ON ORDERED PAIRS OF CARDINAL NUMBERS

APPROVED:

George Joji
Major Professor

Minor Professor

Director of the Department of Mathematics

Robert G. Toulouse
Dean of the Graduate School
ON ORDERED PAIRS OF CARDINAL NUMBERS

THESIS

Presented to the Graduate Council of the North Texas State College in Partial Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

John Dean Dickinson, B. A.

Denton, Texas
January, 1957
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I. INTRODUCTION</strong></td>
<td>1</td>
</tr>
<tr>
<td>Assumptions</td>
<td></td>
</tr>
<tr>
<td>The Genesis of Real and Complex Numbers</td>
<td></td>
</tr>
<tr>
<td>Cantor's Transfinite Cardinals</td>
<td></td>
</tr>
<tr>
<td>Order Types</td>
<td></td>
</tr>
<tr>
<td><strong>II. PAIRS OF CARDINAL NUMBERS</strong></td>
<td>23</td>
</tr>
<tr>
<td>Features of a Set</td>
<td></td>
</tr>
<tr>
<td>Algebras</td>
<td></td>
</tr>
<tr>
<td><strong>III. ASSOCIATED SETS</strong></td>
<td>38</td>
</tr>
<tr>
<td>Sets and Pairs of Cardinals</td>
<td></td>
</tr>
<tr>
<td>Functions</td>
<td></td>
</tr>
<tr>
<td>Topologies</td>
<td></td>
</tr>
<tr>
<td>Measures</td>
<td></td>
</tr>
<tr>
<td><strong>BIBLIOGRAPHY</strong></td>
<td>441</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Assumptions

Gödel has shown (4,6,7) that if mathematics is consistent without the Axiom of Choice (or, equivalently, the assumption that every set can be well-ordered), then mathematics is consistent with this axiom; and, similarly, for the continuum hypothesis. Without fears of inconsistency then the following is presented with no care given to avoid an infinity of choices. The logic (7) that is needed is assumed and used without comment. Sets are arbitrarily assumed never to be members of themselves. This convention is introduced (5) to avoid the paradox of Russell which states: a set $A$ is either a member of itself, $A \in A$, or it is not, $A \notin A$. Let $R$ denote the set of all sets which are not members of themselves. Then if $R \in R$, it follows $R \notin R$. If $R \notin R$ it follows that $R \in R$. Hence it cannot be that $R \in R$ or that $R \notin R$. Some assumptions are made without mention. Others are noted later.

The remainder of this chapter outlines the main features of several theories which are of interest.
The Genesis of Real and Complex Numbers

Integers

It is assumed (3) that a set \( P \) of objects called natural numbers has the following properties:

Axiom 1: \( 1 \) is a natural number.

Axiom 2: For each \( x \) there exists exactly one number, called the successor of \( x \), which will be denoted by \( x' \).

Axiom 3: We always have \( x' \) is not the same as \( 1 \), \( x' \neq 1 \).

Axiom 4: If \( x' \) is the same as \( y' \), \( x' = y' \), then \( x = y \).

Axiom 5: Let there be given a set \( M \) of natural numbers, with the following properties:

a) \( 1 \) belongs to \( M \)

b) If \( x \) belongs to \( M \), then so does \( x' \).

Then \( M \) contains all the natural numbers.

Addition of natural numbers is defined to be the process of assigning to every pair \( x, y \), in exactly one way another natural number called \( x + y \) such that

\[
x + 1 = x' \quad \text{for every } x
\]

\[
x + y' = (x + y)' \quad \text{for every } x \text{ and every } y.
\]

It can be shown that such a function exists and is unique.

The following may also be proved:

\[
x + (y + z) = (x + y) + z \quad \text{(associative law)}
\]

\[
x + y = y + x \quad \text{(commutative law)}
\]

\[
x + z = y + z \text{ implies that } x = y.
\]
Multiplication is defined in the following way. To every pair of numbers \(x, y\), assign in exactly one way a natural number called \(x \cdot y\) such that

\[
x \cdot 1 = x \quad \text{for every } x
\]

\[
x \cdot y' = x \cdot y + x \quad \text{for every } x
\]

\(x \cdot y\) is called the product of \(x\) and \(y\), or the number obtained from multiplication of \(x\) by \(y\). It can be shown that such a function exists, is unique and has the following properties:

\[
x(yz) = (xy)z \quad \text{(associative)}
\]

\[
xy = yx \quad \text{(commutative)}
\]

\[
xz = yz \quad \text{implies that } x = y.
\]

The third fundamental concept (1) in the system \(P\) is that of order. This can be defined in terms of addition by stating that \(a\) is greater than \(b\) \((a > b\) or \(b < a\)) if the equation \(a = b + x\) has a solution for \(x\) in \(P\). The following are the basic properties of this relation:

a) \(x > y\) and \(y > z\) imply \(x > z\) \((\text{transitivity})\)

b) For every ordered pair \((x, y)\) one and only one of the following holds: \(x > y\), \(x = y\), \(x < y\) \((\text{trichotomy})\).

c) In any non-vacuous set of natural numbers there is a least number, that is, a number \(m\) of the set such that \(m \leq s\) for all \(s\) in the set.
Fractions

A fraction \( \frac{x_1}{x_2} \) is defined (3) to be an ordered pair of natural numbers \( x_1, x_2 \). Two fractions are equivalent, that is, \( \frac{x_1}{x_2} \sim \frac{y_1}{y_2} \) if \( x_1 y_2 = x_2 y_1 \). An order is established as follows:

\[
\frac{x_1}{x_2} > \frac{y_1}{y_2} \quad \text{if} \quad x_1 y_2 > x_2 y_1 \quad (> \text{means greater than})
\]

\[
\frac{x_1}{x_2} < \frac{y_1}{y_2} \quad \text{if} \quad x_1 y_2 < x_2 y_1 \quad (< \text{means less than}).
\]

If \( \frac{x_1}{x_2} \), \( \frac{y_1}{y_2} \), are arbitrary, then exactly one of

\[
\frac{x_1}{x_2} \sim \frac{y_1}{y_2}, \quad \frac{x_1}{x_2} > \frac{y_1}{y_2}, \quad \frac{x_1}{x_2} < \frac{y_1}{y_2}
\]

is the case. Transitivity of ordering holds. If

\[
\frac{x_1}{x_2} < \frac{y_1}{y_2}, \quad \frac{y_1}{y_2} < \frac{z_1}{z_2}
\]

then

\[
\frac{x_1}{x_2} < \frac{z_1}{z_2}.
\]

Addition is defined in the following way. By \( \frac{x_1}{x_2} + \frac{y_1}{y_2} \) is meant the fraction \( \frac{x_1 y_2 + y_1 x_2}{x_2 y_2} \). It is true that the class of the sum depends only on the classes to which the "summands" belong. The commutative and associative laws also hold. Thus

\[
\frac{x_1}{x_2} + \frac{y_1}{y_2} \sim \frac{y_1}{y_2} + \frac{x_1}{x_2} \quad \text{(commutative)}
\]

\[
\left( \frac{x_1}{x_2} + \frac{y_1}{y_2} \right) + \frac{z_1}{z_2} \sim \frac{x_1}{x_2} + \left( \frac{y_1}{y_2} + \frac{z_1}{z_2} \right) \quad \text{(associative)}
\]

If \( \frac{x_1}{x_2} > \frac{y_1}{y_2} \), then \( \frac{y_1}{y_2} + \frac{u_1}{u_2} \sim \frac{u_1}{u_2} \) has a solution \( \frac{u_1}{u_2} \).

The specific \( \frac{u_1}{u_2} \) constructed in the proof of this is denoted
by \( \frac{x_1}{x_2} - \frac{y_1}{y_2} \) and is called the difference \( \frac{x_1}{x_2} \) minus \( \frac{y_1}{y_2} \).

Multiplication is defined as follows. By \( \frac{x_1}{x_2} \cdot \frac{y_1}{y_2} \) is meant the fraction \( \frac{x_1 y_1}{x_2 y_2} \). The class of the product thus depends only on the classes to which the "factors" belong. Multiplication of fractions can be shown to obey the commutative, associative and distributive laws:

\[
\frac{x_1}{x_2} \cdot \frac{y_1}{y_2} \sim \frac{y_1}{y_2} \cdot \frac{x_1}{x_2} \quad \text{(commutative)}
\]

\[
\left( \frac{x_1}{x_2} \cdot \frac{y_1}{y_2} \right) z \sim \frac{x_1}{x_2} \left( \frac{y_1}{y_2} z \right) \quad \text{(associative)}
\]

\[
\frac{x_1}{x_2} \left( \frac{y_1}{y_2} + \frac{z_1}{z_2} \right) \sim \frac{x_1}{x_2} \frac{y_1}{y_2} + \frac{x_1}{x_2} \frac{z_1}{z_2} \quad \text{(distributive)}.
\]

Rational Numbers and Integers

A rational number \( X \) is defined (3) to be the set of all fractions which are equivalent to some fixed fraction. \( X = Y \) (\( X \) equals \( Y \)) if the two sets consist of the same fractions. Otherwise \( X \neq Y \). \( X \) is greater than \( Y \), \( X > Y \), if for a fraction \( \frac{x_1}{x_2} \) of the set \( X \), and for a fraction \( \frac{y_1}{y_2} \) of the set \( Y \),

\[
\frac{x_1}{x_2} > \frac{y_1}{y_2}.
\]

\( X \) is less than \( Y \), \( X < Y \), if for a fraction \( \frac{x_1}{x_2} \) of the set \( X \), and for a fraction \( \frac{y_1}{y_2} \) of the set \( Y \),

\[
\frac{x_1}{x_2} < \frac{y_1}{y_2}.
\]

For any given \( X, Y \), exactly one of

\( X > Y, X < Y, X = Y \)

must be the case. Transitivity of ordering holds, that is if \( X > Y, Y > Z \), then \( X > Z \).
Addition is defined as follows. By \( X + Y \) is meant the class which contains a sum of a fraction from \( X \) and a fraction from \( Y \). Addition is commutative and associative, that is:

\[
X + Y = Y + X \quad \text{(commutative)}
\]
\[
(X + Y) + Z = X + (Y + Z) \quad \text{(associative)}.
\]

If \( X > Y \), \( Y + U = X \) has exactly one solution \( U \). This \( U \) is denoted by \( X - Y \) and is called the difference \( X \) minus \( Y \).

The product of \( X \) by \( Y \), \( X \cdot Y \), means the class which contains a product of a fraction from \( X \) by a fraction from \( Y \). This operation, called multiplication, satisfies the commutative, associative and distributive laws as follows:

\[
XY = YX \quad \text{(commutative)}
\]
\[
(XY)Z = X(YZ) \quad \text{(associative)}
\]
\[
X(Y + Z) = XY + XZ \quad \text{(distributive)}.
\]

The equation \( YU = X \) in which \( X \) and \( Y \) are given has exactly one solution \( U \). A rational number is called an integer if the set of fractions which it represents contains a fraction of the form \( \frac{X}{1} \). The integers thus defined satisfy the five axioms of the natural number.

Since \( >, =, < \), sum and product all correspond to the earlier concepts, the integers have all the properties of the natural numbers. In fact, the rational numbers include subsets with operations isomorphic to both natural numbers with operations and fractions with operations so, henceforth, only rational numbers will be employed.
Dedekind Cuts

A set of rational numbers, or lower numbers, is called a cut (3), or lower class, if

a) it contains a rational number, but does not contain all rational numbers;

b) every rational number of the set is smaller than every rational number not belonging to the set;

c) it does not contain a greatest rational number.

The set of all rational numbers which are not contained in the lower class is called the upper class and its elements are called upper numbers.

Using Greek letters to denote cuts

$$\xi = \eta$$

if the sets are identical. Otherwise,

$$\xi \neq \eta$$

If $\xi$ and $\eta$ are cuts then $\xi > \eta$ ( $\xi$ is said to be greater than $\eta$ ) if there exists a lower number for $\xi$, which is an upper number for $\eta$. $\xi$ is said to be less than $\eta$, $\xi < \eta$, if there exists an upper number for $\xi$ which is a lower number for $\eta$. Thus an order is established and it can be shown that for any given $\xi$, $\eta$, exactly one of $\xi > \eta$, $\xi = \eta$, $\xi < \eta$, is the case. Transitivity of ordering also holds. That is, if $\xi > \eta$, $\eta > \zeta$, then $\xi > \zeta$. 
Addition of cuts is defined as follows:

a) Let $\xi$ and $\eta$ be cuts. Then the set of all rational numbers which are representable in the form $X + Y$, where $X$ is a lower number for $\xi$ and $Y$ is a lower number for $\eta$, is itself a cut.

b) No number of this set can be written as a sum of an upper number for $\xi$ and an upper number for $\eta$.

This set is called the sum of $\xi$ and $\eta$, $\xi + \eta$. It can be shown that if $\xi > \eta$, then $\eta + \nu = \xi$ has exactly one solution $\nu$. This $\nu$ is denoted by $\xi - \eta$ and is called the difference $\xi$ minus $\eta$.

The sum follows the commutative and associative laws:

$$\xi + \eta = \eta + \xi \quad \text{(commutative)}$$

$$\xi + (\eta + \zeta) = (\xi + \eta) + \zeta \quad \text{(associative)}$$

Multiplication of cuts is defined as follows:

a) Let $\xi$ and $\eta$ be cuts. Then the set of all rational numbers which are representable in the form $XY$, where $X$ is a lower number for $\xi$ and $Y$ is a lower number for $\eta$, is itself a cut.

b) No number of this set can be written as a product of an upper number for $\xi$ and an upper number for $\eta$.

This set is called the product of $\xi$ and $\eta$, $\xi \cdot \eta$.

Multiplication is commutative, associative and distributive with respect to addition:
\[ \xi \eta = \eta \xi \quad \text{(commutative)} \\
(\xi \eta) \zeta = \xi (\eta \zeta) \quad \text{(associative)} \\
\xi (\eta + \zeta) = \xi \eta + \xi \zeta \quad \text{(distributive)}.
\]
The equation \( \eta \nu = \xi \), where \( \xi \) and \( \eta \) are given, has exactly one solution \( \nu \). This \( \nu \) is denoted by \( \frac{\xi}{\eta} \) and is called the quotient of \( \xi \) by \( \eta \).

### Rational Cuts and Integral Cuts

By defining (5) a rational cut \( X^* \) to be the set of all rational numbers \( < X \) where \( X \) is a rational number and an integral cut \( x^* \) to be the set of all rational numbers \( < x \) where \( x \) is an integer, the rational numbers and integers with their respective addition and multiplication become isomorphic to a subset of the rational cuts and their operations.

Therefore, it suffices to speak only of rational cuts and their operations.

### Real Numbers

Real numbers are defined (3) in the following way. The cuts will be called positive numbers. The rational numbers and integers will be called positive rational numbers and integers. A new number \( 0 \) (zero) is created, distinct from the positive numbers. Negative numbers are created and are distinct from all previous numbers. The set of all positive numbers, of \( 0 \), and of all negative numbers, will be called real numbers.
An order relation is easily introduced on the set of real numbers. Properties of trichotomy and transitivity can be proven on the set. And an addition and multiplication can be uniquely defined and these operations exhibit the properties of commutativity, associativity and multiplication is distributive with respect to addition. Unique solutions to certain equations allow useful subtraction and quotient operations.

**Dedekind's Fundamental Theorem**

The following is the fundamental theorem of Dedekind's (3): Let there be given any division of all real numbers into two classes with the following properties:

1) There exists a number of the first class and also one of the second class.

2) Every number of the first class is less than every number of the second class.

Then there exists exactly one real number $X$ such that every $X < Y$ belongs to the first class and every $X > Y$ to the second class.

**Complex Numbers**

The next extension the number system received was provided by the complex number which is defined in the following way. A complex number is defined to be a pair of real numbers $x, y$, in a definite order. A complex number is denoted by $x + iy$. Two complex numbers, $x + iy$ and $u + iv$, are
considered as equal, \((x + iy) = (u + iv)\), if, and only if, \(x = u\) and \(y = v\); otherwise, they are considered unequal, \((x + iy) \neq (u + iv)\).

Addition of two complex numbers is defined in the following way. If
\[
Z_1 = x + iy, \quad Z_2 = u + iv,
\]
then sum of \(Z_1\) and \(Z_2\) is
\[
Z_1 + Z_2 = (x + u) + i(y + v).
\]
The commutative and associative laws of addition hold in complex numbers. The equation \(Z_1 + Z_2 = Z_3\) has exactly one solution \(Z_2\) for any \(Z_1\) and \(Z_3\). The \(Z_3\) here is denoted by \(Z_3 - Z_1\) and is called \(Z_3\) minus \(Z_1\).

Multiplication is defined as follows. If
\[
Z_1 = x + iy, \quad Z_2 = u + iv,
\]
then the product of \(Z_1\) and \(Z_2\) is
\[
Z_1 \times Z_2 = Z, \quad Z = (xu - yv) + i(xv + yu).
\]
The commutative and associative laws of multiplication hold. Multiplication is distributive with respect to addition.

Establishing an order relation for the complex numbers presents an unusual problem. Graphically, a complex number is represented as follows:

![Diagram of a complex number on the complex plane]
Thus, a complex number is characterized by: 1) the lengths \( x \) and \( y \), 2) angle AOB, and 3) length of OB, which is equal to \( \sqrt{x^2 + y^2} \). Choice of any of these quantities for ordering purposes leads to difficulties. For instance, if the length OB is used, the complex number, \((-1 + 10)\), has length \( +1 \). Then \(-1 > 0\) in this case.

It is true, however, that the set of complex numbers \( \{x + 10\} \) with its addition and multiplication is isomorphic to the set of real numbers with its addition and multiplication even though ordering by magnitudes is not preserved under the isomorphism.

A more general order will be defined later, eliminating intuitive notions such as greater than.

Cantor's Transfinite Cardinals

**Definition 1.1.** Let A and B be two sets. A one-to-one correspondence between A and B is a rule which associates with each element \( a \in A \) exactly one element \( b \in B \), and under which each element \( b \in B \) corresponds to exactly one element \( a \in A \).

**Definition 1.2.** Two sets, A and B, are equivalent or have the same power if, and only if, it is possible to establish a one-to-one correspondence between A and B.

Two finite sets are equivalent if, and only if, they consist of the same number of elements; so, the concept of equivalence is a direct generalization of the concept of
having the same number of elements for finite sets.

**Definition 1.3.** A set $A$ is said to have power $a$ if, and only if, $A$ and $N$ are equivalent, where $N$ is the set, $\{1,2,3,\ldots\}$, of natural numbers.

**Definition 1.4.** A set $C$ is said to have power $c$ if, and only if, $C$ and $Z$ are equivalent, where $Z$ is the set of all real numbers.

The terms "two sets are equivalent", "two sets have the same power", "a set has power $a$" and "a set has power $c$" have been defined. G. Cantor defined (5) the power of a set as "that general idea which remains with us when, thinking of this set, we abstract from all properties of its elements as well as from their order." He denoted the power of a set $A$ by $\bar{A}$. The two lines indicated a "double abstraction." The following definition is more appropriate.

**Definition 1.5.** The power or cardinal number (sometimes shortened to cardinal), $m$, of a set is an arbitrary symbol assigned to a family of sets $\mathcal{M}$ such that two sets $A$ and $B$ fall into $\mathcal{M}$ if, and only if, $A$ and $B$ are equivalent. This arbitrary symbol is the power or cardinal number of every set of the given family.

**Definition 1.6.** The cardinal $\alpha$ of a set $A$ and $\beta$ of a set $B$ are equal, $\alpha = \beta$, if, and only if $A$ and $B$ are equivalent. Otherwise, $\alpha \neq \beta$.

Now, equivalent sets do have the same cardinal and the cardinal of the set of all natural numbers will be
denoted by \( a \). The cardinal of the set of all real numbers will be denoted by \( c \). Also, the concept of the number of elements of a finite set is a particular case of the more general concept of cardinal. 0 is the cardinal of the void set, and 1 is the cardinal of an arbitrary set consisting of exactly one element.

**Definition 1.7.** Let \( A \) and \( B \) be sets having cardinal \( \alpha \) and \( \beta \) respectively. \( \alpha \) is less than \( \beta \), \( \alpha < \beta \), if, and only if: 1) \( \alpha \neq \beta \), and 2) there is a subset \( B^* \) of \( B \) equivalent to the set \( A \).

Then \( a < c \). The problem whether there exists a cardinal \( m \) such that

1) \( a < m \)

and

2) \( m < c \)

is unsolved. A cardinal satisfying both conditions might be described as "between \( a \) and \( c \)." The assertion that there is no such cardinal \( m \) is one form of the continuum hypothesis.

**Definition 1.8.** Let \( A \) and \( B \) be sets having cardinal \( \alpha \) and \( \beta \) respectively. \( \alpha \) is greater than \( \beta \), \( \alpha > \beta \), if, and only if: 1) \( \alpha \neq \beta \); and 2) there is a subset \( A^* \) of \( A \) equivalent to the set \( B \).

A set whose cardinal is zero or a positive integer is called a finite set, and the cardinal is called a finite cardinal. Any other set is called an infinite set, and the cardinal of an infinite set is called a transfinite cardinal.
The following two theorems have been proven but the proofs are omitted here.

Theorem 1. If \( A \) is countable (that is \( A \) has cardinal \( a \) or has a finite cardinal) and if \( B \subseteq A \) (every element of \( B \) is an element of \( A \)), then \( B \) is countable.

Theorem 2. Any infinite set has a subset with cardinal \( a \).

Theorem 3. If \( \beta \) is a transfinite cardinal such that \( \beta \neq a \), then \( a \) is less than \( \beta \).

Proof: Let set \( A \) have cardinal \( a \), and let set \( B \) have cardinal \( \beta \); then by Theorem 2 there is a subset of \( B \) with cardinal \( a \). Since \( a \neq \beta \), then \( a < \beta \) by definition.

For this reason \( a \) is sometimes said to be the smallest transfinite cardinal.

Theorem 4. For any cardinal number \( m \), there is a cardinal number \( n \) greater than \( m \).

Proof: Let set \( A \) have cardinal \( m \) and let \( \mathcal{C} \) have cardinal \( n \) and be the class of all subsets of \( A \). There exists a subclass \( \mathcal{C}' \) of \( \mathcal{C} \) consisting of one-element sets which is equivalent to \( A \). Thus, either \( m = n \) or \( n > m \). To show that \( n \neq m \), let \( D \) be any subset of \( A \). To each element \( d \in D \) let there correspond a set \( A_d \in \mathcal{C} \).

Illustrated thus:
Now define a subset $D'$ of $D$ such that for each $d \in D$, $d \in D'$ if, and only if, $d \notin A_d$. To show that $D'$ is not null assume that $D'$ is null. $D'$ is an $A_d$, say $A_{d_1}$, since it is a subset of $A$. Now, $A_{d_1}$ corresponds to some $d$, say $d_1$. But $d_1$ is not in a null set, $A_{d_1}$. Hence, $d_1$ is in $D'$, a contradiction. Then $D' \in C$, but $D'$ is not any of the sets $A_d$. Hence, no such correspondence can exhaust the class and $m \neq n$. So, $n > m$.

If a set $A$ has cardinal $\alpha$, the set of all subsets of $A$ is generally said to have cardinal $2^\alpha$. By theorem 4, then, $2^\alpha > \alpha$.

**Definition 1.9.** The sum $m + n$ of two cardinal numbers is the cardinal number of the set $M + N$, where $M$ and $N$ are sets and $m$ and $n$ the corresponding cardinals.

It is easily shown that addition of cardinal numbers is commutative and associative. Also

\[
m + a = a \quad \text{(m is an integer)}
\]

\[
a + a = a
\]

\[
a + a + \ldots = a
\]

\[
c + a = c.
\]

The definition of a sum of an infinite sequence of cardinals is given as follows. If $m_1, m_2, m_3, \ldots$ is an infinite sequence of cardinal numbers and $M_i (i = 1, 2, \ldots)$ are sets such that the cardinal of $M_i$ is $m_i$, then the sum of the infinite series $m_1 + m_2 + \ldots$ is the cardinal number of the set $M_1 + M_2 + \ldots$. It may be shown that
\[ a = 1 + 1 + 1 + \ldots \]
\[ a = 1 + 2 + 3 + \ldots \]
\[ a = a + a + a + \ldots \]
\[ c = c + c + c + \ldots \]

**Definition 1.10.** The product \( m \cdot n \) of the cardinal numbers \( m \) and \( n \) is the cardinal number of the set \( \mathbb{Q} \) consisting of all pairs \((x, y)\) where \( x \in M \), \( y \in N \), and where the cardinals of \( M \) and \( N \) are \( m \) and \( n \) respectively.

It can be shown that multiplication of cardinal numbers is commutative and associative. It turns out that

\[ a \cdot a = a \]
\[ m \cdot n = n + n + \ldots + n \quad (m \text{ terms where } m \text{ is a finite cardinal}) \]
\[ m \cdot a = a \]
\[ m \cdot c = c \]
\[ a \cdot n = n + n + \ldots \]
\[ a \cdot c = c + c + \ldots = c \]
\[ c \cdot c = c \]

The definition of a product of two cardinal numbers may be extended to an infinite sequence. It follows readily that if \( m_1 \), \( m_2 \), \( \ldots \), \( m_n \) are given cardinal numbers and \( M_1 \), \( M_2 \), \( \ldots \), \( M_n \) are sets such that the cardinal of \( M_k \) is \( m_k \) for \( k = 1, 2, \ldots, n \), then the cardinal number \( m_1 \cdot m_2 \cdot \ldots \cdot m_n \) is the cardinal of the set of all combinations \((x_1, x_2, \ldots, x_n)\), where \( x_k \in M_k \) for \( k = 1, 2, \ldots, n \). A similar extension suffices for the infinite product. It can be shown that
\[ c = 2.2.2. \ldots \]
\[ c = a.a.a. \ldots \]
\[ c = c.c.c. \ldots \]

Let \( P \) and \( Q \) be two given sets. If with every element of \( P \) there is related an element of \( Q \), where the same element of \( Q \) may be related with several elements of \( P \), we obtain a mapping of the set \( P \) on the set \( Q \).

**Definition 1.11.** Let \( m \) and \( n \) be cardinal numbers of sets \( M \) and \( N \) respectively; then the power, \( m^n \), is defined (8) to be the cardinal of the set of all mappings of the set \( N \) on the set \( M \). It can be shown that for any three cardinal numbers \( m, n, p \):

\[ m^{n+p} = m^n \cdot m^p \]
\[ (mn)^p = m^p \cdot n^p \]
\[ (m^n)^p = m^{np} \]

\( m^n = m \cdot \ldots \cdot m \) (\( n \) factors for finite \( n \))

\( m^a = m \cdot m \ldots \)

\( c^a = a^a = 2^a = c \) and if \( p > n \),

then \( m^p \geq m^n, m^{n+p} = m^p, \) and \( m^n + m^p = m^p \).

**Order Types**

The specific order relation given on page 3 is defined for sets in general as follows.

**Definition 1.12.** A set \( S \) is said to have \( R \) as a relation if for every pair \((a, b)\) of elements of \( S \), the phrase "\( a \) is in the relation \( R \) to \( b \)" is meaningful, being
true or false depending solely on the choice of a and b. If a is in the relation R to b, then the notation aRb is used. If a is not in the relation R to b, then the notation a $\not\in$ b is used.

**Definition 1.13.** A relation R on the set S is:

a) reflexive if aRa for each $a \in S$; b) symmetric if whenever aRb, then also bRa; c) transitive if whenever aRb and bRc, then also aRc.

**Definition 1.14.** An ordered set is a set S in which there is defined a relation R satisfying the postulates:

a) (Transitivity) If aRb and bRc, then aRc.

b) (Trichotomy) For any a, b in S exactly one of the following: aRb, bRa or $a = b$ (a and b are identical) holds.

Two examples of ordered sets, one of which was mentioned earlier, are the set of natural numbers ordered according to:

1) increasing magnitude, \{1, 2, 3, ...\}, and

2) decreasing magnitude, \{...3, 2, 1\}.

Here the relation R coincides with the familiar less than, $<$, and greater than, $>$, relations.

**Definition 1.15.** Two ordered sets A and B are said to be similar if, and only if, there exists a one-to-one correspondence between their elements which leaves the order relations unchanged. Thus, if $a_1$, $a_2$ are any two elements of A and $b_1$, $b_2$ their corresponding elements in B, then the relation $a_1Ra_2$ implies $b_1Rb_2$. 


It turns out that the relation of similarity is reflexive, symmetrical and transitive.

**Definition 1.16.** Divide all ordered sets into classes assigning two sets to the same class if, and only if, they are similar. Then, sets belonging to the same class are said to be of the same ordinal type.

The ordinal type of a set $E$ is denoted by Cantor (2) by $\overline{E}$. If $n$ is a natural number, then all ordered sets consisting of $n$ elements are easily seen to be similar to the set of the first $n$ numbers. Therefore we let $\alpha$ be the symbol for the corresponding ordinal type. The ordinal type of the class which contains the set of all natural numbers in their familiar order, $\{1, 2, 3, \ldots\}$, is denoted by $\omega$. Generally, if $\alpha$ be a given type, then the type reversed in order is denoted by $\alpha^*$. The order type of the set, $\{\ldots, 3, 2, 1\}$, is denoted by $\omega^*$.

There is a set property which further enlightens the study of ordered sets and is defined in the following way.

**Definition 1.17.** An ordered set $M$ is said to be well-ordered, if $M$ itself, as well as every nonempty subset of $M$, has a first element under the order given for its elements by $M$. The empty set is also regarded as well-ordered.

It turns out that of sets having a particular order type, either every set or none is well-ordered. This makes the following definition useful.
Definition 1.18. An ordinal number is an order type which is represented by well-ordered sets.

For finite sets, cardinal number, order type, and ordinal number coincide. With infinite sets, to a cardinal number there may correspond many order types. For example, as seen above, the set of all natural numbers has but one cardinal, \( a \), but two order types have already been mentioned for this set. Whether there are always ordinal numbers among these order types depends on whether every set can be well-ordered.

Assuming every set of cardinal numbers can be well-ordered the following theorem has been proven (2).

Theorem 5. Every set \( K \) of cardinal numbers \( m \), ordered according to the greater than relation (Definition 1.8, on page 14), is well-ordered. If \( m \) is a cardinal in \( K \), there exists a cardinal \( g \) greater than \( M \). In fact, there is a definite next cardinal greater than cardinal number \( m \).

If the set of all transfinite cardinals which are less than a given cardinal number \( n \), is ordered according to the greater than relation, it is customary to denote the smallest transfinite cardinal by \( \kappa_0 \), the next by \( \kappa_1 \), etc.

Earlier, the smallest transfinite cardinal was denoted by \( a \), and hence \( \kappa_0 = a \). The power of the set of all real numbers was denoted by \( c \). Obviously \( c \geq \kappa \). The question whether the equality sign holds is another way of stating the continuum problem.
CHAPTER BIBLIOGRAPHY


CHAPTER II

PAIRS OF CARDINAL NUMBERS

Features of a Set

It is seen that the set of transfinite cardinals exhibits the following properties:

a) There is a smallest cardinal of the set;

b) There is always a next larger cardinal;

c) Cardinals are not classed as odd or even;

d) Addition of cardinals does not produce a larger one;

e) Exponentiation can produce a larger cardinal.

Properties a and b hold in sets of integers which, of course, are finite cardinals. An investigation of a set of ordered pairs of cardinal numbers extending the set of transfinite cardinals is then in order and will be made. All along the way the underlying concept will be the extension of the integers and this can serve as a source of the axioms and a guide to consistency.

A transfinite cardinal will be denoted by a lower case English letter and an ordered pair as follows: \((a_1, a_2)\). The basic idea of "ordered pair" is that \((a_1, a_2)\) is not the same as \((a_2, a_1)\). Defining an order relation between elements of a set of ordered pairs will be dependent upon which cardinals are paired as well as the
prudent choice of an operation. A binary operation and the set of all integers suffices as the foundation of the rational numbers \((1)\). Consequently, only a binary operation and the set of all transfinite cardinals are under consideration presently. Investigations employing operations other than the binary type are exceedingly sparse in the literature. Sets of particular transfinite cardinals are considered in some cases.

Let \(S\) be the set of all ordered pairs of cardinal numbers \((a_1, a_2)\). More appropriately, consider \(S\) to be the space of interest and the essential properties of the space can hopefully be developed later.

Algebras

**Algebra of Cardinal Numbers**

An algebraic system \(\mathcal{A}\) can be regarded (2) as an ordered triple whose first term is the set of cardinals, say \(A\), and whose remaining terms are the operations \(+\) and \(\sum\), that is \((A, +, \sum)\)

where \(+\) is an operation performed on couples \(a, b\), and the second on infinite sequences \(a_\circ, a_1, \ldots, a_i, \ldots\). The results would be

\[
a + b \quad \text{and} \quad \sum_{i=0}^{\infty} a_i
\]

respectively.
An algebraic system, \( A = (A, +, \sum) \), which satisfies the following postulates is called a cardinal algebra.

I (Finite Closure Postulate). If \( a, b \in A \), then \( a + b \in A \).

II (Infinite Closure Postulate). If \( a_0, a_1, \ldots, a_i, \ldots \in A \), then
\[
\sum_{i=0}^{\infty} a_i \in A.
\]

III (Associative Postulate). If \( a_0, a_1, \ldots, a_i, \ldots \in A \), then
\[
\sum_{i=0}^{\infty} a_i = a_0 + \sum_{i=0}^{\infty} a_{i+1}.
\]

IV (Commutative-Associative Postulate). If \( a_0, a_1, \ldots, a_i, \ldots, b_0, b_1, \ldots, b_i, \ldots \in A \), then
\[
\sum_{i=0}^{\infty} (a_i + b_i) = \sum_{i=0}^{\infty} a_i + \sum_{i=0}^{\infty} b_i.
\]

V (Postulate of the Zero Element). There is an element \( z \in A \) such that \( a + z = z + a \) for every \( a \in A \).

VI (Refinement Postulate). If \( a, b, c_0, c_1, \ldots, c_i, \ldots \in A \) and
\[
a + b = \sum_{i=0}^{\infty} c_i,
\]
then there are elements \( a_0, a_1, \ldots, a_i, \ldots, b_0, b_1, \ldots, b_i, \ldots \in A \) such that
\[
a = \sum_{i=0}^{\infty} a_i, b = \sum_{i=0}^{\infty} b_i, \text{ and } c_n = a_n + b_n
\]
for \( n = 0, 1, 2, \ldots \).

VII (Remainder, or Infinite Chain, Postulate). If \( a_0, a_1, \ldots, a_i, \ldots, b_0, b_1, \ldots, b_i, \ldots \in A \) and if
\[ a_n = b_n + a_{n+1} \quad \text{for } n = 0, 1, 2, \ldots, \text{ then there is an} \]
\[ \text{element } c \in A \text{ such that} \]
\[ a_n = c + \sum_{i=0}^{\infty} b_{n+i} \quad \text{for } n = 0, 1, 2, \ldots. \]

The significance of postulate VII is clarified by the following. The sequence of remainders
\[ r_n = \sum_{i=0}^{\infty} b_{n+i} \]
of an infinite sum \( b_n \) has a specific relation to the sequence, \( b = b_o + b_1 + \ldots + b_i + \ldots. \)
That is,
\[ \sum_{i=0}^{\infty} b_{n+i} = b_n + \sum_{i=0}^{\infty} b_{n+1+i} \quad n = 0, 1, 2, \ldots. \]
or
\[ r_n = b_n + r_{n+1}. \]
By postulate VII every other sequence, \( a_o, a_1, \ldots, a_i, \ldots \), which satisfies the same condition can be obtained from the sequence, \( r_o, r_1, \ldots, r_i, \ldots \), by adding a constant term \( c \).

Now, by formal arithmetic (which is not necessarily true in the previous case) if
\[ b = b_o + b_1 + b_2 + \ldots \]
and
\[ r_n = \sum_{i=0}^{\infty} b_{n+i} \]
then
\[ r_n = b_n + r_{n+1}. \]
Let
\[ a = a_o + a_1 + a_2 + \ldots \]
and
\[ a_n = b_n + a_{n+1}. \]
Then, the only possible relation between \( a_n \) and \( r_n \) is
\[
a_n = K + r_n = K + \sum_{i=0}^\infty b_{n+i}
\]
where \( K \) is positive, negative or zero. The element, \( c \), in postulate VII behaves very similarly to \( K \). Postulate VII does not state that the equation
\[
a_n = c + \sum_{i=0}^\infty b_{n+i}
\]
can be solved for \( c \), but it does state that there is an element \( c \) such that the condition given is true.

These postulates can form the basis of an abstract algebra which is similar to the set of cardinal numbers with the operation of cardinal addition. In fact, the two operations \( + \) and \( \sum \) are not independent operations, and each can be defined by means of the other.

**Algebra of Pairs of Cardinal Numbers**

We have seen that addition of transfinite cardinals does not in general give rise to new transfinite cardinals. So, even though the above set is closed with respect to this operation, at least a part of the flavor of the algebra described above is lost. Consider the algebra \( a \) on pairs of transfinite cardinals, that is
\[
a = \{3, \oplus, \circ\}
\]
where
\[
S = \{(a_1, a_2), (b_1, b_2), \ldots ; \ldots \}.
\]
The operations \( \oplus \) and \( \circ \) are defined as follows.
Definition 2.1. The process of assigning to every two pairs \((a_1, a_2), (b_1, b_2) \in S\) another pair, called \((a_1, a_2)\) plus \((b_1, b_2)\), written \((a_1, a_2) \oplus (b_1, b_2)\), such that

\[(a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_1, a_2 + b_2),\]

where \(\oplus\) is the cardinal addition defined on page 16, is called pair addition.

Definition 2.2. The process of assigning to every two pairs \((a_1, a_2), (b_1, b_2) \in S\) another pair, called \((a_1, a_2)\) times \((b_1, b_2)\), written \((a_1, a_2) \odot (b_1, b_2)\), such that

\[(a_1, a_2) \odot (b_1, b_2) = (a_1^{b_1}, a_2^{b_2}),\]

where \(a^b\) is the cardinal power operation defined on page 17 is called pair multiplication.

It can be seen that both operations are well-defined since each is based on a well-defined operation on another set; and, the only additional act required is placing two resulting elements in parenthesis.

There exist certain reasons for selecting these operations. These reasons may or may not affect the interest at this point. It will be recalled that the following operations in integers

\[a + b\]
\[a \cdot b\]
\[a^b\]

do not represent completely different actions on the part
of the operator. Each operation can give rise to a third and different integer and the associative and commutative laws hold for the first two. For real numbers the three paralleling operations are algebraically different. They are seen to be more nearly independent. For instance, how would multiplication correspond to $a^{\frac{1}{2}}$? For real numbers these three operations do readily produce different numbers in some cases, and the associative, commutative and distributive laws apply in much the same way as for integers. Transfinite cardinal operations that are of interest here are

$$a + b$$
$$a \cdot b$$
$$a^b.$$  

The first two, $+$ and $\cdot$, as mentioned earlier, do not in general give rise to a third and different transfinite cardinal. The power operation may easily give rise to a "new" transfinite cardinal. This addition and multiplication is also associative and commutative. It will be seen in the following that one of the chosen operations for the algebra of pairs does not in general produce a third and different element in the set. However, this operation is associative and commutative. The remaining operation in the following readily produces "new" elements but fails to be associative or commutative.
Theorem 2.1. (Commutative Law). For every \((a_1, a_2),\) \((b_1, b_2) \in S,\) \((a_1, a_2) \oplus (b_1, b_2) = (b_1, b_2) \oplus (a_1, a_2).\)

Proof:
\[(a_1, a_2) \oplus (b_1, b_2) = (a_1 + b_1, a_2 + b_2)\]
\[(b_1, b_2) \oplus (a_1, a_2) = (b_1 + a_1, b_2 + a_2)\]
since the commutative law holds for +,
\[a_1 + b_1 = b_1 + a_1\]
and \[a_2 + b_2 = b_2 + a_2.\]
Therefore, the commutative law holds for \(\oplus.\)

Theorem 2.2. (Associative Law). For every \((a_1, a_2),\) \((b_1, b_2),\) \((c_1, c_2) \in S,\)
\[(a_1, a_2) \oplus [(b_1, b_2) \oplus (c_1, c_2)] = [(a_1, a_2) \oplus (b_1, b_2)] \oplus (c_1, c_2).\]

Proof: The proof is similar to the previous proof.

Theorem 2.3. The commutative law does not in general hold for the operation \(\odot.\) That is, for some \((a_1, a_2),\) \((b_1, b_2) \in S,\) \((a_1, a_2) \odot (b_1, b_2) \neq (b_1, b_2) \odot (a_1, a_2).\)

Proof: The following is an example in which the commutative law does not hold:
\[(\mathcal{R}_o, \mathcal{R}_o) \odot (c, c) = (\mathcal{R}_o^c, \mathcal{R}_o^c)\]
\[(c, c) \odot (\mathcal{R}_o, \mathcal{R}_o) = (c \mathcal{R}_o \mathcal{R}_o^c) = (c, c)\]
\[(\mathcal{R}_o^c, \mathcal{R}_o^c) \neq (c, c).\]

Theorem 2.4. The associative law does not hold in general for the operation \(\odot.\) That is, for some \((a_1, a_2),\) \((b_1, b_2), (c_1, c_2) \in S,\)
\[
[(a_1, a_2) \circ (b_1, b_2)] \circ (c_1, c_2) \neq (a_1, a_2) \circ [(b_1, b_2) \circ (c_1, c_2)].
\]

Proof:
\[
[(a_1, a_2) \circ (b_1, b_2)] \circ (c_1, c_2) = \left[ (a_1, a_2) \circ (b_1, b_2) \right] \circ (c_1, c_2)
= \left( a_1^{b_1} a_2^{b_2}, c_1, c_2 \right)
\]
\[
(a_1, a_2) \circ \left[ (b_1, b_2) \circ (c_1, c_2) \right] = \left( a_1^{b_1} c_1, a_2^{b_2} c_2 \right)
\]
but \( (a_1^{b_1} c_1, a_2^{b_2} c_2) \) in general.

The following is an example in which the above equality sign does not hold. Let \( a \) be the smallest transfinite cardinal and let \( c \) be the next larger cardinal.
\[
(a^a)^a = a^{a^a} = a^a = c
\]
\[
a^{a^2} = a^c
\]
\[
c \neq a^c
\]

Theorem 2.5. The operation \( \circ \) is in general distributive with respect to the operation \( \oplus \). Or for every
\[
(a_1, a_2), (b_1, b_2), (c_1, c_2) \in S
\]
\[
(a_1, a_2) \circ \left[ (b_1, b_2) \oplus (c_1, c_2) \right] = (a_1, a_2) \circ (b_1, b_2) \oplus (a_1, a_2) \circ (c_1, c_2).
\]
Proof:
\[
(a_1, a_2) \circ \left[ (b_1, b_2) \oplus (c_1, c_2) \right] = (a_1^{b_1 + c_1}, a_2^{b_2 + c_2}).
\]
\[
[(a_1, a_2) \circ (b_1, b_2)] \oplus [(a_1, a_2) \circ (c_1, c_2)] = (a_1^{b_1} + a_1^{c_1}, a_2^{b_2} + a_2^{c_2}).
\]
and \( a_1^{b_1 + c_1} \) is in general the same as \( a_1^{b_1} + a_1^{c_1} \). This is seen since for any three transfinite cardinals, \( m, n, p, \)}
as indicated on page 10, if \( p \geq n \) then

\[
m^{n+p} = m^p
\]

and

\[
m^n + m^p = m^p
\]

Hence

\[
m^{n+p} = m^n + n^p.
\]

Theorem 2.6. The operation \( \oplus \) is not in general distributive with respect to the operation \( \circ \). Or, for some \((a_1, a_2), (b_1, b_2), (c_1, c_2) \in S\)

\[
(a_1, a_2) \oplus [(b_1, b_2) \circ (c_1, c_2)] \neq [(a_1, a_2) \oplus (b_1, b_2)] \circ [(a_1, a_2) \oplus (c_1, c_2)].
\]

Proof:

\[
(a_1, a_2) \oplus [(b_1, b_2) \circ (c_1, c_2)] = (a_1 + b_1^{c_1}, (a_2 + b_2^{c_2}).
\]

\[
[(a_1, a_2) \oplus (b_1, b_2)] \circ [(a_1, a_2) \oplus (c_1, c_2)]
\]

\[
= [(a_1 + b_1)^{(a_1 + c_1)}, (a_2 + b_2)^{(a_2 + c_2)}].
\]

In general

\[
a_1 + b_1^{c_1} 
eq (a_1 + b_1)^{(a_1 + c_1)},
\]

as seen in the following example. Let \( a \) be the smallest transfinite cardinal and let \( c \) be the next larger cardinal.

\[
c + c^a \neq (c + c)^{(c + a)}
\]

since \( c + c^a = c \)

but \( (c + c)^{(c + a)} = c^c \)

and \( c \neq c^c \).

Logical sum and product operations in set theory are distributive with respect to each other. Multiplication is
distributive with respect to addition in high school algebra. This particular algebra \( a \) described above behaves similarly, that is, multiplication, \( \circ \), is distributive with respect to addition, \( \oplus \), but \( \oplus \) is not distributive with respect to \( \circ \).

One feature of a common to other familiar algebras is the presence of an identity element with respect to the operation \( \oplus \). The element \( (\mathcal{R}_0, \mathcal{R}_0) \) is such an identity element. However, this element is not unique for each member of \( S \). For any two elements \((a_1, a_2), (b_1, b_2) \in S\), \((b_1, b_2) \oplus (a_1, a_2) = (a_1, a_2)\) if \( b_1 \) is not greater than \( a_1 \) and \( b_2 \) is not greater than \( a_2 \). Of course, \( (\mathcal{R}_0, \mathcal{R}_0) \) is the only identity element for all elements of \( S \) with respect to \( \oplus \). Identity elements for the operation \( \circ \) are present. However, since this operation is not commutative different results are found for right and left hand multiplication. For any element \((a_1, a_2) \in S\) such that \( a_1 \neq \mathcal{R}_0 \) and \( a_2 \neq \mathcal{R}_0 \), the element \( (\mathcal{R}_0, \mathcal{R}_0) \) is a right hand identity element. In this case

\[(a_1, a_2) \circ (\mathcal{R}_0, \mathcal{R}_0) = (a_1, a_2).\]

In general \((\mathcal{R}_n, \mathcal{R}_n)\) is a right hand identity element for \((\mathcal{R}_m, \mathcal{R}_m)\) where \( m = n + 1, n + 2, \ldots \). There are no left hand identity elements.

Ordering of the elements of \( S \) in \( a \) follows the definition of an ordered set given on page 19. An ordering relation \( R \) which behaves as that definition requires is
sometimes called a strong partial ordering of a set. If
a relation is transitive and reflexive simultaneously, it
is called a weak partial ordering of a set.

**Definition 2.3.** Let \((a_1, a_2)\) and \((b_1, b_2)\) be two
elements of \(S\). \((a_1, a_2)\) is less than \((b_1, b_2)\), written
\((a_1, a_2) < (b_1, b_2)\), if, and only if: 1) \((a_1, a_2) \neq \)
\((b_1, b_2)\) and 2) \(a_1 < b_1\) if \(a_1 \neq b_1\), or \(a_2 < b_2\) if \(a_1 = b_1\),
and \(a_2 \neq b_2\).

The above definition describes a strong partial order-
ing of the set \(S\).

**Definition 2.4.** Let \((a_1, a_2)\) and \((b_1, b_2)\) be two
elements of \(S\). \((a_1, a_2)\) is greater than \((b_1, b_2)\), written
\((a_1, a_2) > (b_1, b_2)\), if, and only if: 1) \((a_1, a_2) \neq (b_1, b_2)\),
and 2) \((a_1, a_2)\) is not less than \((b_1, b_2)\).

Generally, if \(\{A, R_1, R_2, \ldots\}\) and \(\{B, R_1, R_2, \ldots\}\)
are two algebraic systems where \(A\) and \(B\) are sets of elements
and the relation or operation \(R_1\), corresponds to \(R_1\), a one-
to-one correspondence between the elements of \(A\) and the
elements of \(B\) is called an isomorphism provided that cor-
responding operations and relations are preserved under the
correspondence.

Consider the algebraic system \(\{K, R_1, R_2, R_3\}\) where
each element of \(K\) is a transfinite cardinal,

\[ R_1 = \text{cardinal addition by definition 1.9, page 16,} \]
\[ R_2 = \text{cardinal power by definition 1.11, page 18,} \]
and \[ R_3 = \text{the less than relation of definition 1.7,} \]
The set of transfinite cardinals $K$, may be put in a one-to-one correspondence with the set $\{(a_1, a_2)\}$ a sub-
set $S^*$ of $S$ where $a_1$ is equal to each element of $K$ and $a_2 = \bigcap \alpha$. Now, the algebraic system $\{S^*, r_1, r_2, r_3\}$
where $r_1$ = pair addition for definition 2.1, page 28,
$r_2$ = pair multiplication by definition 2.2, page 28,
and $r_3$ = the less than relation of definition 2.3, page 34,
is not isomorphic to the system $\{K, R_1, R_2, R_3\}$.

For example, let $(a_1, a_2) \in S^*$ correspond to $a_1 \in K$,
$a_1 \rightarrow (a_1, a_2),$
and $b_1 \rightarrow (b_1, a_2).$

Then $(a_1 + b_1) \rightarrow [(a_1 + b_1), (a_2 + a_2)],$
$a_1^{b_1} \rightarrow (a_1^{b_1}, a_2^{a_2})$ which is not in $S^*$, and
$a_1 < b_1 \rightarrow (a_1, a_2) < (b_1, a_2).$

The cancellation laws do not hold in general for trans-
finite cardinals; that is, for each $m, n, p, \in K$, $m$ may not
equal $n$ even though $m + p = n + p$, or $m \cdot p = n \cdot p$. A direct
consequence of this is that in the set $S$

$$(a_1, a_2) \oplus (b_1, b_2) = (c_1, c_2) \oplus (b_1, b_2)$$
does not imply

$$(a_1, a_2) = (c_1, c_2).$$

For transfinite cardinals if

$m < n$ and $p < q,$
then $m + p < n + p.$

This follows from the facts that
\[ m + p = \text{larger of } m \text{ or } p, \text{ say } p \]

\[ n + q = \text{larger of } n \text{ or } q, \text{ say } n. \]

Therefore, \( n > q > r \)

or \( n + q > m + p. \)

Similarly for pairs if

\[ (a_1, a_2) < (b_1, b_2) \text{ and } (c_1, c_2) < (d_1, d_2), \]

then \( (a_1, a_2) \oplus (c_1, c_2) < (b_1, b_2) \oplus (d_1, d_2), \)

or \( (a_1 + c_1, a_2 + c_2) < (b_1 + d_1, b_2 + d_2). \)

there are three possibilities:

1) \( a_1 < b_1 \) and \( c_1 < d_1 \)

in which case the above is true;

2) \( a_1 < b_1 \) and \( c_1 = d_1 \)

\[ a_2 < b_2 \text{ and } c_2 < d_2 \]

again the above is true;

3) \( a_1 < b_1 \)

\[ c_1 = d_1 \text{ and } c_2 < d_2 \]

and the above is true.

Many other algebras can be produced which are founded on \( S \) such as coset algebras. However, other facets of the fundamental idea are being neglected. These are examined or mentioned in the following chapter.
CHAPTER BIBLIOGRAPHY


ASSOCIATED SETS

Sets Associated with Pairs of Cardinals

One trouble with studies of transfinite cardinals is that sets inside the class of sets which describe a particular cardinal \( m \) are not clearly identifiable. For example, two sets which are clearly different, the set of points in a line segment and the set of points in an area segment of a Cartesian plane, represent the same cardinal \( c \). Other properties such as dimension and triangle postulates point to basic differences in these two sets. Another way of stating the trouble is that a cardinal has the fundamental property of being invariant under the operation of choosing any set in a certain class.

The set \( S \) of ordered pairs of cardinal numbers which concerned us above represent a refinement allowing a greater detailed description of the associated sets. Each element \( s \in S \) is closely related to two classes of sets, thus

\[
\begin{align*}
    s_1 &= (a_1, a_2) \\
    s_2 &= (b_1, b_2)
\end{align*}
\]
If a pair of cardinals which make up an element \( s \in S \) are to form useful descriptions of sets, it is simple to fill the element \( s \) with more information than a single cardinal number \( m \) can contain. For instance, for each set \( A \) let \( s \) contain the cardinal of \( A \) in the first position and the cardinal of the closure of \( A, \bar{A} \), in the second position. In this case each element \( s \) is associated with two sets which have a certain relation (5) with each other, e.g., each element of \( A \) is a limit point of \( A \) or a point of \( A \). It is not known if this relation is invariant under operations of \( \oplus \) and \( \circ \) in the algebra of cardinal pairs mentioned above, or conversely. Note that for all closed sets \( A = \bar{A} \) and the resulting element \( s \in S \) has a symmetrical property about it. Thus if \( A = \bar{A} \), the cardinal \( m \) of \( A \) and \( \bar{m} \) of \( \bar{A} \) are equal and \( s = (m, m) \).

Not only does this "two element lattice point" offer a more detailed description of a particular set, but this idea suggests a new study. This new study will concern cardinal functions on sets and topologies and measures of resulting spaces.

Functions

A class of sets \( X \) is said (4) to be completely additive if it satisfies the following postulates:
1) $\emptyset \in \mathcal{X}$ \hspace{1em} ( $\emptyset$ is the null set).

ii) If $A \in \mathcal{X}$, then the complement of $A$ is in $\mathcal{X}$.

iii) If $A$ is any sequence of sets from $\mathcal{X}$, then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{X}.$$ 

This definition from measure theory forms the basis for a function theory (2) of correspondences between sets and ordered pairs of transfinite cardinal numbers.

More specifically, let us define an $s$-valued ($s \in S$) function $\sigma$ as a completely additive set function provided the following postulates are satisfied:

1) The domain of $\sigma$ is a completely additive class $\mathcal{X}$ of sets.

ii) If $A$ is a sequence of disjoint sets from $\mathcal{X}$, then

$$\sum_{n=1}^{\infty} \sigma(A_n)$$

is defined in the set $S$ of transfinite pairs and

$$\sigma\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \sigma(A_n).$$

The previous order relation defined in $S$, $<$, will allow classification of these functions in the following manner. A function $\sigma$ is called non-increasing if

$\sigma(A) \leq \sigma(B)$ whenever $A$ contains $B$, $A \supset B$, and non-decreasing if $\sigma(A) \geq \sigma(B)$ whenever $A \supset B$.

Continuing in this direction an investigation of "bounded" and "unbounded" functions with definitions paralleling real function theory definitions is needed.
Structure Theorems of this set of functions might show basic differences when compared with the Jordan Decomposition Theorem (5) of real function theory. Possibilities of extension of the theory in terms of transfinite induction are not without merit.

**Topologies**

A usual definition of a topology (3) of a space (set) $X$ is that $\mathcal{J}$ is a family of subsets of $X$ and;

1) $\emptyset$ and $X$ are in $\mathcal{J}$; ($\emptyset$ is the null set)

2) The logical sum of the sets of any subfamily of $\mathcal{J}$ is a member of $\mathcal{J}$;

3) The common part (logical product) of any finite number of sets of $\mathcal{J}$ is a set of $\mathcal{J}$.

If a topology $\mathcal{J}$ is given for $X$, then $X$ is called a topological space and the sets of $\mathcal{J}$ are the open subsets of $S$. If $A$ is any set of $X$, then the logical sum of all open subsets of $A$ is called the interior of $A$. Interiors are closely related to neighborhoods. A complement of an open set with respect to the space $X$ is called a closed set with respect to the topology $\mathcal{J}$. From this point on the complexity increases greatly with other topological definitions (6) of continuity, compactness, and limit point inter alia.

The object here is merely to remind the reader that topological properties of the space $S$ of transfinite pairs exist and no investigation is made in this paper.
Measures

Mathematics and quantity have gone hand in hand through the years. Few people realize how completely existing mathematics fails to give quantities even for the number of elements in mathematics. The set \( S \) for transfinite pairs has been assumed to be well-ordered, and this has a great deal of bearing on the counting process. However, such a simple set as the set of real numbers has not been written out in sequence form. No effort is devoted here to writing \( S \) out in sequence form. The next most desirable process would be to assign certain numbers, perhaps in a completely additive function manner to subsets \( S \). This "density" number might in some ways resemble the measure (1) of real function theory.
CHAPTER BIBLIOGRAPHY


BIBLIOGRAPHY

Books

Halmos, P. R., Measure Theory, New York, D. Van Nostrand Company, Inc., 1950


Tarski, Alfred, Cardinal Algebra, New York, Oxford University Press, 1949