**The Trapped Electron Instability in Tokamaks:
Analytic Solutions of the Two-Dimensional
Eigenvalue Problem**

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ABSTRACT

The two-dimensional eigenvalue equation for the trapped electron instability in tokamaks is solved analytically by both a perturbation technique and a more exact method of matched asymptotic expansions. The important physical effect is shown to be the spatial localization of the trapped electron term caused by the difference between the pitch of the magnetic field and that of the mode. This localization results in a completely new form for the dispersion relation and does not impede magnetic shear stabilization. Coupling of neighboring rational surfaces is shown to result in an increase of only $\pi/2$ (at most) in the growth rate.

I. INTRODUCTION

The dissipative trapped electron instability is a version of the drift wave in a tokamak that is destabilized by trapped electrons. When the shear of the magnetic field and the finite ion gyration effects are taken into account, the dispersion relation is obtained as the eigenvalue equation associated with a radial second order differential equation for the fluctuating electrostatic potential. Because of the fast trapped electron bounce motion, the nonadiabatic trapped electron response tends to be flutelike and introduces coupling among different poloidal eigenmodes. Hence, the full eigenmode equation is a second order differential equation in the radial coordinate and an integral equation in the poloidal coordinate. Analytic investigations of this differential equation to date have been one-dimensional. For the nonadiabatic trapped electron response, one generally either assumes a flutelike mode structure aligned with the magnetic field and solves the radial equation¹ or ignores the radial structure and solves for the poloidal mode variation.² In the first approach, the growth rate due to the trapped electrons is found in the limit of an isolated rational surface; the second approach completely neglects the stabilizing effect of shear and the spatial localization of the destabilizing trapped electron term.

In view of the possibly important effects of the dissipative trapped electron instability on the transport in a tokamak plasma, a more careful study of the two-dimensional eigenmode equation is necessary. Numerical investigations of the two-dimensional mode structure problem have been undertaken and preliminary results obtained.^{3,4} We remove the shortcomings of previous analytic treatments by presenting two

analytic solutions of the two-dimensional eigenvalue problem which retain both the full spatial structure of the destabilizing trapped electron coupling term and the magnetic shear. The simpler perturbation theory approach is presented first. Only the most unstable radial eigenmode is considered in order to simplify the mathematics. A technique similar to that employed on the collisionless drift wave⁵ allows us to obtain the eigenvalue equation for all even and odd spatial eigenmodes for orbiting growth rates. The validity of the perturbation theory result is then investigated.

As expected, the coupling of rational surfaces results in larger growth rates; surprisingly, however, the enhancement above the growth rate in the absence of coupling is found to be only a factor of $\pi/2$. The reason for this small increase is that spatial localization of the destabilizing trapped electron coupling terms is found to result in far more important effects. By limiting the spatial extent of the destabilizing term, this localization tends to reduce the growth rate. With x_t the effective radial extent of the mode and Δ the spacing between rational surfaces of poloidal mode numbers m and $m \pm 1$ for a single toroidal mode number l , the spatial localization results in a reduction in the growth rate of order $(\Delta/x_t) \ln(x_t/\Delta)$ for $\Delta \ll x_t$. The inequality $\Delta \ll x_t$ is satisfied for the most unstable modes.

In addition, the spatial localization can result in increased growth rates because the localization is caused by the trapped electrons responding to the helicity of the magnetic field, $\exp(-ilq\theta)$, rather than the helicity of the mode, $\exp(im\theta)$, where q is the spatially varying safety factor and θ is the poloidal angle. One may recall that drift

instabilities can occur whenever a phase difference exists between the adiabatic and nonadiabatic electron responses. Then, because $\exp [i(m - 1q)\theta] \approx \exp (-ix\theta/\Delta) \sim \exp (-ix_t/\Delta) \neq 1$, both the real and the imaginary parts of the nonadiabatic trapped electron response, rather than just the imaginary part, are destabilizing.

Finally, the spatial localization of the trapped electron coupling term allows magnetic shear to convect energy away from the destabilizing region near the rational surface out to the region of strong ion Landau damping, just as occurs in the isolated rational surface models. This shear stabilization is observed, in fact, in the most recent numerical work of Ross and Miner.⁴ Consequently, shear continues to provide an important stabilizing mechanism.

Sections II and III give a derivation of the equation to be solved and its perturbation theory solution. The full solution is presented in Sec. IV. In Secs. III, IV, and V the two solutions are compared with each other and with previous results. Some mathematical details are relegated to the Appendix.

II. DERIVATION OF THE TWO-DIMENSIONAL EIGENVALUE EQUATION

We work in a toroidal geometry with circular concentric flux surfaces. The equilibrium magnetic field can be written as $\vec{B} = (B_0 R_0/R)\{\hat{\zeta} + [\epsilon/q(r)]\hat{\theta}\}$, where $\epsilon = r/R_0$, $R = R_0(1 + \epsilon \cos \theta)$, R_0 is the major radius of the magnetic axis, $B \approx B_0(1 - \epsilon \cos \theta)$, q is the safety factor, and r , θ , and ζ are the radial, poloidal, and toroidal variables, respectively. In the following calculation, order ϵ corrections are neglected; $\epsilon^{1/2}$ corrections, which are associated with the fraction of trapped particles,

are retained. The lowest order equilibrium distribution functions are assumed to be Maxwellian, with the density N and the temperature T_j functions of the canonical angular momentum,

$$F_j = N(M_j/2\pi T_j)^{3/2} \exp(-M_j v^2/2T_j), \quad j = i, e, \quad (1)$$

where M_j is the mass and v is the velocity. The subscript j denotes the species.

For the perturbed distribution function f and the electrostatic potential Φ , we can Fourier analyze in the θ and ζ directions and write

$$[f, \Phi] = \sum_m [f_m(r), \Phi_m(r)] \exp(-i\omega t - il\zeta + im\theta), \quad (2)$$

where we can consider only a single toroidal mode number l without a loss of generality because no toroidal coupling occurs. We now pick a particular rational surface $r = r_m$, which corresponds to the integer m such that $q(r_m) = m/l$, and define for convenience a radial variable $x = r - r_m$. The separation between the m/l rational surface and the $(m+1)/l$ rational surface is denoted by Δ with $\Delta = (lq')^{-1}$, where the prime denotes the radial derivative. We further assume, and justify a posteriori, that the shear length is constant over the region of interest so that Δ does not vary.

Neglecting the drift motion of the ions (which gives rise to an additional coupling of different m modes on the order of ∇B and curvature drift over the diamagnetic drift, an order ϵ correction) and ion collisions, we can write the perturbed ion density as

$$\tilde{N} = \sum_m N_m \exp(-i\omega t + im\theta - il\zeta),$$

where

$$N_m/N = (-e/T_e \omega) (\{\omega \tau - [(\omega \tau + \omega_*) \Gamma_0 - \eta \omega_* b (\Gamma_0 - \Gamma_i)]\} \Phi_m - D[(\rho_i^2 \partial^2 \Phi_m / \partial r^2 + k_{\parallel}^2 T_i / M \omega^2) S \Phi_m]) , \quad (3)$$

with

$$k_{\parallel} = (m - lq)/qR_0 ,$$

$$D = (\omega \tau + \omega_*) (\Gamma_0 - \Gamma_i) + \eta_i \omega_* (\Gamma_0 + 2b\Gamma_i - 2b\Gamma_0) ,$$

$$S = D^{-1} [(\omega \tau + \omega_*) \Gamma_0 + \eta_i \omega_* (\Gamma_0 - b\Gamma_0 + b\Gamma_i)] .$$

Further,

$$\omega_* = (lq/r)(cT_e/eBL_n) ,$$

$$L_n^{-1} = -\partial \ln N / \partial r ,$$

$$\tau = T_e / T_i ,$$

$$\eta_j = d \ln T_j / d \ln N ,$$

$$k = m/r ,$$

$$b = k^2 \rho_i^2 ,$$

$$\rho_i^2 = -Mc^2 T_i / e^2 B^2 ,$$

and $\Gamma_n = I_n(b) \exp(-b)$ with $n = 0, 1$ where I_n is a modified Bessel function of the first kind.

To calculate the perturbed electron density $\tilde{n}$, we write the equation for the nonadiabatic response of the trapped electrons, g , defined by

$$f_e = (e\Phi/T_e)F_e + g:$$

$$[\omega - \omega_d + i\nu_{\text{eff}} + \frac{iv_{\parallel}}{qR_0} \left(\frac{\partial}{\partial \theta} - ilq \right)] = - \frac{e\Phi}{T_e} F_e (\omega - \omega_*^T) , \quad (4)$$

$$\text{where } \omega_d = \omega_*(L_n/R_0)(v^2/v_e^2), \quad \omega_*^T = \omega_* \left\{ 1 + \eta_e [v^2/v_e^2] - (3/2) \right\} ,$$

$v_e^2 = 2T_e/M_e$, and $\nu_{\text{eff}} = \nu_e (v_e/v)^3/\epsilon$ with ν_e the electron-ion collisional deflection frequency for a thermal electron. Because the bounce term $(\partial/\partial\theta - ilq)$ is the dominant term in Eq. (4), the lowest order response g must be of the form

$$g = h \exp(ilq\theta) , \quad (5)$$

where h is independent of θ . For trapped electrons, we can bounce-average the next order equation to obtain h ,

$$h = - \frac{\omega - \omega_*^T}{\omega - \omega_d + i\nu_{\text{eff}}} F_e \frac{e}{T_e} \bar{\Phi} , \quad (6)$$

where $\bar{\Phi} = [\oint d\theta \Phi \exp(-ilq\theta)/|v_{\parallel}|]/\oint d\theta/|v_{\parallel}|$ and $\oint$ represents the loop integral between the turning points of the banana orbit in the θ direction. Neglecting resonant electron effects, the nonadiabatic response for the circulating electrons to the same order is zero and there is no difficulty in satisfying the periodicity condition in θ .

We define a pitch angle variable κ such that $\kappa^2 = [\frac{1}{2} v^2 - \mu B_0(1 - \epsilon)] / 2\epsilon\mu B_0$, where μ is the magnetic moment $v_\perp^2/2B$. This variable κ is zero for an electron trapped in the bottom of the magnetic well and unity at the trapped-untrapped boundary. Using κ , $\bar{\Phi}$ can be written as

$$\bar{\Phi} = \int_{-\theta_0}^{\theta_0} \frac{d\theta \Phi \exp(-ilq\theta)}{(\kappa^2 - \sin^2 \theta/2)^{1/2}} / 4K(\kappa) \quad ,$$

where $\theta_0 = 2 \sin^{-1} \kappa$ and K is the complete elliptic integral of the first kind,

$$K(\kappa) = \int_0^1 dt [(1 - t^2)(1 - \kappa^2 t^2)]^{-1/2} \quad .$$

From Eqs. (5) and (6), integrating g over the trapped electrons only, the perturbed electron density can be written as

$$\begin{aligned} \tilde{n} &= (e/T_e) [N\bar{\Phi} - \int_T d^3v F_e \left(\frac{\omega - \omega_*}{\omega - \omega_d + i\nu_{\text{eff}}} \right) \bar{\Phi} \exp(ilq\theta)] \\ &= (eN/T_e) \left\{ \bar{\Phi} + (\epsilon/2)^{1/2} \left[\frac{(\omega - \omega_*)}{\omega} A_2 - \frac{\eta_e \omega_*}{\omega} A_1 \right] \right. \\ &\quad \left. \int_{\kappa_0}^1 \frac{d\kappa^2 \bar{\Phi} \exp(ilq\theta)}{[\kappa^2 - \sin^2(\theta/2)]^{1/2}} \right\} \quad , \end{aligned} \quad (7)$$

where

$$d^3v \approx 2\pi(\epsilon/2)^{1/2} v^2 dv d\kappa^2 [\kappa^2 - \sin^2(\theta/2)]^{-1/2} \quad ,$$

$$\kappa_0 = \sin(\theta/2) \quad ,$$

$$\Phi = \sum_m \Phi_m \exp(im\theta) ,$$

$$A_1 = -2\pi^{-1/2} \int_{v_*}^{\infty} \frac{dt t^{1/2} (t - 3/2) \exp(-t)}{1 - (\omega_* L_n / \omega R_0) t + i(v_e / \epsilon \omega) t^{-3/2}}$$

and

$$A_2 = -2\pi^{-1/2} \int_{v_*}^{\infty} \frac{dt t^{1/2} \exp(-t)}{1 - (\omega_* L_n / \omega R_0) t + i(v_e / \epsilon \omega) t^{-3/2}}$$

with $v_* = \sqrt{2}(\nu_e q R_0 / \epsilon^{3/2} \nu_e)$, the ratio of the effective collision frequency to the trapped electron bounce frequency for a thermal electron. In obtaining Eq. (7), we must insert an additional multiplier of 2 in order to count electrons with $v_{||} \leq 0$.

Equations (3) and (7) can be combined using the quasineutrality condition to give an integro-differential equation for the fluctuation potential Φ . Multiplying this equation by $\exp(-im\theta)$, integrating over θ , and inverting the order of the θ and κ^2 integrations by noting that

$$\begin{aligned} & \int_{-\pi}^{\pi} d\theta \exp[i(lq - m)\theta] \int_0^1 \frac{d\kappa^2 \Phi}{[\kappa^2 - \sin^2(\theta/2)]^{1/2}} \\ &= \int_0^1 d\kappa^2 \Phi \int_{-\theta_0}^{\theta_0} \frac{d\theta \exp[i(lq - m)\theta]}{[\kappa^2 - \sin^2(\theta/2)]^{1/2}} \end{aligned}$$

yields the equation to be solved, namely

$$\frac{\partial^2 \Phi_m}{\partial x^2} - (\Lambda - \mu^2 x^2) \Phi_m - H \int_0^1 \frac{d\kappa^2}{K(\kappa)} \sum_{p=-\infty}^{\infty} B_{m,m+p} \Phi_{m,m+p} = 0 . \quad (8)$$

In Eq. (8), $\mu = (\Omega/\omega)(k/L_S)S^{1/2}$, $L_S^{-1} = (\epsilon/q) \partial \ln q / \partial r$,

$$\Lambda = \rho_i^2 D^{-1} \{ \omega[1 + \tau(1 - \Gamma_0)] - \omega_*[\Gamma_0 - \eta_i b(\Gamma_0 - \Gamma_i)] \},$$

$$B_{m,m+p} = \frac{1}{2\pi} \int_{-\theta_0}^{\theta_0} \frac{d\theta \exp [i(lq - m)\theta]}{[k^2 - \sin^2 (\theta/2)]^{1/2}} \\ \times \int_{-\theta_0}^{\theta_0} \frac{d\theta \exp [i(p + m - \eta)\theta]}{[\kappa^2 - \sin^2 (\theta/2)]^{1/2}} \quad (9)$$

and

$$H = \frac{1}{4} \rho_i^{-2} D^{-1} (\epsilon/2)^{1/2} \{ [(\omega - \omega_*)/\omega] A_2 - \eta_e (\omega_*/\omega) A_1 \}.$$

The matrix element $B_{m,m+p}$ can be evaluated analytically by assuming $\kappa \ll 1$, that is, by taking a model valid for the electrons trapped in a parabolic magnetic well. Employing $q(r) = q(r_m) + (r - r_m)q'(r_m) + \dots = (m/l) + xq' + \dots$ yields the result

$$B_{m,m+p} \approx 2\pi J_0(2\kappa x/\Delta) J_0[2\kappa(x - p\Delta)/\Delta], \quad (10)$$

where J_0 is the Bessel function of the zeroth order.

Equation (8) can be reduced to the usual radial differential equation for the trapped electron instability if we keep only $B_{m,m}$ from Eq. (9), approximate $K(\kappa)$ by $\pi/2$ (K is not very different from $\pi/2$ except near $\kappa = 1$), and take the limit $\Delta \rightarrow \infty$. In the following calculation, we continue to approximate $K(\kappa)$ by $\pi/2$ in the final step and assume $\kappa \ll 1$ to obtain Eq. (10); because Eq. (8) involves only an integral over $K(\kappa)$ and $B_{m,m+p}$

the answer is not expected to be sensitive to this approximation. However, the elimination of the other two approximations leads to fundamental changes in the eigenvalue equation. In particular, it can easily be seen that the least inspired approximation is the usual $\Delta \rightarrow \infty$ limit. We first identify the two scale lengths in Eq. (8): Δ , the distance over which the trapped electron response $B_{m,m+p}$ varies substantially, and x_t , the turning point distance of the problem in the absence of the trapped electron term,

$$x_t \equiv |\mu^{-1/2}| = |\rho_i \left(\tau \frac{L_S}{L_n} \frac{\omega}{\omega_*} \right)^{1/2} S^{1/4}| \dots$$

The $\Delta \rightarrow \infty$ limit becomes appropriate for the $\Delta \gg x_t$ or $k\rho_i \ll (L_n\omega_*/L_S\omega)^{1/2} < 1$. However, in the usual one-dimensional calculation of trapped electron instability, the most unstable modes are found to be around $k\rho_i \sim 1$. This inequality therefore implies that for a self-consistent calculation of the growth rate, the spatial variation of the trapped electron term must be retained by consideration of the limit $\Delta \ll x_t$. The perturbation theory solution of Eq. (8) in this $\Delta \ll x_t$ limit for the most unstable mode is presented in Sec. III. A more exact solution by a method of matched asymptotic expansions for all the radial modes is presented in Sec. IV.

III. PERTURBATION THEORY SOLUTION OF THE TWO-DIMENSIONAL EIGENVALUE EQUATION

An examination of Eq. (8) results in several useful observations. Because the most unstable modes occur for $k\rho_i \sim 1$, which corresponds to very large m 's, neglect of the p corrections to m for Λ , μ , and H results

in a system of coupled differential equations of the form of Eq. (8), in which $\Phi_m(x)$ and $\Phi_{m+p}(x + \Delta p)$ are indistinguishable except for the slow spatial variations of Λ , μ , H , and Δ . Consequently, if we also neglect this slow spatial structure we are led to the symmetry property

$$\Phi_m(x) = \Phi_{m+p}(x + \Delta)a^p.$$

If the periodic x variation repeated every J surface, then $a = \exp(i-2\pi i/J)$, with $i = 0, 1, \dots, J - 1$. As a result⁶

$$\Phi_{m+p}(x + p\Delta) = \Phi_m(x) \exp(i2\pi jp/J). \quad (11)$$

The relative insensitivity of our results to the number of rational surfaces retained justifies employing Eq. (11) in our perturbation theory solution. Our full solution in Sec. IV, however, does not depend on this symmetry property.

This symmetry property is also implicit in the numerical solutions in Eq. (8).⁴ Numerically, one must truncate the infinite system of differential equations to a finite one by coupling only modes from poloidal mode numbers $m - p$ to $m + p$, with only the m th mode coupled symmetrically. The solution of such a finite system does not satisfy Eq. (11); however, as p becomes larger more poloidal modes are retained, and there is a tendency for Eq. (11) to be better satisfied. Thus, it is not difficult to visualize that Eq. (11) is indeed the property of the solution of the infinite system described by Eq. (8) in the absence of the slow spatial variations of Λ , μ , H , and Δ .

Next, we note that because the coupling matrix $B_{m,m+p}$ is a product of Bessel functions, the asymptotic behavior of $B_{m,m+p}$ for large x goes

like $1/x$. Thus, the coupling term in Eq. (8) behaves somewhat like the resonant electron term for the collisionless drift wave calculation.⁵ Consequently, we expect that near marginal stability the coupling term can be treated via the perturbation theory method, just as in the critical shear calculation for the collisionless drift wave.⁵ In Sec. IV we examine this assertion in detail by retaining the coupling term exactly and employing a technique similar to that of Ref. 5. This more precise solution removes the perturbation theory approximation and is valid provided $\Delta|\partial \ln \phi_m / \partial x| < 1$ in the vicinity of $x = 0$. The perturbation theory result is shown to follow from the more exact calculation near marginal stability, provided $k\rho_i > (L_S/L_n)^{1/2}(L_S/qR_0)$. For $k\rho_i < (L_S/L_n)^{1/2}(L_S/qR_0)$ or away from marginal stability, the full eigenvalue equation must be employed.

Treating the coupling term as a perturbation, we solve Eq. (8) to lowest order and obtain the most unstable mode,

$$\phi_m^{(0)}(x) = \hat{\phi} \exp(-i\mu x^2/2) \quad (12a)$$

From Eq. (11), we also have

$$\phi_{m+p}^{(0)}(x) = \hat{\phi} \exp[-i\mu(x - p\Delta)^2/2] + i 2\pi j p/J \quad (12b)$$

Including the trapped electron coupling term by perturbation theory yields the eigenvalue equation

$$\Lambda + i\mu = -2\left(\frac{i\mu}{\pi}\right)^{1/2} H \int_0^1 \frac{dk^2}{K(k)} \sum_{p=-\infty}^{+\infty} L_p \quad (13)$$

where

$$L_p = 2\pi \int_0^\infty dx \exp(-i\mu x^2 + i\mu x p \Delta - i\mu p^2 \Delta^2/2 + i 2\pi j p/J) J_0\left(\frac{2\kappa x}{\Delta}\right) J_0\left[\frac{2\kappa}{\Delta}(x - p\Delta)\right] \quad (14)$$

The p terms in the exponent of L_p become important only for $|p| \gtrsim x_t/\Delta$. One might therefore conclude that the number of rational surfaces to which Φ_m can effectively couple is of order x_t/Δ . However, the p dependence of $J_0[2\kappa(x - p\Delta)/\Delta]$ in L_p results in a p cutoff of order unity. Anticipating this p cutoff from $J_0[2\kappa(x - p\Delta)/\Delta]$, we neglect the p dependence of the exponent of L_p . Furthermore, employing the integral representations for Bessel functions, we note that

$$J_0(t - \rho) = \sum_{n=0}^{\infty} \epsilon_n J_n(\rho) J_n(t) \quad ,$$

with $\epsilon_n = 1$ for $n = 0$ and 2 for $n \neq 0$, so that the expression for L_p becomes

$$L_p \sim 2\pi(\Delta/2\kappa) \sum_{n=0}^{\infty} \epsilon_n J_n(2\kappa p) \int_0^\infty dt \exp(-\alpha t^2) J_0(t) J_n(t) \exp(i 2\pi j p/J) \quad ,$$

where $\alpha = i\mu (\Delta/2\kappa)^2$. Consequently, we must evaluate the integral of $\exp(-\alpha t^2) J_0(t) J_n(t)$ in the limit $\alpha < 1$ ($\Delta < x_t$). The details of evaluating the integral L_p in this limit are relegated to the Appendix; Eq. (A7) is the result we require for $\Delta < x_t$. Retaining only the leading term from Eq. (A7), the $(-1)^{n/2} \pi^{-1} \ln(x_t/\Delta)$ term, and performing the sum over n , we obtain

$$\begin{aligned}
L_p &\approx (\Delta/\kappa) \ln(x_t/\Delta) \sum_{n=0}^{\infty} \epsilon_n \cos(n\pi/2) J_n(2\kappa p) \exp(i 2\pi j p/J) \\
&\approx (\Delta/\kappa) \ln(x_t/\Delta) \cos(2\kappa p) \exp(i 2\pi j p/J) .
\end{aligned} \tag{15}$$

Inserting Eq. (15) into Eq. (13), using $K(\kappa) \approx \pi/2$, and performing the κ integration yield

$$\Lambda + i\mu = -8\pi^{-1} (i\mu/\pi)^{1/2} H\Delta \ln(x_t/\Delta) \sum_{p=-\infty}^{\infty} \frac{\sin 2p}{2p} \exp(i 2\pi j p/J) , \tag{16}$$

The $\sum \sin 2p/2p \exp(ip\beta)$ that appears in Eq. (16) exhibits the p truncation mentioned earlier. This truncation occurs because the p variation of $J_0[2\kappa(x - p\Delta)/\Delta]$ in Eq. (14) results in a localization of the coupling of ϕ_m to the rational surfaces within a distance of approximately Δ from $x = 0$. Consequently, the neglect of the p variation compared to m in the exponent of L_p is justified, as is the neglect of the slow spatial structure of Λ , μ , and H .

Noting that

$$\sum_{p=-\infty}^{\infty} \frac{\sin 2p}{2p} \exp(ip\beta) = \frac{\pi}{2} ,$$

we find that Eq. (16) gives the final expression for the eigenvalue equation as obtained from perturbation theory, namely.

$$\Lambda + i\mu = -4(i\mu/\pi)^{1/2} H\Delta \ln(x_t/\Delta) . \tag{17}$$

Equation (17) is quite different from the usual radial result,

$$\Lambda + i\mu = -16\pi^{-1} H , \tag{18}$$

which can be recovered from Eq. (13) by retaining only the $p = 0$ term, replacing L_p by

$$B_{m,m} \int_0^\infty dx \exp(-i\mu x^2) \bigg|_{\substack{m = lq \\ \Delta \rightarrow \infty}} = \frac{1}{2} \left(\frac{\pi}{i\mu} \right)^{1/2} \times \int_{-\theta_0}^{\theta_0} \frac{d\theta}{[\kappa^2 - \sin^2(\theta/2)]^{1/2}}$$

and inverting the order of the κ^2 and θ integrations. If Eq. (14) is employed instead of the preceding approximations, the coefficient of H in Eq. (18) is found to be 4 instead of $16/\pi$; the error is a result of the approximation implicit in the Bessel function form for

$$\int_{-\theta_0}^{\theta_0} \frac{d\theta \exp[i(lq - m)\theta]}{[\kappa^2 - \sin^2(\theta/2)]^{1/2}} \bigg|_{m = lq}$$

In Sec. IV, we solve Eq. (8) assuming only that $\Delta |\partial \ln \Phi_{m+p} / \partial x| < 1$ in the vicinity of $x = 0$. This more exact calculation results in the full eigenvalue equation for all the radial eigenmodes.

IV. MATCHED ASYMPTOTIC EXPANSION SOLUTION OF THE TWO-DIMENSIONAL EIGENVALUE EQUATION

In this section, we show that near marginal stability, the eigenvalue equation obtained by the perturbation theory solution is a good approximation to the full eigenvalue equation associated with Eq. (8) only if $k\rho_i > (L_S/L_n)^{1/2} (L_S/qR_0)$.

To obtain the full eigenvalue equation, we solve Eq. (8) by a technique similar to that employed for the collisionless drift instability.⁵ The technique employs two forms of Eq. (8), the first valid in the outer

region away from the rational surface of interest and the second valid in the inner region about the same rational surface. In the outer region, the coupling term in Eq. (8) may be rigorously treated as a small correction and an iterated solution obtained. In the inner region, the coupling term dominates and cannot be treated by iteration. Instead, the inner region form of the differential equation may be employed to obtain an expression for the logarithmic derivative of Φ_m which is matched to the corresponding form from the outer region to obtain the full eigenvalue equation.

We are interested in solving Eq. (8) in the limit in which many rational surfaces are coupled, $\Delta \ll x_t$. Furthermore, we justify a posteriori that only the first few p terms in Eq. (8) are significant. As a result, we may employ $\Phi_{m+p}(x) \approx \Phi_m(x) \exp(i 2\pi j p/J)$ to write Eq. (8) as

$$\frac{\partial^2 \Phi_m}{\partial x^2} - \left[\Lambda - \mu^2 x^2 + 2\pi H \int_0^1 \frac{d\kappa^2}{K(\kappa)} J_0(2\kappa x/\Delta) \times \sum_{p=-\infty}^{\infty} \sum_{n=0}^{\infty} \exp(i 2\pi j p/J) \epsilon_n J_n(2\kappa p) J_n(2\kappa x/\Delta) \right] \Phi_m = 0 \quad (19)$$

In writing Eq. (19), we employ $J_0(t - \rho) = \sum_{n=0}^{\infty} \epsilon_n J_n(\rho) J_n(t)$ with $\epsilon_n = 1$ for $n = 0$ and $\epsilon_n = 2$ for $n \neq 0$.

To obtain the equation appropriate in the outer region, we assume $\kappa x/\Delta > 1$ so that

$$J_0(2\kappa x/\Delta) J_n(2\kappa x/\Delta) \approx \frac{\cos(n\pi/2)}{2\pi\kappa x/\Delta}$$

may be employed. Then noting that

$$\sum_{n=0}^{\infty} \epsilon_n \cos(n\pi/2) J_n(2\kappa p) = \cos 2\kappa p, \quad (20)$$

we find that in the outer region Eq. (19) may be written as

$$\frac{\partial^2 \Phi_m}{\partial x^2} - \left[\Lambda - \mu^2 x^2 + \frac{2H\Delta}{x} \right] p \sum_{p=-\infty}^{\infty} \exp(i 2\pi j p/J) \int_0^1 \frac{d\kappa \cos 2\kappa p}{K(\kappa)} \Phi_m = 0. \quad (21)$$

We can further simplify this equation by noting that the κ integration in Eq. (21) can be performed approximately to obtain

$$\int_0^1 \frac{d\kappa \cos 2\kappa p}{K(\kappa)} \approx \frac{\sin 2p}{\pi p} \quad (22)$$

because $K(\kappa) \approx \pi/2$ for $\kappa < 1$ and $K(\kappa) \approx \frac{1}{2} \ln(1/1 - \kappa)$ for $\kappa \approx 1$. The resulting $(\sin 2p)/p$ dependence then provides the justification for our previous assumption that only the first few p terms in Eq. (8) are significant. In fact, summing over all p gives $\sum (\sin 2p) \exp(ip\beta)/2p = \pi/2$, while retaining only $p = 0$ gives $[(\sin 2p)/2p] \Big|_{p=0} = 1$. Consequently, with Eq. (22) and $\sum (\sin 2p) \exp(ip\beta)/2p = \pi/2$, Eq. (21) yields the equation in the outer region, namely,

$$\frac{\partial^2 \Phi_m}{\partial x^2} - \left(\Lambda - \mu^2 x^2 + \frac{2H\Delta}{x} \right) \Phi_m = 0. \quad (23)$$

A. Even modes

At this point, we observe that in the outer region, the x dependence of the trapped electron contribution to Eq. (23) is proportional to $1/x$. As a result, the form of Eq. (23) is exactly the same as the form in the

outer region for the collisionless drift instability. Therefore, the expressions from Ref. 5 for the eigenvalue equation and the logarithmic derivative of Φ_m [Eqs. (18) and (20) of Ref. 5] may be employed, and converting these expressions to our notation yields

$$1 + \frac{C\Gamma\left(\frac{1}{4} - \frac{i\Delta}{4\mu}\right)}{\sqrt{2}\Gamma\left(\frac{3}{4} - \frac{i\Delta}{4\mu}\right)} \quad (24)$$

and as $x \rightarrow \Delta$

$$\frac{1}{\Phi_m} \frac{\partial \Phi_m}{\partial x} = C(i2\mu)^{1/2} + 2H\Delta \ln [x(2\mu)^{1/2}] \quad , \quad (25)$$

where only the dominant terms have been retained. In Eqs. (24) and (25), the constant C is the relative constant between the two homogeneous solutions of Eq. (23) and is determined by obtaining the expression for the logarithmic derivative of Φ_m from the form of Eq. (19) appropriate for the inner region about the rational surface of interest.

In Ref. 5, further corrections to Eq. (24) are evaluated. By retaining only the dominant terms, we are assuming that $x_t \Delta / \rho_i^2 < 1$ or $k\rho_i > (L_S/L_n)^{1/2} (L_S/qR_0)$. Because $(L_S/L_n)^{1/2} (L_S/qR_0)$ is typically of order unity or greater and because the most unstable modes have $k\rho_i \sim 1$, this restriction is removed in Sec. IVC, in which these corrections are neglected in order to make a convenient comparison with the perturbation theory result.

The width of the inner region must be large enough to extend out to the $1/x$ region for the trapped electron term. Because $\Delta \ll x_t$, this $1/x$ dependence becomes appropriate for $x \ll x_t$. If we consider a mode

even in x so that $\phi'_m(x=0) = 0$, then the lowest order expression for the logarithmic derivative of ϕ_m is obtained by integrating Eq. (19) from $x = 0$ to some x for which $2\kappa x \gg \Delta$; that is,

$$\frac{1}{\phi_m} \frac{\partial \phi_m}{\partial x} \approx + 2\pi H \int_0^1 \frac{d\kappa^2}{K(\kappa)} \sum_{p=-\infty}^{\infty} \sum_{n=0}^{\infty} \epsilon_n J_n(2\kappa p) \times \int_0^x dx' J_0(2\kappa x'/\Delta) J_n(2\kappa x'/\Delta) \quad (26)$$

Contributions to ϕ'_m/ϕ_m from Λ and $\mu^2 x^2$ have been neglected in Eq. (26) because they will result in x and x^3 contributions which will be small because $x \ll x_t$. For $2\kappa x \gg \Delta$, we show in the Appendix that

$$\int_0^x dx' J_0(2\kappa x'/\Delta) J_n(2\kappa x'/\Delta) \approx (\Delta/2\pi\kappa) \cos(n\pi/2) \ln(2\kappa x/\Delta) \quad .$$

Consequently, neglecting the κ dependence in the logarithm, using Eqs. (22) and (24), and again employing $\sum (\sin 2p) \exp(ip\beta)/2p = \pi/2$, we find that Eq. (26) yields

$$\frac{1}{\phi_m} \frac{\partial \phi_m}{\partial x} = +2H\Delta \ln\left(\frac{2x}{\Delta}\right) \quad (27)$$

Equating Eqs. (25) and (27) gives the constant C as

$$C = + \frac{2H\Delta}{(i2\mu)^{1/2}} \ln \frac{x_t}{\Delta} \quad (28)$$

As a result, the eigenvalue equation (24) may be written as

$$1 + \frac{H\Delta}{\pi(i\mu)^{1/2}} \Gamma\left(\frac{1}{4} - \frac{i\Lambda}{4\mu}\right) \Gamma\left(\frac{1}{4} + \frac{i\Lambda}{4\mu}\right) \times \sin\left[\frac{\pi}{4}\left(1 + \frac{i\Lambda}{\mu}\right)\right] \ln\left(\frac{x_t}{\Delta}\right) = 0 \quad (29)$$

Equation (29) is the full eigenvalue equation associated with Eq. (19), provided $x_t\Delta/\rho_i^2 < 1$. To obtain the perturbation theory result, Eq. (17), from Eq. (29), we let $\Lambda + i(2n+1)\mu = \sigma$ with $n = 0, 2, 4, \dots$ and solve for σ , assuming $|\sigma/\mu| < 1$ to obtain

$$\Lambda + i(2n+1)\mu = \frac{4n!}{2n[(n/2)!]^2} (i\mu/\pi)^{1/2} \times H\Delta \ln(x_t/\Delta) \quad (30)$$

This reduction occurs because the gamma function has simple poles where its argument is zero or a negative integer. Because $\mu = \text{Im } \sigma$ at marginal stability, Eq. (30) is expected to be only qualitatively correct away from $\text{Im } \omega = 0$, and for $\text{Im } \sigma \gg \mu$ the full eigenvalue equation (29) should be employed. Furthermore, Eq. (30) is only valid near marginal stability for $k\rho_i > (L_S/L_n)^{1/2}(L_S/qR_0)$ because the order $x_t\Delta/\rho_i^2$ corrections to Eq. (29) were neglected. These order $x_t\Delta/\rho_i^2$ corrections cannot be obtained by the standard perturbation theory treatment.

B. Odd modes

We can also recover the full eigenvalue equation for the odd spatial modes. The crucial observation is that for an odd spatial mode the dominant spatial behavior of Φ_m for $x \lesssim \Delta$ is $\Phi_m \propto x$. However, employing $\Phi_m \propto x$ and integrating Eq. (19) from $x = 0$ out to $\Delta \ll x \ll x_t$ does not generate the leading term of the homogeneous solution of the outer equation, Eq. (23), which has a constant leading term. As a result, in the outer region only the homogeneous solution having a leading term

proportional to x is permitted [the $\chi_0^{(1)} + \chi_1^{(1)}$ of Ref. 5], and the coefficient of the other homogeneous solution [$\chi_0^{(0)} + \chi_1^{(0)}$ of Ref. 5] must vanish ($C \rightarrow \infty$ in Ref. 5). Consequently, the eigenvalue equation for the odd modes is recovered from Eq. (18) of Ref. 5 by retaining only the terms containing $C(C \rightarrow \infty)$. In our notation, Eq. (18) of Ref. 5 is

$$1 + \frac{H\Delta}{2(i\mu)^{1/2}} \sum_{s=0}^{\infty} \frac{\Gamma\left(s - \frac{i\Lambda}{4\mu} + \frac{3}{4}\right)}{\left(s + \frac{1}{2}\right) \Gamma\left(s - \frac{i\Lambda}{4\mu} + \frac{5}{4}\right)} = 0. \quad (31)$$

To obtain the perturbation theory result from Eq. (31), we let $\Lambda + i(2n+1)\mu = \sigma$ with $n = 1, 3, 5, \dots$ and solve for σ by assuming $|\sigma/\mu| < 1$. Because $\Gamma[S - (i\Lambda/4\mu) + (3/4)]$ behaves like a simple pole when its argument is near zero or a negative integer, only the $0 \leq S \leq (n-1)/2$ terms in the sum of Eq. (31) are significant in this $|\sigma/\mu| < 1$ limit. For $|\sigma/\mu| < 1$, Eq. (31) therefore reduces to the perturbation theory result

$$\Lambda + i(2n+1)\mu = -4 \left(\frac{i\mu}{\pi}\right)^{1/2} \times H\Delta \sum_{l=0}^{(n-1)/2} \frac{(2l)!}{2^{2l}(l!)^2 (n-2l)}, \quad (32)$$

where $n = 1, 3, 5, \dots$. Because $\text{Im } \sigma > |\mu|$ away from marginal stability, Eq. (32) is expected to be valid near $\text{Im } \omega \rightarrow 0$ only; away from marginal stability Eq. (31) should be employed.

C. Even modes revisited

In obtaining the eigenmode equation for the even modes, we retained only the dominant terms. For $x_t \Delta / \rho_i^2 \geq 1$, the next correction from Eq. (18)

of Ref. 5 should be retained. This correction is the same as the term with the sum in Eq. (31) as compared to unity. Hence, a still more complete eigenvalue equation for the even modes is

$$\begin{aligned} & \frac{H\Delta \ln(x_t/\Delta)}{\pi(i\mu)^{1/2}} \Gamma\left(\frac{1}{4} - \frac{i\Lambda}{4\mu}\right) \Gamma\left(\frac{1}{4} + \frac{i\Lambda}{4\mu}\right) \sin\left[\frac{\pi}{4}\left(1 + \frac{i\Lambda}{\mu}\right)\right] \\ & \times \left\{ 1 + \left[\frac{H\Delta}{2(i\mu)^{1/2}} \sum_{s=0}^{\infty} \frac{\Gamma\left(s - \frac{i\Lambda}{4\mu} + \frac{3}{4}\right)}{\left(s + \frac{1}{2}\right) \Gamma\left(s - \frac{i\Lambda}{4\mu} + \frac{5}{4}\right)} \right] \right\} + 1 = 0 \quad (33) \end{aligned}$$

Because $H \sim \rho_i^{-2}$, this correction is of order $x_t \Delta / \rho_i^2$. For

$\Lambda + i(2n+1)\mu = \delta$ and $|\delta/\mu| < 1$, this correction reduces to

$$\frac{H\Delta\pi\Gamma\left(\frac{n+1}{2}\right)}{2(i\mu)^{1/2}\Gamma\left(\frac{n}{2}+1\right)}$$

compared to unity. Therefore, for $n = 0$ and $|\delta/\mu| < 1$, Eq. (33) reduces to

$$\Lambda + i\mu = -4H\Delta(i\mu/\pi)^{1/2} \ln(x_t/\Delta) - 2\pi H^2 \Delta^2 \ln(x_t/\Delta) \quad (34)$$

Consequently, the imaginary part of this term is destabilizing for $(\text{Re } H)(\text{Im } H) < 0$, while the real part results in a destabilizing shift in the real part of the frequency for $(\text{Re } H)^2 > (\text{Im } H)^2$. For $b \sim 1$, this correction becomes important when $(\Delta^2/\rho_i^2) \ln(x_t/\Delta) \gtrsim 1$, that is, when $k^2 \rho_i^2 > (L_S/qR_0)^2 \ln(x_t/\Delta)$. Away from marginal stability, the correction would be expected to become important when $|H\Delta\mu^{-1/2}| \sim x_t \Delta / \rho_i^2 \gtrsim 1$,

or $k\rho_i \geq (L_S/L_n)^{1/2}(L_S/qR_n)$. Because the fastest growing modes are expected to have $k\rho_i \sim 1$, Eq. (33) will normally result in more accurate eigenvalues than Eq. (29).

The correction to Eq. (29) occurs because the trapped electron term of Eq. (8) results in modifications of the eigenfunctions so that they are no longer simply the even Hermite polynomials. The $x_t\Delta/\rho_i^2$ correction in Eq. (33) is generated because the departure from the even Hermite polynomials extends into the outer region. This departure is also the reason that the $x_t\Delta/\rho_i^2$ corrections cannot be found from the standard perturbation theory treatment.

V. DISCUSSION

Equations (17), (29), and (33) are qualitatively distinct from the usual result for the most unstable mode, given by Eq. (18). Because of the factor $1 + i$ at the beginning of the right-hand side of Eq. (17), the real parts of A_1 and A_2 contribute to the growth rate as well as the imaginary parts. Because the more exact calculation gives the same effect near marginal stability, the contribution of the real part of the A 's is not due to the inaccuracy of the perturbative treatment. Therefore, the nature of the instability is significantly different from that indicated by the usual result. One may recall that physically the instability is due to the phase difference between the trapped electron response and the fluctuation, whether it is provided by collisions, a drift resonance, or some other mechanism. In our case, an additional phase difference is introduced because the orbits of trapped electrons follow the pitch of the field line rather than the pitch of the mode

(the pitches are the same only on the mode rational surface.) This phase difference is neglected when $\Delta \gg x_t$ is assumed. Phase shifts will be present for $x_t \gtrsim \Delta$ as can be seen from the eigenvalue equation for arbitrary x_t/Δ ,

$$\lambda + i\mu = -(8H/\pi) \int_0^1 d\kappa^2 \kappa^{-1} I_0(\kappa^2/2i\mu\Delta^2) \exp(-\kappa^2/2i\mu\Delta^2) .$$

The right side of this expression is normalized so that the constant agrees Eq. (18) for $x_t/\Delta \ll 1$. Within a multiplier of $2/\pi$, this expression is obtained by neglecting $\exp(i\mu x p - i\mu p^2 \Delta^2/2 + i2\pi p j/J)$ in Eq. (14), using the Schloemilch series $\sum_{p=-\infty}^{+\infty} J_0[2\kappa(x - p\Delta)/\Delta] = 1/\kappa$, and performing the x integration. The discrepancy of a factor of two between this technique and that presented earlier is due to the approximations of Eq. (A7).

The difference between the pitch of the field line and that of the mode also results in a spatial localization of the destabilizing trapped electron term to within about a distance Δ of the rational surface under consideration. This spatial localization accounts for the difference in the magnitudes of the right-hand sides of Eqs. (17) and (18) and would be expected to result in a reduction in the growth rate from the usual result by a factor of order $(\Delta/x_t) \ln(x_t/\Delta)$.

This reduction is far more significant than the increase in the growth rate due to the coupling of the rational surfaces. The increase in the growth rate found by keeping all p terms in $\sum (\sin 2p)/2p \exp(ip\beta)/2p$ (as opposed to keeping just the $p = 0$ term) is only a factor of $\pi/2$. (This factor seems to be consistent with the

numerical work of Ross and Miner.⁴) This increase by only a factor of $\pi/2$ indicates that the coupling of rational surfaces via the trapped electrons is not as strong as is commonly believed. Although the coupling matrix elements $B_{m,m+p}$ of Eq. (8) overlap many rational surfaces, the superposition of their effects converges rapidly because of cancellation beyond the neighboring rational surfaces. This rapid convergence is probably responsible for the rapid convergence of the numerical eigenvalue to a constant value as the number of equations in the truncated subsystem of Eq. (8) is increased.⁴

Another conclusion that can be drawn from Eq. (17) is that shear stabilization is unaffected by the poloidal coupling induced by trapped electrons. This result seems contrary both to the results of Taylor,⁷ who considers poloidal coupling that is not due to trapped electrons, and to the general feeling that the wave energy may be reflected by the next mode rational surface when the ion Landau damping of a particular poloidal mode occurs farther than Δ from the rational surface. However, the survival of shear stabilization is a result of the spatial localization about a rational surface of the trapped electron induced coupling. Therefore, this survival is not at variance with the model of Taylor,⁷ as he does not consider spatially localized coupling terms. Our conclusion that shear stabilization is not defeated by the trapped electron induced coupling of poloidal modes agrees with the observation of Ross and Miner⁴ based on their numerical results.

Finally, by solving the two-dimensional eigenvalue problem without recourse to perturbation theory, we have found that the perturbation theory results for the eigenvalue equations, Eqs. (30) and (32), are valid

only near marginal stability and only if $k\rho_i > (L_S/L_n)^{1/2}(L_S/qR_0)$. Away from marginal stability, Eq. (31) must be employed for the odd radial modes in order to obtain accurate growth rates. For the even modes, Eq. (29) may be employed when $k\rho_i > (L_S/L_n)^{1/2}(L_S/qR_0)$. Equation (33) must be used for the even modes for arbitrary growth rates if $k\rho_i \lesssim (L_S/L_n)^{1/2}(L_S/qR_0)$.

This extension of the work of Rosenbluth and Catto⁵ can also be employed to obtain the general eigenvalue equation for all radial modes for the collisionless drift wave,⁷ an improvement over the method of considering only the most unstable mode ($n = 0$) at marginal stability, which was used in Ref. 5.

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APPENDIX

In this appendix we evaluate the integrals

$$P_n(t_0) = \int_0^{t_0} dt J_0(t) J_n(t) \quad (A1)$$

and

$$Q_n(\alpha) = \int_0^\infty dt \exp(-\alpha t^2) J_0(t) J_n(t) \quad , \quad (A2)$$

where $\text{Re } \alpha > 0$ and $t_0 \gg 1$.

For $\text{Re } \epsilon > 0$ we note that⁹

$$\int_0^\infty dt t^{-\epsilon} J_0(t) J_n(t) = \frac{\Gamma(\epsilon) \Gamma\left(\frac{n+1-\epsilon}{2}\right)}{2^\epsilon \Gamma\left(\frac{1-n+\epsilon}{2}\right) \left[\Gamma\left(\frac{n+1+\epsilon}{2}\right)\right]^2} \quad (A3)$$

Letting $\epsilon \rightarrow 0$ and noting that the gamma function has poles at zero and the negative integers, we obtain

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_0^\infty dt t^{-\epsilon} J_0(t) J_n(t) \\ \approx \begin{cases} \frac{1}{2}(-1)^{(n-1)/2} & n \text{ odd} \\ \frac{(-1)^{n/2}}{\pi\epsilon} \left\{ 1 - \epsilon[\gamma + \ln 2 + 2\psi\left(\frac{n+1}{2}\right)] \right. & \begin{matrix} n = 0 \\ n \text{ even} \end{matrix} \end{cases} \quad (A4) \end{aligned}$$

Next, we note that for $t_0 \gg 1$,

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \int_{t_0}^{\infty} dt t^{-\epsilon} J_0(t) J_n(t) &\approx \lim_{\epsilon \rightarrow 0} \frac{\cos(n\pi/2)}{\pi} \int_{t_0}^{\infty} \frac{dt}{t^{1+\epsilon}} \\
&= \frac{\cos(n\pi/2)}{\pi\epsilon} (1 - \epsilon \ln t_0) .
\end{aligned} \tag{A5}$$

Combining Eqs. (A4) and (A5) gives the expression for $P_n(t_0)$,

$$P_n(t_0) = \begin{cases} \frac{1}{2} (-1)^{(n-1)/2} & n \text{ odd} \\ \frac{(-1)^{n/2}}{\pi} \left\{ \ln t_0 - [\gamma + \ln 2 + 2\psi\left(\frac{n+1}{2}\right)] \right\} & \begin{matrix} n = 0 \\ n \text{ uneven} \end{matrix} \end{cases} . \tag{A6}$$

To evaluate $Q_n(\alpha)$ we simply note that for $|\alpha^{-1/2}| \gg t_0 \gg 1$, we may write

$$\begin{aligned}
Q_n(\alpha) &\approx \int_0^{t_0} dt J_0(t) J_n(t) + \int_{t_0}^{\infty} dt \exp(-\alpha t^2) J_0(t) J_n(t) \\
&\approx P_n(t_0) + \frac{\cos(n\pi/2)}{\pi} \int_{t_0}^{\infty} \frac{dt}{t} \exp(-\alpha t^2) \\
&\approx P_n(t_0) + \frac{\cos(n\pi/2)}{2\pi} [\ln(1/\alpha t_0^2) - \gamma] \\
&\approx \begin{cases} \frac{1}{2} (-1)^{(n-1)/2} & n \text{ odd} \\ \frac{(-1)^{n/2}}{\pi} \left\{ \ln \alpha^{-1/2} - \frac{3}{2} \Gamma + \ln 2 + 2\psi\left(\frac{n+1}{2}\right) \right\} & \begin{matrix} n = 0, \\ n \text{ even} \end{matrix} \end{cases} .
\end{aligned} \tag{A7}$$

The preceding form of $Q_n(\alpha)$ is appropriate for $\alpha < 1$ and in Eq. (A7), $\psi(x) = d \ln \Gamma(x)/dx$ and γ is Euler's constant.

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69. Dr. David A. Dingee, Fusion Programs, Battelle-Northwest, Battelle Boulevard, Richland, WA 99352
70. CTR Reading Room, c/o Prof. C. S. Liu, Dept. of Physics and Astronomy, Univ. of Maryland, College Park, MD 20742
71. CTR Reading Room, c/o Prof. T. Kamnash, 103 Research Admin. Bldg., N. Campus, Univ. of Michigan, Ann Arbor, MI 48105
72. CTR Reading Room, c/o Dr. Ravi N. Sudan, Phillips Hall, Cornell Univ., Ithaca, NY 14850
73. Prof. Marshall N. Rosenbluth, Institute for Advanced Study, Princeton, NJ 08540
74. CTR Reading Room, c/o Prof. R. Gross, Plasma Research Lab., Columbia Univ., New York, NY 10027
75. CTR Reading Room, c/o Prof. Roy Gould, California Inst. of Tech., M.S. 116-81, Pasadena, CA 91125
76. Dr. Nicholas A. Krall, Science Applications, Inc., P.O. Box 2354, 1200 Prospect St., La Jolla, CA 92037
77. CTR Reading Room, c/o Dr. Jay P. Boris, Plasma Physics, Naval Research Laboratory, Washington, D.C. 20390
78. Professor A. Simon, Dept. of Mechanical & Aerospace Sciences, University of Rochester, Rochester, NY 14627
79. CTR Library, c/o Dr. Alan F. Haught, United Technologies Research Labs, East Hartford, CT 06108
80. Dr. H. K. Forsen, Exxon Nuclear Co., Inc., 777-106th Avenue, NE, C-000777, Bellevue, WA 98009
81. Dr. George Vahala, Physics Dept., College of William & Mary, Williamsburg, VA 23185
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90. Dr. Claude Mercier, Service du Theorie des Plasmas, Centre d'Etudes Nucléaires, Fontenay-aux-Roses (Seine) France
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