A THEOREM ON THE CONVERGENCE OF A CONTINUED FRACTION

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A THEOREM ON THE CONVERGENCE OF A CONTINUED FRACTION

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CHAPTER I

INTRODUCTION

When using the word "number" we will mean a complex number unless otherwise stated or unless the size of the number is mentioned.

If a is a number not equal to zero, then

(i) a divided by zero is equal to infinity.

(ii) a multiplied by infinity is equal to infinity.

If a is a number not equal to infinity, then a divided by infinity is equal to zero.

If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are sequence of numbers then we will call the sequence

$$f_1 = \frac{A_1}{B_1} = \frac{a_1}{b_1}, \qquad f_2 = \frac{A_2}{B_2} = \frac{a_1}{b_1 + \frac{a_2}{b_2}},$$

$$f_3 = \frac{A_3}{B_3} = \frac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3}}}, \ldots,$$

$$f_n = \frac{A_n}{B_n} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots + \cfrac{a_n}{b_n}}}}, \ldots$$

the approximents of the continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}.$$

If at most a finite number of B_p vanish and the sequence f_n has a finite limit then the continued fraction is said to converge to this limit or to be convergent.[1]

Since a continued fraction may be considered as a sequence of linear fractional transformation[2] we will prove some general theorems of linear fractional transformations.

<u>Theorem 1.1</u>--If $|Z - Q| \leq R$ and $w = bZ$, $b \neq 0$, then there exists a number c and a real positive number r such that $|w - c| \leq r$. Also if $|w - c| \leq r$, then $|Z - Q| \leq R$.

Let $c = bQ$ and $r = R|b|$. Since $w = bZ$, then $Z = \frac{w}{b}$. If $|Z - Q| \leq R$ then $\left|\frac{w}{b} - Q\right| \leq R$, or $|w - bQ| \leq R|b|$, or $|w - c| \leq r$. Therefore w is in a circle with center $c = bQ$ and radius $r = R|b|$

If $|w - c| \leq r$, then since $w = bZ$, $c = Qb$, and $r = R|b|$ we have $|bZ - Qb| \leq R|b|$, or $|Z - Q| \leq R$.

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[1] H. S. Wall, <u>Continued Fractions</u>, p. 16.

[2] <u>Ibid</u>., p. 19.

<u>Theorem 1.2</u>--If $|Z - Q| \geq R$ and $w = bZ$, $b \neq 0$, then there exists a number c and a real positive number r such that $|w - c| \geq r$. Also if $|w - c| \geq r$, then $|Z - Q| \geq R$.

This can be proved by the same method as used in Theorem 1.1.

<u>Theorem 1.3</u>--If $w = bZ$, $b \neq 0$, and $\bar{a}Z + a\bar{Z} \leq 2p$, $a \neq 0$, then there exists a number $c \neq 0$ and a real number q such that $\bar{c}w + c\bar{w} \leq 2q$. Furthermore if $\bar{c}w + c\bar{w} \leq 2q$, then $\bar{a}Z + a\bar{Z} \leq 2p$

Let $\bar{c} = \overline{ab}$, $c = ab$, and $q = p\,|b|^2$. Since $w = bz$, then $Z = \frac{w}{b}$ and $\bar{Z} = \frac{\bar{w}}{\bar{b}}$.

$$\text{If} \qquad \bar{a}Z + a\bar{Z} \leq 2p \qquad \text{then} \quad \frac{\bar{a}w}{b} + \frac{a\bar{w}}{\bar{b}} \leq 2p,$$

$$\text{or} \qquad b\bar{b}\left[\frac{\bar{a}w}{b} + \frac{a\bar{w}}{\bar{b}}\right] \leq 2p\,|b|^2,$$

$$\text{or} \qquad ab\bar{w} + \overline{ab}w \leq 2p\,|b|^2,$$

Therefore w is in the half plane $\bar{c}w + c\bar{w} \leq 2q$.

If $\bar{c}w + c\bar{w} \leq 2q$, then since $w = bZ$, $\bar{w} = \overline{bZ}$, $\bar{c} = \overline{ab}$, $c = ab$, and $q = p\,|b|^2$, we have $\bar{a}\,\bar{b}\,b\,Z + a\,b\,\bar{b}\,\bar{Z} \leq 2p\,|b|^2$, or $\bar{a}Z + a\bar{Z} \leq 2p$.

<u>Theorem 1.4</u>--If $|Z - Q| \leq R$ and $w = Z+H$, then there exists a number c and a real number r such that $|w - c| \leq r$. Also if $|w - c| \leq r$, then $|Z - Q| \leq R$.

Let $c = Q + H$ and $r = R$. Since $w = Z \quad H$, then $Z = w - H$. If $|Z - Q| \leq R$, then $|(w - H) - Q| \leq R$ or, $|w - H - Q| \leq R$, or

$|w - (H\ Q)| \leq R$, or $|w - c| \leq r$. Therefore, if Z is in the circle $|Z - Q| \leq R$, then w is in the circle $|w - c| \leq r$.

If $|w - c| \leq r$, then since $w = Z + H$, $c = Q + H$, and $r = R$, we have $|(Z+H) - (Q+H)| \leq R$, or $|Z - Q| \leq R$.

<u>Theorem</u> <u>1.5</u>--If $|Z - Q| \geq R$ and $w = Z + H$ then there exists a number c and a real number r such that $|w - c| \geq r$, also if $|w - c| \geq r$, then $|Z - Q| \geq R$.

This can be proved in the same method as used in Theorem 1.4.

<u>Theorem</u> <u>1.6</u>--If $w = Z + H$, $a \neq 0$ and $\bar{a}Z + a\bar{Z}\ \ 2p$ then there exists a number $c \neq 0$ and a real number q such that $\bar{c}w + c\bar{w} \leq 2q$. Furthermore if $\bar{c}w + c\bar{w} \leq 2q$ then $\bar{a}Z + a\bar{Z} \leq 2p$.

Let $\bar{a} = \bar{c}$, $a = c$, and $\mathcal{R}(\bar{a}H) + p = q$. Since $w = Z + H$, then $Z = w - H$ and $\bar{Z} = \bar{w} - \bar{H}$. If $\bar{a}Z + a\bar{Z} \leq 2p$, then

$$\bar{a}\ [w - H] + a\ [\bar{w} - \bar{H}] \leq 2p;$$
$$\bar{a}w - \bar{a}H + a\bar{w} - a\bar{H} \leq 2p;$$
$$\bar{a}w + a\bar{w} - (\bar{a}H\ \ a\bar{H}) \leq 2p;$$
$$\bar{a}w + a\bar{w} - 2\mathcal{R}(\bar{a}H) \leq 2p;$$
$$\bar{a}w + a\bar{w} \leq 2[\mathcal{R}(\bar{a}H) + p];$$

Therefore if Z is in the half plane $\bar{a}Z + a\bar{Z} \leq 2p$ then w is in the half plane $\bar{c}w + c\bar{w} \leq 2q$.

If $\bar{c}w + c\bar{w} \leq 2q$, then since $w = Z + H$, $\bar{w} = \bar{Z} + \bar{H}$, $\bar{c} = \bar{a}$, $c = a$, and $q = \mathcal{R}(aH) + p = q$ we have

$$\bar{a}\ [Z + H] + a\ [\bar{Z} + \bar{H}] \leq 2[\mathcal{R}(aH) + p],$$

or $\quad \bar{a}Z + \bar{a}H + a\bar{Z} + a\bar{H} \leq 2\mathcal{R}(\bar{a}H) + 2p,$

or $\quad \bar{a}Z + a\bar{Z} + 2\mathcal{R}(\bar{a}H) \leq 2\mathcal{R}(\bar{a}H) + 2p,$

or $\quad \bar{a}Z + a\bar{Z} \leq 2p.$

<u>Theorem</u> <u>1.7</u>--If $|Z - Q| \leq R$ and $w = \frac{1}{Z}$,

then

(i) if $|Q| - R = 0$, then there exists a number $a \neq 0$ and a real number q such that $\bar{a}w + a\bar{w} \leq 2q$. Furthermore if $\bar{a}w + a\bar{w} \leq 2q$, then $|Z - Q| \leq R$.

(ii) if $|Q| - R > 0$, then there exists a number c and a real number r such that $|w - c| \leq r$. Also if $|w - c| \leq r$, then $|Z - Q| \leq R$.

(iii) if $|Q| - R \nless 0$, then there exists a number c and a real number r such that $|w - c| \geq r$. Also if $|w - c| \geq r$, then $|Z - Q| \leq R$.

Since $w = \frac{1}{Z}$, then $Z = \frac{1}{w}$. If $|Z - Q| \leq R$,

then

$$\left|\frac{1}{w} - Q\right| \leq R;$$

$$|1 - wQ| \leq R\,|w|;$$

$$[1 - wQ]\,[1 - \overline{wQ}] \leq R^2\,|w|^2;$$

$$1 - \bar{w}\bar{Q} - wQ + w\bar{w}Q\bar{Q} \leq R^2 w\bar{w};$$

$$w\bar{w}\,(Q\bar{Q} - R^2) - wQ - \bar{w}\bar{Q} \leq -1. \quad (1.1)$$

(i) If $|Q| - R = 0$, then $|Q| = R$, or $|Q|^2 = R^2$, and $|Q|^2 - R^2 = 0$.

Let $a = -\bar{Q}$, $\bar{a} = -Q$ and $q = -1/2$.

Thus from equation (1.1) we $-Qw - \bar{Q}\bar{w} \quad -1$.

Hence if Z is in the circle $|Z - Q| \leq R$, then w is in the half plane $\bar{a}w + a\bar{w} \leq 2q$.

Furthermore if $\bar{a}w + a\bar{w} \leq 2q$, then since $a = -\bar{Q}$, $\bar{a} = -Q$ and $q = -1/2$ we have $-Qw - \bar{Q}\bar{w} \leq -1$. Since $|Q|^2 - R^2 = 0$, then $w\bar{w}(|Q|^2 - R^2) - Qw - \bar{Q}\bar{w} \leq -1$. Thus $\left|\frac{1}{w} - Q\right| \leq R$, and since $Z = \frac{1}{w}$, then $|Z - Q| \leq R$.

(ii) If $|Q| - R > 0$, then $|Q|^2 - R^2 > 0$. Let $c = \dfrac{\bar{Q}}{|Q|^2 - R^2}$ and $r = \dfrac{R}{|Q|^2 - R^2}$. The equation (1.1) becomes

$$w\bar{w} - \frac{Q}{|Q|^2 - R^2}w - \frac{\bar{Q}}{|Q|^2 - R^2}\bar{w} + \frac{|Q|^2}{[|Q|^2 - R^2]^2} \leq \frac{|Q|^2}{[|Q|^2 - R^2]^2} - \frac{1}{|Q|^2 - R^2};$$

$$\left|w - \frac{\bar{Q}}{|Q|^2 - R^2}\right|^2 \leq \frac{|Q|^2 - |Q|^2 + R^2}{[|Q|^2 - R^2]^2} = \frac{R^2}{[|Q|^2 - R^2]^2}.$$

Thus
$$\left|w - \frac{\bar{Q}}{|Q|^2 - R^2}\right| \leq \frac{R}{|Q|^2 - R^2}.$$

Therefore if Z is in the circle $|Z - Q| \leq R$, then w is in the circle $|w - c| \leq r$.

If $|w - c| \leq r$, then since $Z = \frac{1}{w}$, $c = \dfrac{\bar{Q}}{|Q|^2 - R^2}$ and $r = \dfrac{R}{|Q|^2 - R^2}$ we have

$$\left| \frac{1}{Z} - \frac{\bar{Q}}{Q^{2} - R^{2}} \right| \leq \frac{R}{|Q|^{2} - R^{2}} ;$$

$$\left| |Q|^{2} - R^{2} - \bar{Q}Z \right| \leq R|Z| ;$$

$$\left[|Q|^{2} - R^{2}\right]^{2} - \left[|Q|^{2} - R^{2}\right] \bar{Q}Z - \left[|Q|^{2} - R^{2}\right] Q\bar{Z} + |Q|^{2} |Z|^{2} \leq R^{2}|Z|^{2} ;$$

$$\left[|Q|^{2} - R^{2}\right]^{2} - \left[|Q|^{2} - R^{2}\right] \bar{Q}Z - \left[|Q|^{2} - R^{2}\right] Q\bar{Z} |Z|^{2} \left[Q^{2} - R^{2}\right] \leq 0 ;$$

$$|Q|^{2} - R^{2} - \bar{Q}Z - Q\bar{Z} + |Z|^{2} \leq 0 ;$$

$$|Z|^{2} - \bar{Q}Z - Q\bar{Z} + |Q|^{2} \leq R^{2} ;$$

$$|Z - Q|^{2} \leq R^{2} ;$$

$$|Z - Q| \leq R.$$

(iii) If $|Q| - R < 0$, then $|Q|^{2} - R^{2} < 0$, and $R^{2} - |Q|^{2} > 0$.

Let $c = \dfrac{\bar{Q}}{|Q|^{2} - R^{2}}$ and radius $r = \dfrac{R}{R^{2} - |Q|^{2}}$.

The equation (1.1) becomes

$$w\bar{w} - \frac{Q}{|Q|^{2} - R^{2}} w - \frac{\bar{Q}}{|Q|^{2} - R^{2}} \bar{w} \geq \frac{-1}{|Q|^{2} - R^{2}} ;$$

$$ww - \frac{Q}{|Q|^{2} - R^{2}} w - \frac{\bar{Q}}{|Q|^{2} - R^{2}} \bar{w} + \frac{Q^{2}}{\left[|Q|^{2} - R^{2}\right]^{2}}$$

$$\geq \frac{1}{R^2 - |Q|^2} + \frac{|Q|^2}{[R^2 - |Q|^2]^2} .$$

Hence $\left| w - \frac{\bar{Q}}{|Q|^2 - R^2} \right| \geq \frac{R}{R^2 - |Q|^2}$. Therefore if Z

is in the region $|Z - Q| \leq R$, then w is in the region $|w - c| \geq r$.

If $|w - c| \geq r$, then since $Z = \frac{1}{\bar{w}}$, $c = \frac{\bar{Q}}{|Q|^2 - R^2}$,

and $r = \frac{R}{R^2 - |Q|^2}$ we have

$$\left| \frac{1}{Z} - \frac{\bar{Q}}{|Q|^2 - R^2} \right| \geq \frac{R}{R^2 - |Q|^2} .$$

$$\left| \frac{1}{Z} - \frac{\bar{Q}}{|Q|^2 - R^2} \right| \geq \frac{-R}{|Q|^2 - R^2}$$

$$\left| |Q|^2 - R^2 - \bar{Q}Z \right| \geq + R|Z|$$

$$[|Q|^2 - R^2]^2 - [|Q|^2 - R^2]\bar{Q}Z - [|Q|^2 - R^2] Q\bar{Z} + |Q|^2|Z|^2 \geq R^2|Z|^2$$

$$[|Q|^2 - R^2]^2 - [|Q|^2 - R^2]\bar{Q}Z - [|Q|^2 - R^2] Q\bar{Z} + [|Q|^2 - R^2] |Z|^2 \geq 0$$

$$|Q|^2 - R^2 - \bar{Q}Z - Q\bar{Z} + |Z|^2 \leq 0$$

$$|Z|^2 - \bar{Q}Z - Q\bar{Z} + |Q|^2 \leq R^2$$

$|Z - Q|^2 \leq R^2$ Thus $|Z - Q| \leq R$

<u>Theorem</u> <u>1.8</u>--If $|Z - Q| \geq R$ and $w = \frac{1}{Z}$ then

(i) if $|Q| - R = 0$, then there exists a number $a \neq 0$ and a real number q such that $\bar{a}w \quad a\bar{w} \quad 2q$. Furthermobe if $\bar{a}w + a\bar{w} \leq 2q$, then $|Z - Q| \geq R$.

(ii) if $|Q| - R > 0$, then there exists a number c and a real number r such that $|w - c| \geq r$. Also if $|w - c| \geq r$, then $|Z - Q| \geq R$.

(iii) if $|Q| - R \angle 0$, then there exists a number c and a real number r such that $|w - c| \leq r$. Also if $|w - c| \leq r$, then $|Z - Q| \geq R$.

Since $w = \frac{1}{Z}$, then $Z = \frac{1}{w}$. If $|Z - Q| \geq R$ then $\left|\frac{1}{w} - Q\right| \geq R$. Then by the same method as used in Theorem 3.1 we get $w\bar{w}(Q^2 - R^2) - \overline{wQ} - wQ \geq -1$. (1.2)

(i) If $|Q| - R = 0$, then $|Q|^2 - R^2 = 0$. Let $a = \bar{Q}$, $\bar{a} = Q$ end $q = 1/2$. Then from equation (1.2) we have

$$- Qw - \overline{Qw} \geq - 1$$

$$Qw + \overline{Qw} \leq 1$$

$$\bar{a}w + a\bar{w} \leq 2q$$

Hence if Z is in the region $|Z - Q| \geq R$, then w is in the half plane $\bar{a}w + a\bar{w} \leq 2q$.

If $\bar{a}w + a\bar{w} \leq 2q$, then since $a = \bar{Q}$, $\bar{a} = Q$, and $q = 1/2$, we have $Qw + Q\bar{w} \leq 1$,

or $- Qw - Q\bar{w} \geq - 1$. Since $|Q|^2 - R^2 = 0$, then $w\bar{w}(|Q|^2 - R^2)$

$- Qw - \overline{Q}\overline{w} \quad - 1.$ Thus $\left|\frac{1}{w} - Q\right| \geqq R$, and since $Z = \frac{1}{w}$,

then $|Z - Q| \geqq R$.

(ii) If $|Q| - R > 0$, then $|Q|^2 - R^2 > 0$. Let $c = \dfrac{\overline{Q}}{|Q|^2 - R^2}$

and $r = \dfrac{R}{|Q|^2 - R^2}$. Then by the same method used in

Theorem 3.1 (ii) we have $\left|w - \dfrac{\overline{Q}}{|Q|^2 - R^2}\right| \geqq \dfrac{R}{|Q|^2 - R^2}$

Therefore if Z is in the region $|Z - Q| \geqq R$, then w is in the region $|w - c| \geqq r$.

If $|w - c| \geqq r$, then by the same method as used in Thworem 3.1 (ii) we have $|Z - Q| \geqq R$.

(iii) If $|Q| - R < 0$, then $|w - c| \leqq r$. Also if $|w - c| \leqq r$, then $|Z - Q| \geqq R$. This can be proved by the same method as used in Theorem 3.1 (iii).

<u>Theorem 1.9</u>--If $w = \frac{1}{Z}$ and $a\overline{Z}\ \ \overline{a}Z \leq 2p$, $a \neq 0$ then

(i) if $p = 0$, there exists a number $b \neq 0$ and a real number q such that $\overline{b}w + b\overline{w} \leq 2q$. Furthermore if $\overline{b}w + b\overline{w} \leq 2q$, then $\overline{a}Z + a\overline{Z} \leq 2p$.

(ii) if $p > 0$, then there exists a number c and a real number r such that $|w - c| \geq r$. Also if $|w - c| \geq r$, then $\overline{a}Z + a\overline{Z} \leq 2p$.

(iii) If $p < 0$, then there exists a number c and a real number r such that $|w - c| \leq r$. Also if $|w - c| \leq r$, then $\overline{a}Z + a\overline{Z} \leq 2p$.

Since $w = \frac{1}{Z}$, then $Z = \frac{1}{w} = \frac{\overline{w}}{|w|^2}$ and $\overline{Z} = \frac{w}{|w|^2}$.

If $$\overline{a}Z + a\overline{Z} \leq 2p,$$

then $$\overline{a}\frac{\overline{w}}{|w|^2} + a\frac{w}{|w|^2} \leq 2p$$

$$\overline{a}\overline{w} + aw \leq 2pw\overline{w} \qquad (1.3)$$

(i) If $p = 0$, Let $p = q$, $a = \overline{b}$, and $\overline{a} = b$. Thus from equation (1.3) we get $aw\ \ \overline{a}\overline{w} \leq 0$. Thus w is in the half plane $\overline{b}w + b\overline{w} \leq 2q = 0$.

Furthermore if $\overline{b}w\ \ b\overline{w}\ \ 2q$, then since $q = p = 0$, $b = a$ and $b = a$ we have

$$aw + \overline{a}\overline{w} \leq 2p = 0 = 2p/w/^2,$$

or $$\overline{a}\frac{\overline{w}}{|w|^2} + a\frac{w}{|w|^2} \leq 2p.$$

Since $Z = \frac{\overline{w}}{|w|^2}$, then $\overline{a}Z + a\overline{Z} \leq 2p$.

(ii) If $p > 0$. Let $c = \frac{\bar{a}}{2p}$, and $r = \frac{|a|}{2p}$. From equation (1.3) we have

$$2pw\bar{w} - aw - \bar{a}\bar{w} \geq 0$$

$$ww - \frac{a}{2p} w - \frac{\bar{a}}{2p} \bar{w} + \frac{a\bar{a}}{4p^2} \geq \frac{a\bar{a}}{4p^2}$$

$$\left| w - \frac{\bar{a}}{2p} \right|^2 \geq \frac{|a|^2}{4p^2}, \quad \text{or} \quad \left| w - \frac{\bar{a}}{2p} \right| \geq \frac{a}{2p}.$$

Therefore if Z is in the region $\bar{a}Z \quad a\bar{Z} \quad 2p$, then w is in the region $|w - c| \geq r$.

If $|w - c| \geq r$, then since $w = \frac{1}{Z}$, $c = \frac{\bar{a}}{2p}$ and $r = \frac{|a|}{2p}$

we have $\left| \frac{1}{Z} - \frac{\bar{a}}{2p} \right| \geq \frac{a}{2p}$,

$$|2p - \bar{a}Z| \geq |a|\,|Z|$$

$$4p^2 - 2p\bar{a}Z - 2pa\bar{Z} + |a|^2\,|Z|^2 \geq |a|^2|Z|^2$$

$$\bar{a}Z + a\bar{Z} - 2p \leq 0$$

$$\bar{a}Z + a\bar{Z} \leq 2p.$$

(iii) If $p < 0$. Let $c = \frac{\bar{a}}{2p}$ and $r = \frac{|a|}{2p}$. Then $\left| w - \frac{\bar{a}}{2p} \right| \leq \frac{|a|}{2p}$ may be proved by the same method as used in (ii) above. Therefore if Z is in the region $\bar{a}Z + a\bar{Z} \leq 2p$, then w is in the circle $|w - c| \leq r$.

Also $|w - c| \leq r$, then $aZ + aZ \leq 2p$. This can be proved by the same method as used in (ii) above.

Theorem 1.11--If $|Z - Q| \leq R$ and if $w = \frac{\alpha Z + \beta}{\gamma Z + \delta}$, where $\alpha\delta - \gamma B \neq 0$, then

(i) if $\gamma = 0$, then there exists a number c and a real number r such that $|w - c| \leq r$. Also if $|w - c| \leq r$, then $|Z - Q| \leq R$.

(ii) if $\gamma \neq 0$, then

a) if $|\gamma^2 Q + \gamma\delta| - |\gamma|^2 R = 0$, then there exists a number $a \neq 0$ and a real number p such that $\bar{a}w + a\bar{w} \leq 2p$. Also if $\bar{a}w + a\bar{w} \leq 2p$, then $|Z - Q| \leq R$.

b) if $\gamma^2 Q + \gamma\delta - |\gamma|^2 R > 0$, then there exists a number c and a real number r such that $|w - c| \leq r$, Also if $|w - c| \leq r$, then $|Z - Q| \leq R$.

c) if $|\gamma^2 Q + \gamma\delta| - |\gamma|^2 R < 0$, then there exists a number c and a real number r such that $|w - c| \geq r$. Also if $|w - c| \geq r$, then $|Z - Q| \leq R$.

(i) If $\gamma = 0$, then $w = \frac{\alpha}{\delta} Z + \frac{\beta}{\delta}$. Let $w_1 = \frac{\alpha}{\delta} Z$ and $w = w_1 \ \frac{B}{\delta}$. Since $|Z - Q| \leq R$, then by Theorem 1.1 there is a c_1 and an r_1 such that $|w_1 - c_1| \leq r_1$. By Theorem 1.4 there is a c and an r such that $|w - c| \leq r$.

If $|w - c| \leq r$, then $|Z - Q| \leq R$ may be proved by the same method of reasoning as above.

(ii) :- If $\gamma \neq 0$. Let $w_2 = \gamma^2 Z$, $w_3 = w_2 + \gamma\delta$, $w_4 = 1/w_3$, $w_5 = (\beta\gamma - \alpha\delta)w_4$, and $w = w_5 \ \frac{\alpha}{\gamma}$. Since $|Z - Q| \leq R$, then by Theorem 1.1 there is a c_2 and an r_2 such that $|w_2 - c_2| \leq r_2$, where $c_2 = \gamma^2 Q$ and $r_2 = |\gamma|^2 R$. By Theorem 1.4 there is a c_3

and an r_3 such that $|w_3 - c_3| \leq r_3$, where $c_3 = \gamma^2 Q$ and $r_3 = |\gamma|^2 R$.

a) If $|c_3| - r_3 = 0$, then by Theorem 1.7 (i) there is an $a_4 \neq 0$ and a p_4 such that $\bar{a}_4 w_4 + a_4 \bar{w}_4 \leq 2p_4$. By Theorem 1.3 there is an $a_5 \neq 0$ and a p_5 such that $\bar{a}_5 w_5 + a_5 \bar{w}_5 \leq 2p_5$. By Theorem 1.6 there is an $a \neq 0$ and a p such that $\bar{a}w + a\bar{w} \leq 2p$.

If $\bar{a}w + a\bar{w} \leq 2p$, then $|Z - Q| \leq R$ may be proved by the same method of reasoning as above.

b) If $|c_3| - r_3 > 0$, then by Theorem 1.7(ii) there is a c_4 and an r_4 such that $|w_4 - c_4| \leq r_4$. By Theorem 1.1 there is a c_5 and an r_5 such that $|w_5 - c_5| \leq r_5$. By Theorem 1.4 there is a c and an r such that $|w - c| \leq r$.

If $|w - c| \leq r$, then $|Z - Q| \leq R$ may be proved by the same method of reasoning as above.

c) If $|c_3| - r_3 < 0$, then by Theorem 1.7(iii), there is a c_4 and an r_4 such that $|w_4 - c_4| \geq r_4$. By Theorem 1.2 there is a c_5 and an r_5 such that $|w_5 - c_5| \geq r_5$. By Theorem 1.5 there is a c and an r such that $|w - c| \geq r$.

If $|w - c| \geq r$, then $|Z - Q| \leq R$ may be proved by the same method of reasoning as above.

<u>Theorem</u> <u>1.12</u>:- If $|Z - Q| \geqq R$ and if $w = \dfrac{\alpha Z + \beta}{\gamma Z + \delta}$,

where $\alpha\delta - \gamma\beta \neq 0$, then

(i) if $\gamma = 0$, then there exists a number c and a real number r such that $|w - c| \geqq r$. Also if $|w - c| \geqq r$, then $|Z - Q| \geqq R$.

(ii) if $\gamma \neq 0$, then

a) if $\gamma^2 Q + \gamma\delta - |\gamma|^2 R = 0$, then there exists a number $a \neq 0$ and a real number p such that $\bar{a}w + a\bar{w} \leqq 2p$. Also if $\bar{a}w + a\bar{w} \leqq 2p$, then $|Z - Q| \geqq R$.

b) if $|\gamma^2 Q + \gamma\delta| - |\gamma|^2 R \quad 0$, then there is a number c and a real number r such that $|w - c| \geqq r$. Also if $|w - c| \geqq r$, then $|Z - Q| \geqq R$.

c) if $\gamma^2 Q + \gamma\delta - |\gamma|^2 R > 0$, then there exists a number c and a real number r such that $|w - c| \leqq r$. Also if $|w - c| \leqq r$ then $|Z - Q| \geqq R$.

(i) If $\gamma = 0$, then $w = \frac{\alpha}{\delta} Z + \frac{B}{\delta}$. Let $w = \frac{\alpha}{\delta} Z$ and $w = w_1 , + \frac{\beta}{\delta}$. Since $|Z - Q| \geqq R$, then by Theorem 1.2 there is a c_1 and an r_1 such that $|w_1 - c_1| \geqq r_1$. By Theorem 1.5 there is a c and an r such that $|w - c| \geqq r$.

If $|w - c| \geqq r$, then $|Z - Q| \geqq R$ may be proved by the same method of reasoning as above.

(ii) If $\gamma \neq 0$. Let $w_2 = \gamma^2 Z$, $w_3 = w_2 \gamma\delta$, $w_4 = 1/w_3$, $w_5 = (\beta\gamma - \alpha\delta) w_4$, and $w = w_5 + \frac{\alpha}{\gamma}$. Since $|Z - Q| \geqq R$, then by Theorem 1.2 there is a c_2 and an r_2 such that $|w_2 - c_2| \geqq r$, where $c_2 = \gamma^2 Q$ and $r_2 = |\gamma|^2 R$. By Theorem 1.5 there is a c_3 and an r_3 such that $|w_3 - c_3| \geqq r_3$, where $c_3 = \gamma^2 Q + \gamma\delta$ and $r_3 = |\gamma|^2 R$.

a) If $|c_3| - r_3 = 0$, then by Theorem 1.8(i) there is an $a_4 \neq 0$ and a p_4 such that $\bar{a}_4 w_4 + a_4 \bar{w}_4 \leqq 2p_4$. By Theorem 1.3 there is an $a_5 \neq 0$ and a p_5 such that $\bar{a}_5 w_5 + a_5 \bar{w}_5 \leqq 2p_5$. By Theorem 1.6 there is an a 0, and a p such that $\bar{a}w + a\bar{w} \leqq 2p$.

If $\bar{a}w + a\bar{w} \leqq 2p$, then $|Z - Q| \geqq R$ may be proved by the same method of reasoning as the above.

b) If $|c_3| - r_3 > 0$, then by Theorem 1.8(ii) there is a c_4 and an r_4 such that $|w_4 - c_4| \geqq r_4$. By Theorem 1.2 there is a c_5 and an r_5 such that $|w_5 - c_5| \geqq r_5$. By Theorem 1.5 there is a c and an r such that $|w - c| \geqq r$.

If $|w - c| \geqq r$, then $|Z - Q| \geqq R$ may be proved by the same method of reasoning as above.

c) If $|c_3| - r_3 < 0$, then by Theorem 1.8(iii), there is an r_4 such that $|w_4 - c_4| \leqq r_4$. By Theorem 1.1

there is a c_5 and an r_5 such that $|w_5 - c_5| \leq r_5$. By Theorem 1.4 there is a c and an r such that $|w - c| \leq r$.

If $|w - c| \leq r$, then $|Z - Q| \leq R$ may be proved by the same method of reasoning as above.

<u>Theorem 1.13</u>--If $\bar{a}Z + a\bar{Z} \leq 2p$ where $a \neq 0$ and if $w = \dfrac{\alpha Z + \beta}{\gamma Z + \delta}$ where $\alpha\delta - \gamma\beta \neq 0$, then

(i) if $\gamma = 0$, there exists a number $c \neq 0$ and a real number q such that $\bar{c}w + c\bar{w} \leq 2q$. Furthermore if $\bar{c}w + c\bar{w} \leq 2q$, then $\bar{a}Z + a\bar{Z} \leq 2p$.

(ii) if $\gamma \neq 0$, then

a) if $|\gamma|^4 p + \mathcal{R}(\bar{a}\bar{\gamma}^2\gamma\delta) = 0$, then there exists a number $c \neq 0$, and a real number q such that $\bar{c}w + c\bar{w} \leq 2q$. Furthermore if $\bar{c}w + c\bar{w} \leq 2q$, then $\bar{a}Z + a\bar{Z} \leq 2p$.

b) if $|\gamma|^4 p + \mathcal{R}(\bar{a}\bar{\gamma}^2\gamma\delta) > 0$, then there exists a number c and a real number r such that $|w - c| \geq r$. Also if $|w - c| \geq r$, then $\bar{a}Z + a\bar{Z} \leq 2p$.

c) if $|\gamma|^4 p + \mathcal{R}(\bar{a}\bar{\gamma}^2\gamma\delta) < 0$, then there exists a number c and a real number r such that $|w - c| \leq r$. Also if $|w - c| \leq r$, then $\bar{a}Z + a\bar{Z} \leq 2p$.

(i) If $\gamma = 0$, then $w = \dfrac{\alpha}{\delta}Z + \dfrac{\beta}{\delta}$. Let $w_1 = \dfrac{\alpha}{\delta}Z$ and $w = w_1 + \dfrac{\beta}{\delta}$. Since $\bar{a}Z + a\bar{Z} \leq 2p$, then by Theorem 1.3 there is a $c_1 \neq 0$ and a q_1 such that $\bar{c}_1 w_1 + c_1\bar{w}_1 \leq 2q_1$. By Theorem 1.6 there is a $c \neq 0$ and a q such that $\bar{c}w + c\bar{w} \leq 2q$.

If $\bar{c}w + c\bar{w} \leq 2q$, then $\bar{a}Z + a\bar{Z} \leq 2p$ may be proved by the same method of reasoning as above.

(ii) If $\gamma \neq 0$. Let $w_2 = \gamma^2 Z$, $w_3 = w_2 + \gamma\delta$, $w_4 = 1/w_3$,

$w_5 = (\beta\gamma - \alpha\delta)w_4$, and $w = w_5 + \frac{\alpha}{\gamma}$. Since $\bar{a}Z + a\bar{Z} \leq 2p$,

then by Theorem 1.3 there is a $c_2 \neq 0$ and a q_2 such that $\bar{c}_2 w_2 + c_2\bar{w}_2 \leq 2q_2$, where $c_2 = a\gamma^2$ and $q_2 = |\gamma|^4 p$. By Theorem 1.6 there is a $c_3 \neq 0$ and a q_3 such that $\bar{c}_3 w_3 + c_3\bar{w}_3 \leq 2q_3$ where $c_3 = a\gamma^2$ and $q_3 = |\gamma|^4 p + \mathcal{R}(\bar{a}\,\bar{\gamma}^2\gamma\delta)$

a) If $q_3 = 0$ then by Theorem 1.9 (i) there is a c_4 and a q_4 such that $\bar{c}_4 w_4 + c_4\bar{w}_4 \leq 2q_4$. By Theorem 1.3 there is a c_5 and a q_5 such that $\bar{c}_5 w_5 + c_5\bar{w}_5 \leq 2q_5$. By Theorem 1.6 there is a $c \neq 0$ and a q such that $\bar{c}w + c\bar{w} \leq 2q$.

If $\bar{c}w + c\bar{w} \leq 2q$, the $\bar{a}Z + a\bar{Z} \leq 2p$ may be proved by the same method of reasoning as above.

b) If $q_3 > 0$, then by Theorem 1.9 (ii) there is a c_4 and an r_4 such that $|w_4 - c_4| \geq r_4$. By Theorem 1.2 there is a c_5 and an r_5 such that $|w_5 - c_5| \geq r_5$. By Theorem 1.5 there is a c and an r such that $|w - c| \geq r$.

If $|w - c| \geq r$, then $\bar{a}Z + a\bar{Z} \leq 2p$ may be proved by the same method of reasoning as above.

c) If $q_3 < 0$, then by Theorem 1.9 (iii) there is a c_4 and an r_4 such that $|w_4 - c_4| \leq r_4$. By Theorem 1.1 there is a c_5 and an r_5 such that $|w_5 - c_5| \leq r_5$. By Theorem 1.4 there is a c and an r such that $|w - c| \leq r$.

If $|w - c| \lneqq r$, then $\bar{a}Z + a\bar{Z} \leq 2p$ may be proved by the same method of reasoning as above.

CHAPTER II

CONVERGENCE OF A CONTINUED FRACTION

__Theorem__ __2.1__:- If $0<K<1$ and $|a_{2p}| \leq K^2$ and $|a_{2p-1}| \leq (1-K)^2$, where $p = 1, 2, 3, \ldots$, then the continued fraction

$$\cfrac{1}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{1+\cdots}}}}} \qquad (2.1)$$

Converges

Let f_n be the nth approximate of the continued fraction (2.1). Let $t_p(u) = \frac{1}{1+a_pu}$ and $T_p(u) = t_1\ t_2\ t_3 \ldots t_p(u)$,

where $p = 1, 2, 3, \ldots$. Then

$T_1(u) = t_1(u) = \frac{1}{1\ \ a_1u}$, and $T_1(0) = 1 = f_1$.

$T_2(u) = t_1\left[t_2(u)\right] = t_1\ t_2(u) = \cfrac{1}{1+\cfrac{a_1}{1+a_2u}}$ and

$$T_2(0) = \frac{1}{1+a_1} = f_2.$$

$$T_3(u) = t_1\ t_2\ t_3\ (u) = \cfrac{}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+a_3 u}}} \quad \text{and}$$

$$T_3(0) = \cfrac{1}{1+\cfrac{a_1}{1+a_2}} = f_3.$$

$$T_p(u) = t_1\ t_2\ t_3 \cdots t_{p-1}\ t_p(u) = \cfrac{1}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cdots \cfrac{1}{1+\cfrac{a_{p-1}}{1+a_p u}}}}}}$$

and $$T_p(0) = \cfrac{1}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cdots \cfrac{1}{1 \quad a_{p-1}}}}}} = f_p \text{ where } p = 1,2,3, \ldots.$$

If M is a set of numbers, let $T_p(M)$ and $t_p(M)$ be the sets of numbers which can be obtained from the elements u of M by transformations $T_p(u)$ and $t_p(u)$ respectively, where p = 1, 2, 3, ...

<u>Lemma</u> (<u>2.1</u>). If $w = \dfrac{1}{1+a_{2p+1}u}$, 0 K 1,

and $|u| \leq \dfrac{1}{1-K}$, then $w \leq \dfrac{1}{K}$.

Since $w = \dfrac{1}{1+a_{2p+1}u}$, then $|u| = \left|\dfrac{1-w}{a_{2p+1} \quad w}\right|$.

Since $|u| \leq \dfrac{1}{1-k}$ then $\dfrac{1-w}{a_{2p+1} \, w} \leq \dfrac{1}{1-K}$ and

$$|1-w| \leq \frac{|a_{2p \quad 1} w|}{(1-K)} .$$

$$w\bar{w} - w - \bar{w} + 1 \leq \frac{\left|a_{2p+1}\right|^2 w\bar{w}}{(1 - K)^2}$$

$$\left[1 - \frac{\left|a_{2p+1}\right|^2}{(1 - K)^2}\right] w\bar{w} - w - \bar{w} \leq -1.$$

$$ww - \frac{(1 - K)^2}{(1 - K)^2 - \left|a_{2p\ \ 1}\right|^2}\, w - \frac{(1 - K)^2}{(1 - K)^2 - \left|a_{2p+1}\right|^2}\, \bar{w} \leq \frac{(1 - K)^2}{(1 - K)^2 - \left|a_{2p\ \ 1}\right|^2}$$

$$ww - \frac{(1 - K)^2}{(1 - K)^2 - \left|a_{2p+1}\right|^2}\, w - \frac{(1 - K)^2}{(1 - K)^2 - \left|a_{2p+1}\right|^2}\, \bar{w} + \frac{(1 - K)^4}{\left[(1 - K)^2 - \left|a_{2p+1}\right|^2\right]^2} \leq \frac{(1 - K)^4}{\left[(1 - K)^2 - \left|a_{2p+1}\right|^2\right]^2} - \frac{(1 - K)^2}{(1 - K)^2 - \left|a_{2p+1}\right|^2}$$

$$\left| w - \frac{(1 - K)^2}{(1 - K)^2 - \left|a_{2p+1}\right|^2} \right|^2 \leq \frac{(1 - K)^4}{\left[(1 - K)^2 - \left|a_{2p\ \ 1}\right|^2\right]^2} -$$

$$\frac{\left[(1-K)^4 - (1-K)^2 \left|a_{2p+1}\right|^2\right]}{\left[(1-K)^2 - \left|a_{2p+1}\right|^2\right]^2}$$

$$\left| w - \frac{(1-K)^2}{(1-K)^2 - \left|a_{2p+1}\right|^2} \right|^2 \leq \frac{(1-K)^2 \left|a_{2p+1}\right|^2}{\left[(1-K)^2 - \left|a_{2p+1}\right|^2\right]^2}$$

$$\left| w - \frac{(1-K)^2}{(1-K)^2 - \left|a_{2p+1}\right|^2} \right| \leq \frac{(1-K)\left|a_{2p+1}\right|}{(1-K)^2 - \left|a_{2p+1}\right|^2}$$

Therefore w is in a circle with center $c = \dfrac{(1-K)^2}{(1-K)^2 - \left|a_{2p+1}\right|^2}$

and radius $r = \dfrac{(1-K)\left|a_{2p+1}\right|}{(1-K)^2 - \left|a_{2p+1}\right|^2}$. Now c is real and

since $(1-K)^2 \quad \left|a_{2p+1}\right|$, then c is positive. Hence $|c| + r = c + r$.

$$c + r = \frac{(1-K)^2}{(1-K)^2 - \left|a_{2p+1}\right|^2} + \frac{(1-K)\left|a_{2p+1}\right|}{(1-K)^2 - \left|a_{2p+1}\right|^2}$$

$$= \frac{(1-K)\left[(1-K)\left|a_{2p+1}\right|\right]}{(1-K)^2 - \left|a_{2p+1}\right|^2} = \frac{(1-K)}{(1-K) - \left|a_{2p+1}\right|}$$

But $\left| a_{2p+1} \right| \leq (1 - K)^2$.

Therefore $|c| + r \leq \dfrac{(1 - K)}{(1 - K) - (1 - K)^2} = \dfrac{1}{1 - (1 - K)} = \dfrac{1}{K}$.

Since w is in the circle with center c and radius r, then $|w - c| \leq r$, or $|w| - |c| \leq r$, or $|w| \leq c + r \leq 1/K$.

Let L be the region $|u| \leq \dfrac{1}{1 - K}$ and S be the region $|w| \leq \dfrac{1}{K}$,

then t_{2p+1} (L) is contained in S.

<u>Lemma</u> (<u>2.2</u>) If $w = \dfrac{1}{1 \quad a_{2p}u}$, $0 < K < 1$, and $|u| \leq \dfrac{1}{K}$,

then $|w| \leq \dfrac{1}{1 - K}$.

Since $w = \dfrac{1}{1 + a_{2p}u}$, then $|u| = \left| \dfrac{1 - w}{a_{2p} \; w} \right|$. Since

$|u| \leq \dfrac{1}{K}$, then $\left| \dfrac{1 - w}{a_{2p}w} \right| \leq \dfrac{1}{K}$ and $|1 - w| \leq \dfrac{|a_{2p}| \; |w|}{K}$.

$$w\bar{w} - w - \bar{w} + 1 \leq \frac{|a_{2p}|^2 \; w\bar{w}}{K^2}$$

$$\left[1 - \frac{|a_{2p}|^2}{K^2} \right] w\bar{w} - w - \bar{w} \leq -1$$

$$w\bar{w} - \frac{K^2}{K^2 - |a_{2p}|^2} \, w - \frac{K^2}{K^2 - |a_{2p}|^2} \, \bar{w} \leq - \frac{K^2}{K^2 - |a_{2p}|^2}$$

$$w\bar{w} - \frac{K^2}{K^2 - |a_{2p}|^2} w - \frac{K^2}{K^2 - |a_{2p}|^2} \bar{w} \quad \frac{K^4}{\left[K^2 - |a_{2p}|^2\right]^2} \leq$$

$$\frac{K^4}{\left[K^2 - |a_{2p}|^2\right]} - \frac{K^2}{K^2 - |a_{2p}|^2}$$

$$\left| w - \frac{K^2}{K^2 - |a_{2p}|^2} \right|^2 \leq \frac{K^4}{\left[K^2 - |a_{2p}|^2\right]^2} - \frac{K^4 - K^2 |a_{2p}|^2}{\left[K^2 - |a_{2p}|^2\right]^2}$$

$$\left| w - \frac{K^2}{K^2 - |a_{2p}|^2} \right|^2 \leq \frac{K^2 |a_{2p}|^2}{\left[K^2 - |a_{2p}|^2\right]^2}$$

$$\left| w - \frac{K^2}{K^2 - a_{2p}{}^2} \right| \leq \frac{K |a_{2p}|}{K^2 - |a_{2p}|^2} .$$

Therefore w is in a circle with center $c = \dfrac{K^2}{K^2 - |a_{2p}|^2}$ and radius $r = \dfrac{K |a_{2p}|}{K^2 - |a_{2p}|^2}$. Now c is real and since $K^2 \geqq |a_{2p}|$, then c is positive. Hence $|c| + r = c + r$.

$$c + r = \frac{K^2}{K^2 - |a_{2p}|^2} + \frac{K |a_{2p}|}{K^2 - |a_{2p}|^2}$$

$$= \frac{K\left[K + |a_{2p}|\right]}{K^2 - |a_{2p}|^2} = \frac{K}{K - |a_{2p}|}.$$

But $|a_{2p}| \leq K^2$. Therefore $|c| + r \leq \dfrac{K}{K - K^2} = \dfrac{1}{1 - K}$.

Since w is in the circle with center c and radius r, then $|w - c| \leq r$, or $|w| - |c| \leq r$, or $|w| \leq |c| + r \leq \dfrac{1}{1 - K}$. As $|w| \leq \dfrac{1}{1 - K}$, then t_{2p} (S) is contained in L. Hence

$T_2(S) = t_1 \, t_2$ (S) which is contained in $t_1(L)$ which is contained in S. Suppose $T_{2p}(S)$ is contained in S. Then

$T_{2(p+1)}$ (S) = $T_{2p} \, t_{2p+1} \, t_{2p+2}$ (S) which is contained in

$T_{2p} \, t_{2p+1}$ (L) which is contained in T_{2p} (S) which is contained in S. Therefore T_{2n} (S) is contained in S, where n = 1, 2, 3, ... Let $t_{2n-1} \, t_{2n}(S) = S_1^{2n}$, where n = 1, 2, 3, . . .

and let $t_{2n-2p-1} \, t_{2n-2p} (S_p^{2n}) = S_{(p+1)}^{2n}$, p = 1, 2, 3, . . . , (n - 1). Then $T_{2p}(S) = S_p^{2p}$, where p = 1, 2, 3, ... Let the radius of the region $S_p^{2p} = R_p$, where p = 1, 2, 3, . . . and the radius of the region $S_p^{2n} = R_p^{2n}$, where n = 1, 2, 3, . . . and p = 1, 2, 3, . . . n.

Let $|u - c| \leq r$ be a region contained in S and $w = \dfrac{1}{1 + a_{2p}u}$,

then $|u| = \left|\dfrac{1 - w}{a_{2p}w}\right|$. Therefore $\left|\dfrac{1 - w}{a_{2p}w} - c\right| \leq r$.

Then

$$|1 - w - ca_{2p}w| \leq r\,|a_{2p}w|$$

$$|1 - w(1 \quad ca_{2p})| \leq r\,|a_{2p}|\,|w|$$

$$1 - w(1 + ca_{2p}) - \overline{w}(1 + \overline{ca}_{2p}) + w\overline{w}\,|1 + ca_{2p}|^2 \leq r^2 |a_{2p}|^2\, w\overline{w}$$

$$w\overline{w}\left[|1 \quad ca_{2p}|^2 - r^2 |a_{2p}|^2\right] - w(1 + ca_{2p}) - \overline{w}(1 + \overline{c}a_{2p}) \leq - 1$$

$$w\overline{w} - \frac{1 + ca_{2p}}{|1 + ca_{2p}|^2 - r^2 |a_{2p}|^2}\,w - \frac{1 + \overline{ca}_{2p}}{|1 + ca_{2p}|^2 - r^2 |a_{2p}|^2}\,\overline{w}$$

$$+ \frac{|1 + ca_{2p}|^2}{\left[|1 \quad ca_{2p}|^2 - r^2 |a_{2p}|^2\right]^2} \leq \frac{|1 + ca_{2p}|^2}{\left[|1 \quad ca_{2p}|^2 - r^2 |a_{2p}|^2\right]^2} - \frac{1}{|1 + ca_{2p}|^2 - r^2 |a_{2p}|^2}$$

$$\left| w - \frac{1+\overline{ca}_{2p}}{1+ca_{2p}^{\;2} - r^2\, a_{2p}^{\;\;2}} \right|^2 \leq$$

$$\frac{\left|1\;\; ca_{2p}\right|^2 - \left[\left|1+ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2\right]}{\left[\left|1\;\; ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2\right]^2}$$

$$\left| w - \frac{1+\overline{ca}_{2p}}{\left|1+ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2} \right|^2 \leq \frac{r^2\left|a_{2p}\right|^2}{\left[\left|1+ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2\right]^2}$$

$$\left| w - \frac{1+\overline{ca}_{2p}}{\left|1+ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2} \right| \leq \frac{r\left|a_{2p}\right|}{\left|1+ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2}$$

Therfore w is in a circle with center

$$Q = \frac{1+\overline{ca}_{2p}}{\left|1+ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2}$$

and radius $R = \dfrac{r\left|a_{2p}\right|}{\left|1+ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2}$.

By Lemma (2.1), since $|u - c| \leq r$ is in region S, then $|w - Q| \leq R$ is in region L.

If $|u - Q| \leq R$ is a region in L and $w = \dfrac{1}{1 \quad a_{2p-1} u}$,

then $|u| = \left|\dfrac{1 - w}{a_{2p-1} w}\right|$. Therfore $\left|\dfrac{1 - w}{a_{2p-1} w} - Q\right| \leq R$. By the same procedure as above we get

$$\left| w - \frac{1 + \overline{Q}\,\overline{A}_{2p-1}}{|1 + QA_{2p-1}|^2 - R^2 |a_{2p-1}|^2} \right| \leq \frac{R|a_{2p-1}|}{|1 + QA_{2p-1}|^2 - R^2 |a_{2p-1}|^2}$$

Therefore w is in a circle with center

$$\alpha = \frac{1 + \overline{Q}\,\overline{A}_{2p-1}}{|1 + QA_{2p-1}|^2 - R^2 |a_{2p-1}|^2}$$

and radius $\beta = \dfrac{R|a_{2p-1}|}{|1 \quad QA_{2p-1}|^2 - R^2 |a_{2p-1}|^2}$

Now the ratio of the radii $\dfrac{\beta}{r} = \dfrac{\beta}{R} \cdot \dfrac{R}{r}$.

But
$$\frac{\beta}{R} = \frac{|a_{2p-1}|}{|1 + Qa_{2p-1}|^2 - R^2|a_{2p-1}|^2} \leq$$

$$\frac{|a_{2p-1}|}{\left[1 - |Q|\,|a_{2p-1}|\right]^2 - R^2|a_{2p-1}|^2}$$

Since $|u - Q| \leq R$ is contained in the region L, $|Q| + R \leq \frac{1}{1-K}$

or $|Q| \leq \frac{1}{1-K} - R$. Also $|a_{2p-1}| \leq (1-K)^2$.

Now
$$\frac{\beta}{R} \leq \frac{|a_{2p-1}|}{\left[1 - |Q|\,|a_{2p-1}|\right]^2 - R^2|a_{2p-1}|^2}$$

$$\leq \frac{(1-K)^2}{\left[1 - \left(\frac{1}{1-K} - R\right)(1-K)^2\right]^2 - R^2(1-K)^4}$$

$$= \frac{(1-K)^2}{\left\{1 - \left[1 - R(1-K)\right](1-K)\right\}^2 - R^2(1-K)^2}$$

$$= \frac{(1-K)^2}{\left[1 - (1-K) + R(1-K)^2\right]^2 - R^2(1-K)^4}$$

$$= \frac{(1-K)^2}{\left[K + R(1-K)^2\right]^2 - R^2(1-K)^4}$$

$$= \frac{(1-K)^2}{K^2 + 2R(1-K)^2 + R^2(1-K)^4 - R^2(1-K)^4}$$

$$= \frac{(1-K)^2}{K^2 + 2R(1-K)^2} < \frac{(1-K)^2}{K^2}$$

Thus $\dfrac{\rho}{R} < \dfrac{(1-K)^2}{K^2}$. Also $R = \dfrac{r\left|a_{2p}\right|}{\left|1 + ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2}$

Then $\dfrac{R}{r} = \dfrac{\left|a_{2p}\right|}{\left|1 + ca_{2p}\right|^2 - r^2\left|a_{2p}\right|^2}$. Since $|u - c| \leq r$ is contained in region L, $|c| + r \leq \dfrac{1}{K}$, or $|c| \leq \dfrac{1}{K} - r$, also $\left|a_{2p}\right| \leq K^2$ and $\dfrac{R}{r} \leq \dfrac{\left|a_{2p}\right|}{\left[1 - |c|\left|a_{2p}\right|\right]^2 - r^2\left|a_{2p}\right|^2}$.

$$\frac{R}{r} \leq \frac{K^2}{\left[1 - \left(\frac{1}{K} - r\right)K^2\right]^2 - r^2K^4} = \frac{K^2}{\left[1 - (1 - rK)K\right]^2 - r^2K^4}$$

$$= \frac{K^2}{\left[1 - K + rK^2\right]^2 - r^2K^4} = \frac{K^2}{\left[1 - 2K + 2rK^2 + K^2 - 2rK^3 + r^2K^4\right] - r^2K^4}$$

$$= \frac{K^2}{(1 - 2K + K^2) + 2rK^2(1 - K)} = \frac{K^2}{(1 - K)^2 + 2rK^2(1 - K)}$$

Thus $\dfrac{R}{r} \leq \dfrac{K^2}{(1 - K)^2 + 2rK^2(1 - K)} < \dfrac{K^2}{(1 - K)^2}$

Now $\dfrac{\rho}{r} = \dfrac{\rho}{R} \quad \dfrac{R}{r} < \dfrac{(1 - K)^2}{K^2} \quad \dfrac{K^2}{(1 - K)^2 \quad 2rK^2(1 - K)}$

$$= \frac{(1 - K)}{(1 - K) \quad 2rK^2} < 1.$$

Now R_1^{2p} is the radius of $t_{2p-1}\, t_{2p}\,(S)$. Hence the ratio

$$R_1^{2p} \Big/ \frac{1}{K} < \frac{(1 - K)}{(1 - K) + 2(1/K)K^2} = \frac{1 - K}{(1 - K) + 2K} = \frac{1 - K}{1 + K} < 1$$

$R_{(q+1)}^{2p}$ is the radius of $t_{2p-2q-1}\, t_{2p-2q}\,(S_q^{2p})$ where $p = 1, 2, 3, \cdots$ and $q = 1, 2, 3, \cdots, (p - 1)$. The ratio

$$R_{(q+1)}^{2p} \Big/ R_q^{2p} < \frac{1 - K}{(1 - K) + 2R_q^{2p}\, K^2} < 1.$$

Thus $1/K > R_1^2 = R_1$, $1/K > R_1^4 > R_2^4 = R_2$, $1/K > R_1^6 > R_2^6 > R_3^6 = R_3$, $1/K > R_1^{2n} > R_2^{2n} > R_3^{2n} > \cdots > R_n^{2n} = R_n$. Since $S_n^{2n} = T_{2n}\,(S)$,

$T_{2n}(S) = T_{2(n-1)}\, t_{2n-1}\, t_{2n}(S) = T_{2(n-1)}\left(S_1^{2n}\right)$ which is contained in $T_{2(n-1)}(S) = S_{n-1}^{2(n-1)}$. Thus S_n^{2n} is contained $S_{n-1}^{2(n-1)}$. Then $1/K \geq R_1 \geq R_2 \geq R_3 \geq \cdots \geq R_n$.

Since the sequence $\{R_n\}$ is a monotone non-increasing sequence of positive numbers then the sequence $\{R_n\} \to d \geq 0$. If $d > 0$, then $R_n \geq d$, where $n = 1, 2, 3, \cdots$. Also $1/K \geq R_p^{2n} \geq R_n \geq d$, where $n = 1, 2, 3, \cdots$ and $p = 1, 2, 3, \cdots, n$.

Then $\dfrac{R^{2p}_{(q+1)}}{R^{2p}_q} < \dfrac{(1-K)}{(1-K) + 2R_q^{2p}K^2} < \dfrac{(1-K)}{(1-K) + 2dK^2} = G < 1,$

where $p = 1, 2, 3, \cdots$ and $q = 1, 2, 3, \cdots, (p-1)$. Therefore $R_n \leq 1/K\ G^n$ where $n = 1, 2, 3, \cdots$. There exists a positive integer say p such that G^p Kd. Thus $R_p < \frac{1}{K}$ $G^p < 1/K \cdot Kd = d$. This contradicts $R_n \geq d$. Therefore $d = 0$.

Let c_n be the center of the region S_n^{2n} with radius R_n where $n = 1, 2, 3, \cdots$. If n m M, then S_n^{2n} is contained in S_m^{2m} which is contained in S_M^{2M} and hence $|c_n - c_M| < R_M$

and $|c_m - c_M| < R_M$. If $\epsilon > 0$, there exists an integer $N > 0$, such that $R_N < \frac{\epsilon}{2}$. If $n > m > N$, then $|c_n - c_m| < \epsilon$. Therefore the sequence of centers c_n converges to some number say A.

Suppose there is one region S_p^{2p} which does not contain A, then $|A - c_p| > R_p$. Let $|A - c_p| - R_p = X$. There is an N such that if $n \geq N$, then $|A - c_n| < X$. Let $q = N + p$. Since $q = N + p$, then the region S_q^{2q} is contained in the region S_p^{2p} and $|c_p - c_q| < R_p$. Since $q > N$, then $|A - c_q| < X$. Therefore

$$|A - c_p| = |A - c_q + c_q - c_p| \leq |A - c_q| + |c_q - c_p|$$

$$< X + R_p = |A - c_p| - R_p + R_p = |A - c_p| .$$

Hence $|A - c_p| < |A - c_p|$ which is a contradiction and therefore the region S_p^{2p} contains A. Hence A is in all of the regions S_n^{2n}, where $n = 1, 2, 3, \cdots$.

Suppose that B is a point in all the regions S_n^{2n}, where $n = 1, 2, 3, \cdots$. Since A is in all the regions, $|A - c_n| < R_n$ and $|B - c_n| < R_n$. Thus $|A - B| < 2R_n$. Since

$\{R_n\} \to 0$, then $|A - B| = 0$ and therefore $A = B$. Hence there is only one point contained in all of the regions S_n^{2n}, where $n = 1, 2, 3, \cdots$.

Since zero is in the region S, and $f_{2p} = T_{2p}(0)$ and $T_{2p}(S)$ contains $T_{2p}(0)$, then f_{2p} is in $T_{2p}(S)$. Also zero is in the region L and hence f_{2p+1} is contained in $T_{2p+1}(L)$. But $T_{2p+1}(L) = T_{2p}\ t_{2p+1}(L)$ which is contained in $T_{2p}(S)$. Thus f_{2p} and f_{2p+1} are in $T_{2p}(S) = S_p^{2p}$ which has radius R_p. If $\epsilon > 0$, then there is a positive integer N such that $R_n < \frac{\epsilon}{2}$. If $n > 2N$, then f_n is in the region S_N^{2N}, and $|A - f_n| \le |A - c_N + c_N - f_n| \le |A - c_N| + |c_N - f_n| \le 2R_N < \epsilon$. Therefore the continued fraction (2.1) has the value A.

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