gamma) + (-\alpha^{\vee} - W^{\vee} W^{\vee}) \int_p (r_p + Z) \, dm. \qquad (144)$$

Examination of equations (135), (142), and (144) yields

$$\Delta_p s = \int_p dm = m_p, \text{ and}$$

$$\Delta_p C = \int_p (r_p + Z) \, dm = m_p (r_p + \bar{Z}_p). \qquad (145)$$

From equation (138),

$$- M_p = \left[\int_p Z^{\vee} dm \right] (-\gamma) + \int_p \left[Z^{\vee} (-\alpha^{\vee} - W^{\vee} W^{\vee}) (r_p \; Z) \right] dm,$$

$$= \left[\int_p Z^{\vee} dm \right] (-\gamma) - \int_p \left[Z^{\vee} (\alpha^{\vee}) (r_p + Z) \, dm \right]$$

$$- \int_p \left[Z^{\vee} W^{\vee} W^{\vee} (r_p + Z) \, dm \right]$$

$$= \left[\int_p Z^{\vee} dm \right] (-\gamma) + \left[\int_p Z^{\vee} r_p^{\vee} \, dm \right.$$

$$\left. + \int_p Z^{\vee} Z^{\vee} \, dm \right] \alpha + W^{\vee} \left[\int_p Z \, r_p' \, dm \right.$$

$$+ \int_\rho Z^\vee Z^\vee \Big] \omega + \omega^2 \left[\int_\rho Z^\vee r_p \, dm \right]. \tag{146}$$

From elementary considerations,

$$\int_\rho Z^\vee Z^\vee \, dm = -J_p,$$

$$\int_\rho Z^\vee \, dm = \left[\int_\rho Z \, dm \right]^\vee$$

$$= (m_p \bar{Z}_p)^\vee,$$

$$\int_\rho Z^\vee r_p^\vee \, dm = m_p \, r_p^\vee \, \bar{Z}_p^\vee,$$

$$\int_\rho Z \, r_p' \, dm = mp \, \bar{Z}p \, rp', \text{ and}$$

$$\int_\rho Z \, rp \, dm = m_p \, \bar{Z}_p^\vee \, r_p.$$

Therefore, due to $(-F_p)$ and $(-M_p)$,

$$\Delta_p \, s = m_p, \text{ and}$$

$$\Delta_o \, C = m_p \, (r_p + Z_p) = m_p \, \bar{x}_p, \tag{147}$$

where $\bar{x}_p$ is the vector from 0 to the centroid of the body in the I_p interval. Also,

$$\Delta_p \, b = m_p \, \bar{Z}_p,$$

$$\Delta_p \, Q = m_p \, \bar{Z}_p^\vee r_p^\vee - J_p, \tag{147a}$$

$$\Delta_p \, d = m_p \, \bar{Z}_p^\vee \, r_p.$$

Finally,

$$\underline{X}_p = s_\rho \; (-\gamma) + (-\alpha^\vee - W^\vee W^\vee) \; C_p, \text{ and}$$

$$L_p = b_p^{\vee}(-\gamma) + Q_p\alpha + W^2 H_p W + W^2 d_p, \qquad (148)$$

where

$$s_p = s_{p+1} + \Delta_p s,$$

$$C_p = C_{p+1} + \Delta_p C,$$

$$b_p = b_{p+1} + \Delta_p b,$$

$$Q_p = Q_{p+1} + \Delta_p Q,$$

$$H_p = H_{p+1} + \Delta_p H,$$

$$d_p = d_{p+1} + \Delta_p d,$$

and where from equations (141), (143), and (147)

$$\Delta_p s = m_p,$$

$$\Delta_p C = m_p (r_p + \overline{Z}_p),$$

$$\Delta_p b = m_p \overline{Z}_p + m_p (r_p - x_p) + s_{p+1} (x_{p+1} - x_p),$$

$$\Delta_p d = m_p \overline{Z}_p^{\vee} r_p + m_p (r_p - x_p)^{\vee} (r_p + \overline{Z}_p)$$
$$+ (x_{p+1} - x_p)^{\vee} C_{p+1},$$

$$\Delta_p H = m_p \overline{Z}_p r_p' - J_p + m_p (r_p - x_p)(r_p + \overline{Z}_p)'$$
$$+ (x_{p+1} - x_p) C_{p+1}', \text{ and}$$

$$\Delta_p Q = m_p \overline{Z}_p^{\vee} r_p^{\vee} - J_p + m_p (r_p - x_p)^{\vee} (r_p + \overline{Z}_p)^{\vee}$$
$$+ (x_{p+1} - x_p)^{\vee} C_{p+1}^{\vee}.$$

Formulation of the Matrix of Unit Inertial Loads

If the subscript "p" is dropped, equations (148) may be written as

$$X = - s I \gamma + C^{\vee} \alpha + (-W^{\vee} W^{\vee})\, C, \text{ and} \qquad (149)$$

$$L = - b^{\vee} \gamma + Q \alpha + W^2 d + W^{\vee} H\, W, \qquad (150)$$

where I is the unit matrix. Equation (149) may be written in the form

$$X = A_1 + A_2 + A_3, \qquad (151)$$

where A_1, A_2, and A_3 are column matrices. It follows that

$$A_1 = - s I \gamma = \begin{bmatrix} -s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -s \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \qquad (152)$$

,

$$A_2 = C^{\vee} \alpha = \begin{bmatrix} 0 & -C_3 & C_2 \\ C_3 & 0 & -C_1 \\ -C & C & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \qquad (153)$$

, and

$$A_3 = (-W^{\vee} W^{\vee})\, C = \begin{bmatrix} 0 & W_3 & -W_2 \\ -W_3 & 0 & W_1 \\ W_2 & -W_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -W_3 & W_2 \\ W_3 & 0 & -W_1 \\ -W_2 & W_1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_1 w_3^2 + c_1 w_2^2 - c_2 w_1 w_2 - c_3 w_1 w_3 \\ -c_1 w_1 w_2 + c_2 w_3^2 + c_2 w_1^2 - c_3 w_2 w_3 \\ -c_1 w_1 w_3 - c_2 w_2 w_3 + c_3 w_2^2 + c_3 w_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & c_1 & c_1 & 0 & -c_3 & -c_2 \\ c_2 & 0 & c_2 & -c_3 & 0 & -c_1 \\ c_2 & 0 & c_2 & -c_3 & 0 & -c_1 \end{bmatrix} \begin{bmatrix} w_1^2 \\ w_2^2 \\ w_3^2 \\ w_2 w_3 \\ w_1 w_3 \\ w_1 w_2 \end{bmatrix} .$$

In like manner

$$L = B_1 + B_2 + B_3 + B_4, \tag{155}$$

where

$$B_1 = - b^{\vee}\gamma = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b & 0 & b \\ b_2 & -b_1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} , \tag{156}$$

$$B_2 = Q\alpha = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} , \tag{157}$$

$$B_3 = W^2 d = (W_1{}^2 \quad W_2{}^2 \quad W_3{}^2) \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\begin{bmatrix} d_1 & d_1 & d_1 \\ d_2 & d_2 & d_2 \\ d & d & d \end{bmatrix} \begin{bmatrix} W_1{}^2 \\ W_2{}^2 \\ W_3{}^2 \end{bmatrix} , \qquad (158)$$

$$B_4 = W^{\vee} H W \begin{bmatrix} 0 & -W_3 & W_2 \\ W_3 & 0 & -W_1 \\ -W_2 & W_1 & 0 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}$$

$$\begin{bmatrix} -W_3W_1h_{21} & -W_3W_2h_{22} & -W_3{}^2h_{23} & W_2W_1h_{31} & W_2{}^2h_{32} & W_2W_3h_{33} \\ W_3W_1h_{11} & W_3W_2h_{12} & W_3{}^2h_{13}-W_1{}^2h_{31} & -W_1W_2h_{32} & -W_1W_3h_{33} & \\ -W_2W_1h_{11} & -W_2{}^2h_{12} & -W_2W_3h_{13} & W_1{}^2h_{21} & W_1W_2h_{22} & W_1W_3h_{23} \end{bmatrix}$$

$$\begin{bmatrix} 0 & h_{32}-h_{23} & h_{33}-h_{22} & -h_{21} & h_{31} \\ -h_{31} & 0 & h_{13} & h_{12} & h_{11}-h_{33} & -h_{32} \\ h_{21} & -h_{12} & 0 & -h_{13} & h_{23} & h_{22}-h_{11} \end{bmatrix} \begin{bmatrix} W_1{}^2 \\ W_2{}^2 \\ W_3{}^2 \\ W_2W_3 \\ W_1W_3 \\ W_1W_2 \end{bmatrix} . \qquad (159)$$

Equations (151) and (155) now may be combined to form the complete matrix solution of the inertial loads shown on the following page.

$$
\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ L_1 \\ L_2 \\ L_3 \end{bmatrix}
=
\begin{bmatrix}
-s & 0 & 0 & 0 & -c_3 & c_2 & 0 & c_1 & c_1 & 0 & -c_3 & -c_2 \\
0 & -s & 0 & c_3 & 0 & -c_1 & c_2 & 0 & c_2 & -c_3 & 0 & -c_1 \\
0 & 0 & -s & -c_2 & c_1 & 0 & c_3 & c_3 & 0 & -c_2 & -c_1 & 0 \\
0 & b_3 & -b_2 & q_{11} & q_{12} & q_{13} & d_1 & d_1+h_{32} & d_1-h_{23} & h_{33}-h_{22} & -h_{21} & h_{31} \\
-b_3 & 0 & b_1 & q_{21} & q_{22} & q_{23} & d_2-h_{31} & d_2 & d_2+h_{13} & h_{12} & h_{11}-h_{33} & -h_{32} \\
b_2 & -b_1 & 0 & q_{31} & q_{32} & q_{33} & d_3+h_{21} & d_3-h_{12} & d_3 & -h_{13} & h_{23} & h_{22}-h_{11}
\end{bmatrix}
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ W_1^2 \\ W_2^2 \\ W_3^2 \\ W_2W_3 \\ W_1W_3 \\ W_1W_2 \end{bmatrix}
$$

Equation (160) may be written in the form

$$N = D\ G, \qquad (161)$$

where

$N =$ the 6 X 1 matrix,

$D =$ the 6 X 12 matrix, and

$G =$ the 12 X 1 matrix.

In actual applications, it will be more efficient to compute D independent of G. Then N can be determined for any maneuver merely by selecting the proper G and performing the matrix multiplication, D G.

Henceforth N will be matrix of inertial loads, D the matrix of unit inertial loads, and G the matrix of accelerations and velocities.

Transfer of the Unit Inertial Loads of an Attached Component into the Unit Inertial Loads of the Parent Component

Consider now the case where the inertial loads of a given component are dependent not only on its own inertial properties but also on the inertial properties of attached components. Such a component will be called a parent component.

Define "matrix of effective unit inertial loads" as the unit inertial loads introduced into point (P) of the parent component by the entire attached component under consideration.

The inertial loads of the parent component may be expressed in the form

$$N = (D_0 + D_1 + D_2 + \cdots + D_n)\ G, \tag{162}$$

where

N = the 6 X 1 matrix of inertial loads,

D_0 = the 6 X 12 matrix of unit inertial loads due to the inertial properties of the parent component,

$D_1, D_2, \cdots, D_n$ the 6 X 12 matrices of effective unit inertial loads of components 1, 2, 3, ..., n.

G = the 12 X 1 matrix of velocities and accelerations.

Define F_j as the matrix of unit inertial loads at point (1) in I, of any component, j.

Now for a given interval, I_p, in the parent component and a given attached component, j, it is reasonable to assume that there exists a matrix

$$K_J = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_6 \end{bmatrix}, \tag{163}$$

such that

$$D_J = K_J\ F_J. \tag{164}$$

Substitution of (164) into (162) yields

$$N = (D_0 + K_1F_1 + K_2F_2 + \cdots + K_JF_J + \cdots + K_nF_n)\ G. \qquad (166)$$

Equation (166) shows that once the unit inertial loads for the parent component are computed, the inertial loads due to any airolane maneuver can be computed merely by multiplying by the proper G.

BIBLIOGRAPHY

Haggard, Chester, The Calculation of the Mass Moments of Inertia of Aircraft, Society of Aeronautical Weight Engineers paper number 148, 1956

Society of Aeronautical Weight Engineers, SAWE Weight Handbook, Volume I, 1944

Wood, H. A., Matrices Vectors, and Tensors, SISM-1009/AWN-7-EI, Chance Vought Aircraft Publication, 1948

Wood, H. A., unpublished notes, Chance Vought Aircraft, Incorporated, Dallas, Texas.