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Equilibrium Phase-Space Distributions and Space Charge Limits in Linacs

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EQUILIBRIUM PHASE-SPACE DISTRIBUTIONS AND SPACE CHARGE LIMITS IN LINACS

by

W. P. Lysenko

ABSTRACT

Limits on beam current and emittance in proton and heavy ion linear accelerators resulting from space charge forces are calculated. The method involves determining equilibrium distributions in phase space using a continuous focusing, no acceleration, model in two degrees of freedom using the coordinates r and z . A nonlinear Poisson equation must be solved numerically. This procedure is a matching between the longitudinal and transverse directions to minimize the effect of longitudinal-transverse coupling which is believed to be the main problem in emittance growth due to space charge in linacs. Limits on the Clinton P. Anderson Meson Physics Facility (LAMPF) accelerator performance are calculated as an example. The beam physics is described by a few space charge parameters so that accelerators with different physical parameters can be compared in a natural way. The main result of this parameter study is that the requirement of a high-intensity beam is best fulfilled with a low-frequency accelerator whereas the requirement of a high-brightness beam is best fulfilled with a high-frequency accelerator.

I. INTRODUCTION

We consider here space charge effects in proton and heavy ion linear accelerator beams. For high-current or high-brightness beams, space charge places important limits on beam current and emittance. With a simple model these limits can be calculated. Such calculations can also provide parameter studies for new high-current or high-brightness linacs.

The method used here involves calculating equilibrium distributions in phase space. An equilibrium distribution is a distribution function $f(\vec{x}, \vec{p})$ in phase space $\vec{x}, \vec{p}$ which is constant in time. This procedure is therefore

a type of matching in six-dimensional phase space. The equilibrium distribution we use is a stationary solution to the Vlasov equation using a model with continuous focusing and no acceleration. Azimuthal symmetry is assumed, so we have two degrees of freedom (coordinates r and z). The procedure follows the work of Gluckstern, Chasman, and Crandall¹ who did a similar problem in one degree of freedom (coordinate r). The extension to two degrees of freedom means that a nonlinear Poisson equation must be solved numerically.

Consider the areas occupied by the beam in one of the phase planes xp_x , yp_y , or zp_z , where the coordinates and momenta refer to the nonrelativistic motion relative to the on-axis synchronous particle. With no space charge or other coupling these areas are preserved in the process of acceleration. Emittance is usually defined as the areas in the xx' , yy' , zz' planes where $x' = dx/ds$, etc., in which s is the path length. For the longitudinal plane, the usual quantities are the phase $\Delta\phi$ and energy ΔE of the particle relative to the synchronous particle. These quantities are related to z and z' by

$$z = \frac{\Delta\phi}{2\pi} \beta\lambda$$

and

$$z' = \frac{\Delta\beta}{\beta} = \frac{1}{2} \frac{\Delta E}{E}.$$

(1)

The normalized emittance, which is a conserved quantity in the absence of space charge or other coupling, is defined as

$$\eta_i = \beta\gamma \frac{E_i}{\pi}$$

(2)

for any direction $i = x, y$, or z .

Rationalized MKS units are used.

II. PHYSICS OF BEAMS

We first will look at the physics of beam bunch behavior from the point of view of a plasma physicist. The CERN report by Lawson² is a useful reference for this section.

A. Collision Effects

1. Temperature. The temperature in the direction i at a point $\vec{x}$ at the time t for the distribution f is defined by

$$kT_i = \frac{m^{-1} \int d^3p (p_i - \langle p_i \rangle)^2 f(\vec{x}, \vec{p}, t)}{\int d^3p f(\vec{x}, \vec{p}, t)} . \quad (3)$$

This is the average kinetic energy of particles at this point, relative to the average motion at the point. Note that if all particles at a given point have the same velocity, then the temperature is zero there. As an example, consider the Kapchinsky-Vladimirsky distribution (K-V distribution) in which $f \sim (p_x^2 + p_y^2)$. Then we have the average transverse temperature given by

$$kT_T = \frac{(p_T(\text{max}))^2}{4m} . \quad (4)$$

As an example, let us consider the Clinton P. Anderson Meson Physics Facility (LAMPF) accelerator. For the present, assume the beam bunch to be a uniformly charged sphere of radius 2.5 mm containing 5.2×10^8 particles (corresponding to an average current of 1 mA at 6 per cent duty factor) with a transverse emittance of $2\pi \times 10^{-5} \text{ m} \cdot \text{rad}$. Note that this assumption is not consistent with the assumption of a K-V distribution used to obtain Eq. (4) but using this formula gives $kT_T = 3.85 \times 10^{-18} \text{ J} = 24.1 \text{ eV}$ or $T = 2.79 \times 10^5 \text{ K}$.

2. Debye Length. The additional electrostatic potential $\delta\phi$ due to a test charge in a beam bunch is modified from $\delta\phi \sim r^{-1}$ to $\delta\phi \sim r^{-1} \exp(-r/h)$ by the presence of all the other charged particles in the bunch. The length h is known as the Debye length

$$h = \left(\frac{\epsilon_0 kT}{ne^2} \right)^{1/2} , \quad (5)$$

where T is the temperature and n the number density. In the more usual case of a neutral plasma, h is the length scale for overall neutrality of the plasma and is therefore usually called the Debye shielding distance.^{3,4} For both neutral and nonneutral plasmas, calculations in which interactions between the particles are treated as sums over two-particle collisions, h is used as the cutoff distance because statistical fluctuations in the particle density disappear past this distance.⁵ For the LAMPF parameters we have $h = 0.41 \text{ mm}$. Note that this represents an unusual situation from a plasma

physics point of view in that the beam bunch is only a few Debye lengths in diameter.

3. Self-Collision Time. For a collection of particles interacting with themselves, the time constant for reducing the isotropy in the velocity distribution (sum of collisions to produce a 90° deflection) and the time constant for the kinetic energy distribution to approach the Boltzmann distribution are about equal and called the self-collision time,⁵

$$t_c = \frac{2\pi\epsilon_0^2 n^{1/2} (3kT)^{3/2}}{0.714 ne^4 \ln \Lambda}, \quad (6)$$

where Λ is the average of the ratio of the interaction cutoff distance (use the Debye length) to the impact parameter for a 90° collision

$$\Lambda = \frac{12\pi(\epsilon_0 kT)^{3/2}}{e^3 n^{1/2}}. \quad (7)$$

For the LAMPF parameters, we have $\Lambda = 2.05 \times 10^7$ and $t_c = 12.6$ ms. This is much longer than the 4.4 μ s it takes for a particle to traverse the whole accelerator. Therefore collision effects are negligible in the accelerator beam. A thermal equilibrium distribution will never be reached.

Collision effects may be understood physically as follows. Consider a beam infinitely long and in which the particles are uniformly distributed inside a circle in x-y space and assume that at a given time all orbits are circles in x-y space. Because of the azimuthal symmetry and because at a given radius r the number of particles inside radius r is proportional to r^2 , the space charge force is linear so that the particle orbits remain circular and the distribution is constant in time. But there are only a finite number of particles and the statistical nature of the process is such that the above collective behavior is only approximate. These statistical fluctuations in the forces seen by a particle causing it to deviate from an exactly circular orbit are the collision effects. Because of these effects, the initial distribution, which in the present example is preserved by the collective effects of space charge, will slowly change, tending toward the thermal equilibrium distribution with the time constant t_c .

B. Collision Effects in Particle Tracing

1. Collision Time. In particle tracing codes, a small number of large particles (macroparticles) for which the ratio e/m is the same as the actual particles are used to simulate the actual beam. This simulates the correct single particle and collective motion, but the collision effects are enhanced compared to the actual beam. Let $x = n_m/n_t$, where n_m is the number of macroparticles and n_t the total number of particles in the actual bunch. The self-collision time for the simulation system is given by

$$t_c(x) = \frac{x}{1 + \ln x / \ln \Lambda} t_c, \quad (8)$$

where t_c is the self-collision time for the actual system and Λ has the actual system value. Figure 1 shows the self-collision time as a function of the number of macroparticles $n_m = xn_t$. Note that in order to keep the collision time above 30 μ s we need more than 10^5 macroparticles. If a small number of macroparticles is used, the results may be overwhelmed by collision effects in the simulation which are not present in the actual system. For small numbers of macroparticles, the collision time increases slower than linearly with the number of macroparticles. This may explain some emittance growth effects⁶ which depend on number of macroparticles as $n_m^{-1/2}$. A standard plasma simulation technique such as the particle-in-cell method may be used to increase the collision time. If the collective effects are to be measured to high accuracy as in a beam halo study, then many macroparticles would be needed simply to satisfy the accuracy requirement and collisions would not be a problem. The PIC method is still useful in such a situation for the great saving in computer time it provides. The PIC force calculation

Fig. 1.
Self-collision time as a function of number of macroparticles. Actual number of particles is 5.2×10^8 .

time is proportional to the number of macroparticles whereas a simple sum over particles calculation time is proportional to the square of the number of macroparticles.

2. Thermal Equilibrium Distributions. Because collision effects may be important in particle tracing calculations, let us consider what happens to the particle beam when enough time has passed so that collisions have caused the distribution to evolve into the thermal equilibrium distribution. We will consider two examples of two-dimensional beams and neglect space charge effects. The Hamiltonian is

$$H = \frac{1}{2} (mv^2 + kr^2) . \quad (9)$$

First, consider a K-V distribution

$$f(\vec{x}, \vec{v}) = \frac{mkn_t}{(2\pi)^2 H_0} \delta(H - H_0), \quad (10)$$

where $H_0 = \frac{1}{2}ka^2$. The particle density is uniform out to the maximum radius a . After a time of approximately the collision time, this distribution will evolve into a thermal equilibrium distribution of the same total number of particles and total energy as the initial distribution. This final distribution is

$$f(\vec{x}, \vec{v}) = \frac{mkn_t^3}{(\pi E_t)^2} \exp \left(- \frac{mv^2 + kr^2}{E_t} n_t \right), \quad (11)$$

where n_t and E_t are the total number of particles and total energy, respectively. From this we can calculate the number of particles outside of a given radius r

$$N(r) = n_t e^{-2(r/a)^2}. \quad (12)$$

Now consider the same Hamiltonian but an initial distribution which is uniform in the transverse four-dimensional phase space.

$$f(\vec{x}, \vec{v}) = \begin{cases} \frac{2n_t}{(\pi ab)^2} , & \frac{x^2 + y^2}{a^2} + \frac{v_x^2 + v_y^2}{b^2} < 1 \\ 0 , & \text{otherwise.} \end{cases} \quad (13)$$

The thermal equilibrium distribution with the same energy and number of particles is

$$f(\vec{x}, \vec{v}) = \frac{36kmn_t}{\pi^2 (ka^2 + mb^2)^2} \exp \left(- \frac{6(kr^2 + mv^2)}{ka^2 + mb^2} \right) . \quad (14)$$

The number of particles outside a radius r is given by

$$N(r) = n_t \exp \left(- \frac{6(r/a)^2}{1 + \frac{m}{k}(\frac{b}{a})^2} \right) . \quad (15)$$

Note that a smaller initial velocity spread (b near zero) leads to a smaller ultimate beam radius. A matched beam has $m/k = (a/b)^2$. Table I shows the fraction of particles outside radius r as a function of r/a for a K-V initial distribution and a matched uniform distribution.

We see that the growth of the beam from collisions is fairly small for the uniform initial distribution. The K-V distribution experiences much larger growth but this distribution is not likely to be found in an actual machine.

TABLE I
FRACTION OF PARTICLES OUTSIDE RADIUS r FOR
THE FINAL THERMAL EQUILIBRIUM DISTRIBUTION

r/a	K-V Initial	Matched Uniform Initial
0.5	0.607	0.472
1.0	0.135	4.98×10^{-2}
1.5	1.11×10^{-2}	1.17×10^{-3}
2.0	3.35×10^{-4}	6.14×10^{-6}

TABLE II

TIME AND LENGTH SCALES FOR THE LAMPF BEAM

	<u>time</u>		<u>length</u>
plasma period	53.7 ns	Debye length	0.41 mm
self-collision	12.6 ms	interparticle spacing	5.0 μ m
time of flight	4.4 μ s	beam diameter	5 mm

C. Collective Effects

1. Time and Length Scales. Table II shows some time and length scales for LAMPF beam parameters. The plasma frequency is given by

$$\omega_p^2 = \frac{ne^2}{\epsilon_0 m} \quad (16)$$

The plasma period $2\pi/\omega_p$ gives the time scale for collective behavior. The average interparticle spacing is simply $n^{-1/3}$ where n is the particle number density. The other quantities in the table have been previously discussed. The values of the time scales indicate that self-collisions are not important but that collective effects may be. From the values of the length scales we see that there are many particles in a Debye sphere. As we will note below, this means that a collisionless system is under discussion. The plasma is small because it is only a few Debye lengths in diameter. This means collective effects will make it unlikely that the beam bunch will have a sharp edge.

2. The Vlasov Equation. Whenever there are many particles in a Debye sphere

$$\begin{aligned} n_D &= \frac{4\pi}{3r_1^{1/2} e^3} (\epsilon_0 kT)^{3/2} \\ &= 2.27 \times 10^6 \text{ (LAMPF)}, \end{aligned} \quad (17)$$

the Vlasov equation provides a good description of the evolution of the distribution function. The Vlasov equation results from the BBGKY system whenever correlations (collisions) are neglected.⁷ That is, the two-particle distribution function which gives the probability that a particle is at $\vec{x}_1, \vec{p}_1$

and another is at $\vec{x}_2, \vec{p}_2$ is given by

$$f_2(\vec{x}_1, \vec{p}_1, \vec{x}_2, \vec{p}_2) = f(\vec{x}_1, \vec{p}_1)f(\vec{x}_2, \vec{p}_2) . \quad (18)$$

This approximation neglects, in particular, the fact that two particles cannot occupy the same location. The Vlasov equation is

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{p}} = 0, \quad (19)$$

where

$$\vec{F} = \text{external} + \text{electrostatic self-force}.$$

This equation is nonrelativistic and the external force must be independent of velocity (or depend on the velocity in the manner of the magnetic force).

III. EQUILIBRIUM DISTRIBUTIONS

We will obtain useful results by determining equilibrium distributions in phase space. These are defined to be distributions that are time independent: $df(\vec{x}, \vec{p}, t)/dt = 0$. Actual time-dependent distributions may oscillate about these equilibrium distributions. Equilibrium distributions may be stable or unstable. An unstable equilibrium is one in which perturbations about the equilibrium grow in space or time. If an initial distribution is close to a stable equilibrium, then the final distribution will not be very different from the initial one. Determining an equilibrium distribution is a way to determine the acceptance of an accelerator taking into account space charge in a self-consistent manner. Because there is more than one equilibrium distribution for a given Hamiltonian corresponding to a given structure, the acceptance is not unique.

A. Theorems

1. Stationary Solutions to Vlasov Equation. In order to determine equilibrium distributions, we use the standard technique of finding stationary solutions to the Vlasov equation by making use of the following theorem: If the distribution function is of the form

$$f(\vec{x}, \vec{p}) = F(H),$$

(20)

where H is a single particle constant of the motion in the external and equilibrium electrostatic self field, then f is a stationary (time-independent) solution to the Vlasov equation. Once a model (that is, the Hamiltonian for the single-particle motion) is established, then a constant of the motion is found and various equilibrium distributions can be determined by choosing various functions F . Because the constant of motion depends on the unknown space charge potential, the relation $f(\vec{x}, \vec{p}) = F(H)$ is really an equation for the potential which must be solved before the distribution f is determined. For time-independent systems, the Hamiltonian is itself a constant of the motion and is used for this purpose.

2. Stability Theorem. For the equilibrium distributions determined by $f(\vec{x}, \vec{p}) = F(H)$, where H is the time-independent Hamiltonian there is a sufficient condition for stability: If $F(H)$ is a monotonically decreasing function of H , then the distribution f is stable to small electrostatic oscillations.⁸ This means a system initially near such a distribution will always remain near it.

B. Continuous Focusing r-z Model.

Numerical studies by Chasman⁹ have shown that the main effect of space charge is a longitudinal-transverse coupling. Because of this, we have to study at least two degrees of freedom. We assume azimuthal symmetry and use r-z coordinates. The procedure we use is a generalization of the Gluckstern, Chasman, and Crandall¹ paper which was a one-degree-of-freedom calculation in the coordinate r . A continuous focusing, no acceleration, model is used.

$$H = \frac{p^2}{2m} + \frac{kr^2}{2} + \frac{k_z z^2}{2} + e\phi(r, z) \quad (21)$$

For the functions F , we use exactly the same sequence of functions as in the one-degree-of-freedom case

$$F(H) = \begin{cases} nB(H_0 - H)^{n-1} & , H < H_0 \\ 0 & , H \geq H_0 \end{cases} \quad (22)$$

where B and H_0 are constants. In the limit $n \rightarrow 0$, all particles have the same energy $F \sim \delta(H - H_0)$. For $n > 1$ these functions are decreasing functions of H and therefore lead to stable equilibria. For $n = 0$ in the one-degree-of-freedom case we have the K-V distribution. It is known that there are instabilities in the K-V distribution.¹⁰ In order to obtain the distribution from $f(\vec{x}, \vec{p}) = F(H)$, we must solve the Poisson equation for the space charge potential

$$\nabla^2 \phi = - \frac{en}{\epsilon_0} = - \frac{e}{\epsilon_0} \int d^3p f. \quad (23)$$

In the one-degree-of-freedom case, this equation was solved analytically. Here, we will have to solve this nonlinear partial differential equation numerically. After ϕ is found the distribution is obtained from $f = F(H)$. This distribution will be matched properly, including the proper longitudinal-transverse match, such that it is an equilibrium distribution.

We may calculate the temperature of the matched beam in the transverse and longitudinal directions in terms of the potential $\phi(r, z)$ by using Eq. (3). The result is

$$kT_T = kT_L = \frac{2}{2n + 3} \left(H_0 - \frac{kr^2}{2} - \frac{k_z z^2}{2} - e\phi \right). \quad (24)$$

We see that for an equilibrium distribution the average kinetic energy at every point is the same in the transverse and longitudinal directions.

C. Solving the RZED Problem

1. Normalize Variables. In order to more easily reduce the number of parameters to a minimum, we will normalize the dependent and independent variables of Eq. (23) as follows:

$$\Phi = \frac{e\phi}{H_0 - e\phi_0},$$

$$R^2 = \frac{(2 + \alpha)k}{H_0 - e\phi_0} r^2,$$

and

$$Z^2 = \frac{(2 + \alpha)k}{H_0 - e\phi_0} z^2,$$

(25)

where

$$\alpha = k_z/k \quad (26)$$

is the ratio of the force constants and ϕ_0 is the space charge potential at the bunch center. The equation to solve then becomes

$$\nabla^2 \phi = \begin{cases} -\mu u^{n+1/2}, & u > 0 \\ 0, & u \leq 0 \end{cases}$$

$$u(R,Z) = 1 + \phi_0 - \phi(R,Z) - \frac{R^2 + \alpha Z^2}{2(2 + \alpha)}, \quad (27)$$

where ϕ_0 is the normalized potential at the origin. The boundary condition we will use is $\phi \rightarrow 0$ as $R^2 + Z^2 \rightarrow \infty$. This corresponds to a conducting surface at infinity. The space charge parameter μ is defined by

$$\mu = \frac{2\mu_r + \alpha\mu_z}{2 + \alpha}, \quad (28)$$

where μ_r and μ_z are the ratios of the space charge defocusing forces to the external focusing forces at the origin. The parameter μ is proportional to the central particle density and ranges from zero to unity. This parameter can be written in terms of the space charge and external potentials as¹¹

$$\mu = - \frac{\nabla^2 \phi_{sc}}{\nabla^2 \phi_{ext}} \quad (29)$$

The distribution which is the solution to Eq. (27) has three parameters n , α , and μ . The parameter α describes the accelerator, and n and μ determine the particular distribution.

2. Numerical Solution of Poisson Equation. Equation (27) is a non-linear Poisson equation since the right-hand side source term contains the potential ϕ . Such an equation can be solved with a Poisson solver by iteration. For this purpose, the computer code RZED (r - z equilibrium distribution) was developed. For the initial guess, the no-space charge result $\phi = 0$ was used on the right-hand side. The code POISSON, a descendant of the TRIM

program,¹² was used to solve the Poisson equation in cylindrical coordinates at each iteration. A nonuniform triangular mesh is used in this code. The triangles are stretched out outside the central region to simulate a boundary at infinity. For small μ values, only a few iterations are required. Values of μ greater than 0.9 cause great difficulty. For a finite boundary to take into account beam pipe effects, a rectangular uniform mesh may be used. Faster codes are available for this situation. One more parameter (the beam pipe radius) is required for this problem.

D. Some Checks and Comparisons.

1. Results for $\mu \rightarrow 0$. For small μ values, some results can be obtained analytically. The quantity u of Eq. (27) vanishes at the bunch edge so setting ϕ and u to zero gives the following values for the maximum coordinate values (bunch size)

$$R_m^2 = 2(2 + \alpha)$$

and

$$Z_m^2 = 2(2 + \alpha)/\alpha. \quad (30)$$

We can define a normalized total number of particles as follows by integrating over the right-hand side of the nonlinear Poisson equation (Eq. (27))

$$N = \int_{u>0} \int 2\pi R dR dZ \mu u^{n+1/2}. \quad (31)$$

For small μ we can neglect the potential ϕ in the expression for $u(R, Z)$ to obtain the approximation

$$N = \frac{\mu \pi^2 (2n + 1)! 2^{1/2} (2 + \alpha)^{3/2}}{(n + 2)! n! 2^{2n} \alpha^{1/2}}. \quad (32)$$

The rms value of R is defined as

$$R_{rms} = \left(2\pi \int_{u>0} \int R dR dZ R^2 \mu u^{n+1/2} / N \right)^{1/2}. \quad (33)$$

TABLE III

COMPARISON OF NUMERICAL AND ANALYTIC RESULTS FOR LOW μ CASE
 THE PARAMETER VALUES ARE $\mu = 10^{-4}$, $\alpha = 0.5$, and $n = 1$

<u>Quantity</u>	<u>Analytic Result</u>	<u>Numerical Result</u>
R_m	2.24	2.30
Z_m	3.16	3.20
N	1.949×10^{-3}	1.951×10^{-3}
R_{rms}	1.118	1.118
Z_{rms}	1.118	1.118

A similar definition holds for Z_{rms} . For $\mu \rightarrow 0$ these can be calculated

$$R_{rms} = \left(\frac{2(2 + \alpha)}{n + 3} \right)^{1/2}$$

and

$$Z_{rms} = \left(\frac{2 + \alpha}{\alpha(n + 3)} \right)^{1/2}$$

(34)

Table III compares these analytic approximations with the RZED numerical calculation for the case $\mu = 10^{-4}$, $\alpha = 0.5$, $n = 1$.

Note that the integrated quantities N , R_{rms} , and Z_{rms} are very accurately determined since some of the discretization errors caused by the finite mesh used by the Poisson solver are averaged out. The accuracy of R_m and Z_m is limited by the mesh spacing of 0.1.

2. Compare With One-Degree-Of-Freedom Results. Whenever the longitudinal focusing force is small in relation to the transverse focusing force (α is small), the equilibrium distribution corresponds to a bunch which is long in relation to its radius. At $z = 0$, such a distribution will look like the one degree-of-freedom case done by Gluckstern.¹ Figure 2 shows the quantity

$$g(r) = \frac{H_0 - e\phi - kr^2/2}{H_0 - e\phi_0} \quad (35)$$

Fig. 2.

Normalized potential as a function of $(1-\mu)R^2$. The continuous curves are the one-degree-of-freedom results (Ref. 1) and the discrete points are the long bunch RZED results.

Fig. 3.

Particle density at $z=0$ as function of radius for long bunch case.

plotted as a function of $(1 - \mu)R^2$ for both cases. Our results agree trivially for $\mu = 0$. For nonzero μ , we see that the curves are not straight lines, indicating that the space charge force is nonlinear. The rather poor agreement between the two calculations probably is due to the finite region used in the numerical integration. (The requirements of a long bunch and the boundary far away from the bunch cannot both be satisfied well because of the limited number of points available.)

Another check on our results is possible by considering a long bunch for the $n = 0$ distribution. This approximates the K-V distribution in which all the particles have the same transverse oscillation energy. Figure 3 shows the radial distribution of particles for the $n = 0$ case. An actual K-V distribution would have a constant particle density with a sharp cutoff at the boundary. Also shown on the graph is the density for the $n = 1$ case for comparison. As before, agreement is poor but the general effect is evident.

3. Verify Time Independence by Particle Tracing. A particle tracing code named HOT (harmonic oscillator tracing) was set up to check the time evolution of the distributions we have created. This code assumes a similar model for the accelerator (continuous linear focusing and no acceleration)

and does the space charge force calculation by a simple sum over particles. An equilibrium distribution that was calculated by RZED for the parameter values $\mu = 0.5$, $\alpha = 0.67$, and $n = 1$ was represented by a collection of one thousand macroparticles. This was traced by HOT for an accelerator corresponding to $\alpha = 0.67$. The time independence of the distribution was verified. As a reference, the same distribution was traced with space charge turned off. For this case, the transverse phase space shape varied from $3.0 \text{ mm} \times 19 \text{ mrad}$ initially to $2.3 \text{ mm} \times 24 \text{ mrad}$ at one point. This indicates that matching takes the space charge forces in account properly. In both of these cases, the emittance (about $5.6\pi \times 10^{-5} \text{ m} \cdot \text{rad}$) was constant in time.

E. Practical Results of RZED

1. Relate to Practical Parameters. The numerical calculation performed by RZED requires the three parameters μ , α , and n . Only these are needed to describe the physics of the beam bunch. In order to relate these results to a real beam, we must also specify the number of particles in the bunch and a length scale for the external focusing forces (wavelength of oscillations). The beam bunch calculated by RZED may correspond to large or small bunch dimensions depending on the number of particles in the bunch.

The factor $H_0 - e\phi_0$ in Eq. (25) may be eliminated in favor of the quantity N defined by Eq. (31) by using

$$N = \frac{n_t e^2 (2 + \alpha)^{1/2} k^{1/2}}{\epsilon_0 (H_0 - e\phi_0)^{3/2}}. \quad (36)$$

The number of particles in the bunch n_t is related to the beam current by

$$I = f n_t e, \quad (37)$$

where f is the frequency of the accelerating field (one bunch per cycle is assumed).

Although our results are for a continuous focusing model, we would like to relate these results to an alternating gradient focusing system. This can be done by expressing the force constants in terms of phase shifts per focusing period. Assuming the focusing period to consist of two cells of total length $2\beta\lambda$ the desired relations are

$$k = \frac{1}{4} m (\mu_{shr}^0 f)^2$$

and

$$k_z = \frac{1}{4} m (\mu_{shz}^0 f)^2 .$$

(38)

The superscripts ⁰ indicate that these phase shifts are zero-space charge values (remember that the force constants describe the external forces). The actual phase shifts with space charge in effect are related to these zero-current phase shifts by

$$\mu_{shr} = \mu_{shr}^0 \sqrt{1 - \mu_r}$$

and

$$\mu_{shz} = \mu_{shz}^0 \sqrt{1 - \mu_z} .$$

(39)

Note that the zero current phase shifts may be greater than π and still correspond to a stable system for nonzero currents. For calculating the phase shifts μ_{shr} and μ_{shz} for an alternating gradient system, the computer code PHASE was developed. This calculates the phase shift, taking into account space charge (μ_r and μ_z are inputs) and rf defocusing effects. The space charge effect depends on the phase shift value itself so that an iterative procedure was required. Linear interpolation was used to determine the next guess for the phase shift (a straight iteration process diverges). From these results, the zero-current value needed for our calculations may be obtained from Eq. (39). For focusing systems other than alternating gradient, we interpret μ_{shr} and μ_{shz} as phase shifts per length $2\beta\lambda$. The ratio of force constants $\alpha = k_z/k$ is related to the phase shifts by

$$\alpha = \left(\frac{\mu_{shz}^0}{\mu_{shr}^0} \right)^2 .$$

(40)

The length scale can now be given in terms of N and practical parameters by

$$\begin{pmatrix} r \\ z \end{pmatrix} = \left(\frac{4e^2 n_t}{\epsilon_0 m (\mu_{shr}^0 f)^2 (2 + \alpha) N} \right)^{1/3} \begin{pmatrix} R \\ Z \end{pmatrix} ,$$

(41)

where the actual radial and longitudinal dimensions r and z are related to

the RZED variables R and Z . The maximum transverse momentum occurs at the origin for particles with $H = H_0$. Thus, the maximum transverse angle is

$$x'_{\max} = \frac{p_{T\max}}{mc\beta\gamma} = 2^{1/6} \left(\frac{e^2 n_t \mu_{\text{shr}}^0 f}{\epsilon_0 m N} \right)^{1/3} \frac{(2 + \alpha)^{1/6}}{c\beta\gamma} \quad (42)$$

The transverse emittance $E_T = \pi r x'_{\max}$ is given by

$$\eta = \frac{\beta\gamma E_T}{\pi} = 2^{5/6} \left(\frac{e^2 n_t}{\epsilon_0 m N} \right)^{2/3} \frac{R_m}{c(\mu_{\text{shr}}^0 f)^{1/3} (2 + \alpha)^{1/6}} \quad (43)$$

By making use of Eq. (37) and using the proton values for the constants gives

$$I = 2.02 \times 10^{-7} \frac{(\mu_{\text{shr}}^0)^{1/2} (\eta f)^{3/2} (2 + \alpha)^{1/4} N}{R_m^{3/2}} \text{ amp} \cdot (\text{sec/meter})^{3/2} \quad (44)$$

This gives the relation between the emittance and current for an accelerator described by μ_{shr}^0 , α , and f . The beam physics parameters N and R_m are a function of the space charge parameter μ (and the integer n). The phase shift μ_{shr}^0 is also dependent on μ .

The density distribution in terms of RZED output can be written

$$n = \begin{cases} \frac{\mu \epsilon_0 m (\mu_{\text{shr}}^0 f)}{4e^2} u^{n+1/2}, & u > 0 \\ 0, & u \leq 0 \end{cases} \quad (45)$$

$$u = 1 + \phi_0 - \phi(R, Z) - \frac{R^2 + \alpha Z^2}{2(2 + \alpha)}$$

2. The LAMPF Beam. The parameters for the LAMPF accelerator at the entrance (750 keV) are shown in Table IV.

TABLE IV

LAMPF ACCELERATOR PARAMETERS AT INJECTION

$\beta\gamma$	0.04
$\bar{r}$	201.25 MHz
μ^0_{shr}	53°
α	0.67

Tables V - VII show some RZED output for $\alpha = 0.67$ and $n = 0, 1$, and 2 . Then using Eqs. (37) and (43) we may calculate the transverse emittance as a function of μ . The results are plotted in Fig. 4. These curves correspond to a fixed value of the RZED (beam physics) accelerator parameter α and also fixed values of f , μ_{shr}^0 , and β to specify a particular accelerator. Several curves corresponding to different n values (an RZED distribution parameter) and different currents are shown. Different points on a given curve are different ways to distribute the same number of particles in the given accelerator. The shape of the curves depends only on α and μ . Changing the accelerator parameters f , μ_{shr}^0 and β only changes the scale of the ordinate in the graphs. The emittance depends on the beam current as $I^{2/3}$. All but one of the curves shown correspond to emittances containing 90% of the particles. For 100% of the particles, the emittance is increased by a factor of about 1.4, 1.9, or 2.3 for $n = 0, 1$, and 2 , respectively, independent of the μ value. The 90 per cent results are only approximate, resulting from multiplying the 100 per cent emittance values by the factor $(R_{90\%}/R_m)^2$.

We can relate these results to measured LAMPF beam properties¹³ in the following manner. We assume that the emittance growth in the linac is due to space charge and occurs mainly in the low-energy portion. Therefore we will use the accelerator parameters at 750 keV. It has been observed experimentally that the normalized emittance

Fig. 4.
Transverse emittance as a function of μ for LAMPF parameters. All curves except the one marked 100% are for emittance values containing 90% of the particles.

grows by a factor of about 4.5 by the time the beam has reached 100 MeV. The important assumption is the following. We assume that at 100 MeV the phase space distribution is close to an equilibrium value. This means that if this (100-MeV) distribution had been injected into the accelerator instead of the beam that had actually been injected, there would have been no emittance growth. Actually, measurements¹⁴ show that the bunch length is still twice as long as is necessary for a proper r-z match. Nevertheless, the beam is much more

TABLE V

RZED OUTPUT FOR $\alpha = 0.67$ AND $n = 0$

μ	N	R_m	$R_{90\%}$	Z_m	$Z_{90\%}$
0.2	9.90	2.60	2.10	3.20	2.10
0.3	17.51	2.70	2.25	3.40	2.20
0.4	28.24	2.90	2.35	3.60	2.40
0.5	43.99	3.10	2.50	3.90	2.60
0.6	68.48	3.30	2.75	4.20	2.80

TABLE VI

RZED OUTPUT FOR $\alpha = 0.67$ AND $n = 1$

μ	N	R_m	$R_{90\%}$	Z_m	$Z_{90\%}$
0.05	0.98	2.40	1.75	2.90	1.70
0.1	2.09	2.40	1.80	3.00	1.70
0.2	4.76	2.50	1.85	3.10	1.80
0.3	8.23	2.60	1.90	3.20	1.90
0.4	12.91	2.70	2.00	3.40	2.00
0.5	19.45	2.90	2.10	3.60	2.10
0.6	29.10	3.00	2.25	3.80	2.30
0.65	35.82	3.10	2.30	3.90	2.40
0.7	44.55	3.20	2.40	4.10	2.50
0.75	56.30	3.40	2.55	4.30	2.60
0.80	72.93	3.60	2.65	4.50	2.80

TABLE VII

RZED OUTPUT FOR $\alpha = 0.67$ AND $n = 2$

μ	N	R_m	$R_{90\%}$	Z_m	$Z_{90\%}$
0.1	1.30	2.40	1.60	3.00	1.50
0.2	2.92	2.50	1.65	3.00	1.60
0.3	5.01	2.60	1.70	3.10	1.70
0.4	7.78	2.60	1.80	3.30	1.70
0.5	11.57	2.80	1.85	3.40	1.80
0.6	17.05	2.90	1.95	3.60	2.00
0.75	32.02	3.20	2.20	4.00	2.20

mismatched at 750 keV than at 100 MeV. Also, the beam is closer to the radial space charge limit at 750 keV than at 100 MeV, and this may be more important (i.e., leading to emittance growth) than the r-z mismatch under these extreme conditions. Because of this situation, we take the measured distribution at 100 MeV for comparison with our calculations. The measured value of the emittance for a current of 1.6 mA at 100 MeV is $2 \times 10^{-5} \pi$ m·rad, normalized to 750 keV. This value is for the emittance containing 90% of the particles. By comparing measured and calculated emittance distributions and radial profiles (see next paragraph), we believe that the $n = 2$ distribution is closest to the observed distribution. So taking the $n = 2$, $I = 1.6$ mA curve in Fig. 4, the measured emittance value corresponds to a space charge parameter value of $\mu = 0.25$. Because the density of the beam bunch is unchanged as the current is varied (experimentally we have approximately $E_T \sim I^{1/2}$), we use the $\mu = 0.25$ point on the 17-mA curve of Fig. 4 to predict the emittance at 17 mA. The result is $9.5 \times 10^{-5} \pi$ m·rad. This is in good agreement with the extrapolation to 17 mA of measured emittances. This fact suggests that the observed output emittance in the linac is determined primarily by space charge effects as assumed. A result evident from Fig. 4 is that it will be difficult to achieve smaller emittances, because μ rapidly approaches unity as the emittance is lowered. operation at a high μ value is desired, then the initial distribution must be very accurately prepared to correspond to an equilibrium distribution in order to avoid emittance growth.

Fig. 5.

Particle density at $z = 0$ as a function of radius for LAMPF parameters with $\mu = 0.4$.

Fig. 6.

Particle density integrated over z as a function of radius for LAMPF parameters with $\mu = 0.4$.

By using Eq. (45) we can calculate the particle number density in the bunch. The radial variation of the density at $z = 0$ is shown in Fig. 5 for $n = 0, 1$, and 2 . The other RZED parameter values are $\mu = 0.4$ and $\alpha = 0.67$, and the density and length scales are determined by specifying the LAMPF parameters in Table IV. The function ϕ in Eq. (45) is reconstructed from the RZED output by its polynomial expansion coefficients. Figure 6 shows the radial dependence of the particle number density integrated over z . This corresponds to measured beam profiles.

Let us examine some limits on the LAMPF accelerator performance. Table VIII shows the emittance and maximum beam radius as a function of the current for the $\mu = 0.25$, $n = 2$ distribution. Values for both 90 and 100 per cent of the particles are shown. The current values were chosen such that the beam radii were certain fractions K of the cell length $\beta\lambda$. The bore radius corresponds approximately to $K = 1/8$. If 100 per cent of the beam is required to

be within one-half of the bore ($K = 1/16$) to account for alignment errors or other effects not considered here, then we see that the maximum beam current is 5.33 mA. If a higher brightness source is used, then higher currents are possible. As a practical maximum, consider $\mu = 0.6$. Table IX shows the corresponding results. We see now that for all the beam inside half the bore radius, the maximum current is 15.6 mA. If alignment or other errors are not a problem or if some beam losses are tolerable, then we see that much higher currents - up to hundreds of milliamperes - are possible.

TABLE VIII

EMITTANCE AND BEAM RADIUS AS A FUNCTION OF CURRENT FOR LAMPF
PARAMETERS WITH $\mu = 0.25$ AND $n = 2$

I (mA)	E_T (m·rad)		r (mm)	
	90%	100%	90%	100%
1.6	$1.88 \times 10^{-5}\pi$	$4.34 \times 10^{-5}\pi$	1.64	2.48
5.33	$4.20 \times 10^{-5}\pi$	$9.67 \times 10^{-5}\pi$	2.45	3.71 ($K = 1/16$)
17.0	$9.09 \times 10^{-5}\pi$	$20.95 \times 10^{-5}\pi$	3.60	5.46
18.7	$9.67 \times 10^{-5}\pi$	$22.29 \times 10^{-5}\pi$	3.71 ($K = 1/16$)	5.63
42.7	$16.79 \times 10^{-5}\pi$	$38.69 \times 10^{-5}\pi$	4.89	7.42 ($K = 1/8$)
149.2	$38.67 \times 10^{-5}\pi$	$89.13 \times 10^{-5}\pi$	7.65 ($K = 1/8$)	11.61

TABLE IX

EMITTANCE AND BEAM RADIUS AS A FUNCTION OF CURRENT FOR LAMPF
PARAMETERS WITH $\mu = 0.6$ AND $n = 2$

I (mA)	E_T (m·rad)		r (mm)	
	90%	100%	90%	100%
1.6	$0.85 \times 10^{-5}\pi$	$1.87 \times 10^{-5}\pi$	1.17	1.74
15.6	$3.74 \times 10^{-5}\pi$	$8.28 \times 10^{-5}\pi$	2.50	3.71 ($K=1/16$)
17.0	$4.08 \times 10^{-5}\pi$	$9.02 \times 10^{-5}\pi$	2.57	3.82
51.2	$8.50 \times 10^{-5}\pi$	$18.81 \times 10^{-5}\pi$	3.71 ($K=1/16$)	5.52
125.0	$15.42 \times 10^{-5}\pi$	$34.11 \times 10^{-5}\pi$	5.00	7.43 ($K=1/8$)
410.0	$34.04 \times 10^{-5}\pi$	$75.29 \times 10^{-5}\pi$	7.43 ($K=1/8$)	11.04

3. High Current-Beam Limits. In the previous section we determined matched beam parameters corresponding to a given accelerator. Here we will do an inverse problem, i.e., we answer the following question: Given a desired beam current, what accelerator parameters are required to achieve it? In the present case we will disregard emittance. To do this problem, let us fix the radial size of the beam to be some fraction K of the cell length,

$$r_m = K\beta\lambda. \quad (46)$$

This allows us to compare accelerators operating at different frequencies. Now let us fix the space charge parameter μ so that different accelerators are compared at comparable levels relative to their respective space charge limits. Combining Eq. (46) with Eqs. (37) and (41) gives the following expression for the current:

$$I = \frac{\epsilon_0 mc^3}{4e} \left(\frac{K\beta}{R_m} \right)^3 \left(\mu_{shr}^0 \right)^2 (2 + \alpha)N. \quad (47)$$

The quantities R_m and N are functions of the given value of μ calculated by RZED. Note that there is no explicit frequency dependence and that higher injection energies are desirable if maximum current is required. In practice, the maximum phase shift obtainable for a given type of focusing system is a function of the frequency. If we fix the magnetic gradient in an alternating gradient quadrupole focusing system and scale magnet lengths with the cell length, then we have $\mu_{shr}^0 \sim f^{-2}$. This leads to $I \sim f^{-4}$ so that low-frequency linacs are heavily favored. Actually, fixing the quadrupole pole tip field is more realistic in practice. Then scaling the quadrupole bore with the cell length gives the following law of frequency dependence of phase shift.

$$\mu_{shr}^0 \sim f^{-1}. \quad (48)$$

In this case, $I \sim f^{-2}$ so the conclusion remains that lower frequencies are favored when high currents, regardless of beam quality, are desired. The normalized emittance for this situation can be calculated using Eqs. (37), (43), and (47).

$$\eta = \frac{c}{2^{1/2}} \frac{(K\beta)^2 R_m^{1/3} \mu_{shr}^0 (2 + \alpha)^{1/2}}{f} \quad (49)$$

Using the phase shift law given by Eq. (48) gives $\eta \sim f^{-2}$. At a low enough frequency, large (near π) phase shifts can be attained so that there is no point to going to still lower frequencies if an alternating gradient quadrupole focusing system is used.

The same general results can be obtained with a crude model which does not require any numerical calculations. We assume again that the bunch radius is chosen as in Eq. (46) but instead of calculating a matched distribution we simply take as a rough approximation the bunch to be a uniformly charged sphere. Then the particle number density is given by

$$n = \frac{3n_t}{4\pi r_m^3} \quad (\text{uniform sphere}) \quad (50)$$

The space charge parameter can now be calculated as the ratio of the space charge defocusing force $-ne^2 r / (3\epsilon_0)$ to the external focusing force $-m(\mu_{sh}^0 f)^2 r$.

$$\mu = \frac{ne^2}{3m\epsilon_0 (\mu_{shr}^0 f)^2} \quad (\text{uniform sphere}) \quad (51)$$

Now using Eq. (37) we obtain

$$I = \mu \frac{4\pi\epsilon_0 mc^3}{e} (K\beta)^3 (\mu_{shr}^0)^2, \quad (\text{uniform sphere}) \quad (52)$$

which gives the identical dependence on accelerator parameters as the result obtained from the proper matching done by RZED.

Let us use these results to see how LAMPF rates as a high-current accelerator. Since the transverse phase shift is 53 degrees, the frequency can be reduced to one-third its present value (to 67 MHz) and still preserve stability (phase shift of 159 degrees) for the zero-current case with no change in the quadrupole pole tip fields. Taking the $\mu = 0.6$, $I = 15.6$ mA point as a reference we see that, at the new frequency, the current will increase in value by a factor of 9 to 140 mA. The emittance will increase by the same factor

to $74.5 \times 10^{-5} \pi$ m·rad. Note that the uniform sphere model predicts 2.26 A for the same parameters. The large discrepancy is due to the long tail of the $n = 2$ distribution. If we compare instead the RZED results for 90 per cent of the beam, the current is $51.2 \times 9 = 461$ mA.

4. High-Brightness Beam Limits. In this section, we will look at the maximum beam current that can be achieved for a fixed transverse emittance. As in the previous section, we also fix the μ value. Then combining Eqs. (43) and (37) we get the following expression for the current.

$$I = \frac{\epsilon_0 m c^3 / 2}{25^{1/4} e} \left(\frac{\eta f}{R_m} \right)^{3/2} \left(\mu_{shr}^0 \right)^{1/2} (2 + \alpha)^{1/4} N. \quad (53)$$

The quantities R_m and N are functions of the given value of μ and are calculated by RZED. Note that now we have an explicit frequency dependence but no energy dependence. Using Eq. (48) for the phase shift law gives $I \sim f$. Therefore, the requirement of high-brightness beams favors higher frequency linacs.

We can solve Eq. (43) for n_t and calculate the bunch radius from Eq. (41).

$$\frac{r}{\beta \lambda} = \left(\frac{2}{2 + \alpha} \right)^{1/4} \left(\frac{\eta f R_m}{\mu_{shr}^0 c} \right)^{1/2} \beta. \quad (54)$$

For the phase shift law given by Eq. (48) we have $r/\beta \lambda \sim f$. This means that there is an upper limit to the frequency corresponding to the point at which $r/\beta \lambda$ exceeds approximately 1/8.

Let us use these results to see how the LAMPF accelerator performance scales with frequency. The phase shift of $\mu_{shr}^0 = 53^\circ$ corresponds to a magnetic field gradient of about $G = 79$ T/m (this includes rf defocusing effects). For a quadrupole bore radius of one-eighth of the cell length, this corresponds to a pole tip field of $B_{pole} = 0.59$ T. Since the phase shift is proportional to the magnetic gradient, a factor of ~ 3 improvement in the phase shift is, in principle, possible (pole tip field ≈ 1.8 T). Using the 53° as a reference, however, the phase shift law becomes

$$\mu_{shr}^0 = \frac{(0.925)(201.25 \times 10^6)}{f} \text{ rad} \cdot \text{Hz}. \quad (55)$$

Let us inject beams corresponding to $\mu = 0.6$. Using $R_{90\%}^2/R_m$ for R_m in Eq. (53), we obtain the following formula for the current for the $\alpha = 0.67$, $n = 2$ distribution.

$$I = 3.99 \times 10^{-2} \eta^{3/2} f \quad (\text{m} \cdot \text{Hz})^{-1}. \quad (56)$$

Fixing the normalized emittance at the 17 mA, $\mu = 0.6$ point value of $\eta = 4.1 \times 10^{-5} \text{ m} \cdot \text{rad}$ (from Fig. 4) and setting $r/\beta\lambda = 1/8$ in Eq. (54) gives an upper frequency limit of 860 MHz. Now by using Eq. (56) we see that for this fixed η value, the current varies from 17 mA at 201.25 MHz to a value of 73 mA at 580 MHz. If smaller emittances are desired, the frequency can be increased to higher levels before the limit is reached according to the following relation

$$\eta^{1/2} f = \text{const.} \quad (57)$$

From Eq. (53) we see that increasing the quadrupole gradients by a factor of 3 by raising the pole tip field to 1.8 T will increase the beam current only by a factor of 1.7 because of the square root dependence on the phase shift. Thus pole tip field increases are not as effective as frequency increases except that for large emittances (as at LAMPF or larger) an increase in the pole tip field also raises the upper frequency limit. The electric field dependence enters through the parameter α . It has been found that these results are not very sensitive to the value of α .

As in the high-intensity beam limit calculation, we can form a crude uniformly charged sphere model which reproduces the same general behavior. The bunch radius in terms of the emittance is

$$r_m = \left(\frac{\eta\lambda}{\mu_{\text{shr}}^0 \sqrt{1-\mu}} \right)^{1/2}. \quad (\text{uniform sphere}) \quad (58)$$

The density and the space charge parameter μ are as given by Eqs. (50) and (51). Using Eq. (37) gives

$$I = \frac{4\epsilon_0 \pi^{7/4} c^{3/2} m}{e} \frac{\mu}{(1-\mu)^{3/4}} (\eta f)^{3/2} (\mu_{\text{shr}}^0)^{1/2}. \quad (\text{uniform sphere}) \quad (59)$$

As an example, we take the LAMPF parameters with a beam at $\mu = 0.6$ ($\eta = 4.1 \times 10^{-5}$ By m·rad). Then the RZED calculation gives $I = 17$ mA, whereas the uniformly charged sphere model gives $I = 32$ mA.

IV. DISCUSSION

We have calculated here some limits on beam current and emittance arising from space charge forces. The general results have been utilized in two ways. In the first approach, we specify certain accelerator parameters and calculate the properties of beams that can be achieved with the given accelerator. By a proper choice of the initial phase space distribution, the current and emittance can be varied within certain limits. Fixing the space charge parameter μ will determine practical limits for these quantities. In the inverse problem, we specify a desired beam and calculate the accelerator parameters required to achieve the required emittance and current. We have found that the requirement of high current is most easily met with a low-frequency accelerator, whereas the requirement of high brightness is most easily met with a high-frequency accelerator. In both cases, there are limits on how low or how high, respectively, these frequencies can be.

The method that we have used to relate beam properties to accelerator parameters was to construct equilibrium solutions to the Vlasov equation. This process is a longitudinal-transverse matching with the coupling due to space charge taken into account in a self-consistent manner. Because a Poisson equation must be solved numerically, we restricted consideration to two degrees of freedom (coordinates r and z). There is no unique solution (distribution in phase space) for a given set of accelerator parameters so that the acceptance of the accelerator is not uniquely defined if space charge forces are taken into account. We have determined solutions, however, that we believe are close to actual beam distributions so that our results are practical limits on accelerator performance.

The approximations used in our model are the following. 1) Continuous focusing was used, that is, the external restoring forces are constant with respect to the independent variable. These forces were also assumed to be linear and uncoupled between x , y , and z but these restrictions were not required by the methods used. 2) Azimuthal symmetry was assumed for the Hamiltonian and the solution in order to consider longitudinal-transverse coupling due to space charge. The x - y problem could have been solved just as easily

instead of the r-z problem but it would make sense to do this only if the first restriction could be lifted so that an alternating gradient focusing system could be solved realistically. 3) No acceleration effects were considered. In our model these effects would correspond to a slow variation of the force constants k and k_z with time.

The following points should be considered further. 1) Probably the most important is the effect of acceleration. If a single-particle adiabatic invariant can be determined then it can be used instead of the Hamiltonian H in the equation $f(\vec{x}, \vec{p}) = F(H)$ to produce a distribution which is slowly varying in time through the variation of the force constants. For example, the K-V distribution would just have the usual emittance damping. More generally, it is a question of how much can be done analytically. 2) The ability to do the initial value problem, that is, determining the evolution of a given initial distribution would be of obvious value. 3) Alternating gradient or three-dimensional effects are probably not as important as the above except for high μ values. But high μ values are not practical because the corresponding distributions must be accurately prepared for injection into the accelerator. 4) Some of these effects can be studied with particle tracing computer simulations. Such work may be valuable for determining which effects are large enough to be profitably pursued analytically.

The motivations for this proposed program of study may be listed as follows. 1) We do not know the effects of r-z mismatches. From the comparison of the plasma period to the time the particles are in the accelerator, we know that collective effects have sufficient time to produce changes in the distribution function, but we do not know how large these changes can be. 2) A deliberate mismatch may be desirable. Numerical studies⁹ have indicated that a small longitudinal emittance (smaller than the matched value) may act to reduce the transverse emittance as a result of coupling effects. 3) Not all changes in the distribution function correspond to changes in the emittance. 4) At low μ values, matching is not important. If this practical limit is determined, perhaps the calculations presented here will turn out to be useful mainly for relating accelerator and beam parameters to μ .

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