ELECTRONIC ANALOG COMPUTER STUDY OF EFFECTS OF
MOTOR VELOCITY AND DRIVING VOLTAGE LIMITS
UPON SERVOMECHANISM PERFORMANCE

APPROVED:

Carl T. York
Major Professor

H. C. Parrish
Minor Professor

C. H. Hanson
Director of the Department of Mathematics

Robert B. Toulouse
Dean of the Graduate School

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THESIS

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By

Joe Preston Haynes, B. A.

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CHAPTER I

INTRODUCTION

The object of this thesis is (1) to demonstrate the value of an electronic analog computer for the solution of non-linear ordinary differential equations particularly when a large family of solutions is required; and (2) to obtain as a by-product results of practical applicability to servomechanism selection and analysis.

The Electronic Analog Computer

An analog computer may be defined as a device similar in mathematical description (but often in no way similar physically) to a system under study such that experiments may be performed upon this device and the measured results directly related to the results that would have been obtained had equivalent experiments been carried out upon the system. The system itself might be of mechanical, electrical, hydraulic, chemical, pneumatic or aerodynamic nature (or any combination of these). The associated analog computer may either be a scale model of the system or may have no relation whatsoever beyond a set of mathematical equations. Such special-purpose analog devices date back to about 1814, while the general-purpose electronic analog computer (sometimes called an electronic differential analyzer) is scarcely a decade old.[1] Over 400 installations of electronic analog computing equipment exist in the United States alone, representing a capital investment of 50 million dollars.[2] A typical medium-large installation is shown in Figure 1.

The electronic analog computer consists of a number of each of several basic "building blocks" or computing units. To form an analog of the system to be studied, the differential equations of that system are written and the computing units interconnected in such a way that the mathematical equations describing electrical voltage at various points in the circuits set up are identical (except in scale factors) to the equations to be solved. One computing unit is required for each addition, multiplication, integration, or other mathematical operation in the equation to be solved. Thus, the more extensive the equations to be solved the more equipment is required, but the time required for solution is not increased because all computer units operate simultaneously. The operation of various types of computing units, such as integrating amplifier ("integrator"), summing amplifier ("summer"), coefficient potentiometer ("pot"), limiter and multiplier is briefly explained in the Appendix.

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[1] Jerry Roedel, An Introduction to Analog Computers (Boston, 1953), p. 2.

[2] John M. Carroll, "Analog Computers for the Engineer," Electronics, XXIX (June, 1956), pp. 122-129.

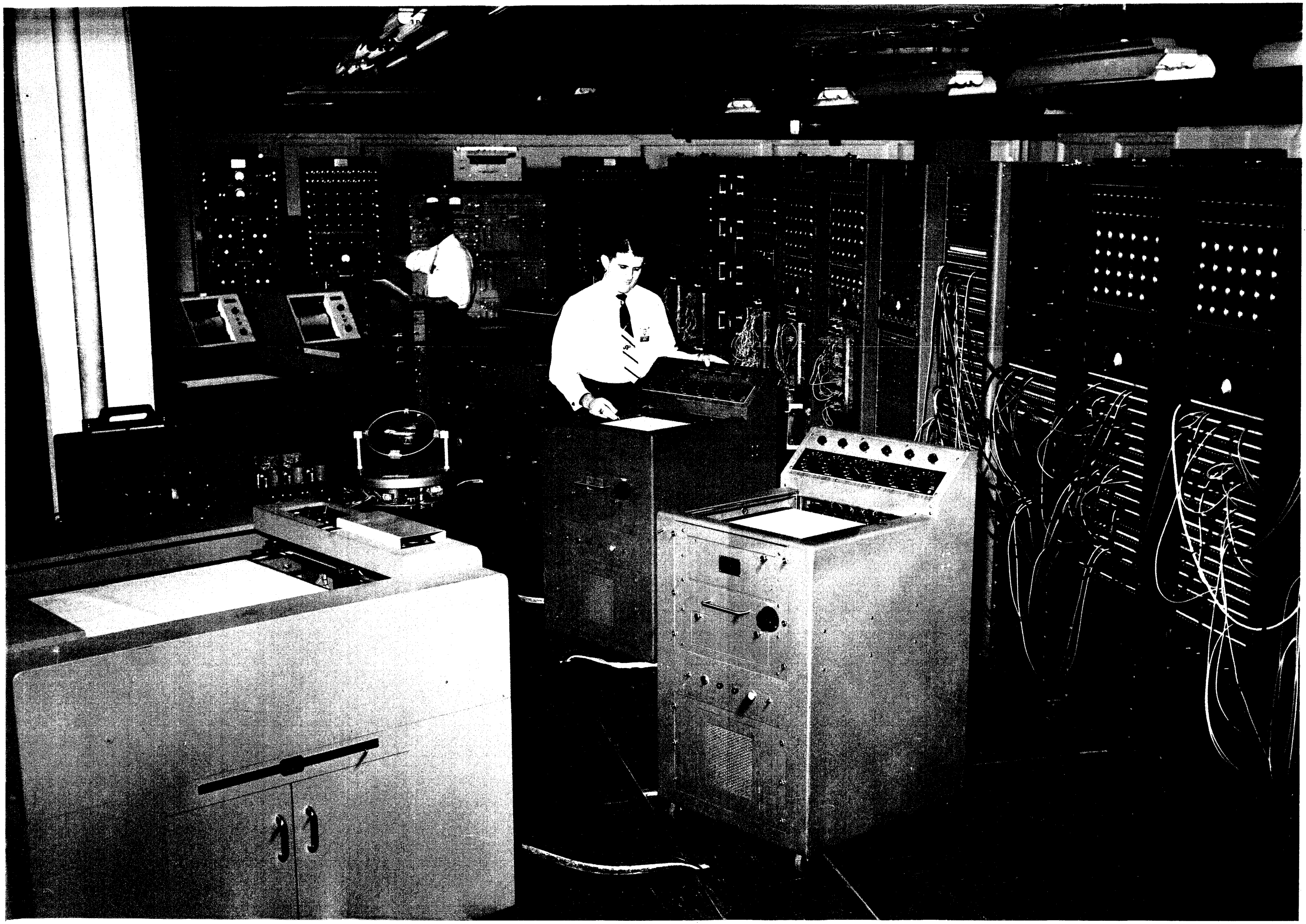
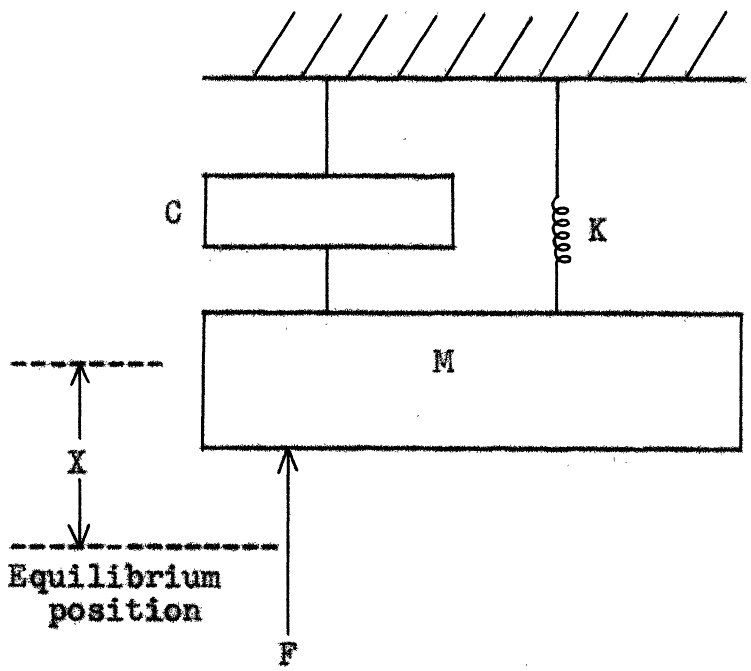


Fig. 1--Chance Vought Aircraft Analog Computer Installation at Dallas, Texas.

The computer solution of a simple problem is outlined below as an introductory example.

Figure 2 depicts a mass "M" suspended from a fixed support by a spring "K" and a dashpot "C". At time equals zero, a constant vertical force "F" is applied to the mass. Find the displacement "X" of the mass as a function of time "t", if M, C, K and F have values indicated.



M = 2 slugs

C = 0.550 lbs. per ft. per sec.

K = 1.21 lbs. per ft.

F = 19.4 lbs.

Fig. 2--Oscillatory mechanical system.

The applicable differential equation is

$$M \frac{d^2x}{dt^2} + C \frac{dx}{dt} + Kx = F \tag{1}$$

or

$$2 \frac{d^2x}{dt^2} + 0.550 \frac{dx}{dt} + 1.21x = 19.4 \tag{2}$$

or $$\frac{d^2x}{dt^2} = -\,0.275\,\frac{dx}{dt} - 0.605x + 9.70 \qquad (3)$$

Note that in equation (3) the highest derivative is isolated on the left-hand side of the equation; this is standard procedure. Figure 3 shows the computer wiring diagram required to solve this equation; symbols used are defined in the Appendix.

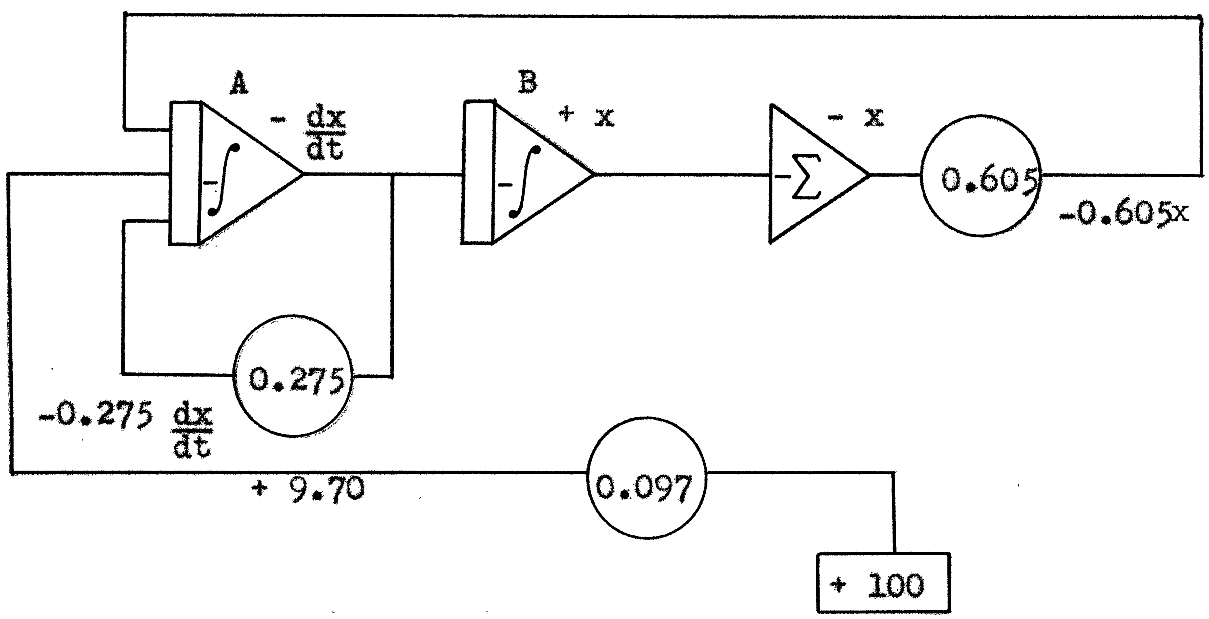


Fig. 3--Analog computer diagram for differential equation of oscillatory mechanical system.

The starting point of the computer wiring diagram of Figure 3 is Integrator "A". If the sum of the inputs to Integrator "A" is assumed to be d^2x/dt^2 the output of this integrator must then be

$$-\int_0^t \frac{d^2x}{dt^2}\,dt = -\,\frac{dx}{dt}.$$

Next all right-hand terms appearing in equation (3) above are produced and connected to the input terminals of Integrator "A".

The circuits set up in accordance with the computer wiring diagram of Figure 3 will obey equation (3). Since it is required to solve for the displacement of "x" of the mass "M" as a function of time, the recorder is connected to the output of Integrator "B". When the "Operate" switch is thrown, the computer begins to solve the equation and the recorder simultaneously plots "x" versus time.

Solution of the equation considered above could have been done easily by hand, of course. But suppose that several hundred solutions of the basic equation must be made for different combinations of numerical coefficients and initial conditions--then the value of the computer is apparent. Or suppose that the differential equation to be solved was linear, and so admitted of a straight-forward manual solution, but was of say, thirtieth order--even a requirement for a single solution would indicate electronic computer applicability. Finally, all physical systems are in some measure non-linear, and where the non-linearities are of sufficient importance to warrant consideration a manual solution of the applicable differential equations in closed form is frequently impossible, while non-linear computer units may be readily included in analog computer circuitry. Because the engineering design and the analysis of practical dynamic system, such as an automatically controlled aircraft or guided missile, often requires the study of a family of several hundred solutions of a heavily non-linear system of differential equations of high degree, the electronic analog computer has become an essential tool in the aircraft and related industries.

Servomechanisms

It is sometimes desired to change electrical voltage to mechanical motion with an accompanying increase in power level, without introducing errors arising from variations of load or power supply. Such a problem can be solved by the use of a servomechanism ("servo"). For example, an electric motor may be made to rotate a shaft on which is mounted a potentiometer with fixed voltage across its end terminals. Rotation of the motor causes motion of a "tap" connection such that the voltage appearing at the "tap" or output of the potentiometer is proportional to the motor shaft position. This output voltage is compared with the original electrical voltage, and the difference made to control the electric motor. This is one type of servo. Such devices are of considerable importance in guided missile and airplane simulators and flight trainers, in gunfire control equipment, and numerous industrial applications. An example is shown in Figure 4.

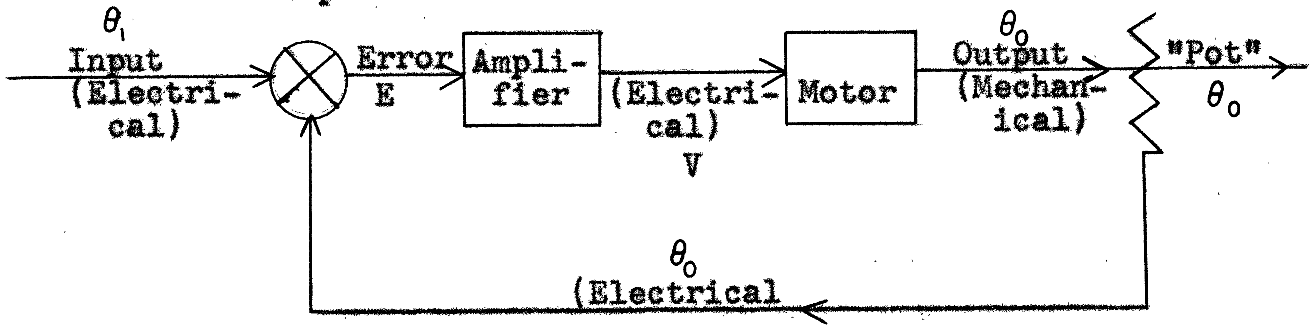


Fig. 4--Block diagram of servomechanism.

At each instant, the comparator yields the difference between input signal θ_1 and output position θ_0. This is the "error" signal E, since for output equals input the signal would be zero. The amplifier accepts the electrical error signal which is at low power level and converts the error signal to a voltage and power level suitable for controlling the motor. The motor will then rotate in a direction to reduce the error to zero. The output of an ideal servo would follow instantaneously all changes in the input to the servo. Because of motor inertia and friction, as well as the finite power available from the amplifier which controls the motor, a practical servo may depart seriously from the ideal.

A linearized set of differential equations[3] describing the servo illustrated in Figure 4 is shown below:

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[3] I. A. Getting, Theory of Servomechanisms, Vol. XXV of M. I. T. Radiation Laboratory Series, edited by Hubert M. James, Nathaniel B. Nichols, and Ralph S. Phillips, 28 vols. (New York, 1947), p. 10.

$$E = \theta_i - \theta_o \tag{4}$$

$$V = k_1 E \tag{5}$$

$$J \ddot{\theta}_o + f \dot{\theta}_o = k_2 V \tag{6}$$

where

variables

E = comparator or error voltage

θ_i = input voltage

θ_o = output mechanical position or voltage

V = amplifier output

constants

J = motor inertia

f = motor friction coefficient

k_1, k_2 = proportionality factors.

However, for economy of power requirements, practical servomechanisms are almost always designed to operate largely outside the linear range of one or more components. The feedback principle fundamental to all servomechanisms insures that the steady-state or static accuracy will not be affected by such non-linearities, but the transient or dynamic performance may be seriously affected. Further discussion is reserved for Chapter II.

CHAPTER II

SPECIFIC PROBLEM TO BE STUDIED AND ITS IMPORTANCE

When choosing a servomechanism for a particular application various practical criteria might be applied, some of the more obvious being availability, cost, size, and power consumption. These factors must be reconciled with accuracy requirements of the servo. If the servo motor and the amplifier will always operate within their linear dynamic ranges, equations (4), (5), and (6) apply and the system is linear and of second order. Dynamic accuracy of the servo can then be measured conveniently by one parameter, the natural frequency of the system. Moreover, the response of the servomechanism to any input signal voltage can be calculated by straight-forward analytical techniques, such as Laplace Transform methods.

As described earlier, however, practical servomechanisms make use of the non-linear dynamic regions of motor and amplifier performance for reasons of power economy. In this case "natural frequency" has a meaning only for the restricted range of performance where linear differential equations apply, and other criteria such as maximum motor speed and acceleration are often employed. Good analytical methods do not exist for handling the non-linear differential equations and thus few results of general applicability to the non-linear servo problem exist. Consequently, there is little understanding in the field as to the implications on performance of these rough criteria. Accordingly, it is the purpose of this thesis to produce a family of servo performance curves which will be illustrative of the capabilities of the electronic analog computer in such problems and which will also be of some immediate practical value in servomechanism selection.

Motor Characteristics

For the "linearized motor" where equation (6) applies, the motor characteristics may be plotted as a family of straight lines as in Figure 5 on the next page.

$$J \ddot{\theta}_o + f \dot{\theta}_o = K_2 V \qquad (6)$$

The distinctly non-linear characteristics of the actual motor are shown in Figure 6[1] for comparison.

The curves in Figure 6 follow no simple analytical expression and in any case represent only one type of motor. Consequently an attempt was made to find a fairly simple means of taking account of the inherent non-linearities of the motor in an approach of reasonably general applicability.

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[1] Harold Chestnut and Robert W. Mayer, Servomechanisms and Regulating System Design, Vol. II, 2 vols. (New York, 1955), p. 224.

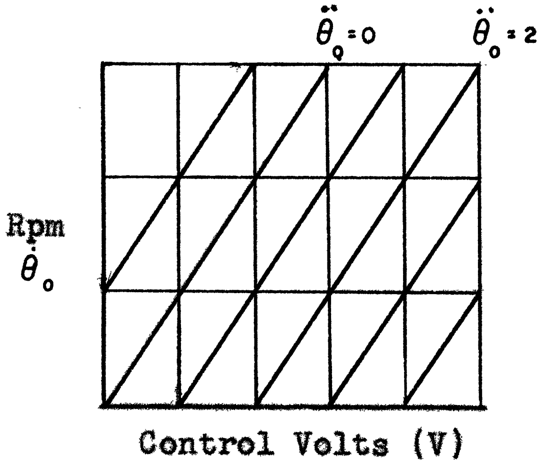

Fig. 5--Motor characteristics of linearized motor.

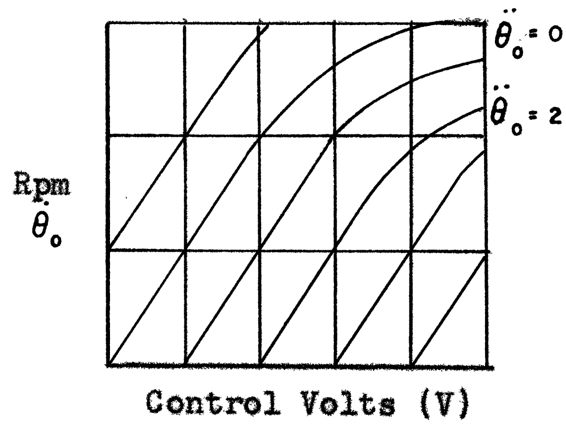

Fig. 6--Non-linear motor characteristics of actual motor.

The motor characteristics typified by Figure 7 were chosen and for low speeds may be represented by equation (6), with all constants set at unity for convenience:

$$\ddot{\theta}_o + \dot{\theta}_o = V \tag{7}$$

$$\text{or } \ddot{\theta}_o = V - \dot{\theta}_o \tag{7a}$$

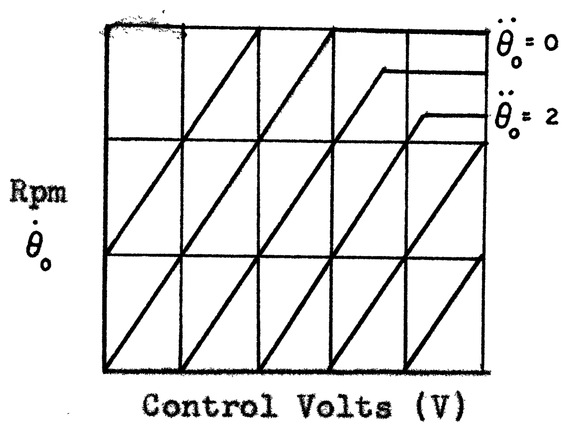

Fig. 7--Motor characteristics chosen for study.

To take account of the horizontal portions of the motor characteristics shown in Figure 7, equation (7a) holds. And

$$\ddot{\theta}_o \leq A\left(1 - \frac{\dot{\theta}_o}{S}\right) \qquad (8)$$

(considering only positive velocities at the present) where A is the maximum acceleration from zero speed and S is the maximum steady-state motor speed.

Amplifier Characteristics

The linear amplifier described by equation (5) actually has limits upon its output, so that it should more accurately be represented by

$$V = \begin{cases} E, & |E| < C \\ C, & E > C \\ -C, & E < -C \end{cases} \qquad (9)$$

where the constants have again been set at unity for convenience. The amplifier characteristic of equation (9) is depicted in Figure 8. Of course, an actual amplifier usually exhibits less sharp transition into the limited-output region.

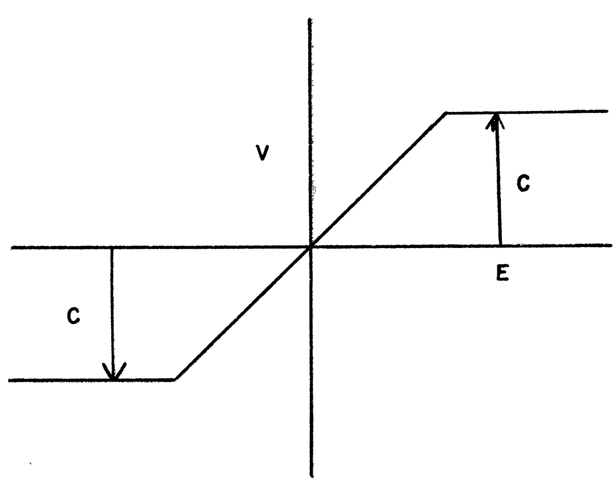

Fig. 8--Amplifier characteristic.

CHAPTER III

THE ELECTRONIC ANALOG COMPUTER PROGRAM

It was desired to produce a compact set of servo performance curves that would have a rather general applicability. This requires that the number of adjustable parameters be held to a minimum, while making it possible to utilize the curves (with proper scaling) for cases not specifically considered. Setting the constants of equations (4) and (5) to unity as was done in equations (7) and (10) results in a dynamic system with natural frequency of one radian per second and damping ratio of 50 per cent. Only the parameters associated with non-linear motor and amplifier characteristics (S and C) were varied in the computer study.

Only one type of servo input was used; a step of 50 volts occurring at time equals zero, as shown in Figure 9.

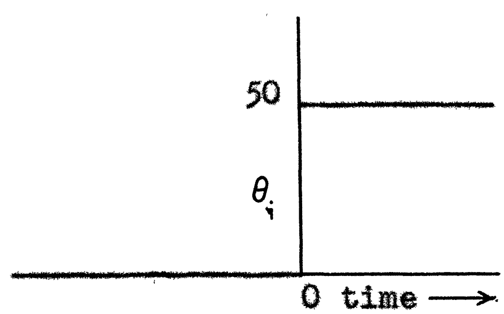

Fig. 9--Step function of 50 volts at time equals zero.

This input was chosen since the step function is the most severe input likely to be encountered by a servo, and also because the response of linear systems to step inputs is well known and the results easily available for comparison with non-linear systems. The value of A was fixed at 50 volts. The equations actually studied on the electronic analog computer were, then,

$$\theta = \begin{cases} 0, & t < 0 \\ 50, & t \geq 0 \end{cases}$$

$$E = \theta_i - \theta_o$$

$$V = \begin{cases} E, & |E| < C \\ C, & E > C \\ -C, & E < -C \end{cases} \qquad (10)$$

$$\ddot{\theta}_o = \begin{cases} (V - \dot{\theta}_o), & -A(1 + \frac{\dot{\theta}_o}{S}) < (V - \dot{\theta}_o) < A(1 - \frac{\dot{\theta}_o}{S}) \\ A(1 - \frac{\dot{\theta}_o}{S}), & (V - \dot{\theta}_o) > A(1 - \frac{\dot{\theta}_o}{S}) \\ -A(1 + \frac{\dot{\theta}_o}{S}), & (V - \dot{\theta}_o) < -A(1 + \frac{\dot{\theta}_o}{S}). \end{cases}$$

The specific program is tabulated in Table I. Note that in all cases $S \leq C$, since the amplifier must be capable of an output voltage at least high enough to cause the motor to run at its maximum speed.

Figure 10 shows the electronic analog computer wiring diagram used to solve the set of equations (10). The switch used to insert the step input is shown, and the coefficient potentiometers controlling the parameters S and C are indicated. The "Brush" Recorder was used to plot automatically time histories of several quantities of interest as indicated

TABLE I

ELECTRONIC ANALOG COMPUTER NON-LINEAR SERVOMECHANISM STUDY

Case No.	C	S	Case No.	C	S
1	50	50.0	22	30	1.875
2	50	40.0	23	30	1.25
3	50	30.0	24	20	20.0
4	50	20.0	25	20	10.0
5	50	10.0	26	20	5.0
6	50	5.0	27	20	2.5
7	50	2.5	28	20	1.875
8	50	1.875	29	20	1.25
9	50	1.25	30	10	10.0
10	40	40.0	31	10	5.0
11	40	20.0	32	10	2.5
12	40	10.0	33	10	1.875
13	40	5.0	34	10	1.25
14	40	2.5	35	5	5.0
15	40	1.875	36	5	2.5
16	40	1.25	37	5	1.875
17	30	30.0	38	5	1.25
18	30	20.0	39	2.5	2.5
19	30	10.0	40	2.5	1.875
20	30	5.0	41	2.5	1.25
21	30	2.5	42	1.25	1.25

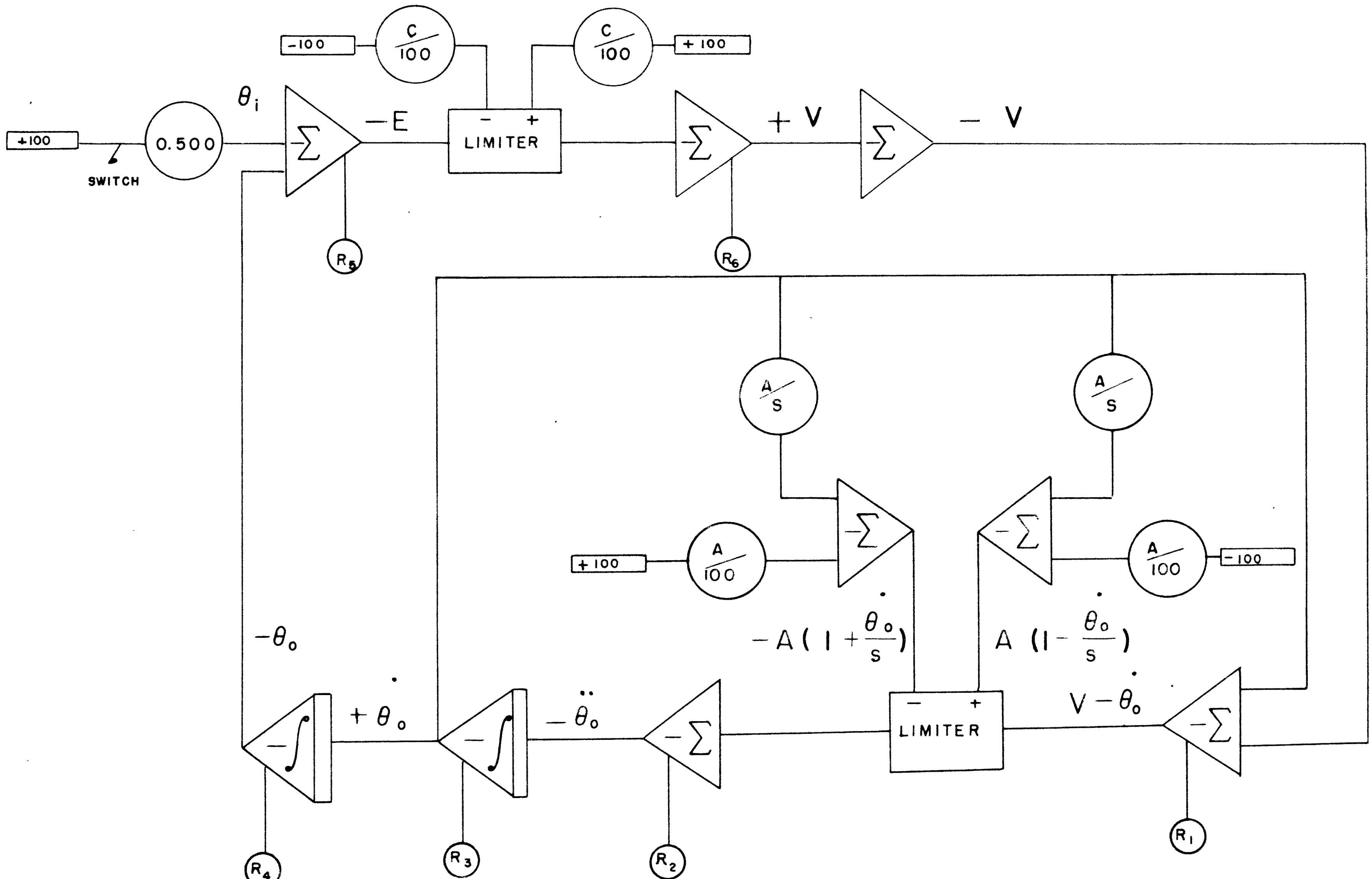

Fig. 10--Analog computer diagram for equation (10).

in Figure 10 by R_i where the subscript i denotes the channel number.

R_1	$V - \dot{\theta}_o$	R_4	$- \theta_o$
R_2	$- \ddot{\theta}_o$	R_5	$- E$
R_3	$+ \dot{\theta}_o$	R_6	$+ V$

Results

A sample "Brush" Recorder result is shown in Figure 11. Such records were obtained for all the cases listed in Table I. They are of use primarily for checking purposes and for more detailed analyses of servo performance than is intended for this thesis. To plot the desired curves, a "Variplotter" plotting board was used since this device produces much larger plots, can plot a whole family of curves on a single sheet with common axes, and can produce cross-plots of dependent variables (rather than only time-histories) if desired. Figures 12, 13, 14, 15, 16, 17 and 18 present the time-histories of servo output in response to a step input for all the cases of Table I. The response for a linear system is essentially the same as Case No. 1.

These curves are of value in themselves, since they make it possible to observe the response to a step input for a specified system; or, conversely, what non-linearities may be tolerated for a given allowable response. However, other servo system criteria of interest may also be calculated

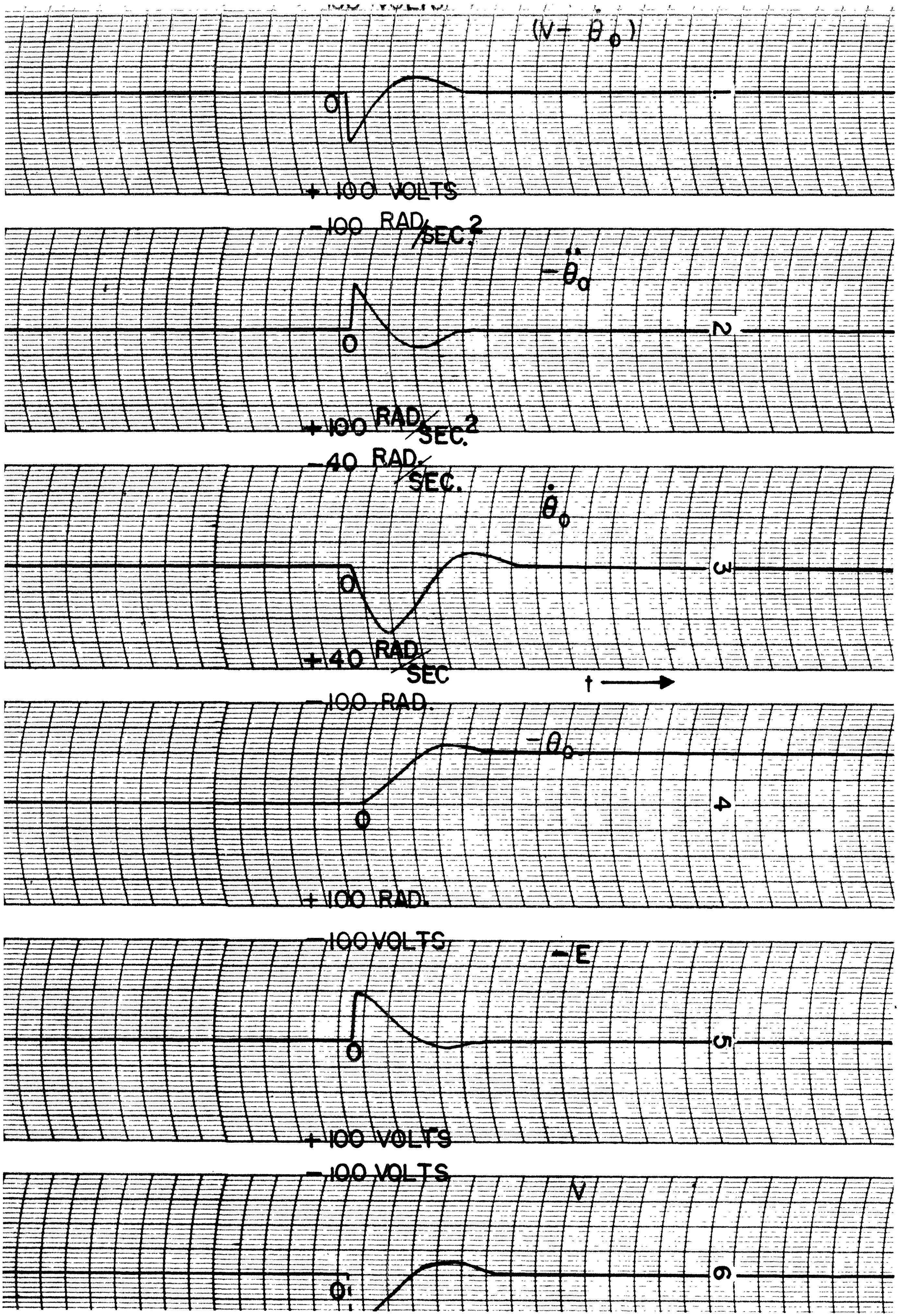

Fig. 11--Brush recorder time history for Case 1.

from the curves or by the analog computer itself. One such criterion applicable to both linear and non-linear systems of any order of complexity is

$$\text{Integral-of-error-squared} = \int E^2 \, dt.$$[1]

It is desired that integral-of-error-squared be made as small as possible. For each case of Table I, integral-of-error-squared was calculated automatically by the analog computer (circuitry not shown in Figure 10) simultaneously with response calculation. Integral-of-error-squared is plotted in Figure 19. The value for the linear case is indicated, so that it can easily be seen for a given degree of non-linearity how much departure from the minimum (i.e., that obtained in the linear case) is incurred.

A common tool in the analysis of non-linear differential equations (especially of second order) is the phase-plane plot, i.e., a plot of output θ_o vs output time-derivative $\dot{\theta}_o$. Only qualitative analysis of a system can be made from such a plot, but because they are fairly easily constructed they are often used. Without additional discussion, phase-plane plots produced automatically on the plotting-board during computer solution of several of the cases of Table I are illustrated in Figures 20 and 21 as a further demonstration of electronic analog computer capabilities.

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[1] Albert C. Hall, The Analysis and Synthesis of Linear Servomechanisms (Cambridge, 1943), p. 23.

Future Recommendations

From the foregoing discussion it may be concluded that basic data have been obtained which will permit the specification of practical servo characteristics required for a given performance description, and which will allow the ready evaluation of a given servo. To be more representative of actual servo inputs, it is recommended the use of random inputs of specified statistical properties be used in future studies rather than a step input as was used in this study. Then the primary calculation to be made would be the integral-of-error-squared for the various cases. Also recommended for future study is the calculation of the average and maximum servo motor power required for both step inputs and random inputs, to relate reduction in power requirements quantitatively to increase in integral-of-error-squared. These power calculations can be made by additional computer elements simultaneous with calculation of servo output response time histories.

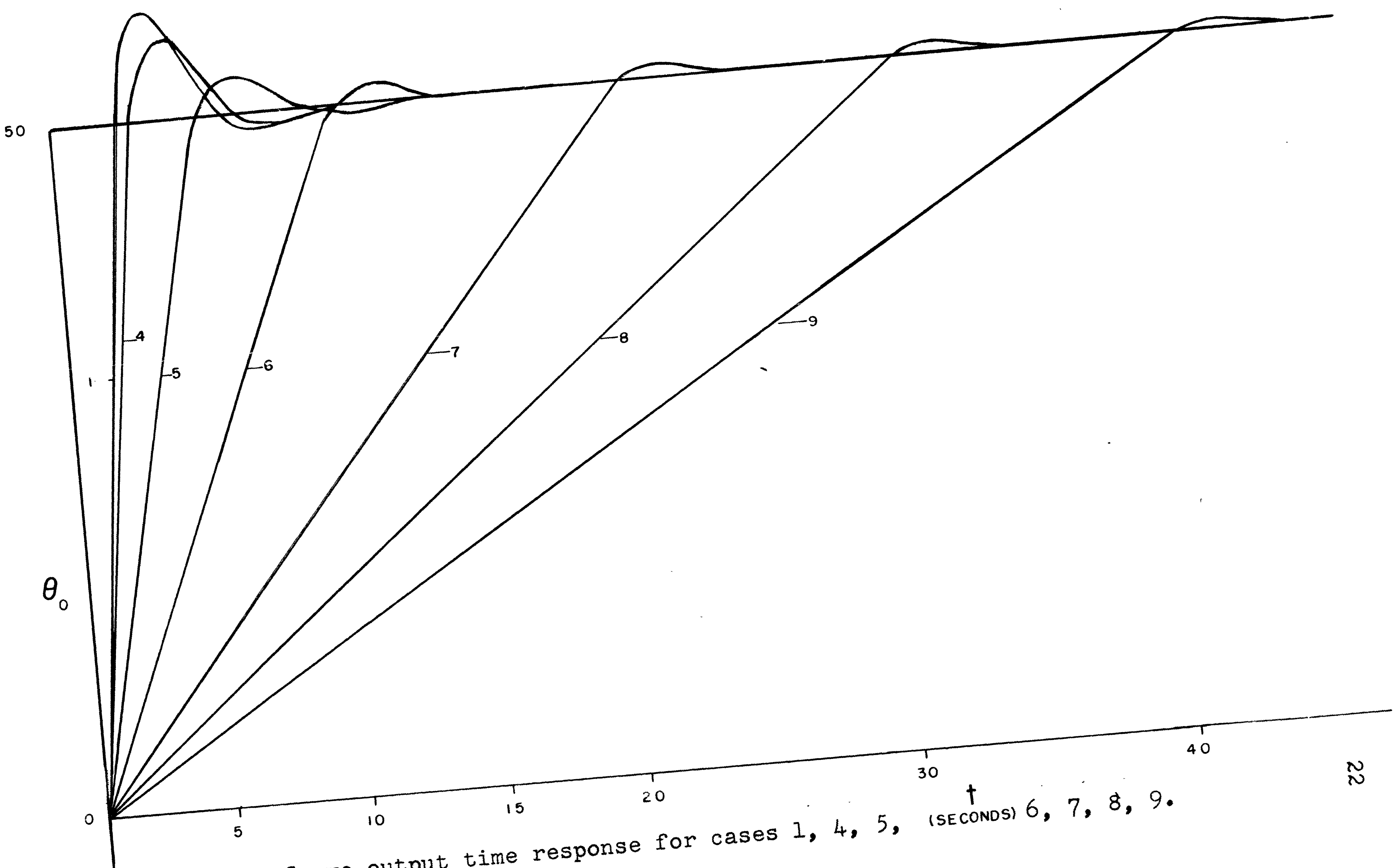


Fig. 12--Servo output time response for cases 1, 4, 5, 6, 7, 8, 9.

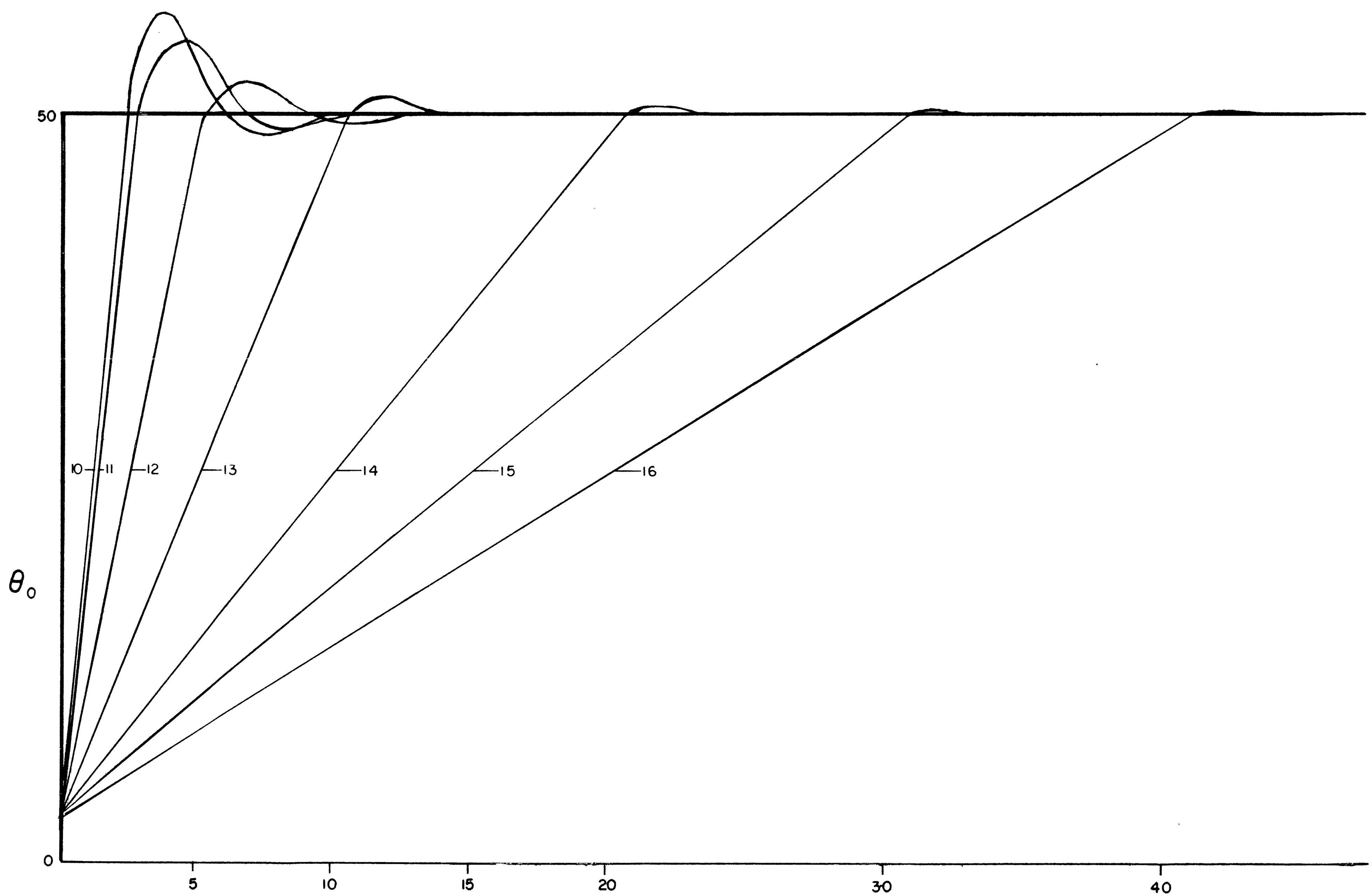


Fig. 13--Servo output time response for cases 10-16.

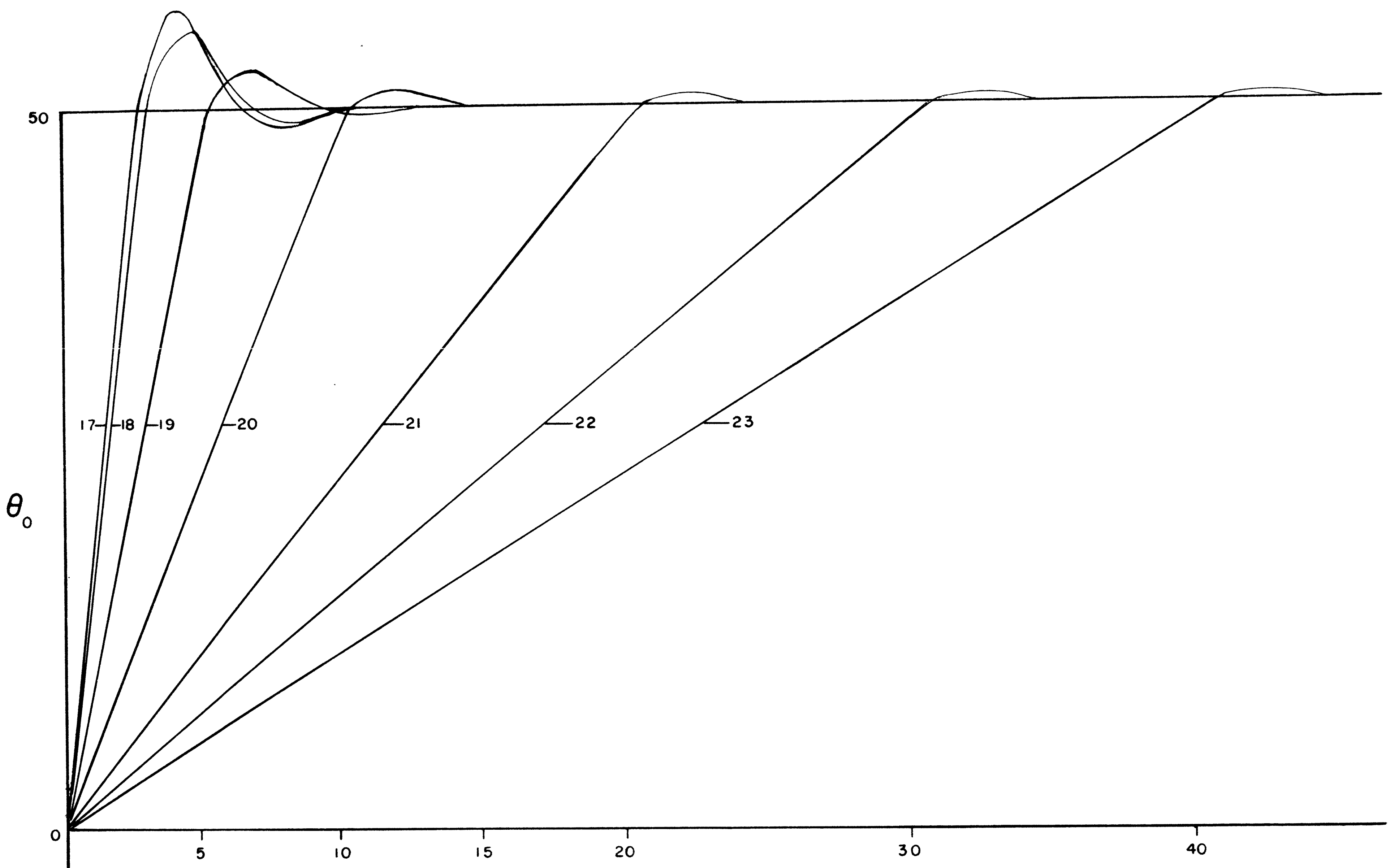


Fig. 14--Servo output time response for cases 17-23.

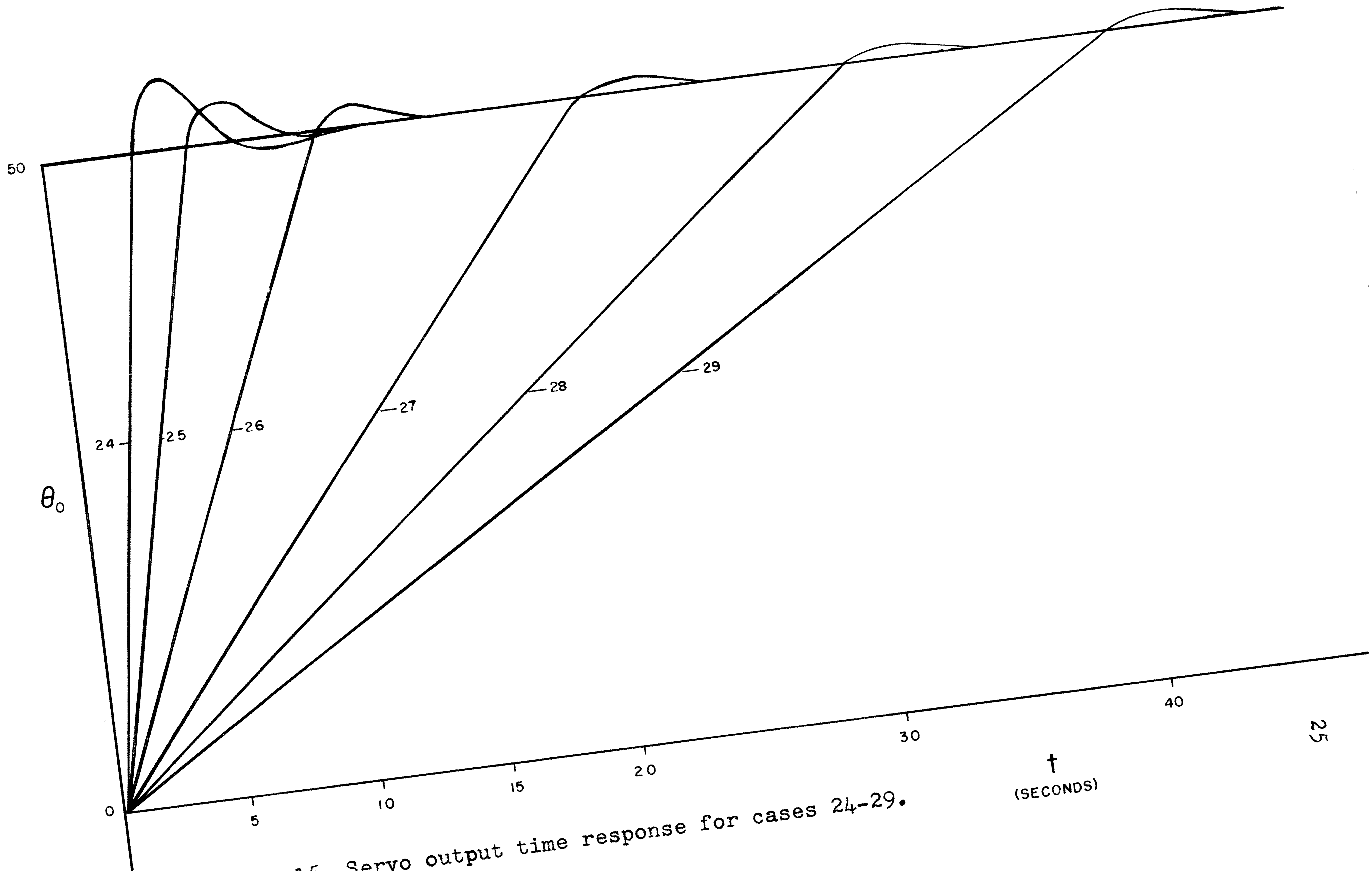


Fig. 15--Servo output time response for cases 24-29.

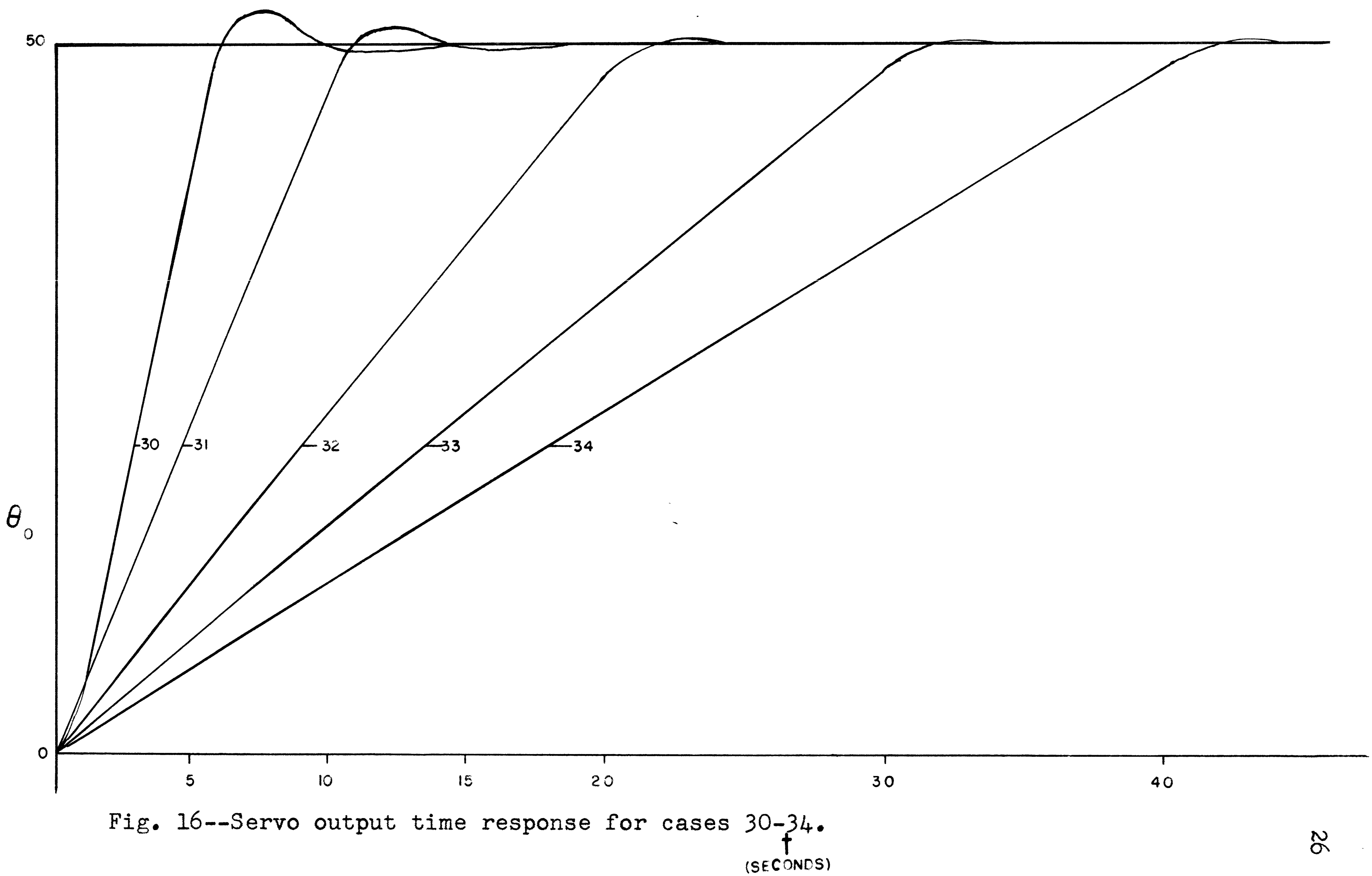


Fig. 16--Servo output time response for cases 30-34.

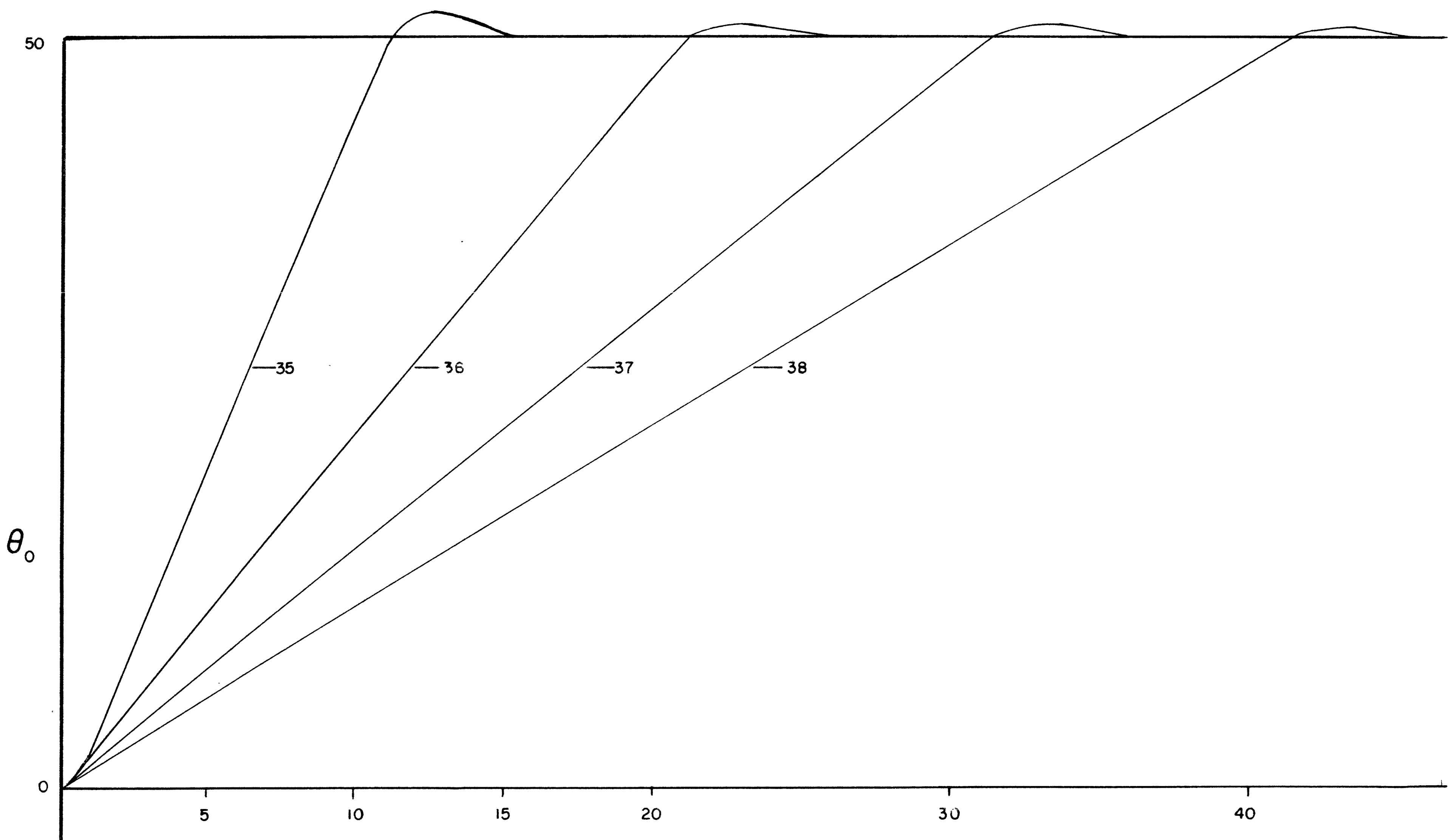


Fig. 17--Servo output time response for cases 35-38.

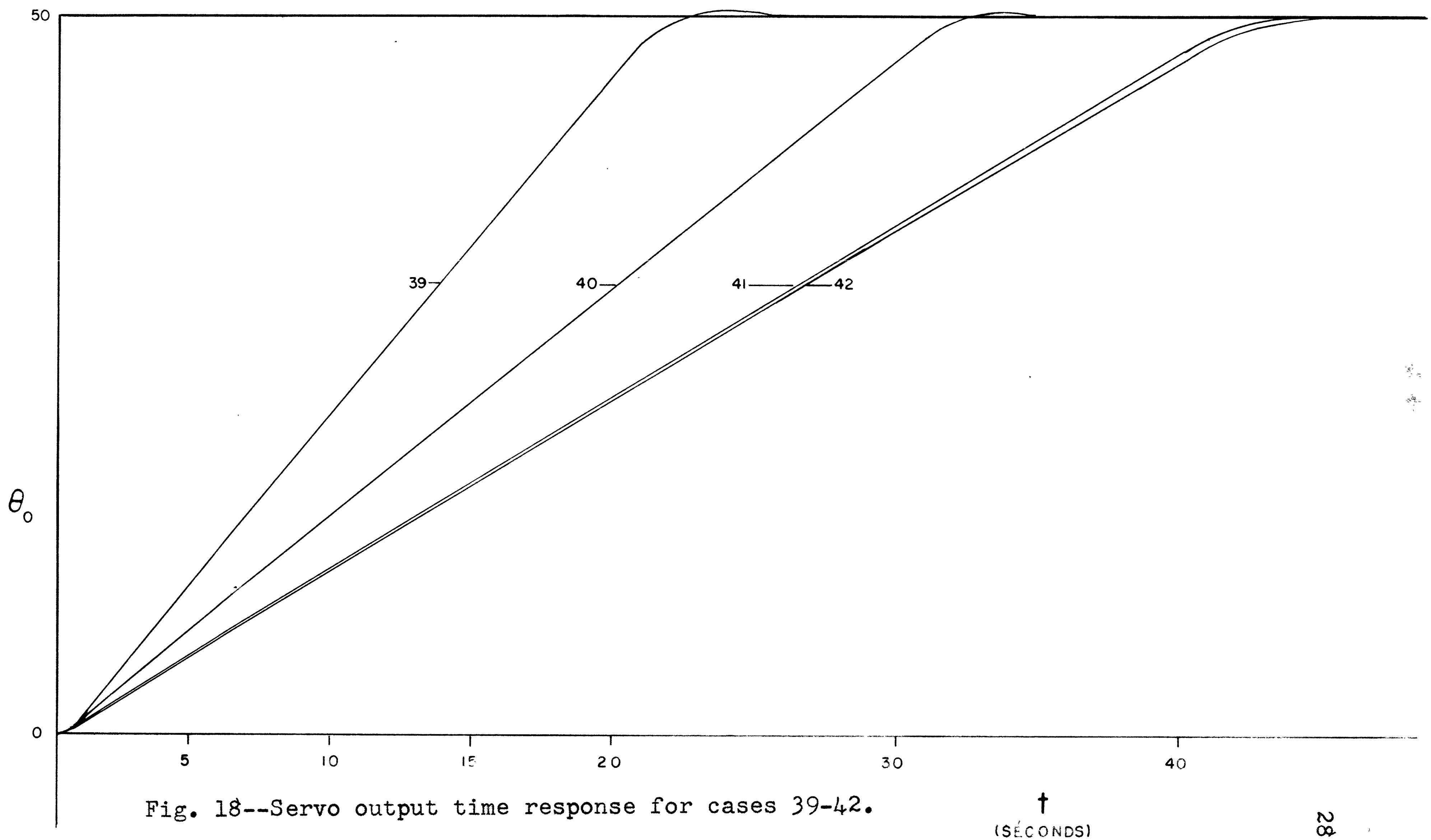


Fig. 18--Servo output time response for cases 39-42.

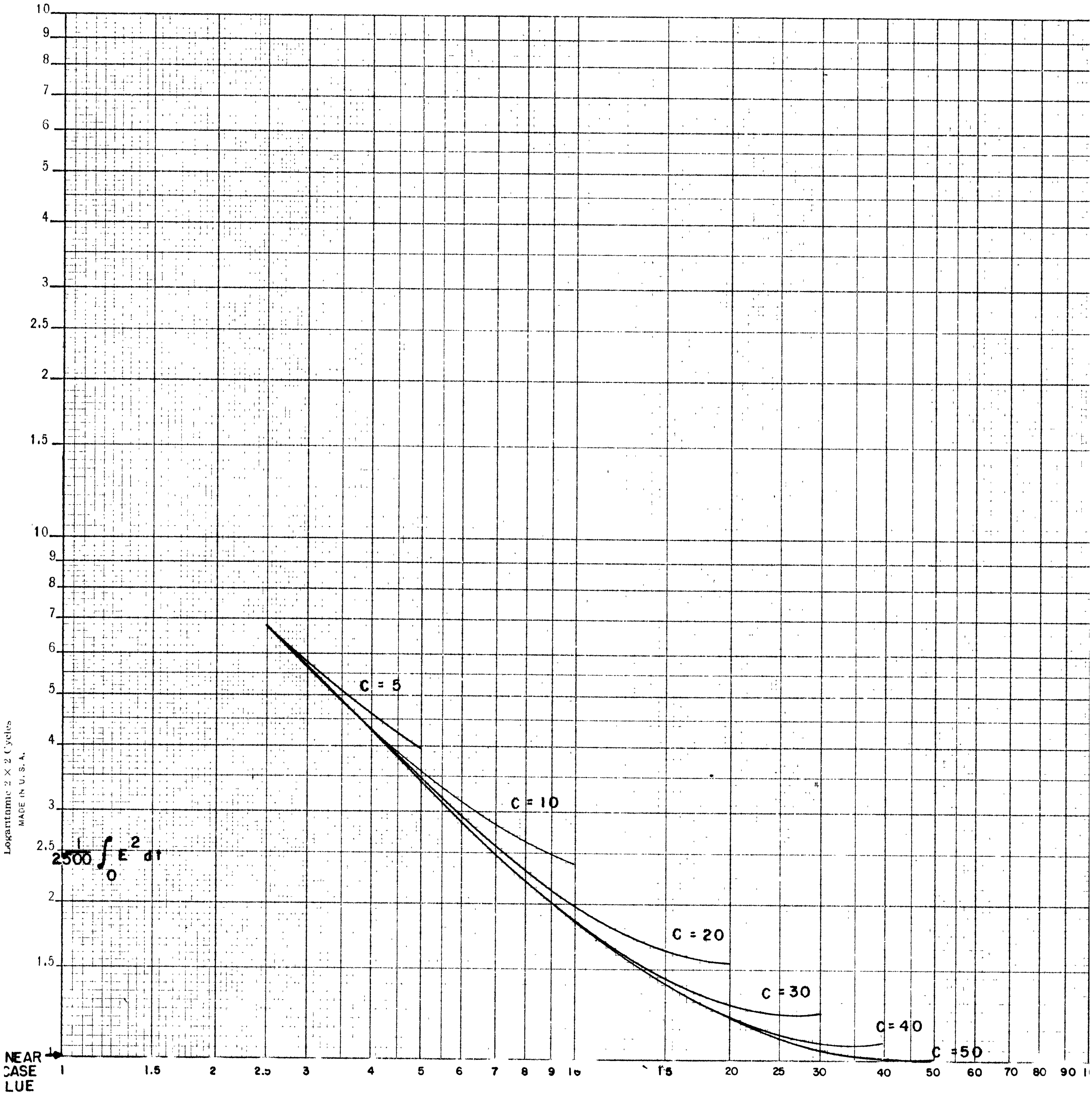


Fig. 19--Integral-of-error-squared curve.

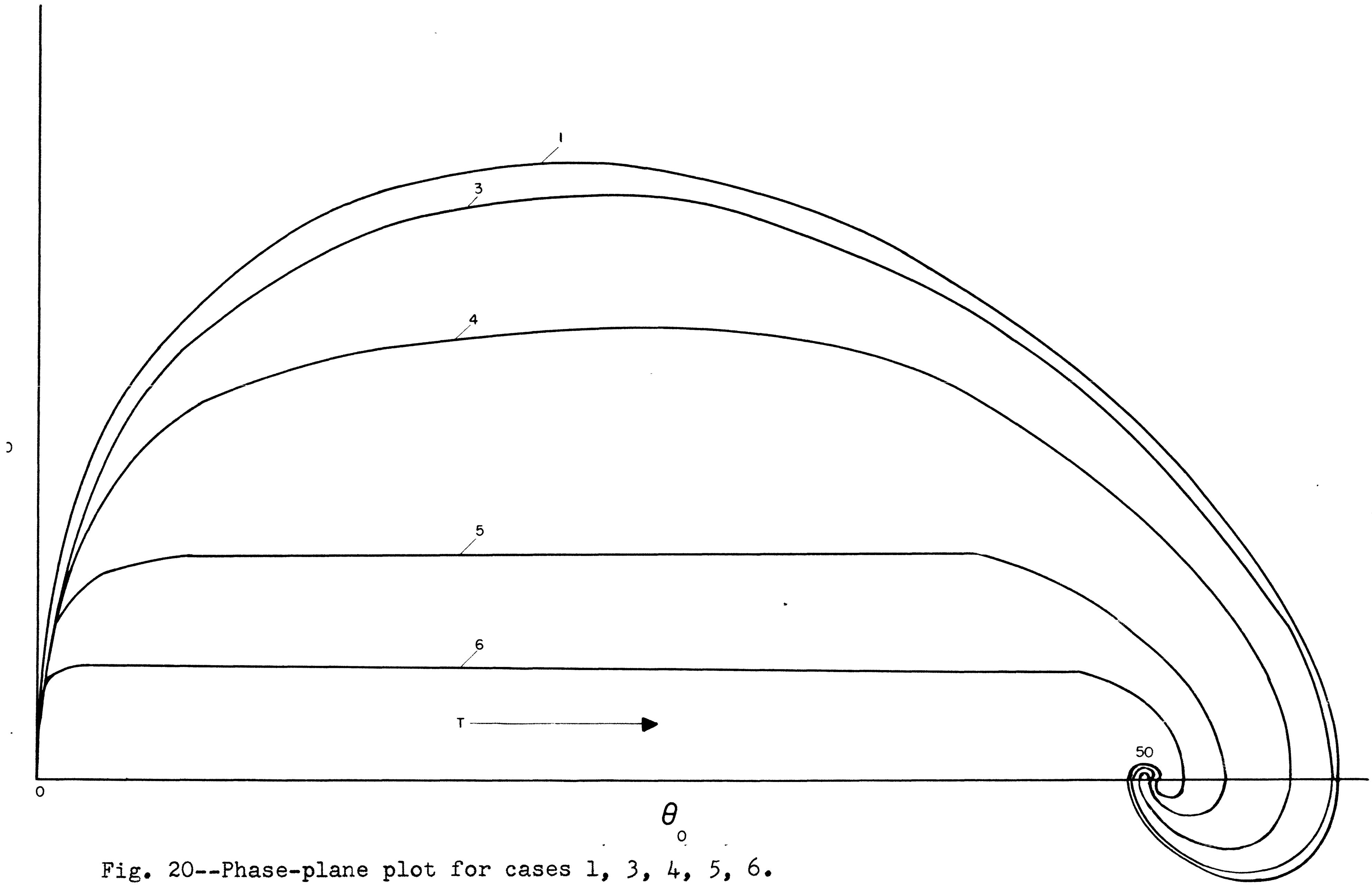


Fig. 20--Phase-plane plot for cases 1, 3, 4, 5, 6.

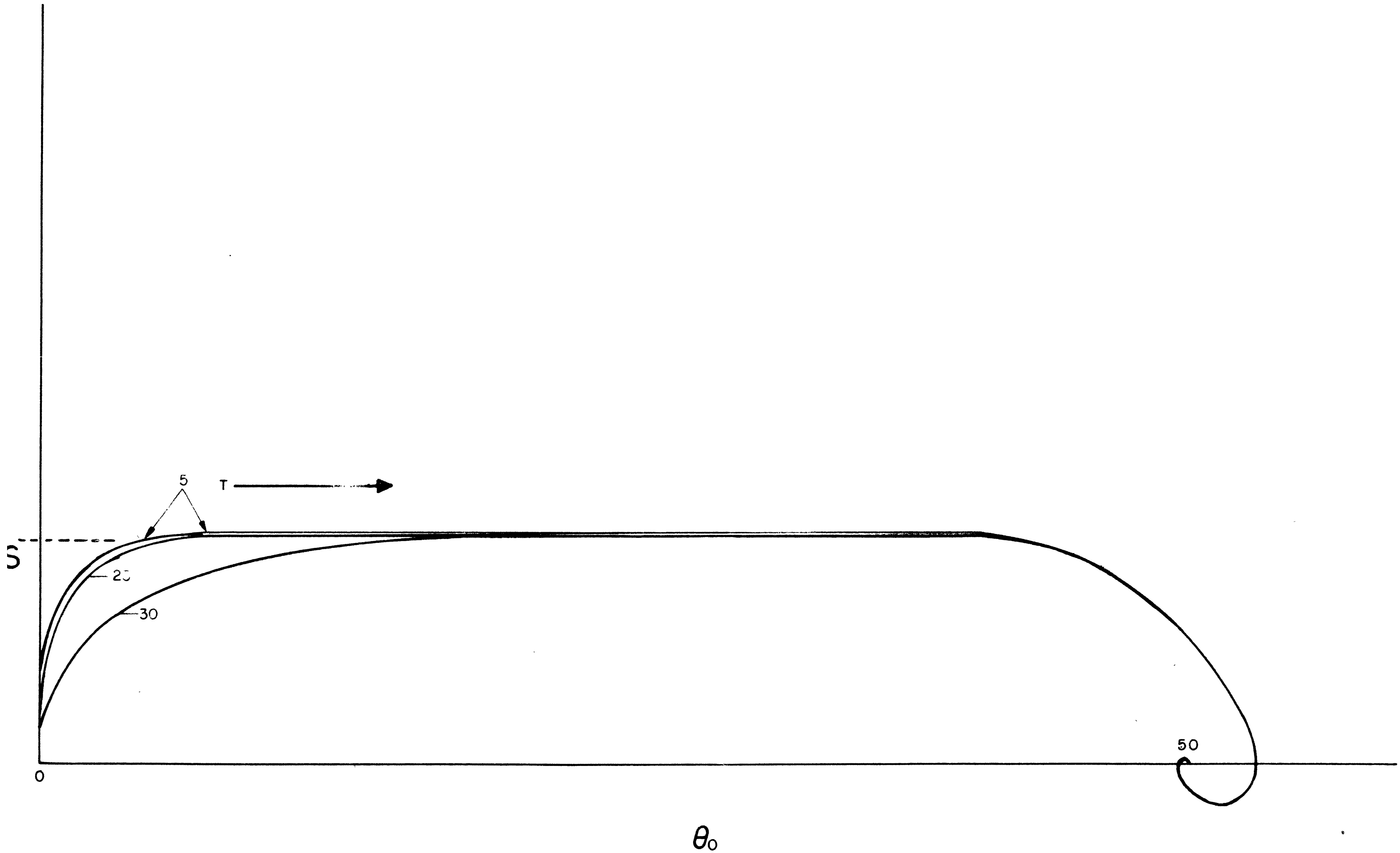
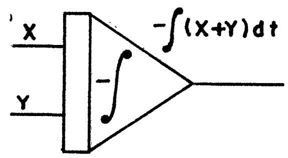
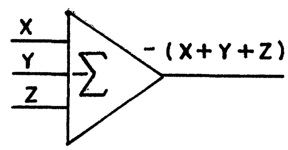
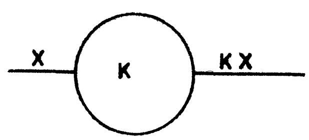


Fig. 21--Phase-plane plot for cases 5, 25, 30.

APPENDIX

ANALOG COMPUTER COMPONENTS AND THEIR FUNCTIONS

The basic components are the integrating amplifier ("integrator"), the summing amplifier ("summer"), the coefficient potentiometer ("pot"), the diode limiter ("limiter"), and the voltage source. To draw a wiring diagram for the computer using these components a set of symbols has become standard. These symbols and their function are listed below:

COMPUTER ELEMENT	SYMBOL	FUNCTION
Integrating Amplifier ("integrator")		Output is minus the time-integral of the sum of the inputs.*
Summing Amplifier ("summer")		Output is minus the sum of the inputs.
Coefficient Potentiometer ("pot")		Output is input multiplied by a positive constant less than 1.**

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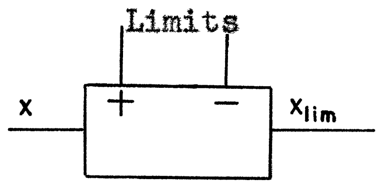
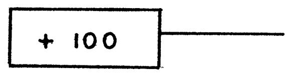
*Integrators also have provisions for inserting "initial conditions."

**Since a "pot" cannot multiply a factor greater than 1, the "integrators" and "summers" have special input terminals which effect multiplication by 4 or 10 while adding or integrating; thus to multiply by 2, a "pot" set at 0.500 is used together with a special input terminal of 4, giving an overall factor of (0.550) X (4) equals 2.

COMPUTER ELEMENT	SYMBOL	FUNCTION
Diode Limiter ("limiter")		Output is input limited at value set on + and - limits.
Voltage Source		Together with "pot" supplies additive constants.

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