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PROPAGATION AND MODE CONVERSION FOR WAVES IN NONUNIFORM PLASMAS

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I. Introduction

A magnetized plasma can usually support two or more modes at the same frequency, often with approximately the same electric polarization. For most choices of plasma parameters these modes will have widely differing wave-numbers, but it may also happen that at some critical set of parameters the wave-numbers will coalesce. Considering now the case for a wave propagating through an inhomogeneous plasma and approaching a region where the local plasma parameters would correspond to such a coalescence of wave-numbers, one may ask whether the incident wave will be reflected or refracted in the same mode, or absorbed, or converted into the complementary mode, or undergo some combination of all these. It is this question to which mode-conversion theory is addressed.

A common situation under which mode conversion may occur is that where $k_x(x)$, i.e., the propagation vector in the direction of the inhomogeneity, calculated from cold-plasma uniform-medium theory using local values of the plasma parameters, diverges at some $x = x_0$. The condition that $k^2 \rightarrow \infty$ is termed a "resonance;" well-known plasma resonances include the ion and electron cyclotron resonances, the upper, lower, and two-ion hybrid resonances, and the Alfvén resonance. Readers familiar with radiofrequency techniques for heating inhomogeneous plasmas will recognize that it is at or near one of these resonances that plasma absorption

of the radiofrequency heating power is usually expected. There is therefore an intimate tie between mode conversion and wave absorption, and it is for this reason that physicists interested in rf plasma heating have devoted great effort to the study of mode-conversion processes.

In the conventional case, propagation of waves through slightly inhomogeneous media may be analyzed by the methods of geometrical optics. As described in Chap. 2.5, these techniques can provide the wave trajectory, the space-time variation of wave amplitude, the conservation of wave energy, and other wave features. The assumptions underlying geometrical optics, as well-known, are that fractional changes in wavelength and frequency over a period in space or time, respectively, shall be small. Generally the validity of these assumptions is guaranteed by the caveat cited above, namely, restriction to "slightly inhomogeneous" media. For wave propagation in plasmas, however, the occurrence of certain phenomena (the parameters for which depend sensitively upon n_e or B_0 , for instance) can cause a dramatic modification in wavelength over a short distance while plasma density or background magnetic-field strength undergo, in the same space interval, only very modest change. An analogy would be the quantum-mechanical description of an electron drifting in an electrostatic field. At some point a small shift in electrostatic potential reduces the available electron kinetic energy to zero, and the wave-function propagation vector at this

critical point changes from real to imaginary. As in the quantum-mechanical analogy, the plasma wave on each side of the critical layer can be described by a WKB approximation, but a special analysis of the region immediately at the critical layer itself is required to join the two asymptotic solutions.

The drifting-electron analysis has an exact analogy in plasmas, namely, for plasma waves approaching a "cutoff," i.e., a point where the propagation vector goes to zero. An illustration would be the familiar electromagnetic plasma wave,

$$k_{xc}^2 = \omega^2 - \omega_{pe}^2, \quad (1)$$

approaching the layer in a gently inhomogeneous plasma where $\omega^2 = \omega_{pe}(x)^2$. The corresponding wave equation to be solved is

$$-\frac{1}{c^2} \frac{d^2 \psi}{dx^2} = \left[\omega^2 - \frac{4\pi n_e(x) e^2}{m_e} \right] \psi, \quad (2)$$

and an analysis of the joining problem may be found in many texts on quantum mechanics.

More interesting than the case of plasma-wave cutoff is that for plasma-wave "resonance," where, in some approximation, $k^2 \rightarrow \infty$. The hybrid resonances of cold-plasma theory furnish several such examples. For a background magnetic field in the z-direction, the cold-plasma (i.e., $\omega/k_z \gg v_{th}^{(e)}$, $k_x \rho_L \ll 1$, etc.) wave equation is

$$\nabla \times (\nabla \times \vec{E}) - \frac{\omega^2}{c^2} \vec{\epsilon} \cdot \vec{E} = 0, \quad (3)$$

$$\vec{\epsilon} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix},$$

$$S = \frac{1}{2} (R + L), \quad D = \frac{1}{2} (R - L),$$

$$R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \left(\frac{\omega}{\omega + \omega_{cs}} \right),$$

$$L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \left(\frac{\omega}{\omega - \omega_{cs}} \right),$$

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2},$$

$$\omega_{ps}^2 = \frac{4\pi n_s q_s^2}{m_s}, \quad \omega_{cs} = \frac{q_s B_0}{m_s c}$$

where s designates particle species (the formulation is still valid when several species of ions are present in the plasma) and q_s is the particle charge including its algebraic sign. The uniform-medium dispersion relation, for $\vec{k} = \hat{x}k_x + \hat{z}k_z$, is found by equating the determinant of the wave equation to zero,

$$\beta n_x^4 - \gamma n_x^2 + \delta = 0, \quad (4)$$

$$\beta = S,$$

$$\gamma = RL + PS - Pn_z^2 - Sn_z^2,$$

$$\delta = P(R - n_z^2)(L - n_z^2),$$

$$\vec{n} = \vec{k}c/\omega,$$

and the cold-plasma hybrid resonances ($n_x^2 \rightarrow \infty$) are given simply by $S = 0$, satisfied by

$$\omega_{UH}^2 \approx \omega_{pe}^2 + \omega_{ce}^2 \quad \text{upper-hybrid resonance} \quad (5)$$

$$\frac{1}{\omega_{LH}^2} \approx \frac{1}{\omega_{ci}^2 + \omega_{pi}^2} + \left| \frac{1}{\omega_{ci}\omega_{ce}} \right| \quad \text{lower-hybrid resonance} \quad (6)$$

$$\omega_{ii}^2 \approx \omega_{ci}\omega_{cj} \frac{x_i\omega_{cj} + x_j\omega_{ci}}{x_i\omega_{ci} + x_j\omega_{cj}} \quad \text{ion-ion hybrid resonance (7)}$$

(high-density limit)

where, in the third case, two different ion species are present in the plasma with relative concentrations x_i and x_j .

The physical nature of the hybrid resonances is a blend of two effects -- collective oscillation and space-charge neutralization. Sheets of electrons oscillating in a neutralizing quasi-stationary ion background will create their own electric field (by Gauss' law). Subject to this first-order E field and to a background zeroth-order B_0 field, the natural frequency for

oscillation is the upper-hybrid frequency. On the other hand the high-density ion-ion hybrid frequency is dominated by the two ion clouds oscillating 180° out of phase with each other, with such amplitude as to cancel the total oscillating ion space charge. The lower-hybrid frequency shows both effects, the second term on the right representing space-charge cancellation of the oscillating ion and electron charge clouds.

Now just as it modifies the nature of the simple Langmuir-Tonks plasma oscillations ($\omega^2 = \omega_{pe}^2$), finite temperature also modifies the hybrid resonances and, in particular, eliminates the divergence in n_x^2 . But where the finite-temperature Bohm-Gross modification [$\omega^2 = \omega_{pe}^2 (1 + 3k^2 kT_e / m_e \omega^2 + \dots)$] to the Langmuir-Tonks dispersion relation only introduces a frequency shift, the corresponding alteration of the hybrid oscillations is more dramatic -- new modes are introduced, the so-called Bernstein modes. The background magnetic field plays an important part in these modes, among its roles, eliminating free-electron streaming perpendicular to $\vec{B}_0$ and the Landau damping which would otherwise have been associated with large k_x values.

Thus it turns out that a magnetized plasma can support two or more modes at the same frequency and it may also happen that, at some critical set of parameters the wave-numbers will coalesce. From the viewpoint of the long wavelength mode, the coalescence layer looks like a resonance ($k_x^2 \rightarrow \infty$), while an approximate dispersion relation for the short wavelength mode would describe the coalescence as a cutoff ($k_x^2 \rightarrow 0$). For

example, the inclusion of lowest-order finite-Larmor radius terms will add a correction to the cold-plasma dispersion relation, (4), which then looks like

$$\alpha n_x^6 - \beta n_x^4 + \gamma n_x^2 - \delta = 0 \quad (8)$$

and, neglecting the contribution from the δ term, approximate solutions for the long and short wavelength modes will be $n_x^2 \approx \gamma/\beta$ and $n_x^2 \approx \beta/\alpha$, in regions where $\beta^2 \gg |4\alpha\gamma|$. Coalescence will occur where $\beta = \beta(x) \rightarrow 0$ and $n_x^2 \approx \pm(-\gamma/\alpha)^{1/2}$.

For a wave propagating through a slightly inhomogeneous plasma and approaching such a coalescence layer, one may ask whether the wave will be reflected or refracted in the same mode, or absorbed, or converted into the complementary mode. In mathematical terms, the first step in the analysis of mode conversion is to reduce the physical representation to an ordinary linear differential equation such as

$$\alpha \frac{d^4 \psi}{dx^4} + \beta(x) \frac{d^2 \psi}{dx^2} + \gamma(x) \psi = 0 \quad (9)$$

where $\beta(x) \rightarrow 0$ at $x = 0$ and, then, to find the coefficients which connect the asymptotic solutions for large $x = |x|$ to those for large $x = -|x|$. In the next three sections we look in some detail at specific wave-propagation situations in which mode conversion is important; this discussion is then followed by an analysis of model mode-conversion equations for several forms of $\gamma(x)$.

II. The Hybrid Resonance

In the discussion of the cold-plasma uniform-medium dispersion relation, Eq. (4), it was mentioned that the condition $\epsilon_{xx} \equiv S \neq 0$ satisfies this relation in the limit $n_x^2 \rightarrow \infty$, and that $S = 0$ is in turn satisfied by the three cold-plasma hybrid resonances listed in Eqs. (5), (6), and (7). The inclusion of finite-Larmor radius effects in $\vec{\epsilon}$ introduces spatial dispersion in the dielectric tensor, i.e., $\vec{\epsilon} = \vec{\epsilon}(\lambda_i, \lambda_e)$ where

$$\lambda_s \equiv \frac{k_x^2 k_{Ts}}{m_s \omega_{cs}^2} = \frac{1}{2} k_x^2 \langle \rho_{Ls}^2 \rangle. \quad (10)$$

A comparison of (8) and (4) then reveals that the dominant contribution, for small values of λ_s , to α will be

$$\alpha \approx - \sum_s \lim_{\lambda_s \rightarrow 0} \frac{\omega^2}{\omega_{cs}^2} \frac{k_{Ts}}{m_s c^2} \frac{\partial}{\partial \lambda_s} \epsilon_{xx}(\lambda_i, \lambda_e). \quad (11)$$

Moreover, the warm-plasma dispersion relations, in regions where $\beta^2 \gg |4\alpha\gamma|$, which one approximates by $n_x^2 \approx \beta/\alpha$, can also be written in forms analogous to the Bohm-Gross relation, namely,

$$\omega^2 = \omega_H^2 + \sum_s x_s \gamma_{Hs} k_x^2 k_{Ts} / m_s \quad (12)$$

where the sums are over the particle species present in relative concentrations x_s , and the subscript H can refer to any of three

hybrid frequencies from Eqs. (5)-(7). γ_{HS} is frequency and parameter dependent; the thermal correction for the upper hybrid frequency is dominated by the electron contribution, for the ion-ion hybrid by the ion contribution, while both ions and electrons contribute to the thermal correction for the lower-hybrid mode.

The thermal effects discussed in the preceding paragraph are just the corrections to first order in λ_i and λ_e to the cold-plasma hybrid resonances. The full electrostatic dispersion relation [taking the divergence of (3) and using $\vec{E} \approx -\nabla\phi + -i\vec{k}\phi$] is valid for all orders of λ_i and λ_e ,

$$\vec{k} \cdot \vec{\epsilon} \cdot \vec{k} = 0. \quad (13)$$

With $k_y \approx 0$ and $k_z \approx 0$, (13) reduces to simply $\epsilon_{xx} = 0$. An evaluation of ϵ_{xx} from the Vlasov equation for the case $k_z = 0$ yields

$$\epsilon_{xx} = 1 - \sum_s \frac{4\pi n_s m_s c^2}{B_0^2} \frac{\alpha(\Omega_s, \lambda_s)}{\lambda_s} \quad (14)$$

where

$$\alpha(\Omega, \lambda) = 2 \sum_{n=1}^{\infty} e^{-\lambda} I_n(\lambda) \frac{n^2}{\Omega^2 - n^2} \quad (15)$$

$$\begin{aligned}
 &= \frac{\lambda}{\Omega^2 - 1^2} + \frac{1 \cdot 3 \lambda^2}{(\Omega^2 - 1^2)(\Omega^2 - 2^2)} \\
 &+ \frac{1 \cdot 3 \cdot 5 \lambda^3}{(\Omega^2 - 1^2)(\Omega^2 - 2^2)(\Omega^2 - 3^2)} + \dots
 \end{aligned} \tag{16}$$

and where $\Omega_S = \omega/\omega_{cS}$. Proof of the expansion in (16), suggested by T. H. Stix (1965), has been given by J. P. M. Schmitt (1974). The first term on the right in (16) is sufficient to lead to Eqs. (5)-(7), while the second term produces the thermal corrections in (12).

For large values of λ and when ω/ω_c is very close to an integer, say j , the summation in (15) is dominated by the single resonant term, leading to an approximate dispersion relation for the very short wavelength ion mode, abbreviating $\lambda = \lambda_i$,

$$\frac{\omega^2}{\omega_{pi}^2} = \left[\frac{j^3 e^{-\lambda} I_j(\lambda)}{\lambda} \right] \frac{\omega_{ci}}{\omega - j\omega_{ci}} \tag{17}$$

and a similar relation for electrons, in which the subscript i is replaced by e . As ω_{pi}^2 varies linearly with density, a maximum on the right-hand side of (17) will determine the minimum density for propagation in this $\omega \approx j\omega_c$ case. The maximum of the quantity in square brackets occurs for $S \equiv I_j'(\lambda)/I_j(\lambda) = 1 + (1/\lambda)$.

Noting that S obeys the first-order differential equation $S' + S^2 + S/\lambda = 1 + (n/\lambda)^2$, a brief calculation in matching coefficients to the asymptotic expansion $S = 1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \dots$ will show that the quantity in square brackets in (17) reaches the maximum value ~ 0.463 when

$$\lambda = \frac{j^3}{3} + \frac{1}{6} + \frac{5}{j^2} + \dots \quad (18)$$

Based on these approximations which apply to the long wavelength [Eq. (4)], intermediate wavelength [three left-most terms in Eq. (8); also Eq. (11)] and the short wavelength [Eqs. (13)-(17)] dispersion relations, one can sketch composite curves which portray, in a WKB sense, the behavior of a wave propagating through an inhomogeneous plasma. Figure 1 is such a sketch for propagation around the upper-hybrid resonance, while Fig. 2 pertains to the lower-hybrid case. For Fig. 2, it is assumed that the cold-plasma dispersion relation predicts propagation over the entire region between the low-density cut-off at $\omega^2 = \omega_{pe}^2$ and the lower-hybrid critical layer, at $\omega^2 = \omega_{LH}^2$. The Stix-Golant condition for "accessibility" in this sense is that

$$\frac{k_z^2 c^2}{\omega^2} \geq 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} \quad (19)$$

Sketches such as Figs. 1 and 2, based on dispersion relations derived for uniform plasmas, are of fundamental importance in understanding the phenomenon of mode conversion. It is possible for wave-energy at very short wave-lengths to follow downward along trajectories such as those sketched in Figs. 1 and 2 and be converted, in a gradual manner, to long wavelengths. For example, it is this "mode-conversion" process which was offered by T. H. Stix (1965) as an explanation for the radiation observed outside a small reflex-discharge column by G. Landauer (1962), the radiation spectrum showing peaks at high harmonics of the electron cyclotron frequency. E. Canobbio and R. Croci (1964) suggested that such a spectrum could result from the excitation of electron Bernstein waves by suprathermal electrons. This short wavelength energy could then propagate inward to the upper hybrid critical layer, reflect and mode-convert at this layer, and then tunnel outward past the right-hand cutoff to reach the detecting apparatus outside the plasma via the electromagnetic QT-X mode.

It is the role of mode-conversion calculations to determine the efficiency of such energy transformations or, more precisely, to evaluate the transmission, reflection, and absorption coefficients for the different stages of the conversion process.

III. The Alfvén Resonance

In a uniform magnetized plasma, there are two low-frequency electromagnetic modes, namely, the shear and the compressional Alfvén waves. With $\vec{B}_0 = \hat{z}B_0$, their dispersion relations are, respectively,

$$k_z^2 c^2 / \omega^2 = 1 + 4\pi \rho c^2 / B_0^2, \quad (20)$$

$$(k_x^2 + k_y^2 + k_z^2) c^2 / \omega^2 = 1 + 4\pi \rho c^2 / B_0^2. \quad (21)$$

With respect to the shear wave, Eq. (20), one notes that the dispersion relation is independent of $k_\perp$ and, in addition, that the group velocity is exactly along $\vec{B}_0$. Energy in this mode therefore does not spread perpendicular to $\vec{B}_0$ and one may envision that wave-motion on each line of force propagates along $\vec{B}_0$ independently of the waves on the adjacent lines. It is this strong degeneracy which underlies the Alfvén resonance. The degeneracy is removed by any of a large number of effects including density gradients, a case explored originally in considerable detail by J. A. Tataronis and W. A. Grossman (1976) and by L. Chen and A. Hasegawa (1974). These authors pointed out that the differential equation describing MHD wave propagation in a nonuniform plasma becomes singular at the Alfvén resonance layer. In the context of this MHD theory, wave energy is absorbed at the resonance

layer by wave phase-mixing and the spectrum of a bounded plasma containing such a singular layer is continuous rather than discrete. Introduction of additional physical effects including $m_e \neq 0$ and finite ion Larmor radius affects the mathematical singularity at the Alfvén resonance layer, but also raises the differential equation from second to fourth or higher order, thus introducing new solutions which describe short wavelength modes. Analysis by mode-conversion theory shows that the wave-energy which the MHD equations had indicated to be absorbed at the Alfvén resonance is, in fact, converted to these new short wavelength modes. The relevant uniform-medium dispersion relation is Eq. (4), but modified to include the most important finite ion-Larmor-radius correction. In detail, the dielectric tensor elements represented by S, D, and P in (4) must be corrected

$$S = \epsilon_{xx}(\lambda) = [\epsilon_{xx} + \lambda \frac{d\epsilon_{xx}}{d\lambda} + \frac{\lambda^2}{2} \frac{d^2\epsilon_{xx}}{d\lambda^2} + \dots]_{\lambda=0} \quad (22)$$

where $\lambda = \lambda_i$ is defined in (10). Writing

$$\lambda_i = \frac{k_x^2 k_{Ti}}{\omega_{ci}^2 m_i} = n_x^2 \frac{k_{Ti}}{m_i c^2} \frac{\omega^2}{\omega_{ci}^2}, \quad (23)$$

it is clear that the lowest-order finite-Larmor-radius correction to S in the coefficient γ in (4) shows up as a correction to the n_x^4 coefficient, namely,

$$\beta \rightarrow S - P \frac{kT_i}{m_i c^2} \frac{\omega^2}{\omega_{ci}^2} \frac{d\epsilon_{xx}}{d\lambda_i} \quad (24)$$

$$\approx 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} \left[1 - \frac{\beta_i m_i}{2Zm_e} \left(1 - \frac{\omega_{ci}^2 - \omega^2}{4\omega_{ci}^2 - \omega^2} \right) \right]$$

where $\beta_i = 8\pi n_i kT_i / B^2$ and where ω is not too near to either ω_{ci} or $2\omega_{ci}$. In the low-frequency range, $\omega \ll \omega_{ci}$, the coefficient in square brackets changes sign when $\beta_i = 8Zm_e/3m_i$. Thus for very low β_i , β is positive and the appropriate sketch for the FLR-corrected Eq. (4) is Fig. 3, while for $\beta_i > 8Zm_e/3m_i$, the appropriate sketch is Fig. 4. The short wavelength mode in the low- β_i case is seen to be a surface wave, while this mode in the higher- β_i case propagates inward from the critical layer toward higher densities.

Finite electron temperature has an effect remarkably similar to ion temperature in changing from the "cold-plasma" behavior sketched in Fig. 3 to "warm-plasma" behavior, Fig. 4. The reason, quite different in this case, may be seen in Eq. (4): the coefficient δ and the dominant terms in the coefficient γ are all proportional to $P \equiv \epsilon_{zz}$. As the electron temperature rises so that $2kT_e/m_e > \omega^2/k_z^2$, the sign of the real part of ϵ_{zz} changes from negative to positive. The net result on the dispersion relation, (4), is the same as changing the sign of the coefficient β . The condition for the electron cold-to-hot transition at the Alfvén resonance layer, where $n_z^2 = S$, is that

$S = (k_z^2 c^2 / \omega^2)_{\text{critical}} = m_e c^2 / 2kT_e$. For $\omega \ll \omega_{ci}$, this implies transition at $\beta_e = 2m_e/n_i$.

Figures 3 and 4 are based on the FLR-modified Eq. (4), which applies to uniform plasmas. The degeneracy associated with the Alfvén shear mode, Eq. (20), is removed in this case not by a plasma inhomogeneity term, but rather by finite ω/ω_{ci} effects which cause $R \neq L$, set (3). In the small- n_x^2 range, the approximate dispersion relation is $n_x^2 \approx \delta/\gamma$ in (4), or

$$n_x^2 \approx \frac{(R - n_z^2)(L - n_z^2)}{(S - n_z^2)} \quad (25)$$

which would predict $n_x^2 \rightarrow \infty$ at the Alfvén resonance, $n_z^2 = S$. However, as discussed in the preceding paragraph, the finite m_e and FLR effects restructure this mathematical singularity and introduce the short-wavelength modes [$n_x^2 \approx \gamma/\beta$, Eq. (4)] indicated in the large- n_x^2 range in Figs. 3 and 4.

Equation (25) shows the two cutoffs ($n_x^2 \rightarrow 0$), occurring at $n_z^2 = R$ and $n_z^2 = L$ (the right- and left-hand cutoffs, respectively), and both cutoffs may be seen in Figs. 3 and 4. The mathematics of the mode-conversion analysis will be seen to be greatly simplified in the case that $|L| \gg |R|$, which removes the right-hand cutoff from the doublet formed by the back-to-back left-hand cutoff ($n_z^2 = L$) and the nearby Alfvén resonance ($n_z^2 = S$). However, the inequality $|L| \gg |R|$ pertains only when $\omega \approx \omega_{ci}$, and in the classic Alfvén regime, namely $\omega \ll \omega_{ci}$, one

has $R = S + D$, $L = S - D$, $|D| \ll S$ leading to a closely spaced cutoff-resonance cutoff triplet. As mentioned earlier, a number of effects destroy the Alfvén degeneracy, and when $\omega \ll \omega_{ci}$, the finite ω/ω_{ci} splitting portrayed in Figs. 3 and 4 can be matched in magnitude by effects due to density gradients, to variation of the magnitude and direction of $\vec{B}_0$, and to cylindrical or toroidal geometry. As an illustration of the manner in which a wave equation to describe a moderately complex mode-conversion situation may be derived, we now examine this "classic Alfvén" situation in some detail.

IV. The Intermediate-Frequency Electromagnetic Wave Equation

We present here a derivation of the electromagnetic wave equation applicable to the intermediate range of frequencies, i.e., frequencies lying within a factor of five or ten, say, higher or lower than the ion-cyclotron frequency. Cold-plasma theory with $m_e = 0$ will be used at first to bring in the effects due to

- (a) Finite ω/ω_{ci} ,
- (b) Density gradient, $n_0 = n_0(x)$,
- (c) Magnetic gradient, $B_0 = B_0(x)$,
- (d) Magnetic shear, $\vec{B}_0 = \vec{B}_0(x)$, and
- (e) Possible presence of several ion species.

Then the arguments leading to Eq. (24) in the preceding section may be invoked to introduce the dominant fourth-order derivative terms due to

- (f) Finite electron mass and
- (g) Finite ion Larmor radius, including $T_i = T_i(x)$.

Extensive as this list is, it still omits other effects which can play significant roles in plasma experiments, including

- (h) Cylindrical geometry, or
- (i) Toroidal geometry, and
- (j) $\vec{B}_0 \cdot \nabla B_0 \neq 0$,
- (k) Finite electron pressure,
- (l) Ion cyclotron damping, and
- (m) Electron Landau and TTMP damping.

Touching briefly on these omitted items, cylindrical geometry, (h), will introduce the familiar singularity at $r = 0$ together with some less dramatic modifications to the coefficients in the wave equation. In addition to these effects, toroidal geometry, (i), will also bring in coupling between waves of different poloidal mode number, the coupling of an m -mode to an $m \pm j$ mode varying in strength as $(r/R)^j$. Item (j) is an unfortunate artifact of real CTR geometries including tokamaks and mirrors, and its proper treatment requires the solution of the wave equation in two or three dimensions. One role played by finite electron pressure was already discussed in connection with Eq. (24). Another role is to introduce ion acoustic waves and drift waves. These are generally low-frequency slow modes with electrostatic polarization which might have significant coupling only under highly specialized circumstances to the intermediate-frequency waves under consideration here. Finally, items (l) and (m) introduce damping effects, which will be discussed in Section VII.

Turning now to the specifics of the derivation of the wave equation which include effects (a) through (e), we choose $\vec{B}_0 = B_0(x) [\hat{y} \sin\psi + \hat{z} \cos\psi]$, $\psi = \psi(x)$, and apply Eq. (3) to this zero-order geometry. The dielectric tensor, $\vec{\epsilon}$, must be oriented with respect to the local $\vec{B}_0$. For this purpose we introduce the unit vectors $\hat{x}$, $\hat{p} = \hat{b} \times \hat{x}$, and $\hat{b} = \vec{B}_0/B_0$, so that

$$\hat{y} = \hat{p} \cos\psi + \hat{b} \sin\psi \quad (26)$$

$$\hat{z} = -\hat{p} \sin\psi + \hat{b} \cos\psi$$

and define $E_x = \hat{x} \cdot \vec{E}$, $E_1 = \hat{p} \cdot \vec{E}$. [In toroidal geometry, $\hat{z}$ would go over to the toroidal direction, $\hat{x}$ to the direction of the minor radius, and $\hat{y}$ to the poloidal direction. The unit vector $\hat{p}$ is perpendicular both to $\hat{x}$ and to the magnetic field direction, $\hat{b}$.]

In slab geometry, the first-order fields vary as

$E(x, y, z) = E(x) \exp(ik_y y + ik_z z - i\omega t)$, and k_y and k_z are assumed to be real and constant in space. The dimensionless wave numbers in the parallel ($\hat{b}$) direction, $n_{||}$, and in the perpendicular ($\hat{p}$) direction, $n_{\perp}$, are

$$n_{\perp} = (k_y \cos\psi - k_z \sin\psi)c/\omega, \quad (27)$$

$$n_{||} = (k_y \sin\psi + k_z \cos\psi)c/\omega.$$

Straightforward evaluation of (3) now displays its $\hat{x}$, $\hat{p}$, and $\hat{b}$ components, respectively,

$$i(n_{\perp} E_1 + n_{||} E_n)' + (n_{||}^2 + n_{\perp}^2) E_x = \hat{x} \cdot \vec{\epsilon} \cdot \vec{E}, \quad (28)$$

$$-E''_1 + \psi'^2 E_1 + n''^2 E_1 - \psi'' E_n - n_1 n_n E_n - 2\psi' E_n + i n_1 E'_x = \hat{p} \cdot \vec{e} \cdot \vec{E}$$

$$-E''_n + \psi'^2 E_n + n_1^2 E_n + \psi'' E_1 - n_1 n_n E_1 + 2\psi' E'_1 + i n_n E'_x = \hat{b} \cdot \vec{e} \cdot \vec{E}$$

where (') denotes differentiation with respect to the dimensionless variable $x\omega/c$. We have also used two identities which follow from (27),

$$n'_1 = -n_n \psi' , \quad (29)$$

$$n'_n = n_1 \psi' .$$

For the longer wavelengths it is justified to let $m_e \rightarrow 0$ which causes $\epsilon_{nn} \equiv P \rightarrow -\infty$, Eq. (3). To keep PE_n , on the right-hand side of the third equation in set (28), finite requires $E_n \rightarrow 0$. This simplification reduces set (28) to just two equations,

$$(i n_1 E_1)' + i D E_1 - T E_x = 0 \quad (30)$$

$$E''_1 + (T + n_1^2 - \psi'^2) E_1 - i n_1 E'_x + i D E_x = 0$$

where $T \equiv S - n_1^2 - n_n^2 = \epsilon_{xx} - (k_y^2 + k_z^2) c^2 / \omega^2$. Eliminating E_x leads to the intermediate-frequency electromagnetic cold-plasma

$m_e = 0$ slab model wave equation, valid for plasmas with several species of ions present,

$$E_1'' + GE_1' + HE_1 = 0, \quad (31)$$

$$G = \frac{(T + n_1^2)'}{T + n_1^2} - \frac{T'}{T},$$

$$H = T - \frac{n_1(n_1'T' + DT')}{T(T + n_1^2)} + \frac{n_1n_1'' - \psi'^2T + n_1D' - n_1'D - D^2}{T + n_1^2}$$

The singularity at $T + n_1^2 = S - n_n^2 = 0$ is the Alfvén resonance condition. The zero at $T = S - n_n^2 - n_1^2 = 0$ turns out to be a much weaker singularity; the two singularities are discussed and compared in Section IX.

Despite its algebraic complexity, one still expects -- away from the singularities and in the absence of dissipative processes -- that Eq. (31) will show conservation of wave energy. The Poynting vector in the x-direction is

$$\begin{aligned} P_x &= \frac{c}{4\pi} (\vec{E} \times \vec{B}) \cdot \hat{x} = \frac{c}{4\pi} E_1 B_n \\ &= \frac{c}{16\pi i} E_1^* (E_1' - i n_1 E_x) + cc \\ &= \frac{c}{16\pi i} [E_1^* E_1' (1 + \frac{n_1^2}{T})] + cc \end{aligned} \quad (32)$$

where the second and third lines are expressed in terms of the E-field complex Fourier amplitudes. Maxwell's induction equation was used to derive the second line, and E_x was evaluated by (30), assuming D and T are both real. In the absence of absorption (i.e., for D and T real), one expects P_x to be conserved. It is easy to show that, for G and H real, the quantity $(E_1^* E_1' - E_1 E_1'^*) \exp \left[\int G d(x_0/c) \right]$ is conserved by Eq. (31), which, on carrying out the integral, is seen to concur with (32) and which thus confirms P_x conservation in this instance.

As one simple check on the validity of Eq. (31), we may apply it to the case of a uniform medium. Then $E_1' \rightarrow -n_x^2 E_1$, $G \rightarrow 0$ and $H \rightarrow T - D^2/(T+n_1^2)$. In this form, Eq. (31) becomes identical to Eq. (25), provided Eq. (25) is appropriately modified for the rotation of axes, namely, replacing n_x^2 in (25) by $n_x^2 + n_1^2$.

In this same vein, it is worth writing down explicitly the simple form which Eq. (31) assumes when there is no shear and when $n_1^2 = 0$. Making use of the identities in (3), one has

$$\frac{d^2 E_1}{dx^2} + \frac{(R - n_n^2)(L - n_n^2)}{S - n_n^2} = 0 \quad (33)$$

which is exactly the wave equation that Eq. (25) would lead us to expect.

The interesting applications of Eqs. (31) and (33) are frequently to media with nonuniformities such that the algebraic complexities dictate their solution by numerical methods. However, analytic approximations may be obtained in certain model

cases and these are discussed in Section IX. Moreover, in conjunction with short-wavelength and cyclotron-harmonic correction terms as described in Section III, Eqs. (31) and (33) may be used to construct model mode-conversion equations, as in Sections V and VI, valid for plasmas with several ion species present, and for frequencies up to several times the ion cyclotron frequency.

V. The Standard Equation

The discussion in the previous sections described the physical environment for mode conversion and outlined means for deriving an appropriate wave equation. However, the inclusion of even a reasonably small number of physical effects can make the resulting equation cumbersome indeed, and analytic tractability demands that one work with simplified equations which model the resonances and cutoffs in a credible fashion. The basic model equation, which we term the "Standard Equation," is in the form of Eq. (9), namely,

$$\psi'''' + \lambda^2 z \psi'' + \gamma \psi = 0 \quad (34)$$

γ is constant, the scaling parameter, λ , is a positive constant, and the independent variable is z .

For real values of λ^2 , z , and γ , there is a conservation law for the Standard Equation analogous to the conservation of P_x , Eq. (32), by Eq. (31). In the case of (34), the conserved quantity is

$$P = \psi'''\psi'^* - \gamma\psi'\psi^* - cc. \quad (35)$$

Conservation of P may be quickly verified by computing dP/dz , using (34) to evaluate ψ'''' . It should be kept in mind that the Standard Equation, (34), and the Tunneling Equation, (47), are just model equations for the mode-conversion process. For these equations, the behavior of the quantity P , (35) and (48), is that which one would expect for wave energy flow. Now for the hot-plasma case, the actual wave-energy flow in the sum of the Poynting flux, (32), plus an "acoustic" energy flux proportional to $\partial(\vec{E}^* \cdot \vec{\epsilon} \cdot \vec{E})/\partial \vec{k} + cc$. The construction of more accurate wave equations which conserve this actual wave-energy flux appears to require the introduction of odd-order derivatives and density-gradient terms [H. H. Kuehl, B. B. O'Brien, and G. E. Stewart (1970); also D. G. Swanson (unpublished)].

Returning to consideration of (34), looking now at large values of $|\lambda^2 z|$, the four asymptotic solutions of (34) are

$$\begin{aligned}\psi_{\text{slow}} &\sim |z|^{-5/4} \exp(\pm 2i\lambda z^{3/2}/3) , \\ \psi_{\text{fast}} &\sim |z|^{1/4} \exp[\pm 2i(\gamma z)^{1/2}/\lambda] .\end{aligned}\tag{36}$$

Recalling the time dependence, suppressed here, that $\psi \sim \exp(-i\omega t)$, one notes that in the regions where λz or γz are positive, these solutions represent waves moving toward or away from the origin, $z = 0$. For negative λz or γz , the solutions represent slow-or-fast decaying-or-growing amplitudes in a region of wave evanescence. These various asymptotic solutions are only valid, however, far to the left and far to the right of the origin. The objective of the mode-conversion analysis, then, is to establish the connection formulas between them.

In general, the analytic determination of the connection formulas requires the mathematical technique of matched asymptotic expansions, and an example of a calculation by this method is given in Section IX. However, Eq. (34), and the more general Eq. (47) below, display coefficients which vary only linearly in z , and for this reason these equations lend themselves to an elegant mathematical treatment by integral transforms. Their mathematical analysis was first carried through by W. Wasow (1950) and by A. L. Rabenstein (1958), while their application to plasma

physics wave theory was initiated by T. H. Stix (1965) and to plasma instability theory by D. Gorman (1966).

Following now through the several steps of the integral transform analysis, we first multiply Eq. (34) through by e^{-pz} . Integrating over z several times by parts leads to the transformed equation

$$p^4 \tilde{\psi} - \lambda^2 \frac{d}{dp} (p^2 \tilde{\psi}) + \gamma \tilde{\psi} = \text{integrated terms} \quad (37)$$

where

$$\tilde{\psi}(p) = \int e^{-pz} \psi(z) dz. \quad (38)$$

It will be seen that the integrated terms need not be retained. Equation (37) is now quickly integrated [it is this step which is facilitated by the linear variation of the coefficients in (34) and (47)],

$$\tilde{\psi}(p) = \frac{\text{const}}{p^2} \exp\left[\frac{1}{\lambda^2} \left(\frac{p^3}{3} - \frac{\gamma}{p}\right)\right]. \quad (39)$$

Inverting the transform to obtain $\psi(z)$,

$$\begin{aligned} \psi(z) &= \text{const} \int_C dz e^{pz} \tilde{\psi}(p) \\ &= \text{const} \int_C du \exp\left[\frac{1}{\lambda^2} \left(-\frac{1}{3u^3} - \frac{\lambda^2 z}{u} + \gamma u\right)\right] \end{aligned} \quad (40)$$

where, for convenience, we have substituted $p = -1/u$. Substitution will immediately verify that (40) is a solution of (34) provided either that the integrand vanishes at the endpoints of the contour, C , or that the contour, C , is closed. Subject only to this restriction, different choices of contour for (40) will produce different solutions, $\psi(z)$, and it is not difficult to find four different contours which produce four linearly independent solutions, $\psi(z)$. The elegance of this transform method for the wave problem appears just at this point: each chosen contour represents a single solution valid for all z values; connection requires only the evaluation of the integral along the chosen contour in the two limits of large $z = \pm|z|$. Now in the evaluation here of what might be two difficult integrals, help comes from complex variable theory which allows us to deform the contour provided the deformation does not carry the contour across any singularities. Therefore, we can adjust the contour in each case for maximum mathematical convenience. In particular, the hills and valleys of the exponent will slide about as z varies, but -- as long as no singularity is crossed -- the contours can be shifted as z is changed to permit integration always along paths of steepest descent.

Writing $u = r(\cos\theta + i \sin\theta)$, the real part of the exponent, $\phi(u)$, in the integrand of (40) is

$$\operatorname{Re} \phi(r, \theta) = \frac{1}{\lambda^2} \left(-\frac{\cos 3\theta}{3r^3} - \frac{\lambda^2 z \cos \theta}{r} + \gamma r \cos \theta \right). \quad (41)$$

Contour maps of $\text{Re}\phi$ are depicted in Figs. 5-8. Near $r = 0$, $\text{Re}\phi$ varies as $-\cos 3\theta$ and is negative in the sectors $-30^\circ < \theta < 30^\circ$, $90^\circ < \theta < 150^\circ$, and $-150^\circ < \theta < -90^\circ$. Similarly, for $r \rightarrow \infty$, $\text{Re}\phi$ is negative for $90^\circ < \theta < 270^\circ$. Valid contours may start in any of these negative pits at $r \rightarrow 0$ or $r \rightarrow \infty$, and end in any other. A closed contour which encircles the origin provides another valid nontrivial possibility.

Distorting the contours, as suggested earlier, to follow along paths of steepest descent, the dominant contribution to the integrals will come from the saddle-point regions along each path. Saddle points in $\text{Re}\phi$ occur where $d\phi/du = 0$; for large $\lambda^2|z|$, there is an inner pair of saddle points at $u_{sp} \approx \pm(-\lambda^2 z)^{-1/2}$ and an outer pair at $u_{sp} \approx (-\lambda^2 z/\gamma)^{1/2}$. In Table I we list the various possibilities in terms of a radicand that is always positive.

Table I lists the values of u , $\phi(u)$, and $\phi''(u)$ for one of the points, u_1 , for each pair of saddle points. The other point, u_2 , appears at $u_2 = -u_1$, and the other values are $\phi(u_2) = -\phi(u_1)$ and $\phi''(u_2) = -\phi''(u_1)$.

Integration through a saddle point along the line of steepest descent then makes a contribution, $\Delta\psi(z)$, to $\psi(z)$, Eq. (40),

$$\Delta\psi(z) = \int_{u_1}^{u_2} du \exp[\phi(u)] \approx e^{i\alpha} \left| \frac{2\pi}{\phi''(u_{sp})} \right|^{1/2} \exp[\phi(u_{sp})] \quad (42)$$

	$z > 0$	$z < 0$	$z/\gamma > 0$	$z/\gamma < 0$
u_1	$\frac{i}{\lambda z^{1/2}}$	$\frac{1}{\lambda (-z)^{1/2}}$	$i\lambda \left(\frac{z}{\gamma}\right)^{1/2}$	$\lambda \left(-\frac{z}{\gamma}\right)^{1/2}$
$\phi(u_1)$	$\frac{2}{3} i\lambda z^{3/2}$	$\frac{2}{3} \lambda (-z)^{3/2}$	$\frac{2i\gamma}{\lambda} \left(\frac{z}{\gamma}\right)^{1/2}$	$\frac{2\gamma}{\lambda} \left(-\frac{z}{\gamma}\right)^{1/2}$
$\phi''(u_1)$	$2i\lambda^3 z^{5/2}$	$-2\lambda^3 (-z)^{5/2}$	$-\frac{2iz}{\lambda^3} \left(\frac{\gamma}{z}\right)^{3/2}$	$-\frac{2z}{\lambda^3} \left(-\frac{\gamma}{z}\right)^{3/2}$

TABLE I

Values of u , $\phi(u)$, and $\phi''(u)$ at the saddle point

$u = u_1$. A conjugate saddle point appears at

$u = u_2 = -u_1$.

where $e^{i\alpha}$ is the inclination of the path of integration at the saddle point. If u_1 and u_2 are close to one another, $e^{i\alpha} \approx (u_2 - u_1)/|u_2 - u_1|$. It is both informative and agreeable to note that (42), with substitutions from Table I, agrees with the asymptotic forms anticipated earlier, in (36). The comparison tells us that the traversal of any single saddle point adds, as a contribution to the integral along the chosen contour, one of the waves in (36), either in its propagating form or as an evanescent growing or decaying mode. Moreover, where the earlier forms in (36) merely indicated the set of possibilities, Eq. (42) plus Table I now give the relative amplitudes and phases for each component.

The simplest illustration of mode conversion using the Standard Equation occurs for the case where γ is a negative number, in which case (34) provides long-wavelength propagating solutions for $z \ll 0$ and short-wavelength propagation when $z \gg 0$. Plots of the contour levels of the real value of the exponent in the integrand, Eq. (40), are shown in Figs. 5 and 6, for $z \ll 0$ and $z \gg 0$, respectively. The same single path of integration, P_I , is shown in the two figures, starting in the $90^\circ < \theta < 150^\circ$ sector as $r \rightarrow 0$ and ending in the $-90^\circ < \theta < 90^\circ$ sector as $r \rightarrow \infty$. For $z \ll 0$, the steepest-descent path takes P_I through the outer upper saddle-point at $u_{sp} \approx i\lambda/z|\gamma|^{1/2}$, while for $z \gg 0$ the steepest-descent path goes through the inner upper saddle-point at $u_{sp} \approx i/|\lambda^2 z|^{-1/2}$. The evaluation by (42) thus gives the connection formula

$$\begin{aligned} \psi(z) &= e^{i\pi/4} \left| \frac{\pi^2 \lambda^6 z}{\gamma^3} \right|^{1/4} \exp \frac{2i(\gamma z)^{1/2}}{\lambda} \\ z < 0 \\ &\leftrightarrow e^{i\pi/4} \left| \frac{\pi^2}{\lambda^6 z^5} \right|^{1/4} \exp \frac{2}{3} i \lambda z^{3/2} = \psi(z) \\ z > 0 \end{aligned} \quad (43)$$

and shows 100% conversion of the right-moving fast wave in the $z \ll 0$ region to a right-moving slow wave in the $z \gg 0$ region. There is no reflection at all into a possible left-moving fast wave.

Similarly, integration along the mirror reflection of P_I on the other side of the real- u axis shows 100% conversion from the left-moving slow wave in the $z \gg 0$ region to a left-moving fast wave in the $z \ll 0$ region.

Other contours may be found for the remaining two linearly independent solutions, but their physical interpretation -- relating purely evanescent modes on each side of $z = 0$ -- is less interesting.

More skill is required to associate contours in the $\gamma > 0$ case with their physical interpretation. The contour levels for the exponent in the integrand of (40) are shown in Figs. 7 and 8, for $z \ll 0$ and $z \gg 0$ respectively. When $z \ll 0$, Table I and Eq. (42) tell us that both the fast wave and the slow wave are evanescent. This situation is manifested in Fig. 7 by the appearance

of all the saddle points on the real-u axis. If wave power is incident from the right, i.e., is moving left from $z \gg 0$ toward $z = 0$, then the only solutions which are reasonable in a physical sense in the $z \ll 0$ region are the decaying solutions. These are the two saddle points on the left-hand side of the real-u axis, Fig. 7. Contour P_I , a steepest-descent path through these points, is also chosen so that the integrand in (40) vanishes at its start ($r \rightarrow 0$) and finish ($r \rightarrow \infty$). For $z \gg 0$, Fig. 8, the steepest-descent route for this same contour passes through both the inner and outer lower saddle points, leading to the connection formula, by Table I and (42),

$$\begin{aligned}
 & - \left| \frac{\pi^2 \lambda^6 z}{\gamma^3} \right|^{1/4} \exp - \frac{2(-\gamma z)^{1/2}}{\lambda} \\
 & + e^{-i\pi/4} \left| \frac{\pi^2}{\lambda^6 z^5} \right|^{1/4} \exp - \frac{2}{3} i \lambda z^{3/2} \\
 & + e^{-i3\pi/4} \left| \frac{\pi^2 \lambda^6 z}{\gamma^3} \right|^{1/4} \exp - \frac{2i(\gamma z)^{1/2}}{\lambda} .
 \end{aligned} \tag{44}$$

For z negative and large, the contribution from the faster decaying term on the left is subdominant and has been neglected. The first and second term on the right represent slow and fast waves, respectively, both with phase velocity to the left. However, substitution into (35) will verify that the directions of power flow for these two waves are in opposite directions (the slow wave is a backward wave) and indeed, that the two power

flows exactly cancel. Again we have 100% mode conversion: in this case, incident fast-wave power from the right reemerges from the resonance region as slow-wave power moving to the right.

The conjugate integration path, the reflection of P_I in the real-u axis, gives the same physical situation but with all wave directions reversed.

Another physical situation of interest is the case where the source of radiofrequency power is physically located in the evanescent region. In this situation rf power must tunnel its way to the critical layer ($z \approx 0$), and may only then emerge as a propagating wave on the far side. Substitution into (35) will show that a single-mode evanescent wave will carry no power and, furthermore, even two evanescent modes which are in time phase will not carry power. But two modes 90° out of time phase can transmit power through an evanescent region. Contour P_{II} in Fig. 7 corresponds to such a situation, and this same contour in Fig. 8 passes only through the outer upper saddle point, corresponding to an outgoing fast wave in the $z \gg 0$ region. The connection formula in this case is

$$\left| \frac{\pi^2}{\lambda^6 z^5} \right|^{1/4} \exp \frac{2}{3} \lambda (-z)^{3/2} + \dots$$

$$\leftrightarrow e^{3\pi i/4} \left| \frac{\pi^2 \lambda^6 z}{\gamma^3} \right| \exp \frac{2i(\gamma z)^{1/2}}{\lambda} \quad (45)$$

The amplitudes of the evanescent components that carry the rf power in the $z \ll 0$ region are, unfortunately, subdominant terms and in this example cannot be computed by the steepest-descent method.

A final remark pertaining to solutions of the Standard Equation, (34). Scaling $\gamma \sim \lambda^2$ and taking the large λ^2 limit minimizes the influence of the ψ''' term. The same limiting process applied to the solution of (34) expressed as a contour integral, namely Eq. (40), eliminates the $1/3u^3$ term in the exponent. The resulting contour integral, with appropriate paths, is precisely the integral representation for the various first-order Bessel functions, and the fast-wave contributions to (43)-(45) may be represented exactly as Hankel functions (in the propagating case) or as modified Bessel functions (in the evanescent case) of argument $2(\pm\gamma z)^{1/2}/\lambda$, to wit,

$$\begin{aligned}
 H_1^{(1)} [2(\gamma z)^{1/2}/\lambda] &\leftrightarrow e^{-3\pi i/4} \left(\frac{\lambda^2}{\pi^2 \gamma z} \right)^{1/4} \exp 2i(\gamma z)^{1/2}/\lambda , \\
 H_1^{(2)} [2(\gamma z)^{1/2}/\lambda] &\leftrightarrow e^{3\pi i/4} \left(\frac{\lambda^2}{\pi^2 \gamma z} \right)^{1/4} \exp -2i(\gamma z)^{1/2}/\lambda , \\
 I_1 [2(-\gamma z)^{1/2}/\lambda] &\leftrightarrow \left(\frac{\lambda^2}{-4\pi^2 \gamma z} \right)^{1/4} \exp 2(-\gamma z)^{1/2}/\lambda , \\
 K_1 [2(-\gamma z)^{1/2}/\lambda] &\leftrightarrow \left(\frac{\pi^2 \lambda^2}{-4\gamma z} \right)^{1/4} \exp -2(-\gamma z)^{1/2}/\lambda .
 \end{aligned}
 \tag{46}$$

The exponentials are simply the leading terms in the asymptotic expansions for the Bessel functions above. However, the double-ended arrows denote here that the exponential fast-wave terms in (43)-(45) may be in this way replaced by the more accurate Bessel-function expressions. Evaluation of the power flow for $\psi \sim z^{1/2} H_1^{(1) \text{ or } (2)} [2(\gamma z)^{1/2}/\lambda]$, using just $P = -\gamma \psi' \psi^* - cc$ [cf. (35)] and making use of the Hankel-function Wronskian, will quickly verify the constancy of P even for small values of the argument.

VI. The Tunneling Equation

The same Laplace-transform technique used to obtain connection formulae for the Standard Equation, (34), can also be applied to more complicated linear differential equations provided the coefficients are no more than first-order polynomials in z . For example, a host of possible applications in ICRF and ion-ion hybrid modes, involving equations up to eighth order, is discussed by F. W. Perkins (1977). One particular model equation which has received considerable attention in mode-conversion studies is

$$\psi'''' + \lambda^2 z \psi'' + \alpha \lambda^2 \psi' + (\lambda^2 z + \gamma) \psi = 0 \quad (47)$$

where α and γ are complex constants. For convenience, we call (47) the Tunneling Equation. It is a close relative of the equation examined in considerable detail by A. L. Rabenstein (1958) which differs from (47) only in that the term in parentheses is simply γ .

For the important case where the coefficient α in (47) is zero, one may again write down a conserved quantity when λ^2 , γ , and z are real, namely,

$$P = \psi^* \psi'''' + \psi^* \psi''' - \psi'^* \psi'' - \gamma \psi^* \psi' - c.c. \quad (48)$$

The primary feature of the Tunneling Equation which distinguishes it, as a model, from the Standard Equation is that it describes not only a resonance phenomenon at $z = 0$, but also introduces a cutoff at $z = -\gamma/\lambda^2$. The sketch in Fig. 9 indicates in a qualitative way the variation of the local index of refraction, k_z , that would be associated with the Tunneling Equation. To the right of $z = (2/\lambda^2)[1 + (1+\gamma)^{1/2}]$ and to the left of the cutoff at $z = -\gamma/\lambda^2$, k_z^2 is positive and the solutions to (47) are propagating waves. Between resonance and cutoff, however, the solutions are evanescent and wave energy is transmitted through this region only by the process of tunneling. The first study of tunneling through such a back-to-back cutoff-resonance pair was the classic work of K. G. Budden (1955) based on the second-order model equation

$$\lambda^2 z \psi'' + (\lambda^2 z + \gamma) \psi = 0 . \quad (49)$$

Budden brought this equation into the standard form for confluent hypergeometric functions and, by introducing a small amount of damping to resolve the ambiguous singularity at $\text{Re } z = 0$, was thereby able to connect the asymptotic solutions. For the right-moving wave which passes the cutoff first and then comes to the singularity, Budden found amplitude coefficients of reflection (R) and transmission (T) (i.e., tunneling)

$$|R| = 1 - \exp(-\pi\eta) \quad (50)$$

$$|T| = \exp(-\pi\eta/2)$$

while for the left-moving wave,

$$|R| = 0 \quad (51)$$

$$|T| = \exp(-\pi\eta/2) ,$$

where $\eta \equiv |\gamma/\lambda^2|$. In both cases $|R|^2 + |T|^2 < 1$, indicating partial absorption at the singularity. In mode-conversion terms, this "absorbed" energy in the dissipationfree limit is actually the wave energy converted into the short-wavelength mode. It will be seen that the reflection and transmission coefficients obtained from the fourth-order Tunneling Equation, (47), are the same as (50) and (51) provided that the definition of η is changed in accordance with Eq. (56) below. These fourth-order equation coefficients then reduce to the Budden result in the limit of large λ^2 , scaling $\gamma \sim \lambda^2$, which is the limit which minimizes the contribution from the ψ''' term in (47).

Applications of the Tunneling Equation have been made by Y. C. Ngan and D. G. Swanson (1977) to the propagation of the fast Alfvén wave and to ion Bernstein waves near resonances. The

equation has also been invoked by E. Lazzaro, G. Ramponi, and G. Giruzzi [1981] to treat absorption near a harmonic of the electron cyclotron frequency, and by D. G. Swanson (1975) to examine mode conversion and tunneling at the two-ion hybrid resonance.

VII. Asymptotic Solutions of the Tunneling Equation

Solving (47) by the Laplace integral technique as before, the inversion equation is

$$f(z) = \int_C dp (1+p^2)^{(\alpha/2)-1} \exp\left[\frac{p^3}{3\lambda^2} + p\left(z - \frac{1}{\lambda^2}\right) + \frac{1+\gamma}{\lambda^2} \tan^{-1}p\right] \quad (52)$$

where C again represents a contour in the complex p-plane which is either closed or is chosen so that the integrand vanishes at both ends. The change of variable $p = \tan u$ brings (52) into a form which avoids the multi-valued $\tan^{-1}p$ term,

$$f(z) = \int_C du \sec^\alpha u \exp\left[\frac{\tan^3 u}{3\lambda^2} + \left(z - \frac{1}{\lambda^2}\right) \tan u + \frac{1+\gamma}{\lambda^2} u\right] \quad (53)$$

The set of contours shown in Fig. 10 represents several physically meaningful solutions to (47) together with others which are convenient or necessary for completeness. Solutions denoted as f_k , $k = 1, 2, 3 \dots$, are associated with the integration of (53) along the contour marked C_k . Clearly only four such solutions will be linearly independent. It will be shown that f_1 and f_2

represent incoming fast waves from opposite sides of the tunneling region, while $f_5 = f_0 + f_1$ represents an incident slow wave. f_0 and f_4 by themselves exhibit exponentially decreasing or increasing evanescent behavior as $z \rightarrow \pm\infty$.

Turning again to integration by the method of steepest descents, the saddle points for the integration appear approximately where the derivative of the exponent in the integrand is zero, namely, where

$$\tan^4 u + \lambda^2 z \tan^2 u + \lambda^2 z + \gamma = 0. \quad (54)$$

For large $|z|$, (54) may be factored into solutions which pertain to slow waves (large values of $p = \tan u$), $\tan^2 u_{sp} \approx -\lambda^2 z + 1$, and to fast waves (small p), $\tan^2 u_{sp} \approx -[1 + (1+\gamma)/\lambda^2 z]$. Putting $u = u_{sp}$ into the exponent in the integrand of (53) immediately gives the dominant factor for the saddle-point contribution: when $\tan^2 u_{sp}$ is pure positive, integration through the saddle-point gives an evanescent (growing or decaying) mode, while $\tan^2 u_{sp}$ pure negative produces an $\exp(\pm 2i\lambda z^{3/2}/3)$ or $\exp(\pm iz)$ variation corresponding to slow or fast wave propagation, respectively. In the latter case, however, from the identity

$$\tan(x + iy) = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y} \quad (55)$$

one notes that when $\tan^2 u_{sp}$ is pure negative, the real part of u_{sp} must be a multiple of $\pi/2$. Putting this $\text{Re } u_{sp}$ into the third term of the integrand exponent in (53) gives rise to the factor

$$\exp(-\eta) \equiv \exp\left[-\frac{\pi}{2} \frac{(1+\gamma)}{\lambda^2}\right] \quad (56)$$

which appears frequently in the connection formulas, and which is the critical parameter in the tunneling formulas. [Compare Eqs. (50) and (51)].

From the form of Eq. (53) it is apparent that most of the integrand is periodic in u since $\tan(u+\pi) = \tan u$ and $\sec(u+\pi) = -\sec u$. The linear term in the exponent leads to a constant factor when $u \rightarrow u+\pi$ and the $\sec^\alpha u$ will lead to a phase shift so that a shift in the u -plane to the left or right by an amount π will result in multiplying the function by the constant factor $\exp \pm 2\tau$, where $\tau \equiv (\pi/2) [(1+\gamma)\lambda^2 + i\alpha]$. Using this translation factor, the contour segments need only be evaluated between $\pm\pi/2$ and the analytic continuation is accomplished by multiplication by the appropriate factor. Note that $\eta = \text{Re}(\tau)$.

One remark needs to be made before discussing the specific choice of contours. As noted in the previous paragraph, the effect of shifting $u \rightarrow u+\pi$ is to change the altitude of the u -plane contours by the factor $\exp(\pm 2\eta)$. When η is small, this difference in elevation may be insufficient to validate the evaluation by steepest descents; when η is large, the retention

of a subdominant term of order $\exp(-|\eta|)$ may be questionable. Nevertheless, the subdominant contribution does have physical meaning in terms of the amplitudes of the allowed waves and therefore, in determining the amplitude of the coefficient in a saddle-point integration, we shall rely -- where necessary -- on self-consistency arguments rather than on the contribution to the integral taken solely from the immediate vicinity of the saddle point. The self-consistency arguments become available when the inhomogeneous Tunneling Equation is treated by the method of Green's functions, as outlined in the next section, and consist in the demand that the inhomogeneous-case solution reduce to the homogeneous-case result as the size of the inhomogeneous term is scaled down to zero.

Contour-levels and sample paths of integration for the Tunneling Equation are shown in Figs. 10-14. The saddle points which correspond to waves with phase velocity to the right are designated f_+ and s_+ , to the left, f_- and s_- , for the fast (f) and slow (s) modes. In this example, only the slow wave shows evanescence (for $z \ll 0$), and σ_+ designates the mode which increases exponentially away from $z = 0$, while σ_- decays with increasing $|z|$. It should be kept in mind that the short-wavelength modes ($s_{\pm}$) are, in fact, backward waves so that energy in these modes propagates in the direction opposite to the phase velocity. This feature may be verified by substitution of (58) into Eq. (48) and comparing the leading terms in P for $\psi \sim f_{\pm}$ and $\psi \sim s_{\pm}$.

Referring now to Figs. 10-14, it is apparent that f_4 grows exponentially as $z \rightarrow -\infty$ since it alone contains σ_+ , so f_4 is not relevant for an unbounded medium. Also, since s_+ represents an incoming slow wave (it is a backward wave), it cannot be present for incoming fast wave solutions. Since all paths ending above the axis contain an s_+ term and all paths ending along the axis contain σ_+ terms, any incident fast wave solution must start and stop below the axis.

For a solution which represents an incoming fast wave from $+\infty$, f_- is the incident wave, and no f_+ term can be allowed on the $-\infty$ side (only a transmitted wave as $z \rightarrow -\infty$). Therefore the contour starts and stops below the axis, so this solution is f_1 . This immediately shows that the reflection in this case is zero ($R_1 = 0$) since there is no f_+ term on either side.

For the solution which represents an incoming wave from $-\infty$ the incident wave is f_+ , so the contour still starts and stops below the axis but must go above the axis to pick up an f_+ term as $z \rightarrow -\infty$. It must not cross the f_- saddle point as $z \rightarrow \infty$ however (only a transmitted wave is permitted) so these constraints force the contour C_2 to cross the branch cut which appears for non-integral α . Thus f_2 is the other incident fast wave solution and here $R_2 \neq 0$, and in fact only for $\alpha = 0$ or $\alpha = 2$ is $|R_2| < 1$, so other values of α are not physical if related only to the homogeneous equation.

For an incident slow wave, only outgoing fast waves are allowed, and the contour must start or stop above the axis to pick up an s_+ term for the incident wave. Outgoing fast waves are f_+ as $z \rightarrow \infty$ and f_- as $z \rightarrow -\infty$, so only solution f_5 has the proper form. Solution f_3 is useful mathematically since $f_3 = f_1^*$ and f_2 may be constructed from f_0 , f_1 , and f_3 by

$$f_2 = (f_0 + f_1)(1 - \exp(-2\tau)) - f_3. \quad (57)$$

We designate the wave-like variation for each of the saddle-point contributions in the following way:

$$\begin{aligned} f_{\pm} &= \frac{\lambda \pi^{1/2}}{(1+\gamma)^{1/2}} \left| \frac{1+\gamma}{\lambda^2 z} \right|^{\alpha/2} \exp(\pm iz), \quad \text{all } z, \\ s_{\pm} &= \frac{\pi^{1/2}}{\lambda^{3/2} z^{5/4}} |\lambda^2 z|^{\alpha/2} \exp\left(\pm \frac{2i\lambda z^{3/2}}{3}\right), \quad z > 0, \\ \sigma_{\pm} &= \frac{i^{(1\mp 1)/2} \pi^{1/2}}{\lambda^{3/2} (-z)^{5/4}} |\lambda^2 z|^{\alpha/2} \exp\left[\pm \frac{2\lambda(-z)^{3/2}}{3}\right], \quad z < 0. \end{aligned} \quad (58)$$

Using these abbreviations, the connection formulas obtained for the various path integrals in Figs. 10-14 are listed in Table II. For the important cases of $\alpha = 0$ or $\alpha = 2$, $|\epsilon|^2 = 1 - \exp(-2\eta)$, determined by the self-consistency argument as described above [D. G. Swanson (1978)].

TABLE II

Formulas for connection between the asymptotic forms for large $|z|$.

$z \rightarrow -\infty$		$z \rightarrow \infty$
$e^{\eta_{\sigma_-}}$	f_0	$e^{\tau}(\epsilon f_+ - s_+) - e^{\tau^*}(\epsilon^* f_- - s_-)$
$\epsilon^* f_-$	f_1	$e^{\tau^*} \epsilon^* f_- - e^{\tau^*}(1 - e^{-2\tau^*})s_-$
$(\epsilon^* f_- + e^{\eta_{\sigma_-}})(1 - e^{-2\tau}) - \epsilon f_+$	f_2	$-e^{-\tau} \epsilon f_+ + e^{-\tau^*}(1 - e^{-2\tau})s_-$
ϵf_+	f_3	$e^{\tau} \epsilon f_+ - e^{\tau}(1 - e^{-2\tau})s_+$
$\epsilon^* f_- - e^{-\eta_{\sigma_+}}$	f_4	$e^{-\tau^*} s_-$
$e^{\eta_{\sigma_-}} + \epsilon^* f_-$	f_5	$e^{\tau}(\epsilon f_+ - s_+) + e^{-\tau^*} s_-$

The amplitude coefficients for reflection, transmission, and for mode conversion can be taken from Table II by inspection. Looking at the fast waves alone, one may quickly verify the modified Budden results, as described by (50) and (51) with the altered definition of η in (56).

Figures 9-14 and Table II all refer to the case where $1 + \gamma > 0$, which is the case appropriate to the modeling of ion-ion hybrid (γ large and positive) and ion cyclotron-harmonic ($\gamma = -2/3$) resonances. Electron cyclotron resonances can lead to consideration of the $1 + \gamma < 0$ case, which shifts the fast-wave saddle-points by $u = \pm\pi/2$. Results for this situation are offered by D. W. Faulconer (1980).

VIII. Localized Absorption

The analysis of both the Standard Equation and the Tunneling Equation have depended heavily on complex-variable theory. It is a simple modification, therefore, to add a small imaginary part to z , which represents physical distance, and it is in this manner that the effects of dissipation can be introduced into the model equations and their asymptotic solutions. See, for instance, B. N. Moore and M. E. Oakes (1972). On the other hand, there are cases of interest where the absorption effects are strongly localized and are not accurately approximated by a simple off-axis evaluation. For example, in numerical studies of the warm-plasma dispersion relation in the vicinity of the two-ion hybrid resonance, it was found by J. Jaquinot, B. D. McVey, and J. E. Scharer (1977) that the ion cyclotron absorption was both localized and sufficiently strong for larger $k_{\parallel}$ to greatly modify the mode-conversion structure.

Another physical application where local modifications to the asymptotic wave equation are needed occurs in the treatment of the $\omega = 2\omega_{ci}$, $k_{\parallel} = 0$ resonance. The wave equation, derived directly from the Vlasov-Maxwell equations assuming $\vec{B}_0(x) = B_0[1 + (x/L)]\hat{z}$, can be put in the form of the Tunneling Equation, with certain correction terms on the right-hand side. These correction terms vanish for large $|x|$, but are important near the origin where coupling occurs [D. G. Swanson, 1981].

A method of solution for these problems is to treat the correction terms, important locally but not in the large- $|x|$ asymptotic limits, as inhomogeneous additions to an otherwise homogeneous linear differential equation. The use of Green's functions then provides a straightforward method for solutions. The Green's functions are readily constructed as linear combinations of solutions to the adjoint equation; the adjoint to the Tunneling Equation, (47), for instance, for the case of a constant-weight-function scalar product, is the same equation save for the replacement of α by $2 - \alpha$. Asymptotic solutions to the adjoint equation satisfying the proper boundary conditions can therefore be obtained in exactly the manner described in the previous sections, and these are sufficient to determine the form of the Green's function. The full solution of the inhomogeneous equation, which is now an integral equation, is then obtained by iteration, starting with homogeneous solutions as the first trial solutions. Exact (not just asymptotic) solutions of the homogeneous adjoint equation are needed and are obtained in the course of the computation. The signal advantage of this Green's function method over direct integration is the complete avoidance of the explosive numerical instabilities which plague the direct integration of the fourth-order wave equation. Details and results for the Green's function method are given by D. G. Swanson (1978, 1980).

IX. Matched Asymptotic Expansions; the Low-Frequency Alfvén Resonance

As a final topic, we return to an examination of the Alfvén resonance. When ω is a finite fraction of ω_{ci} , or larger, the right-hand cutoff ($n_n^2 = R$), left-hand cutoff ($n_n^2 = L$) and Alfvén resonance [$n_n^2 = S \equiv (R+L)/2$] are well separated in frequency or, in the presence of a density gradient (recall that $R, L \sim n_i$), are well separated in physical space. In this circumstance, the mode-conversion problem can be well modeled by the Standard Equation, or the Tunneling Equation, with boundary conditions plausibly set by physical intuition. At low frequencies ($\omega \ll \omega_{ci}$), however, $R \approx S(1 - \omega/\omega_{ci})$, $L \approx S(1 + \omega/\omega_{ci})$, and the cutoffs and resonance form a closely spaced triplet. In the simplest case of no shear and $n_1^2 = 0$, Eq. (33), using $R = S + D$, $L = S - D$, $|D| \ll S$, and $x \sim S - n_n^2$, the pertinent model wave equation would read

$$x\psi'' + \lambda^2(x^2 - 1)\psi = 0 \quad (59)$$

with

$$\lambda^2 = \left| \frac{D^3 \omega^2}{c^2} \frac{1}{(dS/dx)^2} \right| \sim \frac{\omega^3}{\omega_{ci}^3} \quad (60)$$

x here is the physical distance in the direction of the density gradient. The cutoffs occur at $x = \pm 1$, the resonance at $x = 0$, and for small λ it is necessary to consider the entire triplet. [For the case where the cutoffs and resonance are well separated, see the analysis by T. Tang (1970)]. However, due to the x^2 coefficient, the Laplace transform of (59) is also a second-order differential equation and solution by this method offers no simplification.

We turn, therefore, to another powerful mathematical technique, namely matched asymptotic expansions, and examine the more general intermediate-frequency equation, Eq. (31). [For an analysis of the Tunneling Equation by this technique, see Y. C. Ngan and D. G. Swanson (1977).] The parameters are all functions of x and we expand around the Alfvén resonance at $x = x_0$, where $T(x_0) + n_1^2(x_0) = 0$,

$$T + n_1^2 = [T(x_0) + n_1^2(x_0)]'(x - x_0)\omega/c + \dots \quad (61)$$

and substitute

$$\begin{aligned} \xi &= \omega(x - x_0)/\ell c, \\ \ell &= [(T + n_1^2)']^{-1/3}, \\ v &= n_1 \ell, \\ \alpha &= (D^2 + n_1' D - n_1 D' + \psi'^2 T - n_1 n_1'') \ell^4, \\ \beta &= -n_1 (n_1' + D) \ell^3 \tau, \\ \tau &= T' / (T + n_1^2)' \end{aligned} \quad (62)$$

where the prime (') indicates the derivative with respect to $x\omega/c$. In dimensionless form Eq. (31) becomes

$$\frac{d^2 E}{d\xi^2} + \left(\frac{1}{\xi} - \frac{\tau}{\xi - v^2} \right) \frac{dE}{d\xi} + \left[\xi - v^2 - \frac{\alpha}{\xi} + \frac{\beta}{\xi(\xi - v^2)} \right] E = 0 \quad (63)$$

and consider that ℓ , v , α , β , and τ , evaluated at $x = x_0$, are constants. The singularities at $\xi = 0$ and $\xi = v^2$ appear, at first glance, to be of similar strength. But in fact, the singularity at $\xi = v^2$ is quite weak: we recognize that $\tau \approx 1$ and if we put $x = \xi - v^2$, the leading terms in (63) are

$$\frac{d^2 E}{dx^2} - \frac{1}{x} \frac{dE}{dx} + \frac{\beta}{v^2} \frac{E}{x} \approx 0 \quad (64)$$

which is satisfied by

$$E_a = 1 - \mu x - \mu^2 \left(\frac{x^2}{2} + \frac{\mu x^3}{6} \right) \ln x + \dots \quad (65)$$

$$E_b = \frac{x^2}{2} + \frac{\mu x^3}{6} + \dots$$

where $\mu \equiv -\beta/v^2$. Singular behavior begins only at the $x^2 \ln x$ term. Following the analysis of C. F. F. Karney, F. W. Perkins, and Y. C. Sun (1979), we focus just on the Alfvén singularity, $\xi = 0$. In the inner region of (63), i.e., the neighborhood of $\xi = 0$, we substitute $\xi = \epsilon x$, write $E = E_0 + \epsilon E_1 + \dots$, to obtain the sequence of equations,

$$E''_0 + \frac{E'_0}{x} = 0 \quad (66)$$

$$E''_1 + \frac{E'_1}{x} = -\frac{\tau E'_0}{v^2} + \left(\alpha + \frac{\beta}{v^2}\right) E_0, \text{ etc.}$$

Solving and replacing the original ξ variable, we have the inner solutions

$$E_a = 1 + \left(\alpha + \frac{\beta}{v^2}\right) \xi + \dots \quad (67)$$

$$E_b = -\left[\frac{\tau}{v^2} + 2\left(\alpha + \frac{\beta}{v^2}\right)\right] \xi + \left[1 + \left(\alpha + \frac{\beta}{v^2}\right) \xi\right] \ln \xi + \dots$$

We match these inner solutions to outer solutions on each side of the Alfvén resonance at $\xi = 0$. A critical point is the jump in $\ln \xi$ as ξ passes through 0. We recall that $\xi \sim x - x_0 \sim T + n_{\perp}^2 \approx (4\pi\rho c^2/B^2) - k_{\perp} c^2/\omega^2$. From the equation of motion, one may show that the inclusion of a small amount of ion drag is equivalent to $\rho \rightarrow \rho + i[\delta] \text{Sgn } \omega$, suggesting that a small imaginary increment can be assigned to ξ . With this knowledge, we have $\ln \xi^+ - \ln \xi^- = -i\pi \text{Sgn } \omega$.

The outer region of (63) is approximated in low order, substituting $y = v^2 - \xi$, by the Airy equation,

$$\frac{d^2 E}{dy^2} - yE = 0 \quad (68)$$

with solutions $Ai(y)$, $Bi(y)$. An Alfvén wave propagating from the high density region will be reflected, and slightly absorbed, at the Alfvén resonance-triplet, and will be evanescent on the low-density side (where y is positive). It is $Ai(y) \sim \exp(-2y^{3/2}/3)$ which is the appropriate choice for that region, and we therefore match $Ai(y) = c_1 - c_2 y + \dots$ in the overlap region to a linear combination of E_a and E_b . Here, $c_1 = Ai(0) = 0.355$, $c_2 = -Ai'(0) = 0.259$. On the high-density side, the same linear combination of E_a and E_b (taking note of the jump in $\ln \xi$) is matched in the next overlap region again to Airy functions, using $Ai(y) = c_1 - c_2 y + \dots$, $Bi(y) = -3^{1/2}(c_1 + c_2 y + \dots)$. It is now convenient to reexpress this $y < 0$ linear combination of real Airy functions in terms of complex Airy functions,

$$F_{\pm}(y) = Bi(y) \pm iAi(y) \sim \frac{I}{(-\pi^2 y)^{1/4}} \exp\left\{\pm i\left[\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4}\right]\right\} \quad (69)$$

The expression on the right is the asymptotic form for $F_{\pm}$ for large $-y = \xi - v^2$ and, for $\text{Sgn } \omega > 0$, represents waves traveling away from $\xi = 0$ toward higher density (F_+) and toward $\xi = 0$ from higher density (F_-). Carrying through the algebra, one obtains the amplitudes, $A_{\pm}$, for $F_{\pm}$ in the $y < 0$ region which are matched to the original decaying $Ai(y)$ solution in the $y > 0$ region,

$$[Ai(y)]_{y>0} \leftrightarrow [A_+ F_+(y) + A_- F_-(y)]_{y<0} \quad (70)$$

Extending the result obtained by C. F. F. Karney, F. W. Perkins, and Y. C. Sun (1979), the energy absorption coefficient from this calculation is

$$\frac{|A_-|^2 - |A_+|^2}{|A_-|^2} \cdot \text{Sgn} \cdot \omega = \frac{4ba_-^2}{1 + 2ba_-^2 + b^2a_-^2(a_-^2 + 3a_+^2)} \quad (71)$$

$$a_{\pm} = \sigma c_1 \pm c_2(1 + \sigma v^2) \quad ,$$

$$b = \frac{\pi}{2(3)^{1/2}} \cdot \frac{1}{c_1 c_2} \cdot \frac{v^2}{\tau + 2\sigma v^2} \cdot \text{Sgn} \omega \quad ,$$

$$\sigma = \alpha + \frac{\beta}{v^2} \quad .$$

α , β , v , and τ are defined in (62). Reflectivity with very low loss, which would characterize a high-Q eigenmode or cavity resonance, is predicted when a_- goes to zero. This behavior has been confirmed in numerical integration of the wave equation by T. H. Stix (1981). An impressive detailed numerical analysis of the Alfvén resonance problem, including finite electron mass and temperature, finite ion Larmor radius, magnetic shear, and cylindrical geometry has been recently carried out by D. W. Ross, G. L. Chen, and S. M. Mahajan (1981).

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Figure Captions

Figure 1. The square of the index of refraction (n_x^2) is plotted against density. The frequency, which is fixed, is equal to the hybrid frequency at the value of density indicated. For an inhomogeneous plasma with linearly increasing density, the abscissa is proportional to distance. Dotted lines represent extrapolations of approximate dispersion relations beyond their regions of validity. QT-X designates the quasitransverse extraordinary electromagnetic mode. The parallel wave number, $k_{||}$, is assumed small or zero. At the top of the figure, the bulge out to the right corresponds to the maximum of the right-hand side in Eq. (17) which occurs under the conditions of Eq. (18). From T. H. Stix (1965).

Figure 2. Same as Fig. 1, except that it is assumed that the accessibility criterion, Eq. (19), for accessibility to the lower-hybrid resonance is satisfied. From T. H. Stix (1965).

Figure 3. Dispersion curve for Alfvén waves in a cold plasma [$\beta > 0$, Eq. (24)]. The square of the refractive index is plotted against electron density. From T. H. Stix (1981).

Figure 4. Same as Fig. 3, but for a warm plasma [$\beta < 0$, Eq. (24)]. From T. H. Stix (1981).

Figure 5. Lines of constant $\text{Re } \phi(u)$, Eq. (41), in the complex- u plane. Parameters for the plot are $\lambda = 1.0$, $\gamma = -1.0$, $z = -2.5$. The integration contour P_1 passes through the saddle point corresponding to an incoming fast wave. Dotted curves denote that $\text{Re } \phi(u) < 0$.

Figure 6. Same as Fig. 5, but with $z = 2.5$. P_I passes through the saddle point for a slow wave with outgoing group velocity.

Figure 7. Similar to Figs. 5 and 6, but with $\gamma = 1.0$, $z = -2.5$. P_I passes through saddle points for modes which decay rapidly (the inner point) and slowly (the outer point) for $z \ll 0$.

Figure 8. Same as Fig. 7, but with $z = 2.5$. P_I passes through saddle points for an incoming fast wave and a slow wave with outgoing group velocity.

Figure 9. The square of the index of refraction plotted versus distance, for the Tunneling Equation, (47), with $\alpha = 0$, $1 + \gamma > 0$. For this example, $\lambda = 1$, $\gamma = 1$.

Figure 10. Six possible contours of integration in the complex- u plane, for Eq. (53). Only four contours will lead to linearly independent solutions. For nonintegral α , C_2 ends on a different Riemannian sheet from where it starts.

Figure 11. Levels of constant $\text{Re } \phi(u)$, where $\phi(u)$ is the logarithm of the integrand in Eq. (53), plotted in the complex- u plane with $\lambda^2 = 1$, $\gamma = -2/3$, $\eta = \pi/6$, $z = 10$. From Y. G. Ngan and D. G. Swanson (1977).

Figure 12. Saddle points and contours of integration for Fig. 11.

Figure 13. Same as Fig. 11, but with $z = -10$.

Figure 14. Saddle points and contours of integration for Fig. 13.

Fig. 1

Fig. 2

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Fig. 3

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Fig. 4

Fig. 5

Fig. 6

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Fig. 7

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Fig. 8

Fig. 9

Fig. 10

Fig. 12

Fig. 13

Fig. 14