INFINITE MATRICES

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INFINITE MATRICES

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CHAPTER I

INTRODUCTION

This paper will be mostly concerned with matrices of infinite order with elements which lie in Hilbert Space. All the properties of real and complex numbers and all the properties of infinite series and infinite sequences that are not listed below will be assumed.

Theorem 1 - 1: If \(a_1, a_2, \ldots\) is an infinite sequence and \(\sum_{n=1}^{\infty} |a_n|\) converges, then \(\sum_{n=1}^{\infty} a_n\) converges.

Theorem 1 - 2: If \(\sum_{p=1}^{\infty} a_p\) converges, then \(\lim_{n \to \infty} a_n = 0\), and for every \(\varepsilon > 0\), there is a \(N\) such that if \(m, n > N\), then \(\left|\sum_{p=1}^{n} a_p - \sum_{p=1}^{m} a_p\right| < \varepsilon\).

Theorem 1 - 3: If \(M\) is a bounded set of real numbers, then there is a \(\delta\) that is the least upper bound of \(M\).

In the treatment of infinite matrices, certain notations and definitions will be needed. These are:

Definition 1 - 1: If \(A\) is a matrix having a general element \(a_{pq}\), and \(x_1, x_2, \ldots\) is a sequence of numbers and \(y_1, y_2, \ldots\) is a sequence of numbers, then \(A(x, y) = \sum_{p=1}^{\infty} (\sum_{q=1}^{\infty} a_{pq} y_q) x_p = \sum_{p=1}^{\infty} a_{pq} x_p y_q\), provided each series involved converges.
Definition 1 - 2: \( A_n(x,y) = \sum_{p=1}^{n} \left( \sum_{q=1}^{n} a_{pq} y_q x_p \right) \).

Definition 1 - 3: If \( \alpha \) is a vector \( [a_1, a_2, \ldots] \), then \( \alpha' \) is \( (a_1, a_2, \ldots) \).

Definition 1 - 4: If \( c \) is any constant, then \( c\alpha \) is 
\[
\begin{bmatrix}
ca_1 \\
ca_2 \\
\vdots \\
\vdots
\end{bmatrix}.
\]

Definition 1 - 5: \( \alpha' \delta = [a_1, a_2, \ldots] \begin{bmatrix} d_1 \\vdots \\vdots \end{bmatrix} = 
\begin{bmatrix}
d_2 \\
\vdots \\
\vdots
\end{bmatrix} \sum_{p} a_{pq} d_p \).

Definition 1 - 6: If \( z \) is a complex number, then \( \overline{z} \) is defined to be the conjugate of \( z \) and \( z\overline{z} = |z|^2 \).

Definition 1 - 7: If \( z_1 \) and \( z_2 \) are complex numbers, then \( |z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2| \).

In addition to the theorems and definitions listed above, there is another theorem that will be used extensively in the proofs of theorems in the second and third chapters. This theorem is known as Schwarz Inequality, which shall now be proved.
Theorem 1-4: \( \left| \sum_{p=1}^{n} a_p b_p \right| \leq \sqrt{\sum_{p=1}^{n} |a_p|^2 \sum_{p=1}^{n} |b_p|^2} \), where the equality holds if and only if there exist a constant \( k \) such that \( a_p = k \bar{a}_p \), \( p = (1, 2, \ldots, n) \), or \( k \bar{a}_p = b_p \), \( p = (1, 2, \ldots, n) \).

Now if \( \bar{a}_p = 0 \), \( p = (1, 2, \ldots, n) \), then \( \left| \sum_{p=1}^{n} a_p b_p \right| = 0 = \sqrt{\sum_{p=1}^{n} |a_p|^2 \sum_{p=1}^{n} |b_p|^2} \) and \( \bar{a}_p = 0 \bar{b}_p \), \( p = (1, 2, \ldots, n) \). If \( b_p = 0 \), \( p = (1, 2, \ldots, n) \), then \( \left| \sum_{p=1}^{n} a_p b_p \right| = 0 = \sqrt{\sum_{p=1}^{n} |a_p|^2 \sum_{p=1}^{n} |b_p|^2} \) and \( b_p = 0 \bar{a}_p \), \( p = (1, 2, \ldots, n) \).

If there is a positive integer \( j < n \) such that \( a_j \neq 0 \) and a positive integer \( h < n \) such that \( b_h \neq 0 \), then consider \( \sum_{p=1}^{n} \left| \bar{a}_p x + b_p \right|^2 \geq 0 \). The equality does not hold for \( x = 0 \). The equality holds if and only if \( a_p = b_p = 0 \) or \( a_p \neq 0 \) and \( x = -\frac{b_p}{\bar{a}_p} = -\frac{b_j}{\bar{a}_j} \neq 0 \), \( p = (1, 2, \ldots, n) \). Hence if the equality holds then \( b_p = \frac{b_j \bar{a}_p}{\bar{a}_j} \), \( p = (1, 2, \ldots, n) \). Conversely if there is a number \( k \) such that \( b_p = k \bar{a}_p \), then if \( x = -k \), the equality holds. If there is a number \( k \) such that \( \bar{a}_p = k b_p \), then since \( a_j \neq 0 \), \( k \neq 0 \), and if \( x = 1/(-k) \), then equality holds.

It will suffice now to consider the equation (1),
\[ |\bar{a}_1 x + b_1|^2 + \ldots + |\bar{a}_n x + b_n|^2 \geq 0. \]

If inequality holds for all \( x \), then there is at least one member of equation (1) not zero. Now by squaring the left members of equation (1) and collecting terms, the expression \( |x|^2 \sum_{p=1}^{n} |a_p|^2 + \bar{x} \sum_{p=1}^{n} a_p b_p + x \sum_{p=1}^{n} \bar{a}_p b_p + \sum_{p=1}^{n} |b_p|^2 \geq 0. \)
is obtained. Then by transposing the constant term and dividing by \( \sum_{p=1}^{n} |a_p|^2 \) which is not zero since for some \( p, a_p \neq 0 \), the result is \( |x|^2 + \sum_{p=1}^{n} \frac{a_p b_p}{\sum_{p=1}^{n} |a_p|^2} + x \frac{\sum_{p=1}^{n} a_p b_p}{\sum_{p=1}^{n} |a_p|^2} \geq \\
\frac{\sum_{p=1}^{n} |b_p|^2}{\sum_{p=1}^{n} |a_p|^2} \). By adding \( \frac{\sum_{p=1}^{n} a_p b_p}{\sum_{p=1}^{n} |a_p|^2} \) to both sides and factoring, the expression \( x + \sum_{p=1}^{n} \frac{a_p b_p}{\sum_{p=1}^{n} |a_p|^2} \) is obtained.

\[
\frac{\sum_{p=1}^{n} |a_p|^2}{\sum_{p=1}^{n} |a_p|^2} \sum_{p=1}^{n} |b_p|^2
\]

Now if the equality holds, then \( \frac{a_p}{a_p} x = b_p \), \( p = (1, 2, \ldots, n) \). Then substituting in the previous equation gives the result:

\[
| x + (-x) \frac{\sum_{p=1}^{n} a_p b_p}{\sum_{p=1}^{n} |a_p|^2} |^2 = 0 = \frac{\sum_{p=1}^{n} a_p b_p}{\sum_{p=1}^{n} |a_p|^2} \sum_{p=1}^{n} |b_p|^2
\]

from which \( | \sum_{p=1}^{n} a_p b_p | = \sqrt{\sum_{p=1}^{n} |a_p|^2 \sum_{p=1}^{n} |b_p|^2} \).

If the equality does not hold, then let \( x = -\sum_{p=1}^{n} \frac{a_p b_p}{\sum_{p=1}^{n} |a_p|^2} \),

then \( \frac{\sum_{p=1}^{n} a_p b_p}{\sum_{p=1}^{n} |a_p|^2} + \frac{\sum_{p=1}^{n} a_p b_p}{\sum_{p=1}^{n} |a_p|^2} = 0 \) \( | \sum_{p=1}^{n} a_p b_p |^2 - \sum_{p=1}^{n} |a_p|^2 \sum_{p=1}^{n} |b_p|^2 | \)

and hence \( | \sum_{p=1}^{n} a_p b_p | < \sqrt{\sum_{p=1}^{n} |a_p|^2 \sum_{p=1}^{n} |b_p|^2} \).

Therefore, for any value of \( x, | \sum_{p=1}^{n} a_p b_p | \leq \sqrt{\sum_{p=1}^{n} |a_p|^2 \sum_{p=1}^{n} |b_p|^2} \).
**Corollary 1 - 1:** If $\sum_{r=1}^{\infty} |a_p|^2$ converges and $\sum_{r=1}^{\infty} |b_p|^2$ converges, then $|\sum_{r=1}^{\infty} a_p b_p|$ converges and $|\sum_{r=1}^{\infty} a_p b_p| \leq \sqrt{\sum_{r=1}^{\infty} |a_p|^2 \sum_{r=1}^{\infty} |b_p|^2}$, where the equality holds if and only if there exists a constant $k$ such that $a_p = k \overline{b_p}$, $p = (1, 2, \ldots, n.)$, or $k \overline{a_p} = b_p$, $p = (1, 2, \ldots, n.)$. 
CHAPTER II

BOUNDNESS IN HILBERT SPACE

A linear form in infinitely many variables $F(x)$ is defined to be $\sum_{n=1}^{\infty} a_n x_n$. This implies that a vector $\alpha'$ of infinitely many components is multiplied by a vector $X$ of infinitely many coordinates.

**Definition 2 - 1:** $F(x)$ is bounded if there exists a number $N$ such that for each positive integer and all vectors $X' = (x_1, x_2, \ldots)$, $\left| \sum_{p=1}^{\infty} a_p x_p \right| \leq N \sqrt{\sum_{p=1}^{\infty} |x_p|^2}$. The least such $N$ is called the norm of $F(x)$.

By the use of this definition, the following theorem will be proved.

**Theorem 2 - 1:** $F(x)$ is bounded if and only if $\sum_{p=1}^{\infty} |a_p|^2$ converges. If $F(x)$ is bounded, its norm is $\sqrt{\sum_{p=1}^{\infty} |a_p|^2}$.

If $a_p = 0$, $p = (1, 2, \ldots)$ the theorem is trivial since $\left| \sum_{p=1}^{\infty} a_p x_p \right| = 0$ and $\sum_{p=1}^{\infty} |a_p|^2 = 0$ and hence $N = 0$.

Suppose first that $F(x)$ is bounded. Then by definition 2 - 1, there is a number $N$ such that for every $n > 0$,

$\left| \sum_{p=1}^{n} a_p x_p \right| \leq N \sqrt{\sum_{p=1}^{n} |x_p|^2}$.

Now choose $x_p = \bar{a}_p$. Then $\left| \sum_{p=1}^{n} a_p x_p \right| = \left| \sum_{p=1}^{n} |a_p|^2 \right| \leq N \sqrt{\sum_{p=1}^{n} |x_p|^2} = N \sqrt{\sum_{p=1}^{n} |a_p|^2}$. Dividing by $\sqrt{\sum_{p=1}^{n} |a_p|^2}$ gives the result that $\sqrt{\sum_{p=1}^{n} |a_p|^2} \leq N$, and $\sum_{p=1}^{n} |a_p|^2 \leq N^2$ for every $n$.

Therefore, $\sum_{p=1}^{\infty} |a_p|^2$ converges.
Now suppose \( \sum_{p=1}^{\infty} |a_p|^2 \) converges to \( M \), that is,
\[
\sum_{p=1}^{\infty} |a_p|^2 = M.
\]
Then consider the inequality
\[
\sqrt{\sum_{p=1}^{n} |a_p|^2} |x_p|^2 = \sqrt{\sum_{p=1}^{n} |a_p|^2} \sqrt{\sum_{p=1}^{n} |x_p|^2} \leq M \sqrt{\sum_{p=1}^{n} |x_p|^2}.
\]
Let \( |K| < M \), then there is an \( n \) such that
\[
\sqrt{\sum_{p=1}^{n} |a_p|^2} > \sqrt{|K|}.
\]
Let \( x_p = \bar{a}_p \). Then
\[
\sqrt{\sum_{p=1}^{n} |a_p|^2} \sqrt{\sum_{p=1}^{n} |x_p|^2} \leq \sqrt{\sum_{p=1}^{n} |a_p|^2} \sqrt{\sum_{p=1}^{n} |x_p|^2},
\]
but
\[
\sqrt{\sum_{p=1}^{n} |a_p|^2} \sqrt{\sum_{p=1}^{n} |x_p|^2} > \sqrt{|K|} \sqrt{\sum_{p=1}^{n} |x_p|^2}.
\]
Therefore \( M \) is the norm, since \( M \) is the smallest bound of \( \sum_{p=1}^{\infty} |a_p|^2 \), and hence \( F(x) \) is bounded.

A sequence \( x_1, x_2, \ldots \) of complex numbers is called a vector in Hilbert Space if \( \sum_{p=1}^{\infty} |x_p|^2 \) converges, where \( x_1, x_2, \ldots \) are the components of the vector \( X \). The vector is in Hilbert Space if and only if \( \sum_{p=1}^{\infty} |x_p|^2 \) converges. Let \( |X| \) denote \( \sum_{p=1}^{\infty} |x_p|^2 \).

**Definition 2 - 2:** If \( X \) and \( Y \) are vectors with an infinite number of components, \( A \) is an infinite matrix \( (a_{pq}) \), \( \sum_{p=1}^{\infty} a_{pq} y_q \) converges, and \( \sum_{q=1}^{\infty} a_{pq} y_q x_p \) converges, then \( A(x, y) = X^t (AY) = \sum_{q=1}^{\infty} a_{pq} y_q x_p \). Also \( F_n(x, y) = \sum_{p=1}^{n} (\sum_{q=1}^{n} a_{pq} y_q) x_p = \sum_{q=1}^{n} a_{pq} x_p y_q \).

**Definition 2 - 3:** \( F(x, y) \) is bounded if and only if there exists a number \( N \) such that for each \( n \) and any \( X \) and \( Y \),
\[
|\sum_{p=1}^{n} a_{pq} y_q| \leq N \sqrt{\sum_{p=1}^{n} |x_p|^2} \sqrt{\sum_{q=1}^{n} |y_q|^2}.
\]
The smallest such \( N \) will be called the norm of \( F(x, y) \). If \( F(x, y) \) is bounded, then \( A \) will be called a bounded matrix.

**Theorem 2 - 2:** A necessary condition for \( F(x, y) \) to be bounded is that for each \( q \), \( \sum_{p=1}^{\infty} |a_{pq}|^2 \) converges, and for each \( p \), \( \sum_{q=1}^{\infty} |a_{pq}|^2 \) converges.
Let \( k \) be a positive integer and let \( n > k \). If \( a_{kq} = 0 \), \( q = (1, 2, \ldots) \), then \( \sum_{\tilde{q}} |a_{kq}|^2 = 0 \). If for one \( q \), \( a_{kq} \neq 0 \) consider

\[
|F_n(x, y)| = \left| \sum_{\tilde{p}, \tilde{q}} a_{pq} x_p y_q \right|. \]

If \( F(x, y) \) is bounded, then

\[
\left| \sum_{\tilde{p}, \tilde{q}} a_{pq} x_p y_q \right| \leq N \sqrt{\sum_{\tilde{p}} x_p^2 \sum_{\tilde{q}} y_q^2}. \]

Let \( x_p = 1 \) for \( p = k \) and let \( x_p = 0 \), \( p = (1, 2, \ldots, k - 1, k + 1, \ldots, n) \) and let \( y_q = \tilde{a}_{kq} \). Then

\[
F_n(x, y) \leq \left| \sum_{\tilde{p}, \tilde{q}} a_{pq} x_p y_q \right| = \left| \sum_{\tilde{q}} a_{kq} y_q \right| = \left| \sum_{\tilde{q}} a_{kq} \right|^2 \leq N \sqrt{\sum_{\tilde{q}} |a_{kq}|^2},
\]

and therefore

\[
\sqrt{\sum_{\tilde{q}} |a_{kq}|^2} \leq N. \]

Therefore

\[
\sum_{\tilde{q}} |a_{kq}|^2 \leq N^2.
\]

Since this is true for any \( p \), then

\[
\sum_{\tilde{q}} |a_{pq}|^2 \leq N^2.
\]

Similarly, for each \( q \), \( \sum_{\tilde{p}} |a_{pq}|^2 \) converges. In this case, let \( y_q = 1 \) for some \( q = k \) and \( y_q = 0 \), \( q = (1, 2, 3, \ldots, k - 1, k + 1, \ldots, n) \). Then let \( x_p = \tilde{a}_{pk} \), from which

\[
\sum_{\tilde{p}} |a_{pq}|^2 \leq N^2.
\]

This condition is not sufficient to assure boundness, for suppose that \( p = q = n \), each \( a_{pq} = n \), and for \( p \neq q \), \( a_{pq} = 0 \), \( x_p = 1 \), \( p = (1, 2, \ldots) \), and \( y_q = 1 \), \( q = (1, 2, \ldots) \). Then

\[
\sum_{\tilde{p}, \tilde{q}} a_{pq} x_p y_q = \sum_{\tilde{q}} (\tilde{a}_{qk} x_k). \]

If \( N \) is a number, let \( n \) be a positive integer greater than \( 2N \).

Then

\[
\left| \sum_{\tilde{p}, \tilde{q}} a_{pq} x_p y_q \right| = n(n + 1)/2 > Nn = N \sqrt{\sum_{\tilde{p}} x_p^2 \sum_{\tilde{q}} y_q^2}. \]

**Theorem 2 - 3:** (A) A sufficient condition for boundness is that \( \sum_{\tilde{p}, \tilde{q}} |a_{pq}|^2 \) converges. (B) If \( A \) is bounded, then for every \( p \), \( \sum_{\tilde{q}} a_{pq} y_q \) converges for every \( Y \) in Hilbert Space, and the series \( \sum_{\tilde{q}} (\tilde{a}_{pq} y_q) x_p \) converges for every \( X \) in Hilbert Space.
For the proof of part (A), suppose \( \sum_{p,q} |a_{pq}|^2 \) converges. Let \( \sum_{p,q} |a_{pq}|^2 = N^2 \), then \( \sum_{p,q} |a_{pq}|^2 \leq N^2 \). Since for every \( X \) and \( Y \) in Hilbert Space, \( \sum_{p,q} |x_p|^2 \) and \( \sum_{p,q} |y_q|^2 \) converges, then \( \sum_{p,q} |x_p|^2 \sum_{p,q} |y_q|^2 \sum_{p,q} |a_{pq}|^2 \leq N^2 \sum_{p,q} |x_p|^2 \sum_{p,q} |y_q|^2 \).

Now, \( \sum_{p,q} a_{pq} x_p y_q \sum_{p,q} a_{pq} x_p y_q \leq \sum_{p,q} a_{pq} y_q \sum_{p,q} a_{pq} y_q \sum_{p,q} a_{pq} x_p \sum_{p,q} a_{pq} x_p \leq \sum_{p,q} \left( \sum_{p,q} |a_{pq}|^2 \right) \sum_{p,q} |y_q|^2 \). From which \( \sum_{p,q} a_{pq} x_p y_q \sum_{p,q} a_{pq} x_p y_q \leq \sqrt{\sum_{p,q} |x_p|^2} \sqrt{\sum_{p,q} |y_q|^2} \sum_{p,q} |a_{pq}|^2 \sum_{p,q} |y_q|^2 \). Since this is true for any \( n \), then \( \sum_{p,q} a_{pq} x_p y_q \sum_{p,q} a_{pq} x_p y_q \leq N \sqrt{\sum_{p,q} |x_p|^2} \sqrt{\sum_{p,q} |y_q|^2} \).

This result is not necessary for \( F(x,y) \) to be bounded for suppose \( a_{pq} = \delta_{pq} \). Then \( \sum_{p,q} a_{pq} x_p y_q \sum_{p,q} a_{pq} x_p y_q \leq \sqrt{\sum_{p,q} |x_p|^2} \sqrt{\sum_{p,q} |y_q|^2} \). Hence, for any \( N = 1 \), \( F(x,y) \) is bounded, but \( \sum_{p,q} |a_{pq}|^2 \) does not converge.

To prove part (B) of this theorem, the following lemma will be proved.

**Lemma 2-1:** If \( A(x,y) \) is bounded for every \( X \) and \( Y \) in Hilbert Space, then \( \lim A_n(x,y) \) exists.

If \( \varepsilon > 0 \) is chosen, then there is a positive integer \( G_1 \) such that is \( m > n > G_1 \), then \( \sum_{p,q} |x_p|^2 < \frac{\varepsilon^2}{[3N(\sqrt{|Y|} + 1)]^2} \), there is a positive integer \( G_2 \) such that if \( m > n > G_2 \), then \( \sum_{p,q} |x_p|^2 \leq \frac{\varepsilon}{3(N + 1)} \), there is a positive
integer $G_3$ such that if $m > n > G_3$, \[ \sum_{f=0}^{m} |y_q|^2 < \frac{\varepsilon}{3(N+1)} \]
and there is a positive integer $H$ such that if $m > n > H$, \[ \sum_{f=0}^{n} |y_q|^2 < \frac{\varepsilon^2}{[3N(\sqrt{1} + 1)]^2} \]
Let $K = \max[G_1, G_2, G_3, H]$.

Then let $m > n > K$. Let $y_q = 0$, $q = (1, 2, \ldots, n)$, $y_q = y_q$, $q > n$, $x_p = 0$, $p = (1, 2, \ldots, n)$, and $x_p = x_p$, $p > n$.

Now \[ |A_m(x,y) - A_n(x,y)| = \sum_{p=n+1}^{m} \left( \sum_{f=n+1}^{m} a_{pq} y_q \right) x_p + \]
\[ \sum_{p=n+1}^{m} \left( \sum_{f=n+1}^{m} a_{pq} y_q \right) x_p + \sum_{f=n+1}^{m} \left( \sum_{p=n+1}^{m} a_{pq} y_q \right) x_p \]
\[ \leq N \sum_{f=n+1}^{m} |x_p| \sum_{p=n+1}^{m} |y_q| \]
\[ + N \sqrt{\sum_{f=n+1}^{m} |x_p|^2} \sum_{p=n+1}^{m} |y_q|^2 \leq N \sum_{f=n+1}^{m} |x_p| \sum_{p=n+1}^{m} |y_q| \]
\[ + N \sqrt{\sum_{f=n+1}^{m} |x_p|^2} \sum_{p=n+1}^{m} |y_q|^2 \leq N \varepsilon / 3N + \]
\[ \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 = \varepsilon. \]

If \[ \sum_{f=0}^{\infty} \left( \sum_{p=0}^{\infty} a_{pq} y_q \right) x_p \]
converges, then it must be proved that if $\varepsilon > 0$ is chosen, there is a $W$ so that if $m > n > W$, then \[ \sum_{f=n+1}^{m} \left( \sum_{p=n+1}^{m} a_{pq} y_q \right) x_p < \varepsilon. \]

If $\varepsilon > 0$, then since \[ \sum_{f=0}^{\infty} |x_p|^2 \]
converges, there is a $W$ such that if $m > n > W$, then \[ \sum_{f=n+1}^{m} |x_p|^2 < \frac{\varepsilon^2}{[2N(\sqrt{1} + 1)]^2}. \]
Since $\sum_{p=1}^{\infty} |x_p|^2$ converges, there is a $K > |x_p|$, $p = (1, 2, \ldots)$. Also, since $\sum_{j=1}^{\infty} a_{n+i} q_j$ converges for every $p$, there is a $G_1$ such that if $j > G_1$, $|\sum_{i=1}^{\infty} a_{n+i} q_j| < \frac{\varepsilon}{2K(m-n)}$, $i = [1, 2, \ldots, (m-n)]$. Let $G = \max [G_1, G_2, \ldots, G_{m-n}, m+1]$, and let $x'_1 = x'_2 = \cdots + x'_n = 0 = x'_{m+1} = \cdots = x'_G$ and $x'_{n+1} = x'_{n+1} = x'_{n+2} = x'_{n+2} = \cdots = x'_{m} = x'_{m}$.

Now, \[ \left| \sum_{p=1}^{\infty} \left( \sum_{l=1}^{\infty} a_{p} q_j \right) x_p \right|^2 = \left| \sum_{p=1}^{\infty} \left( \sum_{l=1}^{\infty} a_{p} q_j \right) x_p \right|^2 + \left| \sum_{p=1}^{m} \left( \sum_{l=1}^{\infty} a_{p} q_j \right) x_p \right|^2 \leq \left| \sum_{p=1}^{\infty} \left( \sum_{l=1}^{\infty} a_{p} q_j \right) x_p \right|^2 + \left| \sum_{p=1}^{m} \left( \sum_{l=1}^{\infty} a_{p} q_j \right) x_p \right|^2 = \left| \sum_{p=1}^{G} \left( \sum_{l=1}^{\infty} a_{p} q_j \right) x_p \right|^2 + \left| \sum_{p=1}^{m} \left( \sum_{l=1}^{\infty} a_{p} q_j \right) x_p \right|^2 \leq N \sqrt{\sum_{p=1}^{G} |x'_p|^2} \sqrt{\sum_{l=1}^{\infty} |q_j|^2} + \left| \sum_{p=1}^{m} \left( \sum_{l=1}^{\infty} a_{p} q_j \right) x_p \right| \leq N \sqrt{\sum_{p=1}^{m} |x'_p|^2} \sqrt{\sum_{l=1}^{\infty} |q_j|^2} + |K| \left( \sum_{p=1}^{m} a_{p} q_j \right) \leq N \frac{\varepsilon \sqrt{|Y|}}{2N(\sqrt{|Y|} + 1)} + \frac{|K| (m-n)\varepsilon}{2K(m-n)} < \varepsilon/2 + \varepsilon/2 = \varepsilon.

The sum of the preceding series will be called the value of the bilinear form $A(x,y)$, by definition.

**Theorem 2.4:** If $A$ is bounded and $X$ and $Y$ are in Hilbert Space, then $\lim_{n \to \infty} A_n(x,y) = A(x,y)$.
If \( \varepsilon > 0 \) is chosen, there is a positive integer \( H_1 \), such that if \( j > H_1 \), \( \left| \sum_{p=1}^{\infty} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right| < \varepsilon / 3 \).

There is a positive integer \( H_2 \), such that if \( j > H_2 \),
\[
\sum_{i=1}^{j} \left| y_{q} \right|^2 < \frac{\varepsilon^2}{\left(3N(\sqrt{|X|} + 1)\right)^2}.
\]

Since \( x_p \) converges, then there is a \( K \) such that \( K > \left| x_p \right| \), \( p = (1, 2, \ldots) \).

If \( n > H_1 + H_2 \), there is a \( G_1 \) such that if \( l > G_1 \), then
\[
\left| \sum_{q=1}^{\infty} a_{iq} y_q \right| < \frac{\varepsilon}{3nK}, \text{ for } i = (1, 2, \ldots, n). \]

Let \( G = G_1 + G_2 + \ldots + G_n + n \). Let \( x'_p = x_p, p = (1, 2, \ldots, n) \),
\[
x'_p = 0, \quad p > n, \quad y'_q = 0, \quad q = (1, 2, \ldots, n), \quad \text{and } y'_q = y_q, \quad q > n.
\]

Now,
\[
\left| A - A_n \right| = \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p - \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right|
\]
\[
= \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p + \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p - \sum_{p=1}^{n} a_{pq} y_q x_p \right|
\]
\[
= \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p + \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p + \sum_{p=1}^{\infty} a_{pq} y_q x_p \right|
\]
\[
- \sum_{p=n+1}^{\infty} a_{pq} y_q x_p \right| = \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p + \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right|
\]
\[
\leq \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right| + \left| \sum_{p=n+1}^{\infty} a_{pq} y_q x_p \right|
\]
\[
= \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right| + \left| \sum_{p=n+1}^{\infty} a_{pq} y_q x_p \right|
\]
\[
= \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right| + \left| \sum_{p=n+1}^{\infty} a_{pq} y_q x_p \right|
\]
\[
= \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right| + \left| \sum_{p=n+1}^{\infty} a_{pq} y_q x_p \right|
\]
\[
\leq \sum_{p=1}^{n} \left| a_{pq} y_q x_p \right| + \sum_{p=n+1}^{\infty} \left| a_{pq} y_q x_p \right|
\]
\[
\leq N \sqrt{\sum_{p=1}^{n} \left| x_p \right|^2} \sum_{p=n+1}^{\infty} \left| a_{pq} y_q \right|^2 + \sum_{p=n+1}^{\infty} \left| a_{pq} x_p \right|^2 \left| y_q \right|^2 + \left| \sum_{p=1}^{n} \left( \sum_{q=1}^{\infty} a_{pq} y_q \right) x_p \right|.
\]
Now, \[ N \sqrt{\frac{2}{3} \text{ } |X_{p}|^2 + \sum_{\substack{p \geq 1 \atop q \geq 1}} \left| \sum_{\substack{q \geq 1 \atop p \geq 1}} a_{pq} Y_q \right|^2} + \sum_{p \geq 1}^{\infty} |\sum_{q \geq 1}^{\infty} a_{pq} Y_q| |X_p| + \]
\[ \sum_{p \geq 1}^{\infty} |\sum_{q \geq 1}^{\infty} a_{pq} Y_q| |X_p| < N \sqrt{|X| \left( \sum_{p \geq 1}^{\infty} |Y_p|^2 + \sum_{p \geq n+1}^{\infty} \sum_{q \geq q+1}^{\infty} a_{pq} |X_p| + \sum_{p \geq n+1}^{\infty} |\sum_{q \geq q+1}^{\infty} a_{pq} Y_q| \right) \]
matrices $A$ and $B$, and multiplication of a scalar $c$ and a matrix $A$, where the matrices $A$ and $B$ are bounded if their bilinear form is bounded.

**Definition 2-4:** If $c$ is a scalar, then $cA = (ca_{pq})$.

**Definition 2-5:** If $A$ and $B$ are bounded matrices, then $A + B = (a_{pq} + b_{pq})$.

**Definition 2-6:** If $A$ and $B$ are bounded matrices, then $AB = \sum_{r=1}^{\infty} a_{pr} b_{rq}$ provided this sum converges. If it diverges, the product is not defined.

These definitions imply that the norm $N_{cA} = cN_A$, and that the norm $N_{A+B} \leq N_A + N_B$. These two statements may be easily verified; however, they will not be proved in this paper.

**Theorem 2-5:** If $A$ and $B$ are bounded matrices, then $AB$ exist and is bounded. The norm $N_{AB} \leq N_A N_B$.

Now $\sum_{r=1}^{\infty} a_{pr} b_{rq} \leq \sqrt{\sum_{r=1}^{\infty} |a_{pr}|^2} \sqrt{\sum_{r=1}^{\infty} |b_{rq}|^2} \leq \sqrt{\sum_{r=1}^{\infty} |a_{pq}|^2} \sqrt{\sum_{r=1}^{\infty} |b_{pq}|^2} \leq N_A N_B$. Hence $AB$ exist.

In order to complete the proof of this theorem, the following property will be needed.

**Lemma 2-2:** If $A$ is bounded and $\sum_{q=1}^{\infty} |x_q|^2$ converges, then $\sum_{q=1}^{\infty} a_{pq} x_q$ converges for $p = (1, 2, \ldots)$.

If $y_p = \sum_{q=1}^{\infty} a_{pq} x_q$, then $\sum_{q=1}^{\infty} |y_p|^2$ converges.

If $x_q = 0$, $q = (1, 2, \ldots)$, then the lemma is trivial.
Now, if at least one \( x_q \neq 0 \), then \( |y_p|^2 = y_p \overline{y}_p = (\sum_{p=1}^{\infty} a_{pq} x_q) \overline{y}_p \). Now, \( \sum_{p=1}^{n} |y_p|^2 = \sum_{p=1}^{n} (\sum_{q=1}^{\infty} a_{pq} x_q) \overline{y}_p = \sum_{p=1}^{n} \sum_{q=1}^{\infty} a_{pq} x_q |y_p|^2 \leq N_A \sqrt{\sum_{p=1}^{\infty} |x_q|^2 \sqrt{\sum_{p=1}^{n} |y_p|^2}} \). Then dividing the equation by \( \sqrt{\sum_{p=1}^{\infty} |y_p|^2} \) gives the result that \( \sqrt{\sum_{p=1}^{\infty} |y_p|^2} \leq N_A \sqrt{\sum_{p=1}^{\infty} |x_q|^2} \). Therefore, \( \sum_{p=1}^{\infty} |y_p|^2 \leq N_A \sum_{p=1}^{\infty} |x_q|^2 \) converges.

Now, let \( W_r = \sum_{p=1}^{n} a_{pr} x_p \). Then \( W_r \overline{W}_r = (\sum_{p=1}^{n} a_{pr} x_p) \overline{W}_r \), and \( \sum_{r=1}^{\infty} |W_r|^2 = \sum_{r=1}^{\infty} (\sum_{p=1}^{n} a_{pr} x_p) \overline{W}_r = \sum_{r=1}^{\infty} (\sum_{p=1}^{n} a_{pr} x_p) \overline{W}_r \leq N_A \sqrt{\sum_{p=1}^{\infty} |x_p|^2} \sqrt{\sum_{r=1}^{\infty} |W_r|^2} \). Hence \( \sqrt{\sum_{r=1}^{\infty} |W_r|^2} \leq N_A \sqrt{\sum_{p=1}^{\infty} |x_p|^2} \) by lemma 2-2. Therefore, \( \sum_{r=1}^{\infty} |W_r|^2 \) converges.

Similarly, let \( V_r = \sum_{q=1}^{\infty} b_{rq} y_q \), and by a similar process show that \( \sqrt{\sum_{q=1}^{\infty} |V_r|^2} \leq N_B \sqrt{\sum_{q=1}^{\infty} |y_q|^2} \). Therefore, \( \sum_{r=1}^{\infty} |V_r|^2 \) converges.

Now, \( |\sum_{r=1}^{\infty} (\sum_{p=1}^{n} a_{pr} b_{rq}) x_p y_q | = \sum_{k=1}^{\infty} \sum_{p=1}^{n} (a_{pr} b_{rq}) x_p y_q \overline{W_r} \leq \sqrt{\sum_{k=1}^{\infty} |W_r|^2} \sum_{p=1}^{n} |x_p|^2 N_B \sqrt{\sum_{r=1}^{\infty} |V_r|^2} \overline{|y_q|^2} = N_A N_B \sqrt{\sum_{p=1}^{\infty} |x_p|^2 \overline{|y_q|^2}} \). Hence \( \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} |x_p|^2 \overline{|y_q|^2} \) is bounded.

\( N_A N_B \) is a norm for \( AB(x,y) \), but it is not necessarily the smallest such bound. \( N_{AB} \) is also a norm for \( AB(x,y) \) and is the greatest lower bound of all the norms of \( AB(x,y) \). Hence \( N_{AB} \leq N_A N_B \).
Theorem 2-6: Multiplication of bounded matrices obey the associative law, that is, \((AB)C = A(BC)\).

Now \((AB)C = [(a_{pq})(b_{pq})](c_{pq}) = [\sum_{k=1}^{\infty} a_{pr} b_{rq}](c_{pq}) = \sum_{k=1}^{\infty} (\sum_{r,s=1}^{\infty} a_{pr} b_{rs} c_{sq})\). Let \(x_r = a_{pr}\) and \(y_s = c_{sq}\). Then
\[
\sum_{k=1}^{\infty} (\sum_{r,s=1}^{\infty} a_{pr} b_{rs} c_{sq}) = \left[ \sum_{k=1}^{\infty} (\sum_{r,s=1}^{\infty} x_r b_{rs} y_s) \right] = \left[ \sum_{k=1}^{\infty} a_{pr} (\sum_{r,s=1}^{\infty} b_{rs} c_{sq}) \right] = A(BC).
\]

Definition 2-7: A matrix whose kth element is the jth row is the same as the jth element in the kth row will be called symmetric.

Definition 2-8: If \(a_{pq} = a_{qp}\) is real symmetric and \(x_p\) is real, then \(\sum_{p=1}^{\infty} a_{pq} x_p x_q\) is called a real quadratic form.

Definition 2-9: The real symmetric bilinear form \(\sum_{p=1}^{\infty} a_{pq} x_p y_q\) is bounded relative to real Hilbert Space if and only if there is an \(N\) such that
\[
N \sum_{p=1}^{n} |x_p|^2 \leq \sum_{p=1}^{n} a_{pq} x_p x_q \leq N \sum_{p=1}^{n} |y_q|^2
\]
for all \(x_p\)'s and all \(n\).

A real symmetric bilinear form has many properties of a bounded matrix \(A(x,y)\). One such property is given by the following theorem.

Theorem 2-7: If \(\sum_{p=1}^{\infty} |x_p|^2 \leq 1\), \(\sum_{q=1}^{\infty} |y_q|^2 \leq 1\), and \(\sum_{p=1}^{\infty} a_{pq} x_p y_q\) is a real symmetric bilinear form bounded relative to real Hilbert Space, then
\[
N \sum_{p=1}^{n} |x_p|^2 \leq \sum_{p=1}^{n} a_{pq} x_p y_q \leq N \sum_{q=1}^{n} |y_q|^2
\]
for all \(n\).

If \(x_q = y_q\), then the proof is trivial.
Now, \[ \sum_{p,q=1}^{n} a_{pq} x_{p} y_{q} = \frac{1}{4} \left| A_{n}(x+y)(x+y) - A_{n}(x-y)(x-y) \right| = \frac{1}{4} \left| \sum_{p,q=1}^{n} a_{pq} (x_{p} + y_{p})(x_{q} + y_{q}) - \sum_{p,q=1}^{n} a_{pq} (x_{p} - y_{p})(x_{q} - y_{q}) \right| \leq \frac{1}{4} \left[ \sum_{p,q=1}^{n} a_{pq} (x_{p} + y_{p})(x_{q} + y_{q}) \right] \leq \frac{N}{4}[ \frac{1}{p_{i=1}^{n}} x_{p}^{2} + \frac{1}{p_{j=1}^{n}} y_{q}^{2} ] \leq \frac{N}{4}[ \frac{1}{p_{i=1}^{n}} x_{p}^{2} + \frac{1}{p_{j=1}^{n}} y_{q}^{2} ] = \frac{N}{2}[ \frac{1}{p_{i=1}^{n}} x_{p}^{2} + \frac{1}{p_{j=1}^{n}} y_{q}^{2} ] = N \sqrt{\sum_{i=1}^{n} |x_{p}|^{2}} \sum_{j=1}^{n} |y_{q}|^{2}. \]

Note: \( N \leq N \sqrt{\sum_{i=1}^{n} |x_{p}|^{2}} \sum_{j=1}^{n} |y_{q}|^{2} \) not necessarily true.

Suppose \( \gamma = 1 \) then \( N \leq N(\gamma) \). Smallwood

Now, suppose \( A \) is complex, that is, \( A = B + iC \), then
\[ \left| \sum_{p,q=1}^{n} a_{pq} x_{p} y_{q} \right| \leq \frac{1}{4} \left[ \sum_{p,q=1}^{n} (b_{pq} + ic_{pq}) (x_{p} + y_{p})(x_{q} + y_{q}) \right] \leq \frac{1}{4} \left[ (N_{B} + N_{C}) \sum_{i=1}^{n} (x_{p} - y_{p})^{2} + (N_{B} + N_{C}) \sum_{i=1}^{n} (x_{p} - y_{p})^{2} \right] = \frac{1}{2}[ (N_{B} + N_{C}) \sum_{i=1}^{n} |x_{p}|^{2} + (N_{B} + N_{C}) \sum_{i=1}^{n} |y_{p}|^{2} ]. \]

Let \( N = \max[N_{B}, N_{C}] \). Then \( 1/2[(N_{B} + N_{C}) \sum_{i=1}^{n} |x_{p}|^{2} + \sum_{j=1}^{n} |y_{q}|^{2}] \leq N \sum_{i=1}^{n} |x_{p}|^{2} + \sum_{j=1}^{n} |y_{q}|^{2} \] \( \leq 2N \leq 2N \sum_{i=1}^{n} \frac{1}{p_{i=1}^{n}} |x_{p}|^{2} \frac{1}{p_{j=1}^{n}} |y_{q}|^{2}. \) Therefore, if \( N \) for \( A \) real is replaced by \( 2N \), then the theorem holds for \( A \) complex.
CHAPTER III

INVERSES AND RECIPROCALS

This chapter will be devoted entirely to the study of reciprocals of bounded matrices, that is, inverses with respect to multiplication.

Matrices both of finite and infinite order are much similar to real and complex numbers because they have both an identity matrix with respect to addition and multiplication. The identity matrix with respect to addition is obviously the matrix with 0 for each element and is called the "zero" matrix.

**Definition 3 - 1:** The identity or unity matrix with respect to multiplication is denoted by \( I = (\delta_{pq}) \), where for \( p = q \), \( \delta_{pq} = 1 \) and for \( p \neq q \), \( \delta_{pq} = 0 \).

**Definition 3 - 2:** \( BA = I \) if and only if \( B \) is a left hand reciprocal of \( A \).

**Definition 3 - 3:** \( AC = I \) if and only if \( C \) is a right reciprocal of \( A \).

This might suggest that \( B = C \); however, this is not true in all cases. In the discussion to follow, certain restrictions will be made which will determine when \( B = C \), and when \( B = C \) is unique.
Theorem 3 - 1: If \((a_{pq})\) is a bounded matrix, consider the system \(A(X) = Y\), that is, (1) \(\sum_{f_{z_i}} a_{pq} x_q = y_p\), \(p = (1, 2, \ldots)\).

(a) If \((a_{pq})\) has a bounded right reciprocal \(C\), then equation (1) has at least one solution \(X\) in Hilbert Space corresponding to each \(Y\) in Hilbert Space.

(b) If \((a_{pq})\) has a bounded left reciprocal \(B\), then equation (1) has at most one solution \(X\) in Hilbert Space corresponding to each \(Y\) in Hilbert Space.

To prove part (a) in which \(AC = I\), consider \(AX = Y\), and let \(X = CY\). Then \(AX = A[CY] = [AC]Y = IX = Y\). Hence \(X = CY\) is a solution if \(X\) is in Hilbert Space.

It remains to prove \(X\) is in Hilbert Space. Now, \(x_p = \sum_{f_{z_i}} c_{pq} y_q\) and \(\overline{x_p} = \sum_{f_{z_i}} c_{pq} y_q\). Then \(x_p \overline{x_p} = |x_p|^2 = |\sum_{f_{z_i}} c_{pq} y_q|^2\), and \(\sum_{f_{z_i}} |x_p|^2 = \sum_{f_{z_i}} (\sum_{f_{z_i}} c_{pq} y_q) \overline{x_p}\).

By lemma 2 - 2, \(\sum_{f_{z_i}} |x_p|^2 \leq N^2 \sum_{f_{z_i}} |y_q|^2\). Therefore \(X\) is in Hilbert Space. Hence equation (1) has at least one solution \(X\) in Hilbert Space.

In case (b), \(BA = I\), and \(AX = Y\). When the latter equation is multiplied by \(B\), the result is that \(BY = B[AX] = [BA]X = IX = X\). Therefore, \(x_p = \sum_{f_{z_i}} b_{pq} y_q\) is the only solution of equation (1).

Therefore, equation (1) has at most one solution \(X\) in Hilbert Space.
**Theorem 3-2:** If $A$ is bounded and has both a right reciprocal $C$ and a left reciprocal $B$ that are bounded, then $\sum_{i=1}^{\infty} y_p \cdot \frac{1}{x_i} \cdot p_i \cdot q_i$ has exactly one solution $X$ in Hilbert Space for a given $Y$ in Hilbert Space. Moreover, $B = C$.

By theorem 3-1, if $A$ has a bounded right hand reciprocal, then equation (1) has at least one solution in Hilbert Space, and if $A$ has a bounded left hand reciprocal, then equation (1) has at most one solution in Hilbert Space. Hence, if $A$ has both a left and right hand reciprocal, then equation (1) has exactly one solution in Hilbert Space.

Now $BA = I$, and hence $BA - I = 0$. Then $[BA - I]C = [BA]C - IC = B[AC] - C = BI - C = B - C = OC = 0$. Therefore $B = C$.

The uniqueness of the inverses of $A$, called $A^{-1} = B = C$, will be established by the use of the following properties.

**Definition 3-4:** If $A$ is a bounded matrix, the transpose $A'$ means that the rows of $A$ are the columns of $A'$ and the columns of $A$ are the rows of $A'$.

**Lemma 3-1:** If $A$ is a bounded symmetric matrix, then $A = A'$.

**Lemma 3-2:** The transpose of a bounded matrix is bounded.

The above lemmas will not be proved in this paper.
Theorem 3 - 3: If A and B are bounded matrices, then \((AB)' = B'A'\).

Let the matrix \(AB = C\), that is, \((a_{ij})(b_{ij}) = (c_{ij}) = \left( \sum_{k=1}^{\infty} a_{ik} b_{kj} \right)\). Then \((AB)' = C'\). Let \(C' = D = (d_{ij}) = \left( \sum_{k=1}^{\infty} b_{jk} a_{ki} \right)\), and let \(B'A' = EF = G\). Then \(B'A' = EF = \left( \sum_{k=1}^{\infty} e_{ik} f_{kj} \right) = G = (g_{ij})\), where \((e_{ij}) = (b_{ji})\) and \((f_{ij}) = (a_{ji})\). Hence \((g_{ij}) = EF = \left( \sum_{k=1}^{\infty} e_{ik} f_{kj} \right) = \left( \sum_{k=1}^{\infty} b_{jk} a_{ki} \right)\).

Now, \(B'A' = EF = \left( \sum_{k=1}^{\infty} e_{ik} f_{kj} \right) = \left( \sum_{k=1}^{\infty} b_{jk} a_{ki} \right) = (d_{ij}) = D = C' = (AB)'.\)

Theorem 3 - 4: If A is a bounded symmetric matrix, then there are two and only two possibilities:

1. Either A has neither a bounded left reciprocal nor a right bounded reciprocal,

2. Or (2) A has a unique bounded reciprocal \(A^{-1}\) such that \(AA^{-1} = A^{-1}A = I\).

Suppose A has a bounded right reciprocal C. Then, \(AC = I\), and \((AC)' = C'A' = C'A = I' = I\) by lemma 3 - 1 and theorem 3 - 3. Therefore, if C is a bounded right reciprocal, then C' is a bounded left reciprocal. The converse is also true. By theorem 3 - 2, \(C = C'\).

If A has a bounded right reciprocal D, then \(C' = C'(AD) = (C'A)D = ID = D\). Hence \(C = C' = D\).

The results of the above theorem may be stated in a different form.
**Theorem 3 - 5:** If the bounded matrix $A$ has a unique bounded right(left) reciprocal $C(B)$, then $CA = AC = I$.

Now, $AC = I$. If $C$ is unique, it must be proved that $CA = I$. By multiplying $AC = I$ from the right by $A$ and transposing the right side, $[AC]A - A = 0$ is obtained. Now by adding $AC$ to both sides of the equation, the result is that $AC + A[CA] - A = AC = I$. Therefore $A[C + CA - I] = I$.

Now if $C$ is unique, then $C + CA - I = C$, and hence $CA - I = 0$. Therefore $CA = I$. 

BIBLIOGRAPHY


