ELEMENTS OF BOOLEAN ALGEBRA THEORY

APPROVED:

Herbert C. Parrish
Major Professor

George Cope
Minor Professor

[Signature]
Director of the Department of Mathematics

Robert J. Toulousse
Dean of the Graduate School
ELEMENTS OF BOOLEAN ALGEBRA THEORY

THESIS

Presented to the Graduate Council of the North Texas State College in Partial Fulfillment of the Requirements

For the Degree of

MASTER OF ARTS

By

John Bowman Harvill, Jr., B. A.

Denton, Texas
January, 1957
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>iv</td>
</tr>
<tr>
<td><strong>I. INTRODUCTION</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>II. A SET OF POSTULATES FOR A BOOLEAN ALGEBRA AND SOME ELEMENTARY CHARACTERISTIC THEOREMS</strong></td>
<td>6</td>
</tr>
<tr>
<td><strong>III. THE EQUIVALENCE OF VARIOUS REPRESENTATIONS OF BOOLEAN ALGEBRAS</strong></td>
<td>25</td>
</tr>
<tr>
<td><strong>IV. ELEMENTS OF LATTICE THEORY RELATING TO BOOLEAN ALGEBRA</strong></td>
<td>43</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>51</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
</tr>
<tr>
<td>1. Showing Independence of Postulate Set I</td>
<td>10</td>
</tr>
</tbody>
</table>

iv
CHAPTER I

INTRODUCTION

Boolean algebra, more commonly known as the algebra of logic, was originated by an Englishman, George Boole, in his book, Laws of Thought, in 1854. Boole was anticipated in idea by Leibniz, who in about 1700 used the symbol @ to mean addition other than arithmetic addition, and by De Morgan, but he was the first to successfully compile an algebra of logic. A short biography of George Boole is included in the introduction, primarily to show how his background brought about such an original new work. The biography is essentially that of E. T. Bell (1).

George Boole was born on November 2, 1815 at Lincoln, England, into the lowest strata of society England afforded. At an early age he observed that the people of the middle and upper classes had some knowledge of Latin and Greek, so with firm conviction that the acquisition of those two languages would remove him from his own foul lot, he started to study at the age of eight. By the time he was twelve years old he had mastered enough Latin to translate an ode of Horace into English verse. A classical master denied that a boy of twelve could have produced such a translation and
pointed out several technical errors. Boole was humiliated and spent the next two years slaving over the languages.

His father tutored him in mathematics to supplement his normal meager education, and at the age of twenty the son opened an elementary school of his own. In studying the common mathematical texts of the day, he was at first enlightened by them, then contemptible of them. To fill the gaps existing in his textbooks, Boole turned to the masters. It is remarkable that Boole mastered, unaided, the Mécanique céleste of Laplace, one of the most difficult masterpieces ever written, for it is full of gaps and declarations that "it is easy to see." He also made a thorough, understanding study of the excessively abstract Mécanique analytique of Lagrange, in which there is not a single diagram to illuminate the analysis from beginning to end.

The effort was not wasted. Out of his unguided efforts came his first contribution, a paper on the calculus of variations. A further result from the study was his discovery of invariants, one that the mighty Lagrange himself had overlooked.

At the same time several mathematicians, namely Peacock, Herschel, De Morgan, Babbage, and Gregory, were in the process of analyzing algebra. Their investigations led Boole to study the symbols of mathematical operations apart from the things upon which they operate. This study led to his first book, no more than a pamphlet, The Mathematical
Analysis of Logic, in 1848. The book broke new ground and attracted admiration from several noted mathematicians of the time. In 1849 he was appointed Professor of Mathematics at Queen's College in Cork, Ireland. Until this time he had continued in his elementary teaching position.

With his now comparatively light duties, Boole finally published in 1854 his masterpiece, An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities. Boole was thirty-nine at the time, an unusually late age for a work of such profound originality. Boole reduced logic to an extremely easy and simple type of algebra, which presented a symbolic method for logical argument in contrast to the old pre-Boole methods of verbal argument. The daring originality of Boole's project displays one of the most brilliant facets of his mind.

Honored and with fast growing fame, Boole died on December 8, 1864, at the age of fifty. His premature death was due to pneumonia contracted after faithfully keeping a lecture engagement when he was soaked to the skin. He died conscious that he had done great work.

Boolean algebra was ignored for a great while as a subject of mathematical importance. Professor Peano's Formulario Mathematico showed a good deal of thought on the subject, and Ernst Schröder's Vorlesungen über die Algebra der Logik, in 1890, actually had a complete set of postulates for Boolean algebra scattered throughout the book, as did
Whitehead and Russell's *Principia Mathematica* in 1910.

E. V. Huntington, in 1904, published a paper containing the first assemblage of postulate sets for Boolean algebras. In this paper Huntington postulated Boolean algebras by three different methods, all equivalent. In 1912 H. M. Sheffer published a set of postulates for Boolean algebras which, in terms of a new operation "rejection," reduced the minimum number of postulates needed from Huntington's ten to five. Using the same operation, B. A. Bernstein further reduced the number of postulates to four. The latter postulate set is the most refined in the postulational study of Boolean algebra.

Boolean algebra has shown a great unifying influence in mathematics. Garrett Birkhoff (2) shows that a Boolean algebra is equivalent to a complemented distributive lattice, and further that a Boolean algebra is equivalent to several topological spaces. A Boolean algebra is also equivalent to certain types of algebraic rings. Investigations of M. Stone proved one of the fundamental Boolean algebra theorems, "any Boolean algebra is isomorphic to a Boolean algebra of sets."

The most important use of Boolean algebra is the role it plays in modern symbolic logic. The subject is also very important in industry as it provides a mathematical model for researches in electronic switching circuits, and is playing a great part in the development of modern computers.

The primary purpose of this paper is to state a set of postulates for Boolean algebra and show the characteristic
theorems derivable from them, and to unify in one paper the more important methods of representing Boolean algebra and show their equivalence. A proof of a theorem of the principle of duality for Boolean algebra is offered, being derived by purely deductive methods and not characteristically assumed as is done by most authorities.
CHAPTER I BIBLIOGRAPHY


CHAPTER II

A SET OF POSTULATES FOR A BOOLEAN ALGEBRA AND
SOME ELEMENTARY CHARACTERISTIC THEOREMS

A class or aggregate of elements will be denoted by capital letters $K, C, T, \ldots$. The elements of the class, say $K$, will be denoted by small letters $a, b, c, \ldots$ and Greek letters $\alpha, \beta, \gamma, \ldots$. $a \in K$ is defined to mean that $a$ is a member of the class $K$. $a, b, c, \ldots \in K$ is defined to mean that $a, b, c, \ldots$ are all members of the class $K$. $\equiv$ is an equivalence relation which is governed by the following three postulates:

E1. $a = a$.

E2. If $a = b$, then $b = a$.

E3. If $a = b$ and $b = c$, then $a = c$.

$\equiv$ is further defined in that if $\circ$ is an operation of a postulate set and if $a = b$ and $c = d$, then $a \circ c = b \circ d$. Every postulate set has at least two distinct elements and if in any postulate set with class $K$ of elements $a, b, c, \ldots$ occur, it is understood that they are members of the class $K$.

Postulate Set I

Undefined are a class $K$ of elements $a, b, c, \ldots$ and two binary operations $\cup$ and $\cap$. 

6
P1. \((a \cup b) \in K\).

P2. \((a \cap b) \in K\).

P3. There exists an element \(0 \in K\) such that if \(a \in K\),
then \(a \cup 0 = a\).

P4. There exists an element \(1 \in K\) such that if \(a \in K\),
then \(a \cap 1 = a\).

P5. \(a \cup b = b \cup a\). \(\forall (a \cup b), (b \cup a) \in K\)

P6. \(a \cap b = b \cap a\). \(\forall (a \cap b), (b \cap a) \in K\)

P7. \(a \cup (b \cap c) = (a \cup b) \cap (a \cap c)\). \(\forall (a \cup b), (a \cap c),
(a \cap b), a \cup (b \cap c), (a \cup b) \cap (a \cap c) \in K\)

P8. \(a \cap (b \cup c) = (a \cap b) \cup (a \cap c)\). \(\forall (a \cap b), (a \cap c),
(b \cup c), a \cap (b \cup c), (a \cap b) \cup (a \cap c) \in K\)

P9. If the elements 0 and 1 in P3 and P4 exist and are
unique and if \(a \in K\), then there exists an \(a' \in K\)
such that \(a \cup a' = 1\) and \(a \cap a' = 0\).

In Postulate Set I, \(\cup\) will be replaced with \(\cap\), \(\cap\) with
\(\cup\), 0 with 1, and 1 with 0. This gives another set of
postulates, Ia.

Postulate Set Ia

P1a. \((a \cap b) \in K\).

P2a. \((a \cup b) \in K\).

P3a. There exists an element \(1 \in K\) such that if \(a \in K\),
then \(a \cap 1 = a\).

P4a. There exists an element \(0 \in K\) such that if \(a \in K\),
then \(a \cup 0 = a\).
P5a. \( a \land b = b \land a \). \([a \land b], (b \land a) \in \mathcal{J}\)

P6a. \( a \lor b = b \lor a \). \([a \lor b], (b \lor a) \in \mathcal{J}\)

P7a. \( a \land (b \lor c) = (a \land b) \lor (a \land c) \). \([a \land b], (a \land c), (b \lor c), a \land (b \lor c), (a \land b) \lor (a \land c) \in \mathcal{J}\)

P8a. \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \). \([a \lor b], (a \lor c), (b \land c), a \lor (b \land c), (a \lor b) \land (a \lor c) \in \mathcal{J}\)

P9a. If the elements 1 and 0 in P3a and P4a exist and are unique and if \( a \in \mathcal{K} \), then there exists an \( a' \in \mathcal{K} \) such that \( a \land a' = 0 \) and \( a \lor a' = 1 \).

The statements in the brackets are considered part of the postulate sets only when proving the independence of each postulate. When all nine postulates are assumed, the brackets will be disregarded.

Postulate Set Ia will be called the dual set of postulates for Postulate Set I. The dual of any element or statement of equality is found by replacing \( \lor \) with \( \land \), \( \land \) with \( \lor \), \( 0 \) with \( 1 \), and \( 1 \) with \( 0 \). The fact that the dual postulate set of Postulate Set I is equal to Postulate Set I leads to a statement of method of proof for dual theorems:

Given a theorem of Postulate Set I and its proof, replace each \( \lor \) with \( \land \), \( \land \) with \( \lor \), \( 0 \) with \( 1 \), and each \( 1 \) with \( 0 \) in the statement of the theorem, and in each line of the proof. Also add an a postscript to the number of each postulate from Postulate Set I and to each theorem used in each argument of the proof. The a postscripts refer to Postulate Set Ia and certain theorems so designated.
Before developing theorems any further, the consistency and independence of Postulate Set I will be shown.

A postulate set is said to be consistent whenever no one postulate of the set contradicts the others or any conclusion of the others. To show the consistency of this set of postulates some specific system, \((K, \cup, \cap, a')\) must be named where \(K, \cup, \cap,\) and \(a'\) are so interpreted that the postulates are satisfied. If there is such a system, the postulates are consistent and any consequences of the postulates are merely expressions of the properties of the particular system.

One such system satisfying Postulate Set I is as follows: Let \(K\) be the set of all subsets of a plane, including the null set and the whole plane; \(a \cup b\) is the smallest subset containing both \(a\) and \(b\); \(a \cap b\) is the largest subset contained in both \(a\) and \(b\); \(a'\) is the complement of \(a\) with respect to the whole plane. It is readily seen that this system satisfies Postulate Set I.

A postulate set is said to be independent if and only if no one postulate of the set follows as a conclusion of the others. To show the independence of Postulate Set I, let \(K\) be the class of two elements, \(a\) and \(b\), with \(\cup\) and \(\cap\) defined as shown in Figure 1.

The system next to each postulate shows that this particular postulate is independent of the others. In every system, \(a' = b\) and \(b' = a\). In set 1 and set 2, \(x\) is neither \(a\) nor \(b\). In sets 1-8, \(0 = a\) and \(1 = b\). In set 9, \(0 = a\)
<table>
<thead>
<tr>
<th>Postulate</th>
<th>$a \cup a$</th>
<th>$a \cup b$</th>
<th>$b \cup a$</th>
<th>$b \cup b$</th>
<th>$a \cap a$</th>
<th>$a \cap b$</th>
<th>$b \cap a$</th>
<th>$b \cap b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>x</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>2</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>x</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>3</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>4</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>5</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>6</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>7</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>8</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>9</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

Fig. 1--Showing independence of Postulate Set I.

and $l = a$. It has been shown that for each postulate a system can be devised so that this postulate fails and each of the others hold. This shows that the particular postulate can not be derived from the others and the independence of each is proven. Postulate Set I and the proof of consistency and independence is essentially that of Huntington (1).

In the proof of the following theorems, all references to the three equivalence postulates, E1, E2, and E3 will be omitted.
Theorem 1. The element 0 in P3 is unique.

Suppose there are two members \( \alpha, \beta \in K \) that have the property of P3.

\[
\alpha \cup \alpha = \alpha \text{ and } \alpha \cup \beta = \alpha
\]

Hence

\[
\alpha \cup \alpha = \alpha \cup \beta
\]

and

\[
\alpha \cup \alpha = \beta \cup \alpha
\]

Therefore

\[ \alpha = \beta \]

P5

P3

Theorem 1a. The element 1 in P4 is unique.

This theorem follows when the method of proof for dual theorems is applied to Theorem 1.

Theorem 2. The element \( a' \) in P9 is uniquely determined by \( a \).

Choose \( a \) and suppose that \( a_1' \) and \( a_2' \) satisfy P9. Then

\[
a_1 \cap a_1' = 0 \quad \text{and} \quad a_1 \cap a_2' = 0
\]

\[
a_1 \cup a_1' = 1 \quad \text{and} \quad a_1 \cup a_2' = 1
\]

\[
a_1' \cap (a \cup a_2') = a_1' \quad \text{and} \quad a_2' \cap (a \cup a_1') = a_2'
\]

P3

\[
(a_1' \cap a) \cup (a_1' \cap a_2') = a_1'
\]

and

P8

\[
(a_2' \cap a) \cup (a_2' \cap a_1') = a_2'
\]

\[
(a_1 \cap a_1') \cup (a_1 \cap a_2') = a_1'
\]

P6

\[
0 \cup (a_1' \cap a_2') = a_1' \quad \text{and} \quad 0 \cup (a_1' \cap a_2') = a_2' \quad \text{assumption}
\]

P5

\[
(a_1' \cap a_2') \cup 0 = a_1' \quad \text{and} \quad (a_1' \cap a_2') \cup 0 = a_2'
\]

P3

Hence \[ a_1' = a_2' \]
Theorem 3. For every $a \in K$, $a'' = a$.

\[ a \land a' = 0 \quad \text{and} \quad a \lor a' = 1 \]

\[ a' \land a = 0 \]

and \quad \[ a' \lor a = 1 \]

\[ a'' = a \]

P9, T2

Theorem 4. For every $a, b \in K$, if $a = b$, then $a' = b'$.

\[ a \land a' = 0 \quad \text{and} \quad a \lor a' = 1 \]

Assumption

\[ b \land a' = 0 \quad \text{and} \quad b \lor a' = 1 \]

\[ a' = b' \]

P9, T2

Theorem 5. For every $a, b \in K$, if $b = a'$, then $a = b'$.

\[ a' = b \]

Assumption

\[ a'' = b' \]

T4

\[ a = b' \]

T3

Theorem 6. $0' = 1$.

\[ 1 \lor 0 = 1 \]

P3

\[ 0 \lor 1 = 1 \]

P5

and \quad \[ 0 \land 1 = 0 \]

P4

\[ 0' = 1 \]

P9, T2

Theorem 6a. $1' = 0$

\[ 0' = 1 \]

T6

\[ 0'' = 1' \]

T4

\[ 0 = 1' \]

T3
Theorem 7. For every \( a \in K \), \( a \cup a = a \).  
\[
\begin{align*}
 a \cup a &= (a \cup a) \cap 1 \quad \text{P4} \\
 &= (a \cup a) \cap (a \cup a^*) \quad \text{P9} \\
 &= a \cup (a \cap a^*) \quad \text{P7} \\
 &= a \cup 0 \quad \text{P9} \\
 &= a \quad \text{P3}
\end{align*}
\]

Theorem 7a. For every \( a \in K \), \( a \cap a = a \).

This theorem follows when the method of proof for dual theorems is applied to Theorem 7.

Theorem 8. For every \( a \in K \), \( a \cup 1 = 1 \).  
\[
\begin{align*}
 a \cup 1 &= (a \cup 1) \cap 1 \quad \text{P4} \\
 &= (a \cup 1) \cap (a \cup a^*) \quad \text{P9} \\
 &= a \cup (1 \cap a^*) \quad \text{P7} \\
 &= a \cup (a^* \cap 1) \quad \text{P6} \\
 &= a \cup a^* \quad \text{P4} \\
 &= 1 \quad \text{P9}
\end{align*}
\]

Theorem 8a. For every \( a \in K \), \( a \cap 0 = 0 \).

This theorem follows when the method of proof for dual theorems is applied to Theorem 8.

Theorem 9. For every \( a, b \in K \), \( a \cup (a \cap b) = a \).  
\[
\begin{align*}
a \cup (a \cap b) &= (a \cap b) \cup a \quad \text{P5} \\
 &= (a \cap b) \cup (a \cap 1) \quad \text{P4} \\
 &= a \cap (b \cup 1) \quad \text{P8}
\end{align*}
\]
\[ a \cap b = a \cap 1 = a \]

**Theorem 9a.** For every \( a, b \in K \), \( a \cap (a \cup b) = a \).

This theorem follows when the method of proof for dual theorems is applied to Theorem 9.

**Theorem 10.** For every \( a, b \in K \), \( a' \cup (a \cup b) = 1 \).

\[
\begin{align*}
    a' \cup (a \cup b) &= \overline{a' \cup (a \cup b)} \cap 1 \\
    &= \overline{a' \cup (a \cup b)} \cap (a \cup a') \\
    &= \{ a' \cup (a \cup b) \cap a \} \cup \{ \overline{a' \cup (a \cup b)} \cap a' \} \\
    &= \{ a' \cup (a \cup b) \cap a \} \cup \{ a' \cap \overline{a' \cup (a \cup b)} \cap a' \} \\
    &= \{ a' \cup (a \cup b) \cap a \} \cup a' \\
    &= \{ a \cap \overline{a' \cup (a \cup b)} \} \cup a' \\
    &= \{ (a \cap a') \cup \overline{a \cap (a \cup b)} \} \cup a' \\
    &= \{ 0 \cup \overline{a \cap (a \cup b)} \} \cup a' \\
    &= \{ 0 \cup a \} \cup a' \\
    &= \{ a \cup 0 \} \cup a' \\
    &= a \cup a' \\
    &= 1
\]

**Theorem 10a.** For every \( a, b \in K \), \( a' \cap (a \cap b) = 0 \).

This theorem follows when the method of proof for dual theorems is applied to Theorem 10.

**Theorem 11.** For every \( a, b \in K \), \( (a \cup b)' = a' \cap b' \).

\( (a \cup b)' \in K \), and is unique \( \text{P1, P9, T2} \)
Also, (1) \((a \cup b) \cup (a \cup b)' = 1\)

(2) \((a \cup b) \cap (a \cup b)' = 0\)

However,

\[
(a \cup b) \cup (a' \cap b') = \overline{\overline{(a \cup b)} \cup a'} \cap \overline{(a \cup b) \cup b'}
\]

\[
= \overline{a'} \cup (a \cup b') \cap \overline{b'} \cup (b \cup a')
\]

\[
= 1 \cap 1
\]

\[
= 1
\]

And,

\[
(a \cup b) \cap (a' \cap b') = (a' \cap b') \cap (a \cup b)
\]

\[
= \overline{a' \cap b'} \cap \overline{a} \cup \overline{a' \cap b'} \cup b
\]

\[
= \overline{a} \cap (a' \cap b') \cup \overline{b} \cap (b' \cap a')
\]

\[
= 0 \cup 0
\]

\[
= 0
\]

Now, \((a' \cap b')\) satisfies (1) and (2). Hence by P9 and T2

\[
(a \cup b)' = a' \cap b'
\]

**Theorem 11a. For every a, b \in K, (a \cap b)' = a' \cup b'**.

This theorem follows when the method of proof for dual theorems is applied to Theorem 11.

**Theorem 12. For every a, b, c \in K, (a \cup b) \cup c = a \cup (b \cup c)**.

Let \((a \cup b) \cup c = x\) and \(a \cup (b \cup c) = y\)

\[
y \cup x' = y \cup \overline{(a \cup b) \cup c'}
\]

\[
= y \cup \overline{(a \cup b)'} \cap c'
\]

\[
= y \cup \overline{(a' \cap b') \cap c'}
\]

\[
= \overline{y} \cap (a' \cap b') \cap \overline{c'}
\]

\[
\text{Assumption}
\]

\[\text{T11}\]

\[\text{T11}\]

\[\text{P7}\]
\[ (y \cup a^* ) \cap (y \cup b^* ) \cap (y \cup c^* ) \]

Assumption

\[ = (a \cup (b \cup c)) \cup a^* ) \cap (y \cup b^* ) \cap (y \cup c^* ) \]

P5

\[ = (a^* \cup (b \cup c)) \cup (y \cup b^* ) \cap (y \cup c^* ) \]

T10

\[ = (y \cup b^* ) \cap (y \cup c^* ) \]

P6

\[ = (y \cup b^* ) \cap (y \cup c^* ) \]

P4

\[ = y \cup b^* \cap c^* \]

P7

\[ = b^* \cap c^* \cap y \]

P5

\[ = b^* \cap c^* \cup \{(a \cup (b \cup c)) \cup \} \]

Assumption

\[ = b^* \cap c^* \cup \{(a \cup (b \cup c)) \cup \} \]

P5

\[ = b^* \cap c^* \cup \{(a \cup (b \cup c)) \cup \} \]

T11

T10

Also, \( y \cap x^* = \{(a \cup (b \cup c)) \cap (a \cup b) \cup c^* \}

= \{(a \cup (b \cup c)) \cap (a \cup b) \cup c^* \}

T11

\[ = \{(a \cup (b \cup c)) \cap (a \cup b) \cup c^* \}

= \{(a \cup (b \cup c)) \cap (a \cup b) \cup c^* \}

T11

\[ = \{(a \cup (b \cup c)) \cap (a \cup b) \cup c^* \}

P6

\[ = \{(a \cup (b \cup c)) \cap (a \cup b) \cup c^* \}

P8

\[ = \{(a \cup (b \cup c)) \cap (a \cup b) \cup c^* \}

\cup \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

\cup \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

P8

\[ = \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

\cup \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

\cup \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

T10a

\[ = \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

P3

\[ = \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

P8

\[ = \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

P6

\[ = \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

P6

\[ = \{(a \cup b^* ) \cap (a \cup b) \cup c^* \}

P6
= \{a \cup b\} \cap \{a \cup b\}' \cap c'\}

= 0

Since \(y \cup x' = 1\) and \(y \cap x' = 0\), then

\(y' = x'\)  \\
\(y'' = x''\)  \\
\(y = x\)

i.e. \(a \cup b \cap c = a \cup (b \cap c)\)

\text{Assumption}

\textbf{Theorem 12a.} \textbf{For every} \(a, b, c \in K\), \((a \cap b) \cap c = a \cap (b \cap c)\).

This theorem follows when the method of proof for dual theorems is applied to Theorem 12.

\textbf{Theorem 13.} \((a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cup b) \cap (b \cup c) \cap (c \cup a)\).

\((a \cap b) \cup (b \cap c) \cup (c \cap a)\)

= \{a \cap b\} \cup (b \cap c) \cup (c \cap a) \quad \text{T12}

= \{a \cap b\} \cup (b \cap c) \cup (c \cap a) \quad \text{P7}

= \{(a \cap b) \cap (b \cap c)\} \cup (c \cap a) \quad \text{P7}

= \{(a \cap b) \cap (b \cap c)\} \cup (c \cap a) \quad \text{P7}

= \{a \cup (b \cap c)\} \cup (c \cup (a \cap b)) \quad \text{P5}

= \{a \cup (b \cap c)\} \cup (c \cup (a \cap b)) \quad \text{P5}

= \{b \cap (c \cup (a \cap b))\} \quad \text{P6}

= \{b \cap (c \cup (a \cap b))\} \quad \text{T9}

= \{b \cap (c \cup (a \cap b))\} \quad \text{T9}

= \{c \cup (a \cap b)\} \quad \text{P7}

= \{c \cup (a \cap b)\} \quad \text{P7}

= \{c \cup (a \cap b)\} \quad \text{P7}

= \{c \cup (a \cap b)\} \quad \text{P7}
\[
\begin{align*}
&= \mathcal{L}\{c \cup b\} \wedge \{/(c \cup (c \cup a)) \wedge /(c \cup (c \cup b))\} \mathcal{J} \\
&\quad \wedge \mathcal{L}\{a \cup b\} \wedge \{/(a \cup (c \cup a)) \wedge /(a \cup (c \cup b))\} \mathcal{J} & \text{P7} \\
&= \mathcal{L}\{c \cup b\} \wedge \{/(c \cup (c \cup a)) \wedge /(c \cup (c \cup b))\} \mathcal{J} \\
&\quad \wedge \mathcal{L}\{a \cup b\} \wedge \{/(a \cup (c \cup a)) \wedge /(a \cup (c \cup b))\} \mathcal{J} & \text{P5} \\
&= \mathcal{L}\{c \cup b\} \wedge \{/(c \cup (c \cup a)) \wedge /(c \cup (c \cup b))\} \mathcal{J} \\
&\quad \wedge \mathcal{L}\{a \cup b\} \wedge \{/(a \cup (c \cup a)) \wedge /(a \cup (c \cup b))\} \mathcal{J} & \text{T12} \\
&= \mathcal{L}\{c \cup b\} \wedge \{(c \cup a) \wedge (c \cup b)\} \mathcal{J} \\
&\quad \wedge \mathcal{L}\{a \cup b\} \wedge \{(c \cup a) \wedge (a \cup b) \cup c\} \mathcal{J} & \text{T7} \\
&= \mathcal{L}\{c \cup b\} \wedge \{(c \cup a) \wedge (c \cup b)\} \wedge (c \cup a) \mathcal{J} \\
&\quad \wedge \mathcal{L}\{a \cup b\} \wedge \{(a \cup b) \cup c\} \wedge (c \cup a) \mathcal{J} & \text{P6} \\
&= \mathcal{L}\{c \cup b\} \wedge (c \cup a) \mathcal{J} \wedge \mathcal{L}\{(a \cup b) \cup (a \cup b) \cup c\} \mathcal{J} \\
&\quad \wedge (c \cup a) \mathcal{J} & \text{T12a} \\
&= \mathcal{L}\{c \cup b\} \wedge (c \cup a) \mathcal{J} \wedge \mathcal{L}\{(a \cup b) \cup (a \cup b) \cup c\} \mathcal{J} \\
&\quad \wedge (c \cup a) \mathcal{J} & \text{T7a} \\
&= \mathcal{L}\{c \cup b\} \wedge (c \cup a) \mathcal{J} \wedge \mathcal{L}\{(a \cup b) \cup (a \cup b) \cup c\} \mathcal{J} \\
&\quad \wedge (c \cup a) \mathcal{J} & \text{T9a} \\
&= \mathcal{L}\{c \cup b\} \wedge (c \cup a) \mathcal{J} \wedge \mathcal{L}\{(a \cup b) \cup (a \cup b) \cup c\} \mathcal{J} \\
&\quad \wedge (c \cup a) \mathcal{J} & \text{P6} \\
&= (c \cup b) \wedge \mathcal{L}\{(c \cup a) \wedge (c \cup a)\} \wedge (a \cup b) \mathcal{J} & \text{T12a} \\
&= (c \cup b) \wedge (c \cup a) \wedge (a \cup b) \mathcal{J} & \text{T7a} \\
&= (a \cup b) \wedge (c \cup b) \wedge (c \cup a) \mathcal{J} & \text{P6} \\
&= (a \cup b) \wedge (b \cup c) \wedge (c \cup a) \mathcal{J} & \text{P5}
\end{align*}
\]

Definition 1. The **complete dual** of any identity, whether satisfied by all elements of \(K\) or a particular set of elements of \(K\), is found by replacing each \(\cup\) with \(\wedge\), each \(\wedge\) with \(\cup\), and each element with the prime of that element.
**Remark.** The following definition may be extended to any finite number of elements of K. To simplify notation, the function of only two elements is defined.

**Definition 2.** \(f(a, b)\), read "function of a and b," means any expression containing a's and b's which may be obtained by applying the operations \(', \cup, \cap\) a finite number of times.

**Definition 3.** \(F(a, b)\) shall denote the dual of \(f(a, b)\).

**Remark.** The complete dual of \(f(a, b)\) is \(F(a', b')\).

**Theorem 14.** If any identity in a Boolean algebra is true, the complete dual of the identity is also true.

Let \(f(a, b) = g(a, b)\)

then \(\neg F(a, b)' = \neg g(a, b)'\) \hspace{1cm} T4

\(F(a, b)' = F(a', b')\) and \(g(a, b)' = G(a', b')\) \hspace{1cm} T11, T11a

Hence \(F(a', b') = G(a', b')\) \hspace{1cm} (DeMorgan's Theorems)

**Lemma 1.** Every element in K is the prime of some element in K.

\((a')' = a\) \hspace{1cm} T3

**Theorem 15.** If, for every element in K, an identity \(f(a, b) = g(a, b)\) is true in a Boolean algebra, then the dual of the identity is also true for every element in K.

For every \(c, d \in K\), \(f(c, d) = g(c, d)\) \hspace{1cm} Assumption

For every \(c, d \in K\), \(F(c', d') = G(c', d')\) \hspace{1cm} T14

Choose \(a, b \in K\). Let \(c = a', d = b'\) \hspace{1cm} Lemma 1

Then \(c' = a\) and \(d' = b\) \hspace{1cm} T5

Thus, \(F(a, b) = G(a, b)\)
Definition: 4. \( a \leq b \) means \( a \cap b = a \).

Definition: 5. \( a \geq b \) means \( b \leq a \).

Note: 1. \( a \leq b \) is read \( a \) is included in \( b \).

2. \( a \geq b \) is read \( a \) includes \( b \).

Theorem 16. For every \( a, b \in K \), \( a \leq b \) if and only if \( a \cap b^* = 0 \).

Necessity: If \( a \leq b \), then \( a \cap b^* = 0 \).

\[
\begin{align*}
a \cap b^* &= (a \cap b) \cap b^* & \text{Assumption and Def. 4.} \\
&= a \cap (b \cap b^*) & \text{T12a} \\
&= a \cap 0 & \text{P9} \\
&= 0 & \text{T8a}
\end{align*}
\]

Sufficiency: If \( a \cap b^* = 0 \), then \( a \leq b \).

\[
\begin{align*}
a \cap b &= (a \cap b) \cup 0 & \text{P3} \\
&= (a \cap b) \cup (a \cap b^*) & \text{Assumption} \\
&= a \cap (b \cup b^*) & \text{P8} \\
&= a \cap 1 & \text{P9} \\
&= a & \text{P4} \\
\end{align*}
\]

\( a \leq b \) \hspace{1cm} \text{Def. 4}

Theorem 17. For every \( a, b \in K \), \( a \leq b \) if and only if \( a' \cup b = 1 \).

Necessity: If \( a \leq b \), then \( a' \cup b^* = 1 \)

\[
\begin{align*}
a \cap b^* &= 0 & \text{T16} \\
(a \cap b^*)' &= 0' & \text{T4} \\
a' \cup b^* &= 0' & \text{T11a} \\
a' \cup b &= 0' & \text{T3} \\
a' \cup b &= 1 & \text{T6}
\end{align*}
\]
Sufficiency: If $a' \cup b = 1$, then $a \preceq b$.

\[
\begin{align*}
a' \cup b &= 1 & \text{Assumption} \\
(a' \cup b)' &= 1' & \text{T4} \\
a'' \cap b' &= 1' & \text{T11} \\
a \cap b' &= 1' & \text{T3} \\
a \cap b' &= 0 & \text{T6a} \\
a \preceq b & & \text{T16}
\end{align*}
\]

**Theorem 18.** For every $a, b \in K$, $a \preceq b$ if and only if $a \cup b = b$.

**Necessity:** If $a \preceq b$, then $a \cup b = b$

\[
\begin{align*}
a \cup b &= (a \cup b) \cap 1 & \text{P4} \\
&= (a \cup b) \cap (a' \cup b) & \text{Assumption} \\
&= (b \cup a) \cap (b \cup a') & \text{T17} \\
&= b \cup (a \cap a') & \text{P5} \\
&= b \cup 0 & \text{P7} \\
&= b & \text{P9} \\
&= b & \text{P3}
\end{align*}
\]

**Sufficiency:** If $a \cup b = b$, then $a \preceq b$.

\[
\begin{align*}
a \cap b' &= a \cap (a \cup b)' & \text{Assumption} \\
&= a \cap (a' \cap b') & \text{T11} \\
&= (a \cap a') \cap b' & \text{T12a} \\
&= 0 \cap b' & \text{P9} \\
&= b' \cap 0 & \text{P6} \\
&= 0 & \text{T8a} \\
a \preceq b & & \text{T16}
\end{align*}
\]
Theorem 19. For every $a, b \in K, a \leq b$ if and only if $b' \leq a'$.

Necessity: If $a \leq b$, then $b' \leq a'$

- $a \cap b = a$  \hspace{1cm} \text{Def. 4}$
- $(a \cap b)' = a'$ \hspace{1cm} T4
- $a' \cup b' = a'$ \hspace{1cm} T11a
- $b' \cup a' = a'$ \hspace{1cm} P5
- $b' \leq a'$ \hspace{1cm} T18

Sufficiency: If $b' \leq a'$, then $a \leq b$

- $b' \cup a' = a'$ \hspace{1cm} T18
- $(b \cap a)' = a'$ \hspace{1cm} T11a
- $(b \cap a)'' = a''$ \hspace{1cm} T4
- $b \cap a = a$ \hspace{1cm} T3
- $a \cap b = a$ \hspace{1cm} P6
- $a \leq b$ \hspace{1cm} Def. 4

Theorem 20. For every $a \in K, a \leq a$.

- $a \cap a = a$ \hspace{1cm} T7a
- $a \leq a$ \hspace{1cm} Def. 4

Theorem 21. For every $a \in K, 0 \leq a$ and $a \leq 1$.

- $a \cap 0 = 0$ \hspace{1cm} T8a
- $0 \cap a = 0$ \hspace{1cm} P6
- $0 \leq a$ \hspace{1cm} Def. 4

and

- $a \cup 1 = 1$ \hspace{1cm} T8
- $a \leq 1$ \hspace{1cm} T18
Theorem 22. For every \( a, b \in K \), if \( a \preceq b \) and \( b \preceq a \), then \( a = b \).

If \( a \preceq b \), then \( a \land b = a \) \hspace{1cm} \text{Def. 4}

If \( b \preceq a \), then \( b \land a = b \) \hspace{1cm} \text{Def. 4}

and \( a \land b = b \) \hspace{1cm} \text{P6}

Hence \( a = b \)

Theorem 23. For every \( a, b, c \in K \), if \( a \preceq b \) and \( b \preceq c \), then \( a \preceq c \).

If \( a \preceq b \), then \( a \land b = a \) \hspace{1cm} \text{Def. 4}

If \( b \preceq c \), then \( b \land c = b \) \hspace{1cm} \text{Def. 4}

Thus \( a \land (b \land c) = a \)

\( (a \land b) \land c = a \) \hspace{1cm} \text{T12a}

\( a \land c = a \)

\( a \preceq c \) \hspace{1cm} \text{Def. 4}
CHAPTER II BIBLIOGRAPHY

CHAPTER III

THE EQUIVALENCE OF VARIOUS REPRESENTATIONS
OF BOOLEAN ALGEBRAS

The following postulate sets seem to be the most interesting of the various representations for a Boolean algebra. Postulate Set II defines a Boolean ring with unity element; Stabler (4). Postulate Set IV is the unusual set of postulates introduced by Sheffer (3), in 1913. Sheffer's postulates reduced the number of postulates needed to define a Boolean algebra from Huntington's (2) nine postulates to four. The postulates are based on the powerful operation of "rejection." Postulate Set III is the modification of Sheffer's postulates presented by Bernstein (1), further reducing the number of postulates needed from four to three. Postulate Set III is the most economical set of postulates for a Boolean algebra that has been introduced. In this paper the postulates defining a Boolean algebra in terms of the "inclusion" relation have been omitted. The postulate sets and the proof of their equivalence can be found in the paper of Huntington (2).

In this chapter the equivalence of the various postulate sets will be proven. Two postulate sets A and B will be equivalent if the suitable definitions of the operations and
elements of Postulate Set B, the postulates and theorems of Postulate Set A imply the postulates of Set B; and if with suitable definitions of the operations and elements of Postulate Set A, the postulates and theorems of Postulate Set B imply the postulates of Set A.

It will be shown that Postulate Set I implies Postulate Set II; Postulate Set II implies Postulate Set I; Postulate Set I implies Postulate Set III; Postulate Set III implies Postulate Set IV; and Postulate Set IV implies Postulate Set I. This is actually a "circle" of equivalences, i.e.

```
I ←→ II  III → IV
```

It is apparent that any one of the four postulate sets implies any of the others.

In Postulate Sets II, III, and IV, some of the theorems following from each of the sets of postulates will be stated. Proof of these theorems can be found in the respective previously indicated references. The I, II, III, and IV preceding each argument in the proof of the equivalences refer to the particular postulate set the argument is taken from. The sets of undefined elements in each set of postulates contain at least two elements.

Postulate Set II

Undefined are a class T of elements, a, b, c, ... and two binary operations, + and ·.
P1. \( a + b \in T. \)

P2. \( a \cdot b \in T. \)

P3. There exists an element \( \emptyset \in T \) such that if \( a \in T \), then
\[
a + \emptyset = a.
\]

P4. There exists an element \( \Theta \in T \) such that if \( a \in T \), then
\[
a \cdot \Theta = a.
\]

P5. \( a + b = b + a. \)

P6. \( a \cdot (b + c) = (a \cdot b) + (a \cdot c) \)

P7. \( (a + b) \cdot c = (a \cdot c) + (b \cdot c) \)

P8. \( a + (b + c) = (a + b) + c. \)

P9. \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \).

P10. \( a \cdot a = a. \)

P11. For every \( a \in T \) there exists an element \( \overline{a} \in T \) such that
\[
a + \overline{a} = \emptyset.
\]

The following theorems are listed:

**Theorem 1.** \( a + a = \emptyset. \)

**Corollary 1.** If \( a + b = \emptyset \), then \( a = b. \)

**Corollary 2.** If \( a + b = c \), then \( a = c + b. \)

**Theorem 2.** \( a \cdot b = b \cdot a. \)

**Theorem 3.** \( a \cdot \overline{a} = \emptyset. \)

**Theorem 4.** \( a \cdot \emptyset = \emptyset. \)

**Statement 1.** Postulate Set I implies Postulate Set II.

**Definition:** \( a + b = (a \land b') \cup (a' \land b) \); \( a \cdot b = a \land b \);

Class \( T \) = Class \( K; \overline{a} = a' \); \( \emptyset = 0 \); \( \Theta = 1 \).

IIIP1. \( a + b \in T \)

\[
(a \land b') \cup (a' \land b) \in K
\]

\[
a + b \in T
\]

IP1, IP9, IP2

Def.
IIP2. \( a \ast b \in T \)

\[
\begin{align*}
   a \land b &\in K & \text{IP2} \\
   a \ast b &\in T & \text{Def.}
\end{align*}
\]

Note: In the following proof and all such that involve the changing of both sides of an identity at the same time, the arguments on the left side of the page refer to the left side of the identity and the arguments on the right side of the page refer to the right side of the identity.

IIP3. There exists an element \( \varnothing \in T \) such that if \( a \in T \), then

\[ a + \varnothing = a. \]

\[
\begin{align*}
   a &\in a \\
   \text{IP3.} &\quad a \cup \varnothing = a \\
   \text{IP4.} &\quad (a \land 1) \cup \varnothing = a \\
   \text{IT8a.} &\quad (a \land 1) \cup (a' \land 0) = a \\
   \text{IT6.} &\quad (a \land 0') \cup (a' \land 0) = a \\
   \text{Def.} &\quad a + 0 = a \\
   \text{Def.} &\quad a + \varnothing = a
\end{align*}
\]

IIP4. There exists an element \( \Theta \in T \) such that if \( a \in T \), then

\[ a \ast \Theta = a. \]

\[
\begin{align*}
   \text{IP4.} &\quad a \land 1 = a \\
   \text{Def.} &\quad a \ast 1 = a \\
   \text{Def.} &\quad a \ast \Theta = a
\end{align*}
\]

IIP5. \( a + b = b + a. \)

\[
(a' \cup b') \land (a \cup b) = (a' \cup b') \land (a \cup b)
\]

\[
\begin{align*}
   \text{IP5.} &\quad (a' \cup b') \land (b \cup a) = (b' \cup a') \land (a \cup b) & \text{IP5} \\
   \text{IP4.} &\quad \neg[(a' \cup b') \land \top] \land \neg[(b \cup a) \land \top]
\end{align*}
\]
\[ (\{b \cup a\} \cap \bar{J}) \cap (\bar{a} \cup b) \cap \bar{J} \]

\[
\text{IP6} \quad \bar{J} \cap (a' \cup b') \cap (b' \cup a) \cap \bar{J} \\
= \bar{J} \cap (b' \cup a') \cap (a' \cup b) \cap \bar{J} \\
\text{IP9} \quad (\{a \cup a\} \cap (a' \cup b') \cap (b' \cup a) \cap (b \cup b') \cap \\
= (b \cup b') \cap (b' \cup a') \cap (a' \cup b) \cap (a \cup a') \cap \bar{J} \\
\text{IP5} \quad (\{a' \cup a\} \cap (a' \cup b') \cap (b' \cup a) \cap (b \cup b') \cap \\
= (b' \cup b) \cap (b' \cup a') \cap (a' \cup b) \cap (a \cup a') \cap \bar{J} \\
\text{IP7} \quad \{a' \cup (a \cap b') \} \cap \bar{b} \cap (a \cap b') \cap \bar{J} \\
= \{b' \cup (b' \cap a') \} \cap \bar{a} \cap (b' \cap a') \cap \bar{J} \\
\text{IP5} \quad (\{a \cap b'\} \cup a') \cap \{\{a \cap b'\} \cup b' \} \\
= \{b' \cap a'\} \cup b' \cap \{b' \cap a'\} \cup a' \cap \bar{J} \\
\text{IP7} \quad (a \cap b') \cup (a \cap b) = (b \cap a') \cup (b' \cap a) \\
\text{Def.} \quad a + b = b + a \\
\text{IIP6.} \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c).
\]

\[
(a \cap b \cap c') \cup (a \cap b' \cap c) = (a \cap b \cap c') \cup (a \cap b' \cap c) \\
\text{IT12a} \quad (a \cap \{\bar{b} \cap c\}) \cup (a \cap \{b' \cap c\}) \\
= \{a \cap b \cap c'\} \cup \bar{b} \cup \{a \cap b' \cap c\} \cup \bar{b} \\
\text{IP3} \quad a \cap (\{\bar{b} \cap c\} \cup \{b' \cap c\}) \\
= \{b \cup (a \cap b \cap c') \} \cup \{b \cup (a \cap b' \cap c)\} \\
\text{Def.} \quad a \cdot (b + c) = \{b \cap 0\} \cup (a \cap b \cap c') \cup \\
\text{IT8a} \quad \cup (c \cap 0) \cup (a \cap b' \cap c') \\
= \{0 \cup b\} \cup (a \cap b \cap c') \cup \{0 \cup (a \cap b' \cap c)\} \cup \{a \cap b' \cap c\} \cup \{a \cap b \cap c'\} \\
\text{IP6} \quad \{\{a \cap a'\} \cap b\} \cup (a \cap b \cap c') \\
\text{IP9} \quad \cup \{\{a \cap a'\} \cap c\} \cup (a \cap b' \cap c) \\
= \{\{a \cap a'\} \cup \{a \cap b\} \cup (a \cap b' \cap c) \} \\
\text{IT12a}
\]
\[ \lbrack \{ a \land \{ b \land a' \} \} \cup ( \{ a \land b \} \land c' ) \rbrack \]
\[ \cup \lbrack \{ a \land \{ c \land a' \} \} \cup ( a \land \{ c \land b' \} ) \rbrack \]
\[ = \lbrack \{ a \land b \} \land a' \rbrack \cup ( \{ a \land b \} \land c' ) \]
\[ \cup \lbrack \{ a \land c \} \land a' \rbrack \cup ( \{ a \land c \} \land b' ) \]
\[ = \lbrack \{ a \land b \} \land ( a' \cup c' ) \rbrack \cup \lbrack \{ a \land c \} \land ( a' \cup b' ) \rbrack \]
\[ = \lbrack \{ a \land b \} \land ( a \land c ) \rbrack \cup \lbrack \{ a \land c \} \land ( a \land b ) \rbrack \]
\[ = \lbrack \{ a \land b \} \land ( a \land c ) \rbrack \cup \lbrack \{ a \land b \} \land ( a \land c ) \rbrack \]
\[ a \cdot ( b + c ) = ( a \cdot b ) + ( a \cdot c ) \]

**IIP7.** \( (a + b) \cdot c = (a \cdot c) + (b \cdot c) \).

\[ \lbrack a \land b \land c \rbrack \cup \lbrack a \land b \land c \rbrack = \lbrack a \land b \land c \rbrack \cup \lbrack a \land b \land c \rbrack \]
\[ \text{IT12a} \; \lbrack (a \land b') \land c \rbrack \cup \lbrack (a \land b) \land c \rbrack \]
\[ = \lbrack (a \land b \land c) \cup 0 \rbrack \cup \lbrack (a \land b \land c) \cup 0 \rbrack \]
\[ \text{IP3} \]
\[ \text{IP6} \; \lbrack c \land (a \land b') \rbrack \cup \lbrack c \land (a \land b) \rbrack \]
\[ = \lbrack (a \land \{ b \land c \} ) \cup 0 \rbrack \cup \lbrack (a \land \{ b \land c \} ) \cup 0 \rbrack \]
\[ \text{IT12a} \]
\[ \text{IP8} \; c \land \lbrack (a \land b') \rbrack \cup (a \land b) \]
\[ = \lbrack (a \land \{ b \land c \} ) \cup (a \land 0) \rbrack \cup \lbrack (a \land \{ b \land c \} ) \cup (b \land 0) \rbrack \]
\[ \text{IT8a} \]
\[ \text{IP6} \; \lbrack (a \land b') \cup (a \land b) \rbrack \land c \]
\[ = \lbrack (a \land \{ c \land b' \} ) \cup (a \land 0) \rbrack \cup \lbrack (a \land \{ b \land c \} ) \cup (b \land 0) \rbrack \]
\[ \text{IP6} \]

**Def.** \( (a + b) \cdot c \)

\[ \lbrack (a \land \{ c \land b' \} ) \cup (a \land \{ c \land c' \} ) \rbrack \]
\[ \cup \lbrack (a \land \{ b \land c \} ) \cup (b \land \{ c \land c' \} ) \rbrack \]
\[ = \lbrack (a \land c' \land b') \cup (\{ a \land c \} \land c' ) \rbrack \]
\[ \cup \lbrack (a \land \{ b \land c \} ) \cup (\{ b \land c \} \land c' ) \rbrack \]
\[ \text{IT12a} \]
\[(a + b) \cdot c = \bigcup\{a \cap c \cap b'\} \cup \{a \cap c \cap c'\}\]
\[\cup \bigcup\{b \cap c \cap a'\} \cup \{b \cap c \cap c'\}\]  
IP6

\[(a + b) \cdot c = \bigcup\{a \cap c\} \cap (b' \cup c')\]
\[\cup \bigcup\{b \cap c\} \cap (a' \cup c')\]  
IP8

\[(a + b) \cdot c = \bigcup\{a \cap c\} \cap (b \cap c)\]
\[\cup \bigcup\{b \cap c\} \cap (a \cap c)\]  
IP9a

\[(a + b) \cdot c = \bigcup\{a \cap c\} \cap (b \cap c)\] \[\cup \bigcup\{a \cap c\} \cap (b \cap c)\]  
IP6

\[(a + b) \cdot c = (a \cdot c) + (b \cdot c)\]  
Def.

IIP8. \(a + (b + c) = (a + b) + c\).

IT12 \((a \cap b' \cap c') \cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\} \cup \{a \cap b \cap c\}\]
\[= \bigcup\{a \cap b' \cap c'\} \cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\} \]
\[\cup \{a \cap b \cap c\}\]  
IT12

IP5 \((a \cap b' \cap c') \cup \bigcup\{a \cap b \cap c\} \cup \{a' \cap b \cap c'\} \cup \{a' \cap b' \cap c\}\]
\[= \bigcup\{a \cap b' \cap c'\} \cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\} \]
\[\cup \{a \cap b \cap c\}\]  
IT12

IT12 \(\bigcup\{a \cap b' \cap c'\} \cup \{a \cap b \cap c\}\] \[\cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\}\]
\[= \bigcup\{a \cap b' \cap c'\} \cup \{a' \cap b \cap c\}\]
\[\cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\}\]
\[\cup \{a \cap b \cap c\}\]  
IT12

IP3 \(\bigcup\{a \cap b' \cap c'\} \cup 0\] \[\cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\}\]
\[= \bigcup\{a \cap b' \cap c'\} \cup \{a' \cap b \cap c\}\]
\[\cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\}\]
\[\cup \{a \cap b \cap c\}\]  
IP3

IP5 \(\bigcup\{a \cap b' \cap c'\} \cup \{a \cap b \cap c\}\]
\[\cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\}\]
\[= \bigcup\{a \cap b' \cap c'\} \cup \{a' \cap b \cap c\}\]
\[\cup \bigcup\{a' \cap b \cap c'\} \cup \{a' \cap b \cap c\}\]
\[\cup \{a \cap b \cap c\}\]
\[
\{a \land b' \land c'\} \cup \{a' \land b \land c'\}
\]
\[
\cup \{a' \land b' \land c'\} \cup \{a' \land b \land c\} \cup \{a \land b \land c\} \cup \{a \land b' \land c\} \cup 02 \]

**IP5**

**IT8a**
\[
\{a \land 0\} \cup \{a' \land b \land c'\} \cup \{a \land b \land c\} \cup \{a \land 0\} \cup \{a' \land b \land c\}
\]
\[
\cup \{a' \land b' \land c\} \cup \{a \land b' \land c\} \cup \{a \land b \land c\} \cup \{a \land 0\}
\]

**IT8a**

**IT12a**
\[
\{a \land 0\} \cup \{a' \land b' \land c'\} \cup \{a' \land b' \land c\} \cup \{a' \land b \land c\} \cup \{a' \land b' \land c\}
\]
\[
\cup \{a' \land b' \land c\} \cup \{a' \land b \land c\} \cup \{a' \land b \land c\}
\]
\[
\cup \{a' \land b' \land c\} \cup \{a' \land b' \land c\} \cup \{a' \land b \land c\} \cup \{a' \land b' \land c\}
\]

**IT12a**

**IP9**
\[
\{a \land b' \land c'\} \cup \{a' \land b' \land c'\} \cup \{a' \land b' \land c\}
\]
\[
\cup \{a' \land b' \land c\} \cup \{a' \land b' \land c\}
\]
\[
\cup \{a' \land b' \land c\} \cup \{a' \land b' \land c\} \cup \{a' \land b' \land c\}
\]

**IP9**

**IP6**
\[
\{a \land b' \land b'\} \cup \{a \land b' \land c'\} \cup \{a \land b' \land c\}
\]
\[
\cup \{a \land b' \land c\} \cup \{a \land b' \land c\}
\]
\[
\cup \{a \land b' \land c\} \cup \{a \land b' \land c\}
\]
\[
\cup \{a \land b' \land c\} \cup \{a \land b' \land c\} \cup \{a \land b' \land c\}
\]

**IP6**

**IT12a**
\[
\{a \land b' \land b\} \cup \{a \land b' \land c'\} \cup \{a \land b' \land c\}
\]
\[
\cup \{a \land b' \land c\} \cup \{a \land b' \land c\}
\]
\[
\cup \{a \land b' \land c\} \cup \{a \land b' \land c\} \cup \{a \land b' \land c\}
\]

**IT12a**

**IP8**
\[
\{a \land b' \land b\} \cup \{a' \land b \land c'\} \cup \{a' \land b \land c\}
\]
\[
\cup \{a' \land b \land c\} \cup \{a' \land b \land c\} \cup \{a' \land b \land c\}
\]
\[
\cup \{a' \land b \land c\} \cup \{a' \land b \land c\} \cup \{a' \land b \land c\}
\]

**IP8**
\[ \mathcal{A} \{b \cup c\} \cap \{a \cap b\} \cup \{b \cup c\} \cap \{a \cap c\} \]

\[ = \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \cup \mathcal{A} \{a \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \]

\[ = \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \cup \mathcal{A} \{a \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \]

\[ = \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \cup \mathcal{A} \{a' \cup (\{b \cap c\} \cup \{b' \cap c\}) \} \]

\[ = \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \cup \mathcal{A} \{a' \cup (\{b \cap c\} \cup \{b' \cap c\}) \} \]

\[ = \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \cup \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \]

\[ = \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \cup \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \]

\[ = \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \cup \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \]

\[ = \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \cup \mathcal{A} \{a' \cap (\{b \cap c\} \cup \{b' \cap c\}) \} \]
\[ = \bigcup \{a \land b\} \cup \{a' \land b'\} \cup \mathcal{C} \cup \bigcup \{a \land b\} \cup \{a' \land b'\} \cup \mathcal{C} \]

**Def.** \(a + (b + c) = (a + b) + c\)  

**IIP9.** \((a \cdot b) \cdot c = a \cdot (b \cdot c)\).  
\[
(a \land b) \land c = a \land (b \land c) \tag*{IT12a}
\]
\[
(a \cdot b) \cdot c = a \cdot (b \cdot c) \tag*{Def.}
\]

**IIP10.** \(a \cdot a = a\).  
\[
a \land a = a \tag*{IT7a}
\]
\[
a \cdot a = a \tag*{Def.}
\]

**IIP11.** For every \(a \in T\) there exists an element \(\hat{a} \in T\) such that  
\[
a + \hat{a} = \emptyset.
\]
\[
l = l
\]

**IP9** \(a \lor a' = l\)  

**IT7a** \((a \land a) \lor (a' \lor a') = l\)  

**IT3** \((a \land a') \lor (a' \lor a') = l\)  

**Def.** \(a + a' = l\)  

**Def.** \(a + \hat{a} = \emptyset\).

It has been shown that with the proper definitions, Huntington's set of postulates implies the set of postulates for a Boolean ring with unity element.

**Statement 2. Postulate Set II implies Postulate Set I.**

**Definitions:**  
\[
a \lor b = (a + b) + (a \cdot b); \quad a \land b = a \cdot b; \quad 0 = \emptyset; \quad 1 = \emptyset; \quad a' = \hat{a}; \quad \text{Class } K = \text{Class } T\]
IP1. $a \cup b \in K$.

$$(a + b) + (a \cdot b) \in T \quad \text{IIP1, IIP2}$$

$a \cup b \in K \quad \text{Def.}$

IP2. $a \wedge b \in K$.

$$a \cdot b \in T \quad \text{IIP2}$$

$a \wedge b \in K \quad \text{Def.}$

IP3. There exists an element $0 \in K$ such that if $a \in K$, then

$$a \cup 0 = a.$$  

$$a = a \quad \text{IIP3}$$

$$a + \emptyset = a \quad \text{IIT4}$$

$$a + (a \cdot \emptyset) = a \quad \text{IIP3}$$

$$a \cup 0 = a \quad \text{Def.}$$

IP4. There exists an element $1 \in K$ such that if $a \in K$, then

$$a \wedge 1 = a.$$  

$$a \cdot \emptyset = a \quad \text{IIP4}$$

$$a \wedge 1 = a \quad \text{Def.}$$

IP5. $a \cup b = b \cup a$.

$$(a + b) + (b \cdot a) = (a + b) + (b \cdot a) \quad \text{IIP5}$$

$$a \cup b = b \cup a \quad \text{Def.}$$

IP6. $a \wedge b = b \wedge a$

$$a \cdot b = b \cdot a \quad \text{IIT2}$$

$$a \wedge b = b \wedge a \quad \text{Def.}$$
IP7. \( a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \).

\[
\begin{align*}
a \cup (b \cap c) &= (a + [\overline{b} \cdot \overline{c}]) + (a \cdot [\overline{b} \cdot \overline{c}]) & \text{Def.} \\
&= (a + [\overline{b} \cdot \overline{c}]) + (a + [\overline{b} \cdot \overline{c}]) + (\phi + \phi + \phi) & \text{IIP3, IIP8} \\
&= (a) + (b \cdot c) + (a + [\overline{b} \cdot \overline{c}]) + (\phi) + (\phi) + (\phi) & \text{IIP8} \\
&= (a) + (\phi) + (\phi) + (b \cdot c) + (\phi) + (a + [\overline{b} \cdot \overline{c}]) & \text{IIP5, IIP8} \\
&= (a) + ([a \cdot \overline{b} \cdot \overline{c}] + [a \cdot \overline{b} \cdot \overline{c}]) + (a \cdot [\overline{b} \cdot \overline{c}]) + (b \cdot c) & \text{ITL1} \\
&= (a) + (a \cdot b) + (a \cdot b) + (a \cdot c) + (a \cdot c) + (b \cdot c) \\
&+ (a \cdot [\overline{b} \cdot \overline{c}]) + (a \cdot [\overline{b} \cdot \overline{c}]) + (a \cdot [\overline{b} \cdot \overline{c}]) & \text{IIP8} \\
&= (a) + (a \cdot b) + (a \cdot c) + (b \cdot c) + (a \cdot b) + (a \cdot [\overline{b} \cdot \overline{c}]) \\
&+ (a \cdot c) + (a \cdot [\overline{b} \cdot \overline{c}]) + (a \cdot [\overline{b} \cdot \overline{c}]) & \text{IIP5} \\
&= (a) + (a \cdot b) + (a \cdot c) + (b \cdot c) + (a \cdot b) + (a \cdot [\overline{b} \cdot \overline{c}]) \\
&+ (a \cdot c) + (a \cdot [\overline{b} \cdot \overline{c}]) + ([a \cdot \overline{b} \cdot \overline{c}]) & \text{IIP9} \\
&= (a) + (a \cdot b) + (a \cdot c) + (b \cdot c) + (b \cdot a) + (a \cdot [\overline{b} \cdot \overline{c}]) \\
&+ (a \cdot c) + (a \cdot [\overline{b} \cdot \overline{c}]) + ([\overline{b} \cdot \overline{a} \cdot \overline{c}]) & \text{ITL2} \\
&= (a \cdot a) + (a \cdot b) + (a \cdot c) + (b \cdot c) + (b \cdot [\overline{a} \cdot \overline{b}]) + (a \cdot [\overline{b} \cdot \overline{c}]) \\
&+ ([\overline{a} \cdot \overline{b} \cdot \overline{c}] + (a \cdot c \cdot b) + ([\overline{b} \cdot [\overline{a} \cdot \overline{c}]) & \text{ITL10} \\
&= (a \cdot a) + (a \cdot b) + (a \cdot c) + (b \cdot c) + ([\overline{b} \cdot \overline{a} \cdot \overline{c}] + ([\overline{a} \cdot \overline{b} \cdot \overline{c}] + (a \cdot [\overline{a} \cdot \overline{c}]) \\
&+ (a \cdot [\overline{a} \cdot \overline{c}]) + ([\overline{b} \cdot \overline{a} \cdot \overline{c}] + ([\overline{a} \cdot \overline{b} \cdot \overline{c}] + (a \cdot [\overline{a} \cdot \overline{c}]) & \text{IIP9} \\
&= (a \cdot a) + (b \cdot a) + (a \cdot c) + (b \cdot c) + ([\overline{b} \cdot \overline{a} \cdot \overline{c}] + ([\overline{a} \cdot \overline{b} \cdot \overline{c}] + (a \cdot [\overline{a} \cdot \overline{c}]) \\
&+ (a \cdot [\overline{a} \cdot \overline{c}]) + ([\overline{b} \cdot \overline{a} \cdot \overline{c}] + ([\overline{a} \cdot \overline{a} \cdot \overline{c}] + (a \cdot [\overline{a} \cdot \overline{c}]) & \text{ITL2} \\
&= \{[\overline{(a \cdot a)} + (b \cdot a)] + (a \cdot c) + (b \cdot c)] \} \\
&+ \{([\overline{a} \cdot \overline{b} \cdot \overline{c}] \cdot \overline{b} \cdot \overline{a} \cdot \overline{b} \cdot c) \} + \{([a \cdot \overline{a} \cdot \overline{b} \cdot \overline{c}]) + (b \cdot [\overline{a} \cdot \overline{b} \cdot \overline{c}]) \} \\
&+ ([\overline{a} \cdot \overline{b} \cdot \overline{c}] + ([\overline{a} \cdot \overline{a} \cdot \overline{c}] + (a \cdot [\overline{a} \cdot \overline{c}]) \} & \text{IIP8}
\[
= \left\{ \begin{array}{c}
\{(a+b)\cdot a/(a+b)c\} + \{(a-b)(a\cdot c\downarrow) + (a\cdot b\cdot c\downarrow)\} \\
+ \{(a+b)\cdot \overline{a\cdot c\downarrow} + (a\cdot b\cdot \overline{a\cdot c\downarrow})\}
\end{array} \right. \\
\text{IIP7}
\]

\[
= \left\{ \begin{array}{c}
\{(a+b)(a+c)/(a+b)c\} + \{(a+b)\cdot \overline{a\cdot c\downarrow} + (a\cdot b\cdot \overline{a\cdot c\downarrow})\} \\
+ \{(a+b)\cdot \overline{a\cdot c\downarrow} + (a\cdot b\cdot \overline{a\cdot c\downarrow})\}
\end{array} \right. \\
\text{IIP6}
\]

\[
= \left\{ \begin{array}{c}
\{(a+b) + (a\cdot b)/(a+c)\} + \{(a+b)\cdot \overline{a\cdot b\cdot \overline{a\cdot c\downarrow}}\} \\
+ \{(a+b)\cdot \overline{a\cdot c\downarrow} + (a\cdot b\cdot \overline{a\cdot c\downarrow})\}
\end{array} \right. \\
\text{IIP7}
\]

\[
= \left\{ \begin{array}{c}
\{(a+b) + (a\cdot b)/(a+c)\} + \{(a+b)\cdot \overline{a\cdot b\cdot \overline{a\cdot c\downarrow}}\} \\
+ \{(a+b)\cdot \overline{a\cdot c\downarrow} + (a\cdot b\cdot \overline{a\cdot c\downarrow})\}
\end{array} \right. \\
\text{IIP6}
\]

\[
= (a \lor b) \land (a \land c)
\]

\[\text{Def.}\]

\[\text{IP8. } a \land (b \lor c) = (a \land b) \lor (a \land c).\]

\[
a \land (b \lor c) = a \cdot \overline{(b+c)} + (b \cdot c)\downarrow
\]

\[= a \cdot (b+c) + a \cdot (b \cdot c)\]

\[= a \cdot (b+c) + (a \cdot b) \cdot c\]

\[= a \cdot (b+c) + (b \cdot a) \cdot c\]

\[= a \cdot (b+c) + (b \cdot \overline{a \cdot a\downarrow}) \cdot c\]

\[= a \cdot (b+c) + (b \cdot a) \cdot (a \cdot c)\]

\[= a \cdot (b+c) + (a \cdot b) \cdot (a \cdot c)\]

\[= \overline{(a \cdot b) + (a \cdot c)\downarrow} + \overline{(a \cdot b) \cdot (a \cdot c)\downarrow}\]

\[= (a \land b) \lor (a \land c)\]

\[\text{Def.}\]

\[\text{IP9. If the elements 0 and 1 in P3 and P4 exist and are unique}
\]

\[\text{and if } a \in K, \text{ then there exists an } a' \in K \text{ such that}
\]

\[a \lor a' = 1 \text{ and } a \land a' = 0.\]

\[A. \ a \lor a' = 1.
\]

\[a + \overline{a} = \emptyset \]

\[\text{IIP11}
\]

\[(a+ \overline{a} ) + (\emptyset) = \emptyset \]

\[\text{IIP3}\]
(a + a') + (a \cdot a') = \Theta \quad \text{IIT3}
\quad a \cup a' = 1 \quad \text{Def.}

B. a \wedge a' = 0

\quad a \cdot \overline{a} = \Theta \quad \text{IIT3}
\quad a \wedge a' = 0 \quad \text{Def.}

It has been shown that with the proper definitions, the set of postulates for a Boolean ring with unity element implies Huntington's (2) set of postulates. It can therefore be stated that a Boolean algebra is equivalent to a Boolean ring with unity element.

Postulate Set III

Undefined are a class R of elements a, b, c, ... and one binary operation \setminus.

P1. a \setminus b \in R.

Def. 1: \overline{a} = a \setminus a.

P2. (b \setminus a) \setminus (b \setminus a) = a.

P3. \overline{a} \setminus (b \setminus c) = \overline{(b \setminus a) \setminus (c \setminus a)}

Remark: Postulate Sets III and IV are shorter by one postulate than the original postulate sets as stated by Bernstein (1) and Sheffer (3) because at the beginning of Chapter II it was assumed that the set of undefined elements for each postulate set contains at least two elements.

Theorem 1. (\overline{a}) = a.

Def. 2: \overline{a} = (\overline{a})
Theorem 2. \( a \mid (b \mid b) = \bar{a}. \)

Theorem 3. \( \bar{a} \mid (b \mid c) = (\bar{b} \mid a) \mid (\bar{c} \mid a). \)

Statement 3. Postulate Set I implies Postulate Set III.

Definitions: \( a \mid b = a' \wedge b'; \bar{a} = a'; \) Class \( R = \) Class \( K. \)

IIIP1. \( a \mid b \in R. \)

\[
\begin{align*}
a' \wedge b' & \in K \quad \text{IP9, IP2} \\
a \mid b & \in R \quad \text{Def.}
\end{align*}
\]

IIIP2. \( (b \mid a) \mid (\bar{b} \mid a) = a. \)

\[a = a\]

IP3

\[a \cup \emptyset = a\]

IP9

\[a \cup (b \wedge b') = a\]

IP7

\[(a \cup b) \wedge (a \cup b') = a\]

IP5

\[(b \cup a) \wedge (b' \cup a) = a\]

IT3

\[(b'' \cup a'') \wedge (b' \cup a'') = a\]

IT9

\[(b' \wedge a') \wedge (b \wedge a')' = a\]

Def.

\[(b' \wedge a') \mid (b \wedge a') = a\]

IT3

\[(b' \wedge a') \mid (b'' \wedge a') = a\]

Def.

\[(b \mid a) \mid (\bar{b} \mid a) = a\]

IIIP3. \( \bar{a} \mid (b \mid c) = \bar{(b \mid a)} \mid (\bar{c} \mid a) \)

\[a \wedge (b' \cup c) = a \wedge (b' \cup c)\]

IT3

\[a'' \wedge (b' \cup c'') = (a \wedge b') \cup (a \wedge c)\]

IP8

IT11a

\[a'' \wedge (b \wedge c') = (b' \wedge a') \cup (c' \wedge a)\]

IP6

Def.

\[a' \mid (b \wedge c') = (b' \wedge a'') \cup (c'' \wedge a'')\]

IT3

IT3

\[a' \mid (b'' \wedge c') = \bar{(b' \wedge a'')} \wedge (c'' \wedge a'')\]

IP9

Def.

\[\bar{a} \mid (b \mid c) = \bar{(b \mid a)} \mid (\bar{c} \mid a)\]

Def.
It has been shown that Postulate Set I implies Postulate Set III.

Postulate Set IV

Undefined are a Class S of elements, \( a, b, c, \ldots \) and one binary operation \( I \).

P1. \( a I b \in S \).

Def. 1: \( a^* = a I a \).

P2. \( (a^*)^* = a \).

P3. \( a I (b I b^*) = a^* \).

P4. \( \sqrt{(a I (b I c))^*} = (b I a)^* I (a I c)^* \).

Theorem 1. \( a I b = b I a \).

Theorem 2. \( a I a^* = b I b^* \).

Theorem 3. \( (a I b)^* \in S \).

Theorem 3a. \( (a^* I b^*) \in S \).

Theorem 4. There exists an element \( \wedge \in S \) such that if \( a \in S \),
then \( (a I \wedge)^* = a \).

Theorem 4a. There exists an element \( \vee \in S \) such that if \( a \in S \),
then \( a^* I \vee^* = a \).

Theorem 5. \( (a I b)^* = (b I a)^* \).

Theorem 5a. \( a^* I b^* = b^* I a^* \).

Theorem 6. \( \sqrt{(a I (b^* I c)^* I)^*} = (a I b)^* I (a I c)^* \).

Theorem 6a. \( a^* I (b I c)^* = (a I b^*) I (a^* I c^*)^* \).

Theorem 7. If the elements \( \wedge \) and \( \vee \) in T4 and T4a are unique
and if \( a \in S \), then there exists an \( @ \in S \) such
that \( (a I @)^* = \vee \) and \( a^* I (@^*) = \wedge \).
Statement 4. Postulate Set III implies Postulate Set IV.

Definitions: \( a \lor b = a \land b; \ a^* = \overline{a}; \) Class \( S \equiv \) Class \( R. \)

IVP1. \( a \lor b \in S. \)

\[
\begin{align*}
\text{IIIIP1} & \quad a \lor b \in R \\
\text{Def.} & \quad a \lor b \in S
\end{align*}
\]

IVP2. \((a^*)^* = a.\)

\[
\begin{align*}
\text{IIIT1} & \quad (\overline{a}) = a \\
\text{Def.} & \quad (a^*)^* = a
\end{align*}
\]

IVP3. \(a \lor (b \land b^*) = a^*.\)

\[
\begin{align*}
\text{IIIT2} & \quad a \lor (b \land \overline{b}) = \overline{a} \\
\text{Def.} & \quad a \lor (b \land b^*) = a^*
\end{align*}
\]

IVP4. \(\overline{a \lor (b \land c)}^* = (b^* \lor a) \land (c^* \lor a).\)

\[
\begin{align*}
\text{IIIT3} & \quad \overline{a \lor (b \land c)} = (\overline{b} \lor a) \land (\overline{c} \lor a) \\
\text{Def.} & \quad \overline{a \lor (b \land c)}^* = (b^* \lor a) \land (c^* \lor a)
\end{align*}
\]

It has been shown that Postulate Set III implies Postulate Set IV.

Statement 5. Postulate Set IV implies Postulate Set I.

Definitions: \( a \land b = a^* \lor b^*; \ a \lor b = (a \land b)^*; \ a^* = \overline{a}; \)
\( 0 = \land; \ 1 = \lor ; \) Class \( K \equiv \) Class \( S. \)

IPI. \((a \lor b) \in K.\)

\[
\begin{align*}
\text{IVT3} & \quad (a \land b)^* \in S \\
\text{Def.} & \quad (a \lor b) \in K
\end{align*}
\]

IP2. \((a \land b) \in K.\)

\[
\begin{align*}
\text{IVT3a} & \quad (a^* \lor b^*) \in S \\
\text{Def.} & \quad (a \land b) \in K
\end{align*}
\]
IP3. There exists an element 0 in K such that if a in K, then $a \lor 0 = a$.

(aI\lor)* = a  \quad \text{IVT4}

a \lor 0 = a  \quad \text{Def.}

IP4. There exists an element 1 in K such that if a in K, then $a \land 1 = a$.

a*I \lor* = a  \quad \text{IVT4a}

a \land 1 = a  \quad \text{Def.}

IP5. $a \lor b = b \lor a$.

(aI b)* = (bI a)*  \quad \text{IVT5}

a \lor b = b \lor a  \quad \text{Def.}

IP6. $a \land b = b \land a$.

a*I b* = b*I a*  \quad \text{IVT5a}

a \land b = b \land a  \quad \text{Def.}

IP7. $a \lor (b \land c) = (a \lor b) \land (a \lor c)$.

\[ a \lor (b \land c) = (a \lor b) \land (a \lor c) \]

$x(aI b)* \lor* = \lor(x(aI c)* \lor*)  \quad \text{IVT6}

a \lor (b \land c) = (a \lor b) \land (a \lor c)  \quad \text{Def.}

IP8. $a \land (b \lor c) = (a \land b) \lor (a \land c)$.

\[ a \land (b \lor c) = (a \land b) \lor (a \land c) \]

$a*I (a \land b)* \lor* = \lor((a*I b)* \lor (a*I c)* \lor*)  \quad \text{IVT6a}

a \land (b \lor c) = (a \land b) \lor (a \land c)  \quad \text{Def.}

IP9. If the elements 0 and 1 in P3 and P4 exist and are unique and if a in K, then there exists an a' in K such that $a \lor a' = 1$ and $a \land a' = 0$.

(aI \lor a')* = \lor and a*I \land* = \land  \quad \text{IVT7}

a \lor a' = 1 and a \land a' = 0  \quad \text{Def.}

It has been shown that Postulate Set IV implies Postulate Set I.

By statements 1-5 then Postulate Sets I-IV are equivalent PostulatesSets and all are representations for Boolean Algebras.
CHAPTER IV

ELEMENTS OF LATTICE THEORY RELATING TO
BOOLEAN ALGEBRA

Definition 6. A partially ordered system \( U \) is a system with
a class \( W \) of elements, \( a, b, c, d, \ldots \) and a binary relation \( \subset \)
defined on the elements of \( W \) which satisfies LP1, LP2, and
LP3.

LP1. \( a \subset a \). (read \( a \) is contained in \( a \))

LP2. If \( a \subset b \) and \( b \subset a \), then \( a = b \).

LP3. If \( a \subset b \) and \( b \subset c \), then \( a \subset c \).

Definition 7. An upper bound of elements \( a, b \in W \) is an element
\( h \in W \) if and only if \( a \subset h \) and \( b \subset h \).

Definition 8. A least upper bound or join of elements \( a, b \in W \)
is an element \( g \in W \) if and only if \( g \) is an upper bound of
\( a, b \), and if for every upper bound \( h \) of \( a, b \), \( g \subset h \). The join
of \( a, b \) will be denoted with \( a \oplus b \).

Definition 9. A lower bound of elements \( a, b \in W \) is an element
\( i \in W \) if and only if \( i \subset a \) and \( i \subset b \).

Definition 10. A greatest lower bound or meet of elements
\( a, b \in W \) is an element \( j \in W \) if and only if \( j \) is a lower bound
of \( a, b \), and if for every lower bound \( i \) of \( a, b \), \( i \subset j \). The
meet of \( a, b \) will be denoted with \( a \odot b \).
Definition 11. A lattice $L$ is a partially ordered system in which every pair of elements $a, b$ has a least upper bound and a greatest lower bound, i.e.,

LP4. $(a \oplus b) \in W$.

LP5. $(a \odot b) \in W$.

Theorem 1. For every lattice $L$, with class $W$ of elements,

$a, b, c, \ldots (a \oplus b) = (b \oplus a)$ and $(a \odot b) = (b \odot a)$.

The proof of Theorem 1 follows directly from Definitions 7, 8, 9, and 10, and will not be written out.

Theorem 2. For every lattice $L$, with class $W$ of elements,

$a, b, c, \ldots a \oplus (a \odot c) = (a \oplus b) \oplus c$ and $a \odot (b \odot c) = (a \odot b) \odot c$.

The proof of Theorem 2 follows directly from Definitions 7, 8, 9, and 10, and will not be written out.

Definition 12. A lattice $L$ is distributive if and only if the elements of $L$ satisfy LP6, LP7, and LP8 identically.

LP6. $(a \odot b) \oplus (b \odot c) \oplus (c \odot a) = (a \oplus b) \oplus (b \oplus c) \oplus (c \oplus a)$.

LP7. $a \odot (b \odot c) = (a \odot b) \odot (a \odot c)$.

LP8. $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

Remark. LP6 is well defined since Theorem 2 holds.

Definition 13. A lattice is complemented if and only if there exist elements $\neg a \in W$ and $\uparrow c \in W$ such that if $a \in W$, then LP9 is satisfied, and if $a \in W$, there exists an element $\neg a \in W$ satisfying LP10.
LP9. \( \cap \subseteq a \) and \( a \subseteq \cup \).

LP10. \( a \circ a = \cap \) and \( a \oplus a = \cup \).

Summarizing the above definitions, the postulates for a complemented distributive lattice are stated in Postulate Set V.

**Postulate Set V**

Undefined: Class \( \mathbb{W} \) of elements \( a, b, c, \ldots \) and one binary operation \( \leq \).

Defined: Definitions 7, 8, 9, and 10.

LP1. \( a \leq a \).

LP2. If \( a \leq b \) and \( b \leq a \), then \( a = b \).

LP3. If \( a < b \) and \( b < c \), then \( a < c \).

LP4. \( (a \circ b) \leq \mathbb{W} \).

LP5. \( (a \circ b) \leq \mathbb{W} \).

LP6. \( (a \circ b) \circ (b \circ c) \circ (c \circ a) = (a \circ b) \circ (b \circ c) \circ (c \circ a) \).

LP7. \( a \circ (b \circ c) = (a \circ b) \circ (a \circ c) \).

LP8. \( a \circ (b \circ c) = (a \circ b) \circ (a \circ c) \).

LP9. There exist elements \( \cap, \cup \leq \mathbb{W} \) such that if \( a \leq \mathbb{W} \), then \( \cap \subseteq a \) and \( a \subseteq \cup \).

LP10. For every \( a \leq \mathbb{W} \), there exists an element \( a \leq \mathbb{W} \), such that \( a \circ a = \cap \) and \( a \oplus a = \cup \).

**Theorem 3.** \( a \leq b \) if and only if \( a \circ b = a \).

**Necessity.** If \( a \leq b \), then \( a \circ b = a \).

\[
\begin{align*}
a \leq a & \quad \text{LP1} \\
\quad \text{Assumption} \quad a \leq b
\end{align*}
\]
a is a lower bound of $a, b$  

And if $x$ is a lower bound of $a, b$

then $x < a$  

Hence $a$ is a greatest lower bound

of $a, b$, i.e. $a \oplus b = a$  

Sufficiency. If $a \oplus b = a$, then $a < b$.

$a$ is a greatest lower bound of $a, b$

$a \subset b$  

**Theorem 4.** $a \subset b$ if and only if $a \oplus b = b$.

Necessity. If $a \subset b$, then $a \oplus b = b$.

$b \subset b$  

$a \subset b$  

$b$ is an upper bound of $a, b$  

And if $x$ is an upper bound of $a, b$

then $b < x$

Hence $b$ is a least upper bound of

$a, b$, i.e. $a \oplus b = b$  

Sufficiency. If $a \oplus b = b$, then $a \subset b$.

$b$ is a least upper bound of $a, b$

$a \subset b$  

**Statement 6.** Postulate Set I implies Postulate Set V.

Definitions: $(a \oplus b) = (a \cup b)$; $(a \ominus b) = (a \cap b)$; $a \subset b$ means $a \leq b$; $\eta = 0$; $\psi = 1$; $a = a'$; Class $W = Class K$.

LPI. $a \subset a$.

$a \leq a$  

$a \subset a$  

**IT20**

Def.
LP2. If $a \subset b$ and $b \subset a$, then $a = b$.

If $a \subseteq b$ and $b \subseteq a$, then $a = b$.  

If $a \subset b$ and $b \subset c$, then $a = b$.  

Def.  

LP3. If $a \subset b$ and $b \subset c$, then $a \subset c$.

If $a \subseteq b$ and $b \subseteq c$, then $a \subseteq c$.  

If $a \subset b$ and $b \subset c$, then $a \subset c$.  

Def.  

LP4. $(a \oplus b) \in W$.

$(a \cup b) \in K$  

$(a \oplus b) \in W$  

IP1  

Def.  

LP5. $(a \odot b) \in W$.

$(a \wedge b) \in K$  

$(a \odot b) \in W$  

IP2  

Def.  

LP6. $(a \odot b) \oplus (b \odot c) \odot (c \odot a) = (a \oplus b) \odot (b \oplus c) \odot (c \oplus a)$.  

$(a \wedge b) \cup (b \wedge c) \cup (c \wedge a) = (a \cup b) \cap (b \cup c) \cap (c \cup a)$.  

IT13  

Def.  

LP7. $a \ominus (b \odot c) = (a \odot b) \ominus (a \odot c)$.  

$a \ominus (b \wedge c) = (a \wedge b) \ominus (a \wedge c)$  

$a \ominus (b \odot c) = (a \odot b) \ominus (a \odot c)$  

IP7  

Def.  

LP8. $a \ominus (b \odot c) = (a \odot b) \ominus (a \odot c)$.  

$a \ominus (b \cup c) = (a \cup b) \ominus (a \cup c)$  

$a \ominus (b \oplus c) = (a \oplus b) \ominus (a \oplus c)$  

IP8  

Def.  

LP9. There exist elements $\cap, \cup \in W$ such that if $a \in W$, $\cap a$ and $a \cup$.  

There exists $\ominus K$ such that if $a \in K$, $a \ominus 0 = a$.  

$0 \ominus a = a$  

IP3  

IP5
\[ 0 \leq a \]
\[ \neg \leq a \]

And there exists \( l \in K \) such that if \( a \in K \), \( a \land l = a \). [IP4]

\[ a \leq l \]
\[ a \leq \neg \]

[Def. 4]

[Def.]

**LP10.** For every \( a \in W \), there exists an element \( a \in W \), such that \( a \circ a = \neg \) and \( a \circ a = \neg \).

\[ 0 \text{ exists and is unique} \]
\[ 1 \text{ exists and is unique} \]

[IP3, IT1]

[IP4, IT1a]

If 0 and 1 exist and are unique, then if \( a \in K \) there exists an element \( a' \in K \) such that \( a \cup a' = 1 \) and \( a \land a' = 0 \).

[IP9]

For every \( a \in W \), there exists an element \( a \in W \) such that \( a \circ a = \neg \) and \( a \circ a = \neg \).

[Def.]

The fact that Postulate Set I implies LP1, LP2, LP3, LP4, and LP5 shows that a Boolean algebra is a lattice.

Since Postulate Set I also implies LP6, LP7, LP8, LP9, and LP10, a Boolean algebra is a complemented distributive lattice. Postulate Set I implies Postulate Set V.

**Statement 7.** Postulate Set V implies Postulate Set I.

Definitions: \((a \lor b) = (a \circ b); (a \land b) = (a \circ b); a \leq b \) means \( a \subset b; 0 = \neg; 1 = \cup; a' = \neg a; \) Class \( K \equiv \) Class \( W \).

**IPL.** \((a \lor b) \in K \).

\[ (a \circ b) \in W \]

[LP4]

\[ (a \circ b) \in K \]

[Def.]
IP2. \((a \land b) \in K\).
\[
(a \lor b) \in W \\
(a \land b) \in K
\]
LP5
Def.

IP3. There exists an element \(0 \in K\) such that if \(a \in K\),
\[a \lor 0 = a.\]
There exists \(\cap \in W\) such that if \(a \in W\), then
\[
\cap \subset a
\]
\[a \land \cap = a\]
LT4
Def.

IP4. There exists an element \(1 \in K\) such that if \(a \in K\),
then \(a \land 1 = a\).
There exists \(\cup \in W\) such that if \(a \in W\), then
\[
a \subset \cup
\]
\[a \lor \cup = a\]
LT3
Def.

IP5. \((a \lor b) = (b \lor a)\).
\[
(a \land b) = (b \land a)
\]
Def.

IP6. \((a \land b) = (b \land a)\).
\[
(a \land b) = (b \lor a)
\]
Def.

IP7. \(a \lor (b \land c) = (a \lor b) \land (a \lor c)\).
\[
a \land (b \land c) = (a \land b) \lor (a \land c)
\]
LP7
Def.
IP8. \( a \land (b \lor c) = (a \land b) \lor (a \land c) \).
\[ a \circ (b \ominus c) = (a \circ b) \ominus (a \circ c) \] \quad \text{LP8}
\[ a \land (b \lor c) = (a \land b) \lor (a \land c) \] \quad \text{Def.}

IP9. If the elements 0 and 1 in P3 and P4 exist and are unique and if \( a \in K \), then there exists an \( a' \in K \) such that \( a \lor a' = 1 \) and \( a \land a' = 0 \).

For every \( a \in W \), there exists an element \( a \in W \), such that \( a \ominus a = \bot \) and \( a \circ a = \top \). \quad \text{LP10}

For every \( a \in K \), there exists an element \( a' \in K \) such that \( a \land a' = 0 \) and \( a \lor a' = 1 \). \quad \text{Def.}

Hence Postulate Set V implies Postulate Set I, and a Complemented Distributive Lattice is a Boolean algebra. By statements 6 and 7 a Boolean algebra and a complemented distributive lattice are equivalent.
BIBLIOGRAPHY

Books


51
Articles


