ORTHOGONAL FUNCTIONS

APPROVED:

George Copp
Major Professor

H. C. Parrish
Minor Professor

R. H. Hanson
Director of the Department of Mathematics

Robert B. Toulouse
Dean of the Graduate School

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THESIS

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Ivan Harrison, B. S., A. B.

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CHAPTER I

DEFINITION OF ORTHOGONAL FUNCTIONS

In this study the idea of orthogonality of two lines will be generalized to the idea of orthogonality of two functions. In particular, the orthogonality of two lines may be treated from the standpoint of the orthogonality of two vectors in two-dimensional space. The point of intersection of the two lines may be considered as the origin of a rectangular cartesian coordinate system and the coordinates of a point different from the origin on each of the two lines will be the coordinates of the two vectors which are orthogonal. If P is one of these points (a,b), Q is the other point (c,d), and O is the origin, then the triangle POQ is a right triangle. From the law of cosines it follows that

$$\cos 90^\circ = \frac{ac + bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}.$$

Hence $ac + bd = 0$. Conversely, if $\sqrt{a^2 + b^2} \neq 0$, $\sqrt{c^2+d^2} \neq 0$, and $ac + bd = 0$, then the two vectors (a,b) and (c,d) are orthogonal.

If two non-zero vectors (a,b) and (c,d) are orthogonal and (e,f) is a non-zero vector, then there exists numbers p and q not both zero so that $p\cdot(a,b)+q\cdot(c,d)=(e,f)$. This follows from the fact that the two equations $ap+cq=e$ and

$bp + dq = f$ have non-trivial solutions for p and q when $ac + bd = 0$. Hence if two vectors are orthogonal in two-dimensional space, any non-zero vector can be represented by a linear combination of these two vectors. A convenient method for solving $p(a,b) + q(c,d) = (e,f)$ for p and q is to use the vector dot product. If both sides of the equation are multiplied by (a,b), then $p(a,b)\cdot(a,b) + q(c,d)\cdot(a,b) = (e,f)\cdot(a,b)$. Since (a,b) and (c,d) are orthogonal, $(c,d)\cdot(a,b) = ac + bd = 0$. Hence $p(a,b)\cdot(a,b) = (e,f)\cdot(a,b)$. $p(a^2 + b^2) = ae + bf$.

$$p = \frac{ad + bf}{a^2 + b^2}.$$

Similarly $q(c,d)\cdot(c,d) = (e,f)\cdot(c,d)$.

$$q = \frac{ce + fd}{c^2 + d^2}.$$

Since $ae + bf \neq 0$ or $ce + df \neq 0$, there is no vector orthogonal to both (a,b) and (c,d).

If two non-zero vectors, (a,b) and (c,d), are orthogonal, then the two vectors

$$\left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right) \text{and} \left(\frac{c}{\sqrt{c^2 + d^2}}, \frac{d}{\sqrt{c^2 + d^2}}\right)$$

are orthogonal. The length of these vectors is one unit and the dot product of each multiplied by itself is one. These will be called an orthonormal set of vectors. Since any non-zero vector in two-dimensional space can be represented by a linear combination of two orthogonal vectors, any two orthogonal vectors in two-dimensional space will be called a closed set. Since no vector is orthogonal to both of two orthogonal vectors in two-dimensional space, any two orthogonal vectors in two-dimensional space will be called complete.

If two non-zero vectors, (a,b,c) and (d,e,f), in three-dimensional space are orthogonal, then the triangle formed by the origin, the point (a,b,c), and the point (d,e,f) is a right triangle. Again by using the law of cosines, the result

$$\cos 90^\circ = \frac{ad + be + cf}{\sqrt{a^2 + b^2 + c^2}\,\sqrt{d^2 + e^2 + f^2}}$$

can be obtained. Hence $ad + be + cf = 0$. Also if

$$ad + be + cf = 0,$$

then the two vectors are orthogonal.

If (a,b,c), (d,e,f,), and (g,h,i) are three non-zero vectors such that each is orthogonal to the other two and (j,k,l) is a non-zero vector, then there exist numbers x, y, and z not all zero such that

$$x(a,b,c) + y(d,e,f) + z(g,h,i) = (j,k,l).$$

This is true because the three equations

$$ax + dy + gz = j$$

$$bx + ey + hz = k$$

$$cx + fy + iz = l$$

have a non-trivial solution for x, y, and z if

$$ad + be + cf = 0,$$

$$dg + eh + fi = 0,$$

$$ag + bh + ci = 0,$$

and one of the numbers j, k, or l is not zero. Again, if the vector dot product is used, the solution of the equations is simplified to,

$$x(a,b,c)\cdot(a,b,c)+y(d,e,f)\cdot(a,b,c)+z(g,h,i)\cdot(a,b,c)=(j,k,l)\cdot(a,b,c).$$

Hence

$$x=\frac{aj+bk+cl}{a^2+b^2+c^2},$$

$$y=\frac{dj+ek+fl}{d^2+e^2+f^2},$$

and

$$z=\frac{gj+hk+il}{g^2+h^2+i^2}.$$

Since one of these is not zero, the vector (j,k,l) is not orthogonal to all of the other three. Thus any three non-zero mutually orthogonal vectors in three-dimensional space form a complete set of orthogonal vectors. Since any non-zero vector can be represented by a linear combination of three mutually orthogonal vectors, any three mutually orthogonal vectors in three-dimensional space will be called a closed set of orthogonal vectors. The vectors

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\frac{b}{\sqrt{a^2+b^2+c^2}},\frac{c}{\sqrt{a^2+b^2+c^2}}\right),$$

$$\left(\frac{d}{\sqrt{d^2+e^2+f^2}},\frac{e}{\sqrt{d^2+e^2+f^2}},\frac{f}{\sqrt{d^2+e^2+f^2}}\right),$$

and

$$\left(\frac{g}{\sqrt{g^2+h^2+i^2}},\frac{h}{\sqrt{g^2+h^2+i^2}},\frac{i}{\sqrt{g^2+h^2+i^2}}\right)$$

are mutually orthogonal and the magnitude of each is one. Because of this they will be called an orthonormal set.

The idea of orthogonality of two vectors can be generalized to vectors in n-dimensional space where n is any positive integer. Here two non-zero vectors $(x_1,x_2,x_3,\cdots x_n)$ and $(y_1,y_2,y_3,\cdots,y_n)$ will be called orthogonal if and only if

$$\sum_{p=1}^{n} x_p y_p = 0.$$

If a set of mutually orthogonal vectors is such that any non-zero vector in the space can be expressed by a linear combination of the vectors in the set, the set of vectors will be called closed. If a set of mutually orthogonal vectors is such that no other non-zero vector in the space is orthogonal to each of them, the set will be called complete. If a set of mutually orthogonal vectors is such that if $(x_1,x_2,x_3,\cdots,x_n)$ is one of them then

$$\sum_{p=1}^{n} x_p^2 = 1,$$

then the set will be called orthonormal.

A vector in n space can be considered as n ordered pairs of numbers; the first numbers in the pairs being $1,2,3,\cdots,n$ and the second numbers being the first component, second component, third component, $\cdots$, nth component respectively. Similarly a function defined at each number in an interval $[a,b]$ may be considered as a set of ordered pairs of num--bers, the first number in each pair being a number in the interval $[a,b]$ and the second number being the value of the function for this number. In the case of vectors, two non-zero vectors were called orthogonal if and only if the sum of the products of corresponding second numbers was zero. If two functions, f(x) and g(x), are defined on an interval $[a,b]$, it is not possible to sum the products of the corresponding second numbers in general. To extend this idea of non-zero and orthogonality to functions, a function f(x) will be called a non-zero function on an interval $[a,b]$, if and only if

$$\int_a^b [f(x)]^2 \, dx \neq 0.$$

If f(x) and g(x) are non-zero functions defined on the interval $[a,b]$, then they will be called orthogonal if and only if

$$\int_a^b f(x)g(x) \, dx = 0.$$

If a certain group of non-zero functions, defined on an interval $[a,b]$, is considered, then any orthogonal set of these functions will be called closed if any one of the functions can be expressed by a linear combination of this set. Also a set of orthogonal functions will be called complete if no other function in the group is orthogonal to each of them. If a set of orthogonal functions, defined on an interval $[a,b]$, is such that if f(x) is one of them then

$$\int_a^b [f(x)]^2 dx = 1,$$

then the set will be called orthonormal.

CHAPTER II

DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

In the study it is necessary to determine the set of all numbers K for which the differential equation $Y'' - K^2 Y = 0$ has a solution satisfying the conditions $Y(a) = c$ and $Y(b) = d$ when a is less than b or the conditions $Y'(a) = c$ and $Y'(b) = d$. The general solution of the differential equation is

$$Y = Ae^{Kx} + Be^{-Kx}$$

where A and B are arbitrary constants (4, p. 170). The interest here is in those solutions only where Y is not identically zero. Hence, it will be required that either A is not zero or B is not zero.

In order to determine K, A, and B so that $Y(a) = c$ and $Y(b) = d$, the only consideration will be of the two equations

$$Ae^{Ka} + Be^{-Ka} = c \ ,$$

and

$$Ae^{Kb} + Be^{-Kb} = d$$

which are linear in A and B. In order to determine a value for K so that these equations will have a solution different from $A = 0$ and $B = 0$, the determinant solution of these equations for various values of c and d will be investigated.

<u>Case</u> <u>I</u>.--If $c = d = 0$, the determinant of the two equations,

$$e^{Ka}e^{-Kb} - e^{Kb}e^{-Ka} \ ,$$

must vanish (3, p. 106). The resulting equation is

$$e^{Ka}e^{-Kb}-e^{Kb}e^{-Ka}=0 ,$$

$$e^{K(a-b)}-e^{-K(a-b)}=0 ,$$

$$e^{-K(b-a)}-e^{K(b-a)}=0 .$$

Now let $K=P+iQ$ and the equation becomes

$$e^{-(P+iQ)(b-a}-e^{(P+iQ)(b-a)}=0,$$

$$e^{-P(b-a)}e^{-iQ(b-a)}-e^{P(b-a)}e^{iQ(b-a)}=0.$$

Since $e^{x+iy}=e^{x}(\cos y+i\sin y)$ (2, p. 326) and

$$e^{-x-iy}=e^{-x}(\cos y - i\sin y),$$

$e^{-P(b-a)}e^{-iQ(b-a)}-e^{P(b-a)}e^{iQ(b-a)}$ 0 can be written in the form $e^{-P(b-a)}[\cos(b-a)Q - i\sin(b-a)Q]$

$$- e^{P(b-a)}[\cos(b-a)Q \quad i\sin(b-a)Q]=0.$$

By factoring the left hand member of this equation the result is $\left[e^{-P(b-a)}-e^{P(b-a)}\right]\cos(b-a)Q$

$$-\left[e^{-P(b-a)}+e^{P(b-a)}\right]i\sin(b-a)Q=0.$$

Since the left hand member of the equation is equal to zero, the real and imaginary parts are equal to zero. By equating the real part to zero and the imaginary part to zero, the imaginary part becomes

$$\left[e^{-P(b-a)}+e^{P(b-a)}\right]\sin(b-a)Q=0.$$

Since both the exponential terms are positive, their sum is positive. Hence, it follows that

$$\sin(b-a)Q=0,$$

$$(b-a)Q=\pm n\pi,$$

$$Q=\pm\frac{n\pi}{(b-a)}, \qquad (n=1, 2, 3, \cdots).$$

When the real part is considered, the result is

$$\left[e^{-P(b-a)} - e^{P(b-a)}\right]\cos(b-a)Q = 0.$$

When the value of Q is substituted in this equation, it is found that $\cos(b-a)Q = \pm 1$. Hence

$$\left[e^{-P(b-a)} - e^{P(b-a)}\right](\pm 1) = 0,$$

$$e^{-P(b-a)} = e^{P(b-a)},$$

$$-P(b-a) = P(b-a),$$

$$-P = P,$$

$$P = 0.$$

Hence K must be $\pm \frac{n\pi i}{b-a}$ and the general equation is

$$A\left[\cos(\pm \frac{n\pi x}{b-a}) + i \sin(\pm \frac{n\pi x}{b-a})\right]$$

$$+ B\left[\cos(\pm \frac{n\pi x}{b-a}) - i \sin(\pm \frac{n\pi x}{b-a})\right] = y,$$

$$(A + B)\cos(\pm \frac{n\pi x}{b-a}) + (A - B)i \sin(\pm \frac{n\pi x}{b-a}) = y.$$

If $\frac{a}{b-a}$ is a whole number N,

$$\frac{a}{b-a} = N,$$

$$a = bN - aN,$$

$$a(1 + N) = bN,$$

$$a(1 + N) = b(\frac{a}{b-a}),$$

$$1 + N = \frac{b}{b-a},$$

and hence, $\frac{b}{b-a}$ is a whole number. Under these conditions, $A = -B$ and $y = 2iA\sin \pm \frac{n\pi x}{b-a}$. If $A_n = \pm 2iA$, then $y = A_n \sin \frac{n\pi x}{b-a}$, $n = 1, 2, 3, \cdots$.

Case II.--If $c = d$ and d does not equal zero, and either numerator, $c(e^{-Kb} - e^{-Ka})$ or $c(e^{Ka} - e^{Kb})$ in the determinant solution is zero, then $K = \pm \frac{2n\pi i}{b-a}$ and both are zero. Under these conditions, the denominators must be zero. Hence there is a solution for the equations considered if $K = \pm \frac{2n\pi i}{b-a}$. If $K = \pm \frac{(2n-1)\pi i}{b-a}$, the denominator is zero and the numerators are not zero. Hence, $K = \pm \frac{(2n-1)\pi i}{b-a}$ can not be used here. Under the afore conditions,

$$y = (A + B)\cos(\pm \frac{2n\pi x}{b-a}) + i(A - B)\sin(\pm \frac{2n\pi x}{b-a}).$$

If $\frac{a}{b-a}$ is an integer then $\frac{b}{b-a}$ is an integer. Hence, $A = -B$ and $y = A_n \sin(\frac{2n\pi x}{b-a})$, where $A_n = \pm 2iA$. $(n = 1, 2, 3, \cdots\cdot)$.

If K is not $\pm \frac{2n\pi i}{b-a}$, then the numerators of the determinate solution of the equations are not zero and the denominators must be different from zero. Hence, if K is different from $\pm \frac{n\pi i}{b-a}$, there is a unique solution for A and B.

Case III.--If $c = -d$ and d is not zero, then each numerator is zero, if and only if, $K = \pm \frac{(2n-1)\pi i}{b-a}$. Since the denominator is zero for these values of K, there is a solution. Under these conditions,

$$y = (A+B)\cos(\pm \frac{(2n-1)\pi x}{b-a}) + i(A-B)\sin(\pm \frac{(2n-1)\pi x}{b-a}).$$

If $\frac{a}{b-a}$ is an integer, then $A = -B$ and $y = A_n \sin(\frac{(2n-1)\ x}{b-a})$, where $A_n = \pm 2iA$. Since the numerators are not zero but the denominators are zero when $K = \pm \frac{2n\pi i}{b-a}$, these values of K can not be used. If K is not $\pm \frac{n\pi i}{b-a}$, the numerators and the denominators are not zero and there is a unique solution for A and B.

Case IV.--If the absolute value of c is not equal to the absolute value of d, then either c is not zero or d is not zero and both numerators of the determinant solution can not vanish. Hence, for any K not equal to $\pm\frac{n\pi i}{b-a}$ there is a unique solution for A and B, but for $K=\pm\frac{n\pi i}{b-a}$ there is no solution for A and B.

In order to determine values for K, A, and B so that the differential equation $y'' - K^2y = 0$ will have a solution,

$$y = Ae^{Kx} + Be^{-Kx},$$

which satisfies the condition $y'(a) = c$ and $y'(b) = d$ for a less than b, the two equations

$$KAe^{Ka} - KBe^{-Ka} = c$$

and

$$KAe^{Kb} - KBe^{-Kb} = d,$$

will be determined. The determinants used in solving these linear equations in A and B are

$$-K(ce^{-Kb} - de^{-Ka}),$$

$$-K(de^{-Ka} - ce^{-Kb}),$$

and

$$-K^2\left[e^{K(b-a)} - e^{-K(b-a)}\right].$$

If K is not equal to zero, these determinants vanish, if and only if, the corresponding determinants considered for the conditions $y(a) = c$ and $y(b) = d$ vanish.

Case I.--If $c = d = 0$ and $K = \pm\frac{n\pi i}{b-a}$,

$$y = (A+B)\cos\left(\pm\frac{n\pi x}{b-a}\right) + i(A-B)\sin\left(\pm\frac{n\pi x}{b-a}\right).$$

If $\frac{a}{b-a}$ is an integer, then $A = B$ and $y = A_n \cos\frac{n\pi x}{b-a}$, where $A_n = A + B$.

Case II.--If $c = d$ and d does not equal zero, then

$$K = \pm \frac{2n\pi i}{b-a},$$

and

$$y = (A+B)\cos(\pm \frac{2n\pi x}{b-a}) + i(A-B)\sin(\pm \frac{2n\pi x}{b-a}),$$

or K is not $\pm \frac{n\pi i}{b-a}$ and there is a unique solution for A and B. If $K = \pm \frac{2n\pi i}{b-a}$ and $\frac{a}{b-a}$ is an integer, then $A = B$ and

$$y = A_n \cos(\frac{2n\pi x}{b-a})$$

where $A_n = A + B$.

Case III.--If $c = -d$ and d is not zero, then

$$K = \pm \frac{(2n-1)\pi i}{b-a},$$

and

$$y = (A+B)\cos(\pm \frac{(2n-1)\pi x}{b-a}) + i(A-B)\sin(\pm \frac{(2n-1)\pi x}{b-a}),$$

or K is not $\pm \frac{n\pi i}{b-a}$ and there is a unique solution for A and B. If $K = \pm \frac{(2n-1)\pi i}{b-a}$ and $\frac{a}{b-a}$ is an integer, then $A = B$ and

$$y = A_n \cos\frac{(2n-1)\pi x}{b-a}$$

where $A_n = A + B$.

Case IV.--If the absolute value of c is not equal to the absolute value of d, then either c is not zero or d is not zero and both the numerators of the determinant solution can not vanish. Hence for $K = \pm \frac{n\pi i}{b-a}$ there is no solution, but for any K not equal to $\pm \frac{n\pi i}{b-a}$ there is a unique solution for A and B.

CHAPTER III

ORTHOGONALITY UNDER CERTAIN BOUNDARY CONDITIONS

Let Y_1 be a solution of the differential equation

$$Y_1'' - K_1^2 Y_1 = 0$$

and let Y_2 be a solution of the differential equation

$$Y_2'' - K_2^2 Y_2 = 0$$

on the interval from a to b. If the first equation is multiplied by the solution of the second and the second is multiplied by the solution of the first, the resulting equations are

$$Y_1'' Y_2 - K_1^2 Y_1 Y_2 = 0,$$

and

$$Y_2'' Y_1 - K_2^2 Y_1 Y_2 = 0.$$

By subtracting the last equation from the preceeding one the resulting equation is

$$Y_1'' Y_2 - Y_2'' Y_1 + (K_2^2 - K_1^2) Y_1 Y_2 = 0.$$

By integrating this equation over the interval from a to b the result is

$$\int_a^b (Y_1'' Y_2 - Y_2'' Y_1)dx + (K_2 - K_1)\int_a^b Y_1 Y_2 dx = 0,$$

$$\int_a^b \frac{d}{dx}(Y_1' Y_2 - Y_2' Y_1)dx + (K_2^2 - K_1^2)\int_a^b Y_1 Y_2 \, dx = 0,$$

$$(Y_1Y_2 - Y_2Y_1)_a^b + (K_2^2 - K_1^2)\int_a^b Y_1Y_2 \, dx = 0.$$

If K_2^2 is not equal to K_1^2 and $(Y_1'Y_2 - Y_2'Y_1)_a^b = 0$, then

$$\int_a^b Y_1Y_2 \, dx = 0.$$

This by definition makes Y_1 and Y_2 orthogonal on the interval (a,b).

If $(Y_1'Y_2 - Y_2'Y_1)_a^b$ is considered with either the set of boundary conditions

$$Y_1(a) = Y_1(b) = 0$$
$$Y_2(a) = Y_2(b) = 0$$

or

$$Y_1'(a) = Y_1'(b) = 0$$
$$Y_2'(a) = Y_2'(b) = 0$$

the Y_1 and Y_2 are orthogonal on the interval (a,b).

If the first set of boundary conditions are satisfied, the solutions for the equation $Y'' + (\pm \frac{n\pi i}{b-a})^2 Y = 0$ are

$$Y_n = A_n\left[(1 - e^{\frac{2an\pi i}{b-a}})\cos\frac{n\pi x}{b-a} + i(1 + e^{\frac{2an\pi i}{b-a}})\sin\frac{n\pi x}{b-a}\right], \quad n = 1, 2, 3, \cdots .$$

These functions are orthogonal on the interval (a,b). Since each Y_n is a continuous function not identically zero

$$\int_a^b Y_n^2 \, dx \neq 0$$

if A_n does not equal zero. Hence if A_n is equal to one divided by plus or minus the square root of the integral over the interval from a to b of

$$\left[(1 - e^{\frac{2an\pi i}{b-a}})\cos\frac{n\pi x}{b-a} + i(1+ e^{\frac{2an\pi i}{b-a}})\sin\frac{n\pi x}{b-a}\right]^2$$

then

$$\int_a^b Y_n^2 \, dx = 1$$

and the set of functions Y_n, $n = 1, 2, 3, \cdots$, is orthonormal on the interval (a,b). If in addition, $\frac{a}{b-a}$ is an integer,

$$Y_n = \pm \sqrt{\frac{2}{b-a}} \sin\frac{n\pi x}{b-a} \text{ .}$$

Only the positive square root will be considered.

If the second set of boundary conditions hold, the set of functions Y_n is orthogonal on the interval (a,b) where $n = 1, 2, 3, \cdots$. Since each of these functions is continuous and not identically equal to zero if A_n does not equal zero, then

$$\int_a^b Y_n^2 \, dx \neq 0.$$

In particular, if A_n is equal to one divided by plus or minus the square root of the integral over the interval from a to b of

$$\left[(1 - e^{\frac{2an\pi i}{b-a}})\cos\frac{n\pi x}{b-a} + i(1 + e^{\frac{2an\pi i}{b-a}})\sin\frac{n\pi x}{b-a}\right]^2$$

then

$$\int_a^b Y_n^2 \, dx = 1$$

and the functions are orthonormal. If $\frac{a}{b-a}$ is an integer,

$$Y_n = \pm \sqrt{\frac{2}{b-a}} \cos\frac{n\pi x}{b-a} \text{ .}$$

Only the positive square root will be considered.

CHAPTER IV

EXPRESSING FUNCTIONS AS A SERIES OF ORTHOGONAL FUNCTIONS

Now a special case of the equation $Y'' - K^2Y = 0$ will be considered where $a = 0$, $b = L > 0$, and $K = \pm\frac{n\pi i}{b-a}$, $n = 1, 2, 3, \cdots$. If $Y_n(a) = Y_n(b) = 0$, then the solution will be of the form $B_n \sin\frac{n\pi x}{L}$. If $Y_n'(a) = Y_n'(b) = 0$, the solution will be of the form $A_n \cos\frac{n\pi x}{L}$.

If there exist constants $B_1, B_2, B_3, \cdots$ such that

$$f(x) = \sum_{1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

on the interval $(0,L)$ and such that the series

$$\sum_{1}^{\infty} \sin\frac{k\pi x}{L} \quad \sin\frac{n\pi x}{L}$$

can be integrated term by term to obtain

$$\int_0^L \sin\frac{k\pi x}{L} f(x)dx,$$

$$B_k = \frac{2}{L}\int_0^L f(x) \sin\frac{k\pi x}{L} dx, \ k = 1, 2, 3, \cdots.$$

This follows from the fact that

$$\int_0^L \sin\frac{k\pi x}{L} \sin\frac{n\pi x}{L} dx = 0$$

if $k \neq n$ and $\int_0^L \left[\sin\frac{k\pi x}{L}\right]^2 dx = \frac{L}{2}$.

In a similar manner, if there exist constants A_1, A_2, A_3, ... such that $f(x) = \sum_1^{\infty} A_n \cos\frac{n\pi x}{L}$ on the interval (0,L) and such that the series $\sum_1^{\infty} \cos\frac{k\pi x}{L} \cos\frac{n\pi x}{L}$ can be integrated term by term to obtain $\int_0^L \cos\frac{k\pi x}{L} f(x)\,dx$, then

$$A_k = \frac{2}{L}\int_0^L f(x) \cos\frac{k\pi x}{L}\,dx, \quad k = 1, 2, 3, \cdots.$$

This follows from the fact that

$$\int_0^L \cos\frac{k\pi x}{L} \cos\frac{n\pi x}{L}\,dx = 0$$

if $k \neq n$ and

$$\int_0^L \left[\cos\frac{k\pi x}{L}\right]^2 dx = \frac{L}{2}.$$

When a curve is symmetrical with respect to the Y-axis, that is $f(x) = f(-x)$, the function is said to be an even function and $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$. For example 1, x^2, x^4, $\cos(nx)$, and $x \sin(nx)$ are even functions.

When a curve is symmetrical with respect to the origin, that is if $f(-x) = -f(x)$, the function is said to be an odd function. x, x^3, $\frac{1}{x}$, and $x^2 \sin(nx)$ are odd functions. Although all functions are not necessarily even or odd, every function can be expressed as the sum of an even and an odd function by the identity

$$f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right].$$

When $f(x)$ is an odd function defined in the interval $(-L,L)$, $A_n = 0$ and $B_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x)dx$. Hence $f(x)$

corresponds to $\frac{2}{L}\sum_{1}^{\infty}\sin(\frac{n\pi x}{L})\int_{0}^{L}f(z)\sin(\frac{n\pi z}{L})dz$ where the variable of integration is z. This series can be expressed when f(x) is any function defined in the interval (0,L) provided there exist integrals which represent its coefficients. When f(x) is defined only on the interval (0,L), an odd function exists in the interval (-L,L) which is identical to f(x) in the interval (0,L). If that odd function is represented then so is f(x).

When f(x) is an even function defined in the interval (-L,L),

$$A_n = \frac{2}{L}\int_{0}^{L}f(x)\cos(nx)dx \qquad (n=0,\ 1,\ 2,\ \cdots)$$

and

$$B_n = 0 \qquad (n=1,\ 2,\ 3,\ \cdots).$$

Then f(x) corresponds to

$$\frac{1}{L}\int_{0}^{L}f(z)dz+\frac{2}{L}\sum_{1}^{\infty}\cos(nx)dx\int_{0}^{L}f(z)\cos(nz)dz.$$

Since every term of the sine series is an odd function, it must converge to an odd function with the period 2L for all values of x if it converges in the interval (0,L). Also if the cosine series converges, it must represent an even periodic function with the period 2L. If f(x) is such that $f(x+2L)=f(x)$ and f(x) is sectionally continuous on the interval (-L, L) then the series converge at each point where f(x) has right and left hand derivatives. Furthermore the series converge to $\frac{1}{2}\left[f(x+)+f(x-)\right]$ at these points. f(x-) and f(x+) simply mean the limits of f(z) as z approaches x from the left and from the right respectively. Less restrictive conditions on $f(x)$ may be used but they will not be discussed here. (Churchill, p. 70)

It is a simple matter to transpose the axis in such a manner that the interval (a,b) will correspond to an interval $(0,L)$ by letting $u = x-a$ and $L = b-a$ and considering the equation

$$Y_n = Ae^{\frac{n\pi ix}{b-a}} + Be^{\frac{-n\pi x}{b-a}} .$$

If

$$Y_n(a) = Y_n(b) = 0,$$

then

$$Ae^{\frac{n\pi ia}{b-a}} + Be^{\frac{-n\pi ia}{b-a}} = 0 \text{ and } B = -Ae^{\frac{2n\pi ia}{b-a}}.$$

By substituting this value for B

$$Y_n = Ae^{\frac{n\pi ix}{b-a}} - Ae^{\frac{2n\pi ia}{b-a}} e^{\frac{n\pi ia}{b-a}},$$

$$Y_n = A(e^{\frac{n\pi ix}{b-a}} - e^{\frac{2n\pi ia}{b-a}} e^{\frac{-n\pi ia}{b-a}}).$$

When $x = u+a$ is substituted the equation becomes

$$Y_n = A(e^{\frac{n\pi iu}{b-a}} e^{\frac{n\pi ia}{b-a}} - e^{\frac{2n\pi ia}{b-a}} e^{\frac{-n\pi iu}{b-a}} e^{\frac{-n\pi ia}{b-a}}),$$

$$Y_n = A(e^{\frac{n\pi iu}{b-a}} e^{\frac{n\pi ia}{b-a}} - e^{\frac{n\pi ia}{b-a}} e^{\frac{-n\pi iu}{b-a}}),$$

$$Y_n = Ae^{\frac{n\pi ia}{b-a}}(e^{\frac{n\pi iu}{b-a}} - e^{\frac{-n\pi iu}{b-a}}).$$

Let $W_n = Ae^{\frac{n\pi ia}{b-a}}$. Then

$$Y_n = W_n(e^{\frac{n\pi iu}{b-a}} - e^{\frac{-n\pi iu}{b-a}}),$$

$$Y_n = W_n(\cos\frac{n\pi u}{b-a} + i \sin\frac{n\pi u}{b-a} - \cos\frac{n\pi u}{b-a} + i \sin\frac{n\pi u}{b-a}),$$

and $$Y_n = 2W_n i \ \sin\frac{n\pi u}{b-a}.$$

By letting $A_n = 2W_n i$ and substituting L for b-a, the function becomes

$$A_n \ \sin\frac{n\pi u}{L}.$$

If $y'(a) = y'(b) = 0$, then $A\frac{n\pi i}{b-a}e^{\frac{n\pi ix}{b-a}} - B\frac{n\pi i}{b-a}e^{\frac{-n\pi ix}{b-a}} = 0$, and $B = Ae^{\frac{2n\pi ia}{b-a}}$. By substituting this value for B,

$$Y = A(e^{\frac{n\pi ix}{b-a}} + e^{\frac{2n\pi ia}{b-a}}e^{\frac{-n\pi ix}{b-a}}) \ .$$

When $x = u + a$ is substituted, the equation becomes,

$$Y = Ae^{\frac{n\pi ia}{b-a}}(e^{\frac{n\pi iu}{b-a}} + e^{\frac{-n\pi iu}{b-a}}).$$

Let $W_n = Ae^{\frac{n\pi ia}{b-a}}$. Then $Y_n = W_n(e^{\frac{n\pi iu}{b-a}} + e^{\frac{-n\pi iu}{b-a}})$.

$$Y_n = W_n(\cos\frac{n\pi u}{b-a} + i \ \sin\frac{n\pi u}{b-a} + \cos\frac{n\pi u}{b-a} - i \ \sin\frac{n\pi u}{b-a}).$$

$$Y_n = 2W_n(\cos\frac{n\pi u}{b-a}).$$

By letting $B_n = 2W_n$ and substituting L for b-a, the function becomes

$$Y_n = B_n\cos\frac{n\pi u}{L}.$$

The set of non-trivial solutions of the differential equation $Y'' - (\frac{n\pi x}{b-a})^2 Y = 0$, $n = 1, 2, 3, \cdots$, with the boundary condition $Y(a) = Y(b) = 0$ or $Y'(a) = Y'(b) = 0$ is an orthogonal set of functions on the interval (a,b). In particular, if $a = 0$ and $b = L$, these functions are $Y_n = B_n \text{sine}\frac{n\pi x}{L}$ for $Y(0) = Y(L) = 0$ and $Y_n = A_n\cos\frac{n\pi x}{L}$ for $Y'(0) = Y'(L) = 0$. If f(x) is sectionally continuous on the interval (0,L), then there exist constants

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx,\ n = 1,\ 2,\ 3,\ \cdots,$$

such that $\sum_1^\infty B \sin\frac{n\pi x}{L}$ converges to $\frac{1}{2}\left[f(x+) + f(x-)\right]$ at every x such that f(x) has right and left hand derivatives. If $\sum_1^\infty B_n \sin\frac{n\pi x}{L}$ converges for all x in the interval (0,L), then it converges for all x and its sum is an odd periodic function of period 2L.

Under the same conditions on f(x) there exist constants

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L},\ n = 0,\ 1,\ 2,\ \cdots,$$

such that $\sum_1^\infty A_n \cos\frac{n\pi x}{L}$ converges to $\frac{1}{2}\left[f(x) + f(x-)\right]$ at each x where f(x) has right and left hand derivatives. If $\sum_1^\infty A_n \cos\frac{n\pi x}{L}$ converges for all x of (0,L) then it converges for all x and its sum is an even periodic function of period 2L. If f(x) is sectionally continuous on (-L,L), then it is the sum of an even and an odd function. There exist constants A_n and B_n such that $A_0 + \sum_1^\infty \left[A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right]$ converges to $\frac{1}{2}\left[f(x+) + f(x-)\right]$ at every x of (-L,L) such that f(x) has a right and left hand derivative.

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