A DEVELOPMENT OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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A DEVELOPMENT OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

THESIS

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CHAPTER I

INTRODUCTION

Exponential and logarithmic functions are first encountered by a student during high school algebra and trigonometry. At this time the student is given a group of rules which have very little meaning to him. However, he is taught to use the rules and to perform certain operations and calculations by them. As the student advances into college mathematics, he will develop a background which is necessary to understand why the rules are true. There are two questions that the student will usually ask when he enters calculus. First, is there a function \( f(x) \) such that it is differentiable and \( f'(x) = f(x) \)? Second, is there a function \( g(y) \) such that it is differentiable and \( g'(y) = 1/y \)? In this study, we develop these functions and their properties. In the past, these functions have been developed by defining \( L(y) = \int_{1}^{y} 1/x \, dx \) and \( E(x) \) the inverse of \( L(y) \). In this study, we develop \( E(x) \) and its properties from the differential equation \( y'(x) = y(x) \) where \( y(a) = b \). Then we define \( L(y) \) to be the inverse of \( E(x) \) and develop the properties of \( L(y) \).

In this study, we will assume known the definitions of continuity, differentiability, integration, limits, and
monotone functions. By $f_n(x) \rightarrow f(x)$, we will mean $f_n(x)$ converges uniformly to $f(x)$. By $f_n(x) \rightarrow f(x)$, we will mean $f_n(x)$ converges to $f(x)$. When we use number or numbers in this study, we will mean real number or real numbers unless we specify differently.

We shall now state certain theorems without proof which will be referred to in the following chapter.

**Theorem 1.1:** If $(A,B)$ is any Dedekind cut in the reals, then there exists a number $c$ such that every $x > c$ is in $B$ while every $x < c$ is in $A$. \[4, \text{p. 89}\]

**Theorem 1.2:** If $f(x)$ is differentiable at $x = c$, then $f(x)$ is continuous at $x = c$. \[2, \text{p. 97}\]

**Theorem 1.3:** If $f(x)$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then there exists a point $c$ of $(a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. \[2, \text{p. 103}\]

**Theorem 1.4:** Every monotone function $f(x)$ on $[a,b]$ is bounded on $[a,b]$. \[1, \text{p. 279}\]

**Theorem 1.5:** If $f(x)$ is continuous on $[a,b]$, then $f(x)$ is integrable on $[a,b]$. \[5, \text{p. 173}\]

---

1Numbers appearing in brackets with the letter p in this chapter will refer to a book in the bibliography and the corresponding page number of each.
Theorem 1.6: If \( f(x) \) is continuous on \([a,b]\), then at every point in \([a,b]\), \( F(x) = \int_{a}^{x} f(v) \, dv \) possesses a derivative and \( F'(x) = f(x) \). [5, p. 180]

Theorem 1.7: If \( f(x) \) and \( g(x) \) are both differentiable, then

1. \( F(x) = f(x) + g(x) \) is differentiable and \( F'(x) = f'(x) + g'(x) \);
2. \( F(x) = f(x) - g(x) \) is differentiable and \( F'(x) = f'(x) - g'(x) \);
3. \( F(x) = \frac{f(x)}{g(x)} \) is differentiable, provided \( g(x) \neq 0 \), and \( F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \);
4. \( F(x) = f(x) \cdot g(x) \) is differentiable and \( F'(x) = f(x)g'(x) + g(x)f'(x) \). [2, p. 137]

Theorem 1.8: If \( f(x) \) and \( g(x) \) are both integrable on \([a,b]\) and \( f(x) \geq g(x) \) at every point of \([a,b]\), then
\[
\int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx. \quad [5, \text{p. 177}]
\]

Theorem 1.9: If \( \int_{a}^{b} f(x) \, dx \) exists and \( \int_{a}^{b} g(x) \, dx \) exists, then
\[
\int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx = \int_{a}^{b} [f(x) + g(x)] \, dx. \quad [2, \text{p. 141}]
\]
Theorem 1.10: If $f(x)$ is bounded and integrable on $[a,b]$ and $k$ is a constant, then

$$\int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx. \quad [5, \text{p. 176}]$$

Theorem 1.11: If $u(x)$ and $v(x)$ have continuous first derivatives on $[a,b]$, then

$$\int_a^b u \frac{dv}{dx} \, dx = [uv]_a^b - \int_a^b v \frac{du}{dx} \, dx. \quad [3, \text{p. 491}]$$

Theorem 1.12: If $f(x)$ is continuous, $t = \phi(u)$ is monotone, $a = \phi(c)$, $b = \phi(d)$, and $\frac{dt}{du}$ is continuous, then

$$\int_a^b f(t) \, dt = \int_c^d f(\phi(u)) \frac{dt}{du} \, du. \quad [3, \text{p. 507}]$$

Theorem 1.13: If $f(x)$ is a continuous function on $[a,b]$, then

$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx. \quad [2, \text{p. 81}]$$

Theorem 1.14: If $f(x)$ is integrable on $[a,b]$, then so is $|f(x)|$, and

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx. \quad [5, \text{p. 178}]$$

Theorem 1.15: If $\phi(x)$ be a function which at every point of $[a,b]$ has a differential coefficient, which is a continuous function $f(x)$, then

$$\phi(x) - \phi(a) = \int_a^x f(x) \, dx. \quad [3, \text{p. 484}]$$
Theorem 1.16: If \( f_n(x) \), \( n = 1, 2, 3, \ldots \), is a sequence of functions such that \( f_n'(x) \) exists and \( \{ f_n'(x) \} \rightarrow \phi(x) \) on \([a, b]\) and \( f_n'(x) \) is continuous on \([a, b]\), then \( \phi(x) \) is continuous on \([a, b]\).

Let \( \epsilon \) be greater than zero. Since \( \{ f_n'(x) \} \rightarrow \phi(x) \), then there exists an \( N \) such that for every \( n \geq N \) and for every \( x \) in \([a, b]\),

\[
|f_n'(x) - \phi(x)| < \frac{\epsilon}{3}.
\]

Since \( f_n'(x) \) is continuous on \([a, b]\), then there exists a \( \delta > 0 \) such that for every \( |x_1 - x_2| < \delta \),

\[
|f_n'(x_1) - f_n'(x_2)| < \frac{\epsilon}{3}. \quad [5, \text{p. 73}]
\]

It follows that

\[
|\phi(x_1) - \phi(x_2)| = |\phi(x_1) - f_n'(x_1) + f_n'(x_1) - f_n'(x_2) + f_n'(x_2) - \phi(x_2)|
\leq |\phi(x_1) - f_n'(x_1)| + |f_n'(x_1) - f_n'(x_2)| + |f_n'(x_2) - \phi(x_2)|
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

Thus \( \phi(x) \) is a continuous function.

Theorem 1.17: If \( f_n(x) \), \( n = 1, 2, 3, \ldots \), is a sequence of functions such that \( f_n'(x) \) exists and \( \{ f_n'(x) \} \rightarrow \phi(x) \) on \([a, b]\), and if \( f_n'(x) \) is continuous on \([a, b]\) and \( a < y < b \),
\[
\int_a^y \phi(x) \, dx = \lim_{n \to \infty} \int_a^y f_n'(x) \, dx.
\]

Let \( \epsilon \) be greater than zero. There exists an \( N \) such that if \( n > N \), then
\[
\left| \phi(x) - f_n'(x) \right| \leq \frac{\epsilon}{b-a}.
\]

Also
\[
\left| \int_a^y \phi(x) \, dx - \int_a^y f_n'(x) \, dx \right| = \left| \int_a^y [\phi(x) - f_n'(x)] \, dx \right| \\
\leq \frac{\epsilon}{b-a} \cdot \frac{(y-a)}{b-a} \leq \epsilon.
\]

Thus
\[
\int_a^y \phi(x) \, dx = \lim_{n \to \infty} \int_a^y f_n'(x) \, dx.
\]

**Theorem 1.18:** If \( f_n(x) \), \( n = 1, 2, 3, \ldots \), is a sequence of functions such that \( f_n'(x) \) exists, \( \{ f_n(x) \} \to F(x) \) on \([a, b]\), \( \{ f_n'(x) \} \to \phi(x) \) on \([a, b]\), \( f_n'(x) \) is continuous on \([a, b]\), and \( a < x < b \), then \( F'(x) \) exists and \( F'(x) = \phi(x) \).

Let \( G(x) = \int_a^x \phi(v) \, dv = \lim_{n \to \infty} \int_a^x f_n'(v) \, dv \). Since \( \phi(x) \) is continuous, then \( G'(x) \) exists and is equal to \( \phi(x) \). Also
\[
G(x) = \lim_{n \to \infty} \int_a^x f_n'(v) \, dv \\
= \lim_{n \to \infty} \left[ f_n(x) - f_n(a) \right] \\
= F(x) - F(a).
\]

Therefore \( G'(x) = F'(x) = 0 = F'(x) \). Hence \( F'(x) = \phi(x) \).
CHAPTER II

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Theorem 2.1: If $\varphi'(x) = \varphi(x)$ on an interval $[P,Q]$ and $P < a < Q$ and $\varphi(a) = 0$, then $\varphi(x) \equiv 0$ on $[P,Q]$.

Suppose that $\varphi(x) \neq 0$ on $[P,Q]$. Let $c$ be a point of $[P,Q]$ where $\varphi(c) > 0$ and suppose $c > a$. Now consider the interval $(a,c)$. $\varphi(c) > 0$, $\varphi(a) = 0$, and $\varphi'(x) = \varphi(x)$ on $(a,c)$. Thus $\varphi(x)$ is continuous on $[a,c]$. Let B contain all $x \geq c$ and all $x < c$ if $\varphi(x) > 0$ on $[x,c]$; otherwise let $x_0$ belong to A. Evidently for every $x_0$, A contains $x_0$ or B contains $x_0$ and no $x_0$ is in A and B. A contains a and B contains c; so neither is empty. Suppose that B contains $x_1$ and $x_1 < x_2$. If $x_2 \geq c$, then B contains $x_2$. If $x_2 < c$, then $x_1 < c$ and $\varphi(x) > 0$ on $[x_1,c]$. Hence B contains $x_2$. It follows that every $x$ in A < any $x$ in B. Thus $(A,B)$ is a Dedekind cut, and there exists a $\xi$ such that for every $x < \xi$ A contains x while for every $x > \xi$ B contains x. Evidently $[a,c]$ contains $\xi$. Either $\varphi(\xi) > 0$, or $\varphi(\xi) = 0$, or $\varphi(\xi) < 0$.

Suppose $\varphi(\xi) > 0$. Choose $\xi = \varphi(\xi)$. There exists a $\delta > 0$ so that $\delta \leq a - P$, and if $|x - \xi| < \delta$ and $x$ is in the domains of definition of $\varphi(x)$, then

$$|\varphi(x) - \varphi(\xi)| < \varphi(\xi);$$
\[-\varphi(\xi) < \varphi(x) - \varphi(\xi) < \varphi(\xi) ;
\]
\[\varphi(\xi) - \varphi(\xi) < \varphi(x) ;
\]
\[0 < \varphi(x) .
\]

Thus for every \( x \) on \( (\xi - c, c) \), \( \varphi(x) > 0 \). Hence there exists an interval, namely \( [\xi - c/2, c] \), where \( \varphi(x) > 0 \) for every \( x \) on \( [\xi - c/2, c] \), which is a contradiction of our Dedekind cut. Thus \( \varphi(\xi) \neq 0 \).

Suppose \( \varphi(\xi) < 0 \). Choose \( \epsilon = -\varphi(\xi) \). There exists a \( c > 0 \) so that \( c \leq c - \xi, c \leq \xi - P, \) and if \( |x - \xi| < c \)
and \( x \) is in the domain of definition of \( \varphi(x) \), then

\[|\varphi(x) - \varphi(\xi)| < -\varphi(\xi) ;
\]
\[\varphi(\xi) < \varphi(x) - \varphi(\xi) < -\varphi(\xi) ;
\]
\[\varphi(x) < -\varphi(\xi) + \varphi(\xi) ;
\]
\[\varphi(x) < 0 .
\]

Thus for every \( x \) on \( (\xi - c, \xi + c) \), \( \varphi(x) < 0 \). Hence there exists an interval, namely \( [\xi + c/2, c] \), where \( \varphi(x) < 0 \) for one \( x \) on \( [\xi + c/2, c] \), which is a contradiction of our Dedekind cut. Thus \( \varphi(\xi) \neq 0 \). Since \( \varphi(\xi) \neq 0 \) or \( \varphi(\xi) \neq 0 \), then it follows that \( \varphi(\xi) = 0 \).

Either \( \xi \) is the largest in \( A \) or \( \xi \) is the smallest in \( B \). Suppose \( \xi \) is the smallest in \( B \). If this be the case, then \( B \) contains one point of \( [a, c] \) where \( \varphi(x) = 0 \) since \( \varphi(\xi) = 0 \). But \( B \) contains all \( x \) \( \geq c \) and all \( x \) \( < c \) where \( \varphi(x) > 0 \) on \( [x, c] \). Thus \( \xi \) is not the smallest in \( B \). Then
it follows that $\xi$ is the largest in $A$ and $\phi(x) > 0$ on $(\xi, c)$. 
Since $\phi(x)$ is continuous on $[a, c]$ and $\xi$ is an interior point of $(a, c)$, then it follows that $\phi(x)$ is continuous on $(\xi, c)$. 
Also $\phi'(x) = \phi(x)$ on $(\xi, c)$. By Theorem 1.3, there exists a $p$ in $(\xi, c)$ such that $\phi'(p) = \frac{\phi(c) - \phi(\xi)}{c - \xi} = \frac{\phi(c)}{c - \xi} = \phi(p)$. If $c - \xi \geq 1$, then $\phi(p) = \phi(c)$. If $0 < c - \xi < 1$, then $\phi(p) > \phi(c)$. Let $d = \min[\xi + 1/2, c]$. By Theorem 1.3, there exists a $q$ in $(\xi, d)$ such that $\phi'(q) = \frac{\phi(d) - \phi(\xi)}{d - \xi} = \frac{\phi(d)}{d - \xi} = \phi(q)$. But $d - \xi \leq 1/2$. Thus $\phi(d) < \phi(q)$ and $\phi'(d) < \phi'(q)$. By the Theorem 1.3, there exists a $h$ in $(q, d)$ such that $\phi'(h) = \frac{\phi(d) - \phi(q)}{d - q}$. But $\phi(d) - \phi(q)$ is negative since $\phi(d) < \phi(q)$ and $d - q$ is positive. Thus $\phi'(h)$ is negative and $\phi(h)$ is negative. This is a contradiction that $\phi(x) > 0$ on $(\xi, c)$ since $h$ is in $(\xi, c)$. Therefore $\phi(x) \equiv 0$.

A similar proof will hold if $c < a$ or $\phi(c) < 0$.

**Corollary 2.1.1:** If $f'(x) = f(x)$ on $[P, Q]$, $g'(x) = g(x)$ on $[P, Q]$, $P < a < Q$, and $f(a) = g(a)$; then $f(x) \equiv g(x)$ on $[P, Q]$.

Let $\phi(x) = f(x) - g(x)$, then $\phi(a) = f(a) - g(a) = 0$. 
$\phi'(x) = f'(x) - g'(x) = \phi(x)$. By Theorem 2.1, $\phi(x) \equiv 0$. 
Thus $\phi(x) = f(x) - g(x) = 0$. Hence $f(x) \equiv g(x)$.

**Theorem 2.2:** If $\phi'(x) > 0$ on the interval $(P, Q)$, then $\phi(x)$ is monotone increasing on $(P, Q)$. 
Let $\phi(x)$ be a function where $\phi'(x)$ exists on $(P, Q)$, and such that for every $x$ in $(P, Q)$, $\phi'(x) > 0$. Let $x_1$ and $x_2$ be two points in $(P, Q)$ where $x_2 > x_1$. By Theorem 1.3, there exists a $\xi$ in $(P, Q)$ such that $\phi'(\xi) = \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}$. Since $\xi$ is in $(P, Q)$, $\phi'(\xi) > 0$; since $x_2 > x_1$, $x_2 - x_1 > 0$. Then $\phi(x_2) - \phi(x_1) = \phi'(\xi)(x_2 - x_1) > 0$. Thus $\phi(x_2) > \phi(x_1)$.

Since $x_1$ and $x_2$ were arbitrary points of $(P, Q)$, the theorem follows for all points of $(P, Q)$.

**Theorem 2.3:** If $\phi'(x) < 0$ on the interval $(P, Q)$, then $\phi(x)$ is monotone decreasing on $(P, Q)$.

A proof similar to the previous proof will yield $\phi(x)$ monotone decreasing on $(P, Q)$. Since $\phi'(x) < 0$, the results will be $\phi(x_2) - \phi(x_1) < 0$ and $\phi(x_2) < \phi(x_1)$. Thus the theorem will follow.

**Theorem 2.4:** If $F_n(x) = 1 + \sum_{v=1}^{n} \frac{1}{v!} x^v$, $n = 1, 2, 3, \ldots$, and $p$ is a number, then there exists an interval $I$ containing $p$ in its interior and a function $F(x)$ such that $F_n(x)$ converges uniformly to $F(x)$.

If $p$ is a number, let $A$ be a positive integer which is greater than $|p|$. Then $p$ is in $(-A, A)$. Let $\epsilon$ be greater than zero. There exists a positive number $Q$ such that \(\left(\frac{1}{2}\right)^Q < \frac{\epsilon}{A^{2n}}\). Let $N = Q + 2A$. If $n > m > N$ and $x$ is in $(-A, A)$, then
\[ |F_n(x) - F_m(x)| = \left| \sum_{v=1}^{n} \frac{1}{v!} x^v - \sum_{v=1}^{m} \frac{1}{v!} x^v \right| \]
\[ = \left| \sum_{v=m+1}^{n} \frac{1}{v!} x^v \right| \]
\[ \leq \sum_{v=m+1}^{n} \frac{1}{v!} A^v \]
\[ = \frac{A^{m+1}}{(m+1)!} + \frac{A^{m+2}}{(m+2)!} + \ldots + \frac{A^n}{n!} \]
\[ = \frac{A^{m+1}}{(m+1)!} \left[ 1 + \frac{A}{(m+2)} + \frac{A^2}{(m+2)(m+3)} + \ldots + \frac{A^{n-m-1}}{(m+2)(m+3)\ldots(n)} \right] \]
\[ \leq \frac{A^{m+1}}{(m+1)!} \left[ 1 + \frac{A}{2} + \frac{1}{4} + \ldots \right] \]
\[ = \frac{2A^{m+1}}{(m+1)!} \]
\[ = 2 \left[ \frac{\frac{1}{2}}{A \frac{1}{2}} \right] \left[ \frac{1}{2} \frac{A}{2} \right] \]
\[ = 2(\frac{A^{m+1}}{2})^{1-2A} \]
\[ = A^{2A} \left( \frac{1}{2} \right)^{m+1-2A} \]
\[ \leq A^{2A} \left( \frac{1}{2} \right)^{q} < \varepsilon. \]

Thus \( F_n(x) \) is uniformly convergent on \((-A,A)\). Let
\[ \lim_{n \to \infty} F_n(x) = F(x). \] Thus \( F_n(x) \to F(x) \) where
\[ F_n(x) = 1 + \sum_{v=1}^{n} \frac{1}{v!} x^v. \]
Corollary 2.4.1: If \( p \) is a number, then there exists an interval \( I \) containing \( p \) in its interior such that \( F_n'(x) \) converges uniformly where \( F_n(x) = 1 + \sum_{\nu=1}^{n} \frac{1}{\nu!} x^\nu \).

Since \( F_n'(x) = F_{n-1}(x) \) for \( n = 1, 2, 3, \ldots \), and since there is an interval \( I \) containing \( p \) in its interior such that \( F_n(x) \rightarrow F(x) \) on \( I \), then \( F_n'(x) \rightarrow F'(x) \) on \( I \). Let \( \lim_{n \to \infty} F_n'(x) = \emptyset(x) \). Thus \( F_n'(x) \rightarrow \emptyset(x) \).

Theorem 2.5: There is one function, \( F(x) \), such that \( F'(x) = F(x) \) and \( F(0) = 1 \).

If \( F'(x) = F(x) \) and \( F(0) = 1 \), then \( F(x) \) and \( F'(x) \) are continuous and

\[
\int_{0}^{y} F'(x) \, dx = \int_{0}^{y} F(x) \, dx;
\]

\[
F(y) - F(0) = \int_{0}^{y} F(x) \, dx;
\]

\[
F(y) = 1 + \int_{0}^{y} F(x) \, dx.
\]

Let \( F_0(y) = 1 \) and \( F_n(y) = 1 + \int_{0}^{y} F_{n-1}(x) \, dx \). Then

\[
F_1(y) = 1 + \int_{0}^{y} F_0(x) \, dx = 1 + y;
\]

\[
F_2(y) = 1 + \int_{0}^{y} F_1(x) \, dx = 1 + y + \frac{y^2}{2!};
\]

\[
F_3(y) = 1 + \int_{0}^{y} F_2(x) \, dx = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!};
\]

\[
F_k(y) = 1 + \int_{0}^{y} F_{k-1}(x) \, dx
\]

\[= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^k}{k!}.
\]
Suppose that \( F_k(y) = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^k}{k!} \), then

\[
F_{k+1}(y) = 1 + \int_0^y F_k(x) \, dx \\
= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^k}{k!} + \frac{y^{k+1}}{(k+1)!}.
\]

Thus by induction, \( F_n(y) = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} \) and \( F_{n+1}(y) = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} \). By Theorem 2.4 and Corollary 2.4.1, \( F_n(y) \rightarrow F(y) \) and \( F_n'(y) \rightarrow F(y) \). But

\[
F'(y) = \lim_{n \to \infty} F_n'(y) = \lim_{n \to \infty} F_n(y) = F(y).
\]

Thus \( F'(y) = F(y) = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots + \frac{y^n}{n!} + \cdots \). Then \( F(0) = 1 \). By Corollary 2.1.1, \( F(x) \) is the only function such that \( F'(x) = F(x) \) and \( F(0) = 1 \).

**Definition 2.1:** Let \( E(x) \) be the function such that \( E'(x) = E(x) \) and \( E(0) = 1 \).

**Theorem 2.6:** There is only one function, \( y(x) \) such that \( y'(x) = y(x) \) and \( y(a) = b \).

Let \( y(x) = b \cdot E(x - a) \). Then \( y(a) = b \cdot E(a - a) = b \cdot E(0) = b \), and \( y'(x) = b \cdot E'(x - a) = b \cdot E(x - a) = y(x) \). By Corollary 2.1.1, there are not two such functions.

**Theorem 2.7:** If \( p \) is a number and \( q \) is a number and \( F(x) \) exists such that \( F'(x) = F(x) \) and \( F(0) = 1 \), then \( F(p + q) = F(p) \cdot F(q) \).
Let \( \phi(x) = F(p + x) \). Then \( \phi'(x) = F'(p + x) \frac{d(p + x)}{dx} = F(p + x) = \phi(x) \), and \( \phi(0) = F(p) \). Let \( G(x) = F(p) \cdot F(x) \). Then \( G'(x) = F(p) \cdot F'(x) = F(p) \cdot F(x) = G(x) \), and \( G(0) = F(p) \). By Theorem 2.6, there is only one such function. Thus \( \phi(x) = F(p + x) = F(p) \cdot F(x) = G(x) \) for all \( x \). Hence \( \phi(q) = F(p + q) = F(p) \cdot F(q) = G(q) \).

**Theorem 2.8:** If \( F(x) \neq 0 \), \( F'(x) = F(x) \), and \( F(0) = 1 \), then \( F(-x) = \frac{1}{F(x)} \).

By Theorem 2.7, \( F(-x) \cdot F(x) = F(-x + x) = F(0) = 1 \). Thus \( F(-x) = \frac{1}{F(x)} \).

**Theorem 2.9:** If \( E(x) = 1 + \sum_{p=1}^{\infty} \frac{x^p}{p!} \), then \( \lim_{x \to \infty} E(x) = \infty \).

Let \( M \) be greater than zero. Then \( E(M) = 1 + M + \sum_{p=1}^{\infty} \frac{M^p}{p!} > 1 + M > M \). Let \( k = M \). Then for every \( x > k \), \( E(x) > M \). Hence \( \lim_{x \to \infty} E(x) = \infty \).

**Theorem 2.10:** If \( E(x) = 1 + \sum_{p=1}^{\infty} \frac{x^p}{p!} \), then \( \lim_{x \to -\infty} E(x) = 0 \).

By Theorem 2.8, \( E(-x) = \frac{1}{E(x)} \). Let \( M = \frac{1}{-k} > 0 \). If \( x < k \), then \( -x > -k \). By Theorem 2.9, \( E(-x) > -x > -k \). Thus \( E(x) = \frac{1}{-k} = M \). Hence \( \lim_{x \to -\infty} E(x) = 0 \).

**Theorem 2.11:** If \( u \) is greater than zero, there is one and only one \( x \) such that \( E(x) = u \).
Let \( u \) be greater than zero. Since \( E(x) \) is continuous, then as a consequence of Theorem 2.9 and Theorem 2.16 there exists an \( a \) such that \( E(a) > u \) and a \( b \) such that \( E(b) < u \). Since \( E(x) \) is continuous for \( b \leq x \leq a \), there exists an \( x \) such that \( E(x) = u \). Since \( E(x) \) is monotone increasing, then there exists only one \( x \) such that \( E(x) = u \). Hence the theorem follows.

**Theorem 2.12:** If \( p \) is a whole number and \( q \) is a whole number and \( q \neq 0 \), then \( \left[E(x)\right]^\frac{p}{q} = E\left(\frac{xp}{q}\right) \).

By \( \left[E(x)\right]^\frac{p}{q} \), we will mean the same that is commonly applied to \( a^\frac{p}{q} \) where \( a > 0 \), that is, \( a^\frac{p}{q} = \sqrt[q]{a^p} \). First, let \( p \) be zero or a positive whole number and we shall prove

\[ \left[E(x)\right]^p = E(xp). \]

If \( p = 0 \), then \( \left[E(x)\right]^0 = 1 = E(0) = E(x\cdot0) \).

If \( p = 1 \), then \( \left[E(x)\right]^1 = E(x) = E(x\cdot1) \). If \( p = 2 \), then

\[ \left[E(x)\right]^2 = E(x)\cdot E(x) = E(x+x) = E(2x). \]

Suppose that \( \left[E(x)\right]^k = E(xk) \). Then

\[ \left[E(x)\right]^{k+1} = \left[E(x)\right]^k \cdot E(x) = E(xk + x) \]

\[ = E\left[x(k + 1)\right]. \]

Thus by induction \( \left[E(x)\right]^p = E(xp) \).

Next, let \( q \) be a positive number different from zero.

Then

\[ \left[E\left(\frac{A}{q}\right)\right]^q = E\left(\frac{Ax}{q}\right) = E(x). \]

Hence \( E\left(\frac{A}{q}\right) = \sqrt[q]{E(x)} \)

\[ = \left[E(x)\right]^\frac{1}{q}. \]

Now we shall prove that if \( p/q > 0 \), then \( \left[E(x)\right]^\frac{p}{q} = E\left(\frac{xp}{q}\right) \). Suppose there is one rational number, say \( g/z \), such that \( \left[E(x)\right]^\frac{g}{z} \neq E\left(\frac{Ag}{Z}\right) \). Let \( S \) be the set of positive whole numbers, say \( s \), such that \( \left[E(x)\right]^\frac{s}{z} \neq E\left(\frac{As}{Z}\right) \). There is a
smallest number in S. Call the smallest g. Then $g > 1$ because $[E(x)]^{\frac{1}{g}} = E(x/Z)$. Thus $[E(x)]^{\frac{g-1}{g}} = E\left(\frac{x(g-1)}{Z}\right)$. Therefore $[E(x)]^{\frac{1}{g}} = \left[E(x)\right]^{\frac{g-1}{g}} \cdot E\left(\frac{Z}{Z}\right) = E\left(\frac{xZ}{Z} - \frac{x}{Z} + \frac{x}{Z}\right) = E\left(\frac{Z}{Z}\right)$. By induction $[E(x)]^{\frac{p}{q}} = E\left(\frac{xp}{q}\right)$ when $p/q$ is zero or positive. If $p/q$ is negative, then

$$\left[E(x)\right]^{\frac{p}{q}} = \frac{1}{[E(x)]^{-\frac{p}{q}}} = \frac{1}{E\left(-\frac{xp}{q}\right)} = E\left(\frac{xp}{q}\right).$$

Thus the theorem follows.

**Definition 2.2:** $E(1) = e$.

**Theorem 2.13:** If $r = p/q$ is a rational number, then $E(p/q) = \sqrt[q]{e^p}$.

Since $E(p/q) = \left[E(1)\right]^{\frac{p}{q}} = e^{\frac{p}{q}} = \sqrt[q]{e^p}$, then the theorem follows.

**Definition 2.3:** If $Z$ is any number, then

$$[E(x)]^Z = E(xZ).$$

If $a > 0$, then by Theorem 2.11, there is a $c$ such that $E(c) = a$. Then $a^Z = [E(c)]^Z = E(cZ)$ and $a^Z$ is thus defined for all positive numbers $a$ and all real numbers $Z$.

**Remark:** Since $E(cZ)$ is a continuous function of $Z$,

$$E(c \cdot \frac{p}{q}) = \sqrt[q]{a^p},$$

and each irrational number is the limit of a
sequence of rational numbers, then \( f(z) = E(cz) = a^z \) is the only continuous function such that \( f(p/q) = \sqrt[q]{a^p} \).

**Theorem 2.1:** \( E(z) = e^z \).

Since \( E(1) = e \), then \( E(z) = \left[ E(1) \right]^z = e^z \).

**Remark:** If \( a > 0 \), we will now show that the following relationships exist:

\[
\begin{align*}
  a^p \cdot a^q &= a^{p+q}; \\
  \frac{a^p}{a^q} &= a^{p-q}; \\
  (a^p)^q &= a^{pq}.
\end{align*}
\]

By Theorem 2.11, there is an \( x \) such that \( E(x) = a \). Thus

\[
\begin{align*}
  a^p \cdot a^q &= E(xp) \cdot E(xq) = E[x(p + q)] \\
  &= \left[ E(x) \right]^{p+q} = a^{p+q}; \\
  \frac{a^p}{a^q} &= \frac{E(xp)}{E(xq)} = E(xp) \cdot E(-xq) \\
  &= E[x(p - q)] = \left[ E(x) \right]^{p-q} = a^{p-q}; \\
  (a^p)^q &= \left[ E(xp) \right]^q = E(xpq) \\
  &= \left[ E(x) \right]^{pq} = a^{pq}.
\end{align*}
\]

**Definition 2.14:** If \( f(x) \) is defined on the set \( S \), then \( g(y) \) defined on the set \( H \) is an inverse of \( f(x) \) on \( S \) if and only if the following conditions hold:

(1) If \( a \) is contained in \( S \), then \( f(a) \) is contained in \( H \).
(ii) If \( b \) is contained in \( H \), then there is one \( a \) contained in \( S \) such that at least one value of \( f(a) = b \).

(iii) If \( a \) is contained in \( S \) and \( f(a) = b \), then at least one value of \( g(b) = a \).

(iv) If \( b \) is contained in \( H \) and \( g(b) = p \), then \( p \) is contained in \( S \) and at least one value of \( f(p) = b \).

**Remark:** If the properties of Definition 2.4 are satisfied, then it is easily shown that there is only one inverse function.

**Remark:** The set of all numbers will be denoted by \( S \). The set of all positive numbers will be denoted by \( H \).

**Theorem 2.15:** If \( f(x) \) is monotone increasing and continuous on \([a, b]\), then \( f(x) \) has an inverse function \( g(y) \) which is monotone increasing and continuous on \([c, d]\) where \( c = f(a) \) and \( d = f(b) \). Furthermore, \( f(x) \) on \([a, b]\) is the inverse of \( g(y) \) on \([c, d]\).

Let \( c = f(a) \) and \( d = f(b) \). If \( a \leq x \leq b \), then since \( f(x) \) is monotone increasing \( c = f(a) \leq f(x) \leq f(b) = d \).

Thus (i) is satisfied. If \( c \leq y \leq d \), then \( f(a) \leq y \leq f(b) \).

Since \( f(x) \) is continuous, there is an \( a \leq x \leq b \) such that \( f(x) = y \). Thus (ii) is satisfied. If \( a \leq x \leq b \), let
$g[f(x)]$ have $x$ for one of its values. Thus (iii) is satisfied. Since $g[f(x)]$ has a value for $a \leq x \leq b$, then by (ii) $g(y)$ is defined for $c \leq y \leq d$. Furthermore, if $c \leq y \leq d$, let $g(y)$ have $p$ as one of its values only if $a \leq p \leq b$ and $f(p) = y$. Hence (iv) is satisfied and $g(y)$ on $[c,d]$ is the inverse of $f(x)$ on $[a,b]$.

We shall now prove that $f(x)$ on $[a,b]$ is the inverse of $g(y)$ on $[c,d]$. If $c \leq y \leq d$, then $g(y)$ is defined and by (iv) $a \leq g(y) \leq b$. Thus (i) is satisfied. If $a \leq x \leq b$, then by (i) $c \leq f(x) \leq d$. Let $y = f(x)$. Then by (iii) at least one value of $g(y) = x$. Thus (ii) is satisfied. If $c \leq y \leq d$ and $g(y) = x$, then by (iv) $a \leq x \leq b$ and at least one value of $f(x) = y$. Thus (iii) is satisfied. If $a \leq x \leq b$ and $f(x) = p$, then by (i) $c \leq p \leq d$ and by (iii) at least one value of $g(p) = x$. Hence (iv) is satisfied and $f(x)$ on $[a,b]$ is the inverse of $g(y)$ on $[c,d]$.

We shall now prove that $g(y)$ is single-valued on $[c,d]$. Let $c \leq y_1 \leq d$ and $c \leq y_2 \leq d$. Let $x_1 = g(y_1)$ and $x_2 = g(y_2)$. Thus $a \leq x_1 \leq b$ and $a \leq x_2 \leq b$. Suppose $g(y_1) \neq g(y_2)$. Since $f(x)$ is monotone increasing on $[a,b]$, $f(x_1) \neq f(x_2)$. Thus $f[g(y_1)] \neq f[g(y_2)]$ which yields $y_1 \neq y_2$. Hence $g(y)$ is single-valued on $[c,d]$.

We shall now prove that $g(y)$ is monotone increasing on $[c,d]$. Let $y_1$ and $y_2$ be two points in $[c,d]$. Let $x_1 = g(y_1)$ and $x_2 = g(y_2)$. Thus $x_1$ and $x_2$ are in $[a,b]$. Suppose that
\[ y_1 < y_2 \text{ and that } g(y_1) \geq g(y_2). \text{ Since } f(x) \text{ is monotone increasing, then } f(x_1) \geq f(x_2). \text{ Thus } f[g(y_1)] \geq f[g(y_2)] \]
which yields \( y_1 \geq y_2 \). Hence, if \( y_1 < y_2 \), \( g(y_1) < g(y_2) \) and \( g(y) \) is monotone increasing on \([c,d]\).

We shall now prove that \( g(y) \) is continuous on \([c,d]\).

Let \( v \) be a point of \([c,d]\) and let \( u = g(v) \). Let \( \epsilon \) be greater than zero. Let \( x_1 = \max[a, u - \epsilon] \) and \( x_2 = \min[u + \epsilon, b] \). Let \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \). Let \( \delta^\wedge = \min[y_2 - v, v - y_1] \) if \( c < v < d \). Let \( \delta^\wedge = y_2 - v \) if \( v = c \). Let \( \delta^\wedge = v - y_1 \) if \( v = d \). Clearly \( \delta^\wedge \) is greater than zero. Let \( y \) be a point in \([c,d]\) such that \( |y - v| < \delta^\wedge \). If \( y = v \), then \( g(y) = g(v) \) and \( |g(y) - g(v)| = 0 < \epsilon \). If \( y > v \), then \( v < y < v + \delta^\wedge \).

Since \( y \) is contained in \([c,d]\), then \( v < d \), \( \delta^\wedge \leq y_2 - v \), and \( v < y < y_2 \). Since \( g(y) \) is a monotone increasing function, then \( g(v) < g(y) < g(y_2) \). Thus \( u < g(y) < x_2 \leq u + \epsilon \) and \( u - \epsilon < g(y) < u + \epsilon \). Hence \( |g(y) - g(v)| = |g(y) - u| < \epsilon \).

If \( y < v \), then \( v - \delta^\wedge < y < v \). Since \( y \) is in \([c,d]\), then \( v > c \), \( \delta^\wedge \leq v - y_1 \) and \( y_1 \leq y < v \). Since \( g(y) \) is monotone increasing, then \( g(y_1) < g(y) < g(y_2) \). But \( x_1 \geq u - \epsilon \).

Hence \( u - \epsilon < g(y) < g(v) \), \( g(v) - \epsilon < g(y) < g(v) + \epsilon \) and \( |g(y) - g(v)| < \epsilon \). Therefore \( g(y) \) is continuous at \( y = v \).

Since \( v \) could be any point in \([c,d]\), then \( g(y) \) is continuous on \([c,d]\).

**Theorem 2.16:** If \( f(x) \) is monotone increasing and differentiable on \([a,b]\), then \( f(x) \) has an inverse function \( g(y) \)
defined on \([c, d]\), where \(c = f(a)\) and \(d = f(b)\). Furthermore \(g(y)\) is monotone increasing on \([c, d]\). If \(v\) is a point of \([c, d]\), \(u = g(v)\), and \(f'(u) \neq 0\), then \(g(y)\) is differentiable at \(y = v\) and \(g'(v) = \frac{1}{f'(u)}\).

Since \(f(x)\) is differentiable on \([a, b]\), \(f(x)\) must be continuous on \([a, b]\). Since \(f(x)\) is monotone increasing on \([a, b]\), then Theorem 2.14 applies and there must exist an inverse function \(g(y)\) which is defined, monotone increasing, and continuous on \([c, d]\).

Let \(v\) be a point of \([c, d]\) where \(g(v) = u\) and \(f'(u) \neq 0\). Let \(\varepsilon\) be greater than zero. Since \(f'(u) \neq 0\), then there exists a \(\Delta > 0\) such that if \(0 < |x - u| < \Delta\) and \(x\) is in the domain of definition, then

\[
|f'(u) - \frac{f(x) - f(u)}{x - u}| < \min\left[\frac{|f'(u)|}{2}, \frac{|f'(u)|^2}{2}\right] \cdot \varepsilon.
\]

Since \(g(y)\) is continuous at \(y = v\), then there exists a \(\delta > 0\) such that if \(|y - v| < \delta\) and \(y\) is in the domain of definition, then \(|g(y) - g(v)| < \Delta\). Let \(y_o\) be a point of \([c, d]\) such that \(0 < |y_o - v| < \delta\) and let \(x_o = g(y_o)\). Therefore \(y_o = f(x_o)\) and \(v = f(u)\). Since \(|y_o - v| < \delta\), then \(|g(y_o) - g(v)| < \Delta\) or \(|x_o - u| < \Delta\). Since \(g(y)\) is monotone increasing, it follows from \(y_o \neq v\) that \(g(y_o) \neq g(v)\) and hence \(x_o \neq u\). Thus \(0 < |x_o - u| < \Delta\). Hence
\[ \left| f'(u) - \frac{f(x_o) - f(u)}{x_o - u} \right| < \frac{|f'(u)|}{2} \cdot \varepsilon , \]

and

\[ |f'(u)| - \frac{f(x_o) - f(u)}{x_o - u} \leq |f'(u) - \frac{f(x_o) - f(u)}{x_o - u}| \leq \frac{|f'(u)|}{2} \]

or

\[ \left| \frac{f(x_o) - f(u)}{x_o - u} \right| > \frac{|f'(u)|}{2} \]

Then

\[ \left| \frac{1}{f'(u)} - \frac{g(y_o) - g(v)}{y_o - v} \right| = \left| \frac{1}{f'(u)} - \frac{x_o - u}{f(x_o) - f(u)} \right| \]

\[ = \left| \frac{f(x_o) - f(u)}{x_o - u} - f'(u) \right| \]

\[ = \left| \frac{f(x_o) - f(u) - f'(u)}{x_o - u} \right| \cdot |f'(u)| \cdot \left| \frac{f(x_o) - f(u)}{x_o - u} \right| \]

\[ \leq \frac{|f'(u)|^2}{2} \cdot \varepsilon = \varepsilon . \]

Therefore \( g(y) \) is differentiable at \( y = v \) and \( g'(v) = \frac{1}{f'(u)} \).

Since \( v \) could be any point of \([c,d]\), then \( g(y) \) is differentiable on \([c,d]\).
Corollary 2.16.1: If \( f(x) \) is monotone increasing and continuous on the set \( S \) and \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = 0 \), then there exists a monotone increasing and continuous function \( g(y) \) defined on the set \( H \) such that \( g(y) \) is the inverse of \( f(x) \). Furthermore, if \( f(x) \) is differentiable on the set \( S \) and \( f'(x) \neq 0 \), then \( g(y) \) is differentiable on the set \( H \) and \( g'(y) = \frac{1}{f'(x)} \).

If \( y \) is greater than zero, then there is an \( a \) such that \( f(a) < y \) and there is a \( b \) such that \( f(b) > y \). Since \( f(x) \) is monotone increasing and continuous, there is an unique \( x \) such that \( a < x < b \) and \( f(x) = y \). Let \( g(y) = x \). Since \( x \) is unique, it follows that \( g(y) \) is single-valued.

We shall now prove that \( g(y) \) on \( H \) is the inverse of \( f(x) \) on \( S \). If \( u \) is in \( S \), then \( f(u) \) is in \( H \) since \( f(x) \) is monotone increasing and \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = 0 \). If \( v \) is in \( H \), then there is one \( u \) in \( S \) such that at least one value of \( f(u) = v \) since \( f(x) \) is continuous and \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = 0 \). If \( f(u) = v \), then \( g(v) = u \) by definition of \( g(y) \). If \( v \) is in \( H \) and \( g(v) \) has \( p \) as one of its values, then \( p \) is in \( S \) and \( f(p) = v \). Hence \( g(y) \) on \( H \) is the inverse of \( f(x) \) on \( S \).

If \( 0 < c < d \), then there are an \( a \) and \( b \) such that \( a < b \) and \( f(a) = c \) and \( f(b) = d \). There is a function \( \phi(y) \) defined on \([c, d]\) such that \( \phi(y) \) is the inverse of \( f(x) \) on \([a, b]\). By Theorem 2.14, \( \phi(y) \) is monotone increasing and continuous on
[c,d]. Furthermore, \( \psi(y) \) on \([c,d]\) equals \( g(y) \) on \([c,d]\).

Suppose there exists a \( y_i \) in \([c,d]\) such that \( \psi(y_i) \neq g(y_i) \).

Let \( x_1 = \psi(y_i) \) and \( x_2 = g(y_i) \). Then \( f(x_1) = y_i \); \( f(x_2) = y_i \).

Hence \( f(x_1) = f(x_2) \) and \( x_1 = x_2 \) since \( f(x) \) is monotone increasing.

Therefore, if \( 0 \leq c \leq d \), \( g(y) \) is single-valued, monotone increasing, and continuous on \([c,d]\). Furthermore, if \( f(x) \) is differentiable on \([g(c),g(d)]\) and \( f'(x) \neq 0 \), then \( g(y) \) is differentiable on \([c,d]\) and \( g'(y) = \frac{1}{f'(x)} \).

Therefore, if \( f'(x) \neq 0 \) on \( S \), then \( g'(y) = \frac{1}{f'(x)} \) for all \( y \) in \( H \). If \( 0 \leq y_i \leq y_2 \), let \( c = \frac{y_1}{2} \) and \( d = y_2 + 1 \). Since \( g(y) \) is monotone increasing on \([c,d]\), then \( g(y_1) \leq g(y_2) \) and \( g(y) \) is monotone increasing on \( H \).

**Theorem 2.17:** There exists a function \( L(y) \) defined on the set \( H \) such that \( L(y) \) is the inverse of \( E(x) \) defined on the set \( S \). Furthermore \( L(y) \) is single-valued, monotone increasing, continuous, and differentiable on the set \( H \). Also \( L'(y) = 1/y \).

Since all the properties of \( E(x) \) fulfill the properties of \( f(x) \) in Corollary 2.15.1, then \( E(x) \) has an inverse function \( L(y) \) such that \( L(y) \) fulfills the properties of \( g(y) \) in Corollary 2.15.1. Hence \( L(y) \) is single-valued, monotone increasing, continuous and differentiable on the set \( H \). If \( E(x) = y \), then
\[ L'(y) = \frac{1}{E'(x)} = \frac{1}{E(x)} = \frac{1}{y}. \]

**Theorem 2.18:** If \( y > 0 \), then \( L(y) = \int_1^y \frac{1}{t} \, dt \).

By Theorem 2.16, \( L'(y) = \frac{1}{y} \) for all positive \( y \). Thus if \( y > 0 \), then

\[
\int_1^y L'(t) \, dt = \int_1^y \frac{1}{t} \, dt;
\]

\[
L(y) - L(1) = \int_1^y \frac{1}{t} \, dt;
\]

\[
L(y) = \int_1^y \frac{1}{t} \, dt.
\]

**Theorem 2.19:** If \( p \) is a positive number and \( q \) is a positive number, then \( L(pq) = L(p) + L(q) \).

By Theorem 2.11, there exists a \( g \) and a \( h \) such that \( E(g) = p \) and \( E(h) = q \). Hence \( L(p) = g \) and \( L(q) = h \). Thus

\[
L(pq) = L\left[E(g) \cdot E(h)\right] = L[E(g + h)]
\]

\[
= g + h = L(p) + L(q).
\]

**Theorem 2.20:** If \( p \) is a positive number and \( q \) is a positive number, then \( L(p/q) = L(p) - L(q) \).

By Theorem 2.11, there exists a \( g \) and a \( h \) such that \( E(g) = p \) and \( E(h) = q \). Hence \( L(p) = g \) and \( L(q) = h \). Thus

\[
L\left(\frac{p}{q}\right) = L\left[E(g) / E(h)\right] = L[E(g) \cdot E(-h)]
\]

\[
= L[E(g - h)] = g - h = L(p) - L(q).
\]
**Theorem 2.21:** If $p$ is a positive number and $r$ is any number, then $L(p^r) = r \cdot L(p)$.

By Theorem 2.11 there exists a $g$ such that $E(g) = p$. Hence $L(p) = g$. Thus

$$L(p^r) = L\left\{E(g)^r\right\} = L[E(rg)] = rg = r \cdot L(p).$$

**Theorem 2.22:** The $\lim_{y \to \infty} L(y) = \infty$.

Let $M$ be greater than zero. Since $L(y)$ is monotone increasing and $L(1) = 0$, then $L(2) > 0$. Let $n$ be a positive integer such that $n > \frac{M}{L(2)}$. Then $M < n \cdot L(2) = L(2^n)$. Let $k = 2^n$ and choose $y > k$. Since $L(y)$ is monotone increasing, then $L(y) > L(k) = L(2^n) > M$. Hence $\lim_{y \to \infty} L(y) = \infty$.

**Theorem 2.23:** For every $y$ greater than zero,

$$L(1/y) = -L(y).$$

Since $L(1/y) = L(1) - L(y) = -L(y)$, the theorem follows.

**Theorem 2.24:** The $\lim_{y \to 0^-} L(y) = -\infty$.

Let $M$ be greater than zero. Since $\lim_{y \to \infty} L(y) = \infty$, there exists a $k > 0$ such that for every $y > k$, $L(y) > M$. Let $d = 1/k$ and take $y$ such that $0 < y < d$. Since $y < d$, then $1/y > 1/d$ and $1/y > k$. Hence $L(1/y) > M$ and $-L(1/y) < -M$. Thus $L(y) = -L(1/y) < -M$. Hence $\lim_{y \to 0^-} L(y) = -\infty$. 
**Theorem 2.25:** If \( y > -1 \) and \( R_n = \int_0^y \frac{t^n}{(1 + t)^{n+1}} \, dt \), then

\[
L(1 + y) = \int_0^y \frac{dt}{1 + t} = \sum_{p=1}^n \frac{1}{p} \left( \frac{y}{1 + y} \right)^p + R_n.
\]

If \( y > -1 \), then \((1 + y)\) is greater than zero. By Theorem 2.17, \( L'(1 + y) = \frac{1}{1 + y} \). Thus

\[
\int_0^y L'(1 + t) \, dt = \int_0^y \frac{dt}{1 + t};
\]

\[
L(1 + y) - L(1) = \int_0^y \frac{dt}{1 + t};
\]

\[
L(1 + y) = \int_0^y \frac{dt}{1 + t}.
\]

Let \( u_1 = \frac{1}{1 + t} \) and \( v_1 = t \). Then by Theorem 1.11,

\[
\int_0^y \frac{dt}{1 + t} = \left[ \frac{t}{1 + t} \right]_0^y - \int_0^y \frac{-t}{(1 + t)^2} \, dt
\]

\[= \frac{y}{1 + y} + \int_0^y \frac{t}{(1 + t)^2} \, dt.\]

Let \( u_2 = \frac{1}{(1 + t)^2} \) and \( v_2 = \frac{t^2}{2} \). Then by Theorem 1.11,

\[
\int_0^y \frac{t^2}{(1 + t)^2} \, dt = \left[ \frac{t^2}{2(1 + t)^2} \right]_0^y - \int_0^y \frac{-2t^2}{2(1 + t)^3} \, dt
\]

\[= \frac{y^2}{2(1 + y)^2} + \int_0^y \frac{t^2}{(1 + t)^3} \, dt.\]

Let \( u_3 = \frac{1}{(1 + t)^3} \) and \( v_3 = \frac{t^3}{3} \). Then by Theorem 1.11,
\[
\int_0^y \frac{t^2}{(1 + t)^3} \, dt = \left[ \frac{t^3}{3(1 + t)^3} \right]_0^y - \int_0^y \frac{-3t^3}{3(1 + t)^3} \, dt
\]
\[
= \frac{y^3}{3(1 + y)^3} + \int_0^y \frac{t^3}{(1 + t)^3} \, dt.
\]

Thus
\[
\int_0^y \frac{dt}{1 + t} = \frac{y}{1 + y} + \frac{y^2}{2(1 + y)^2}
\]
\[
+ \frac{y^3}{3(1 + y)^3} + \int_0^y \frac{t^3}{(1 + t)^3} \, dt.
\]

Suppose that
\[
\int_0^y \frac{dt}{1 + t} = \sum_{p=1}^K \frac{1}{p} \left( \frac{y}{1 + y} \right)^p + \int_0^y \frac{t^K}{(1 + t)^{K+1}} \, dt.
\]

Let \( u_{K+1} = \frac{1}{(1 + t)^{K+1}} \) and \( v_{K+1} = \frac{t^{K+1}}{k+1} \). By Theorem 1.11,
\[
\int_0^y \frac{t^K}{(1 + t)^{K+1}} \, dt = \left[ \frac{t^{K+1}}{(k+1)(1 + t)^{K+1}} \right]_0^y
\]
\[
- \int_0^y \frac{-(k+1)t^{K+1}}{(k+1)(1 + t)^{K+2}} \, dt
\]
\[
= \frac{y^{K+1}}{(k+1)(1 + y)^{K+1}} + \int_0^y \frac{t^{K+1}}{(1 + t)^{K+2}} \, dt.
\]

Therefore
\[
\int_0^y \frac{dt}{1 + t} = \sum_{p=1}^{K+1} \frac{1}{p} \left( \frac{y}{1 + y} \right)^p + \int_0^y \frac{t^{K+1}}{(1 + t)^{K+2}} \, dt.
\]
Hence, if \( R_n = \int_0^\gamma \frac{t^n}{(1 + t)^{n+1}} \, dt \), then by induction

\[
L(1 + y) = \sum_{p=1}^{n} \frac{1}{p} \left( \frac{1}{1 + y} \right)^p + R_n
\]

for \( n = 1, 2, 3, \ldots \).

**Theorem 2.26:** If \( R_n = \int_0^\gamma \frac{t^n}{(1 + t)^{n+1}} \, dt \) and \( 0 < y \leq k \), then \( 0 < R_n < k/n \).

Let \( f(t) = \frac{t^n}{(1 + t)^{n+1}} \). Thus \( f'(t) = \frac{t^{n-1}(n - t)}{(1 + t)^{n+2}} \).

If \( n = t \), then \( f'(t) = 0 \) and \( f(t) \) has a maximum value at \( n = t \). Thus

\[
R_n = \int_0^\gamma f(t) \, dt \leq \int_0^\gamma \frac{n^n}{(1 + n)^{n+1}} \, dt \leq \frac{n^n}{(1 + n)^{n+1}} \cdot k \leq \frac{n^n \cdot k}{n^{n+1}} = \frac{k}{n}.
\]

Since \( y > 0 \), then \( f(t) = \frac{t^n}{(1 + t)^{n+1}} > 0 \). Since \( f'(t) \) exists, then \( f(t) \) is continuous. Thus let \( g(x) \equiv 0 \) in Theorem 1.16.

Then \( f(t) > g(x) \) and

\[
R_n = \int_0^\gamma f(t) \, dt > \int_0^\gamma g(x) \, dx = 0.
\]

Hence \( 0 < R_n < k/n \).

**Theorem 2.27:** If \( H_n = (-1)^n \int_0^\gamma \frac{t^n}{1 + t} \, dt \) and \(-1 < y \leq 1\), then

\[
L(1 + y) = \sum_{p=1}^{n} (-1)^{p-1} \frac{y^p}{p} + H_n.
\]
By Theorem 2.25, \( L(1 + y) = \int_0^y \frac{1}{1 + t} \, dt \). Thus
\[
L(1 + y) = \int_0^y \left( 1 - \frac{t}{1 + t} \right) \, dt = y - \int_0^y \frac{t}{1 + t} \, dt.
\]

Suppose that
\[
L(1 + y) = \sum_{p=1}^k (-1)^{p-1} \frac{y^p}{p} + (-1)^k \int_0^y \frac{t^k}{1 + t} \, dt.
\]

But
\[
(-1)^k \int_0^y \frac{t^k}{1 + t} \, dt = (-1)^k \int_0^y \left( t^k - \frac{t^{k+1}}{1 + t} \right) \, dt
\]
\[= (-1)^k \frac{y^{k+1}}{k+1} + (-1)^{k+1} \int_0^y \frac{t^{k+1}}{1 + t} \, dt.
\]

Thus
\[
L(1 + y) = \sum_{p=1}^{k+1} (-1)^{p-1} \frac{y^p}{p} + (-1)^{k+1} \int_0^y \frac{t^{k+1}}{1 + t} \, dt.
\]

Hence, if \( H_n = (-1)^n \int_0^n \frac{t^n}{1 + t} \, dt \), then by induction
\[
L(1 + y) = \sum_{p=1}^n (-1)^{p-1} \frac{y^p}{p} + H_n
\]
for \( n = 1, 2, 3, \ldots \).

**Theorem 2.28:** If \( 0 = y \leq 1 \), then \( |H_n| \leq \frac{1}{n+1} \). If \(-1 < y < 0\), then
\[
|H_n| \leq \frac{1}{1 + y} \cdot \frac{|y|^{n+1}}{n + 1}.
\]
Let $0 \leq y \leq 1$. By Theorem 2.27, $H_n = (-1)^n \int_0^y \frac{t^n}{1 + t} \, dt$.

Thus

$$|H_n| = \left| \int_0^y \frac{t^n}{1 + t} \, dt \right| \leq \int_0^1 \left| \frac{t^n}{1 + t} \right| \, dt$$

$$\leq \int_0^1 t^n \, dt = \left[ \frac{t^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}.$$ 

Hence $|H_n| \leq \frac{1}{n+1}$.

Let $-1 < y < 0$. Thus

$$|H_n| = \left| \int_y^0 \frac{t^n}{1 + t} \, dt \right| = \left| \int_0^y \frac{t^n}{1 + t} \, dt \right|$$

$$\leq \int_0^y \frac{|t|^n}{1 + t} \, dt \leq \int_0^y \frac{|t|^n}{1 + y} \, dt$$

$$= \frac{1}{1 + y} \int_0^y |t|^n \, dt = \frac{1}{1 + y} \int_0^y (-t)^n \, dt.$$ 

Let $t = -u$. Then $\frac{dt}{du} = -1$ and $(-t)^n = u^n$. Thus

$$\int_{-y}^{0} (-t)^n \, dt = \int_y^{0} u^n \frac{dt}{du} \, du = -\int_y^{0} u^n \, du$$

$$= \int_0^{-y} u^n \, du = \left[ \frac{u^{n+1}}{n+1} \right]_0^{-y}$$

$$= \frac{(-y)^{n+1}}{n+1} = \frac{|y|^{n+1}}{n+1}.$$ 

Thus

$$\frac{1}{1 + y} \int_y^0 (-t)^n \, dt = \frac{1}{1 + y} \cdot \frac{|y|^{n+1}}{n+1}.$$
Hence

\[ |H_n| \leq \frac{1}{1 + y} \cdot \frac{|y|^{n+1}}{n + 1}. \]

**Theorem 2.29:** The limit \( \lim_{y \to 0} (1 + y)^{\frac{1}{y}} = e. \)

Let \( \epsilon > 0 \) be greater than zero. Since \( E(x) \) is continuous, there exists a \( \delta > 0 \) such that if \( |x - 1| < \delta \), then

\[ |E(x) - E(1)| < \epsilon. \]

Let \( \Delta = \min \left[ \frac{\delta}{2}, \frac{1}{2} \right] \). If \( 0 < |y| < \Delta \), let \( n > \frac{2}{\delta^2 |y|} \). Then by Theorem 2.27,

\[ \left| L(1 + y)^{\frac{1}{y}} - 1 \right| = \frac{1}{y} \left| L(1 + y) - 1 \right| \]

\[ = \left| \frac{1}{y} \sum_{p=1}^{n} (-1)^{p-1} \frac{y^p}{p} + \frac{H_n}{y} - 1 \right| \]

\[ = \left| 1 + y \sum_{p=2}^{n} (-1)^{p-1} \frac{y^{p-2}}{p} + \frac{H_n}{y} - 1 \right| \]

\[ = \left| y \sum_{p=2}^{n} (-1)^{p-1} \frac{y^{p-2}}{p} + \frac{H_n}{y} \right| \]

\[ \leq \left| \frac{y}{p} \sum_{p=2}^{n} \left| \frac{y^{p-2}}{p} \right| \right| + \left| \frac{H_n}{y} \right| \]

\[ \leq \left| y \sum_{p=2}^{n} \left| \frac{1}{p^{p-2}} \right| \right| + \left| \frac{H_n}{y} \right| \]

\[ \leq \left| y \sum_{p=2}^{n} \frac{1}{2^{p-2}} \right| + \left| \frac{H_n}{y} \right| \]

If \( 0 \leq y \leq 1 \), then by Theorem 2.28, \( |H_n| \leq \frac{1}{n + 1} \). Thus
\[ |H_n| < 1/n. \text{ If } -1 < y < 0, \text{ then by Theorem 2.28,} \]
\[ |H_n| \leq \frac{1}{1 + y} \cdot \frac{|y|^{n+1}}{n+1}. \]

Thus
\[ |H_n| \leq \frac{1}{1 - \Delta} \cdot \frac{\Delta^{n+1}}{n+1} < \frac{2\Delta}{n} \leq \frac{1}{n}. \]

Therefore
\[ |L(1 + y)^{\frac{1}{y}} - 1| = |y| + \frac{|H_n|}{|y|} \]
\[ < \frac{\Delta}{2} + \frac{1}{n|y|} < \frac{\Delta}{2} + \frac{\Delta}{2} = \Delta. \]

Hence if \( 0 < |y| < \Delta \), then
\[ |(1 + y)^{\frac{1}{y}} - e| = |E[L(1 + y)^{\frac{1}{y}}] - E(1)| < \epsilon \]

and \( \lim_{y \to 0} (1 + y)^{\frac{1}{y}} = e. \)
BIBLIOGRAPHY


