Muon Spin Polarization Measurements Using Cerenkov Radiation

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INTRODUCTION

This thesis is a summary of the experimental development and testing of the muon polarimeter, a device to be used in an experiment proposed for the PEP facility (PEP 16). The work consisted of two phases. Phase one was the cosmic ray studies, done at Brown during the summer of 1978, using a small model of the actual polarimeter. A second series of tests, done on a much more sophisticated version of the polarimeter, was conducted in March and April of 1979 at the Brookhaven AGS A2 Test Beam. The people involved during the summer were C. Allen, D. Cutts, B. K. Dichter, D. Hermann and B.W. Hughlock. The AGS group included D. Cutts, B.K. Dichter, R.S. Dulude, P.G. Grannis, J. Kirz, R.E. Lanou and M. Sievertz.

Chapter I contains a discussion of the polarimeter concept and a brief description of some of the hardware used in the experimental work. Chapter II summarizes the summer tests and discusses the results obtained. Chapter III is devoted to theoretical calculations of the quantities of interest in the Brookhaven tests. The final Chapter has the description of the work done at Brookhaven.

At the outset I would like my advisor Prof. R.E. Lanou for his help in this work and Prof. D. Cutts for his continued guidance throughout the year.

I. HARDWARE

A. Polarimeter

A scale model of the polarimeter was tested in March and April of 1979 at the Brookhaven AGS. The device was a cube with a metal frame and glass sides (fig 1.1). Each of the 22"x22" glass plates was covered by a waveshifter slab of the same dimensions (for a more thorough discussion of the experimental apparatus see section IB). The waveshifters were coupled by light guides to phototubes so that their light output could be measured. Once the module was filled with a clear, non-scintillating fluid and made light-tight, the device was set to observe the Čerenkov light (see Appendix I) of particles traversing through it. In the experiment, 1,1,1 trichloroethane was used. In addition to the above qualities it also does not depolarize the muons. The intent of the experiment was to measure the spin polarization of muons along their momentum by measuring the angular distribution of the decay electrons using Čerenkov radiation.

The muon decays weakly into an electron and two neutrinos, $\mu \rightarrow e + \nu + \bar{\nu}$ (fig 1.2). Since the weak interactions violate parity, one would expect that the decay is not isotropic but somehow correlated with the muon spin $\vec{s}$, since it is the only direction defined in the muon rest frame before the decay. This is in fact the case as

the angular distribution of decay electrons is given by

$$(1.1) \quad \frac{dN}{d(\cos\theta)} = \frac{1}{2} (1 + \alpha P_\mu \cos\theta)$$

where P_μ is the magnitude of the muon spin at the muon production, $\cos\theta = \hat{P}_\mu \cdot \hat{p}_e$, the cosine of the angle between the muon spin and decay electron's momentum and

$$(1.2) \quad \alpha = \frac{2x - 1}{3 - 2x} \quad ; \quad x = E^e / 52.8 \text{ MeV} .$$

If the electron's energy is not known, it must be averaged over and eq. 1.1 becomes

$$(1.3) \quad \frac{dN}{d(\cos\theta)} = \frac{1}{2} (1 + \frac{1}{3} P_\mu \cos\theta) .$$

Parity violation is contained in the $\cos\theta$ term. Since $\hat{\sigma}_\mu$ is an axial vector, $\hat{\sigma}_\mu \cdot \hat{p}_e$ changes sign under space inversion and thus leads to a noninvariant probability amplitude under a parity operation.

Thus if the angular distribution of decay electrons from a sample of muon decays is experimentally measured, it is possible using eq. 1.1 or 1.3 to infer the average spin polarization of the parent muons. The polarimeter performs such a measurement by identifying the direction of momentum of an electron coming from each decay using the electron's Čerenkov light. The average kinetic energy of decay electrons is 39 MeV, corresponding to a velocity $v = .99991c$, and unless the medium has a refractive index $n < 1.00009$ Čerenkov radiation will be produced.¹

The light is emitted in a cone with an opening angle given by eq. A1.1. Within the polarimeter, the light is transmitted through the trichloroethane and glass and then strikes the thin air layer between glass and waveshifter.

As is shown in Appendix II the presence of glass is irrelevant to the determination of the angle of total internal reflection, which is then given by $\theta_r = \sin^{-1}(n_1/n_2)$ where $n_1 = 1$ is the index of air and $n_2 = 1.44$ is the index of trichloroethane, hence $\theta_r = 44^\circ$. From A1.1, for a particle with $\beta \approx 1$, the Čerenkov angle is $\theta_c = \cos^{-1}(1/1.44) = 46^\circ$. Frequently, a light ray will be internally reflected at this interface and hit another side. If it is transmitted, however, it will strike the waveshifter. Since now the light is travelling from a medium of lower index (air) to one of higher (waveshifter), the behavior at this interface is governed by Fresnel's equations (fig 1.3):

$$\begin{aligned}
 R_p &= \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} ; R_l = \frac{-\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \\
 T_p &= \frac{2\sin\theta_i \cos\theta_t}{\sin(\theta_i + \theta_t) \cos(\theta_i + \theta_t)} \\
 (2.4) \quad T_l &= \frac{2\sin\theta_i \cos\theta_t}{\sin(\theta_i + \theta_t)}
 \end{aligned}$$

where the subscripts p and l refer to the vibrations perpendicular and parallel to the plane of incidence, R is the coefficient of reflection and T the coefficient of transmission.² The transmitted part of the light ray is absorbed in the waveshifter and thus can be eventually measured. The reflected part heads back to the box and the whole process begins again. This goes on until all the light is either absorbed or attenuated. As seen from the consideration of the angle of total internal reflection and eqs. 2.4 the amount of light absorbed in each side

Fig. 1.1

Polarimeter Module

Fig. 1.2

Decay $\mu \rightarrow e + \nu + \bar{\nu}$ $\hat{\sigma}_\mu$ is the muon spin

Fig. 1.3

depends very strongly on the angle at which the Cerenkov light is emitted with respect to that side. That angle, of course, is determined by the direction of momentum of the particle. Thus it is possible to reconstruct, from a pattern of light obtained at each side, the direction of motion of the particle.

B. Other Equipment

This section is devoted to the description of the various pieces of equipment used both in the summer and Brookhaven tests.

1) Scintillation Counters and Waveshifters

Scintillation counters are particle detectors consisting of four parts, scintillator, light guide, photomultiplier tube and a base. Scintillator is a plastic that gives off a small light flash when a charged particle goes through it. Light guide is a clear piece of plastic, often lucite, with an index of refraction high enough so that most of the light that enters it on one end will be kept inside by total internal reflection and propagated to the other end. Phototube converts a small light signal into a sizable electrical pulse. The sensitive part of a phototube is a metal cathode. When light photons strike it, they will kick off photoelectrons due to the photoelectric effect. The electrons are then accelerated through a potential and strike another metal plate. The collisions will knock off more electrons which

again are accelerated through a potential and strike a third plate. This cascading process is repeated 16 to 18 times, depending on the kind of tube. At the final stage the electrical pulse is typically on the order of 10 to 100 millivolts and is proportional to the number of photons that hit the cathode. The operating voltage for a tube is on the order of 2 KV, this voltage is fed into the base which then redistributes it to each of the accelerating stages of the tube. A typical counter is shown in fig. 1.4. The scintillator and light guide are wrapped in reflective foil and then light-tighted with black tape. The foil increases the amount of light seen by the phototube by reflecting back into the light guide some of the light that escaped from it.

Waveshifter is a compound plastic with the property that a part of the light that it absorbs is reemitted at some characteristic frequency. The waveshifter used in our experiments had its frequency in the blue-green. The reemitted light is isotropic, so that only a part of the light, kept in by internal reflection, stays in the material. At Brookhaven, the waveshifters were $1/4 \times 22 \times 22$ ", with the light guide coupled to one $1/4 \times 22$ " side. During the summer tests the waveshifters were $1/4 \times 12 \times 20$ " and $1/4 \times 4 \times 12$ " with a light guide coupled to $1/4 \times 12$ " and $1/4 \times 4$ " sides respectively.

2) Fast Electronics

Fast electronics is a name that describes a family of electronic equipment characterized by a response time on the order of nanoseconds ($\text{nsec} = 10^{-9} \text{sec}$). The usual

Fig. 1.4

Scintillation Counter

output of these devices is a NIM pulse, a negative rectangular pulse of .75V with a width typically between 30 and 100 nsec.

Discriminators accept as input a negative pulse of any shape, such as for example a phototube pulse. When the preset threshold voltage is exceeded by the input signal, the discriminator fires a standard NIM pulse. Most discriminators also have a $\overline{\text{NIM}}$ output, $\overline{\text{NIM}}$ is a positive NIM pulse.

Coincidence units are n-fold AND gates that only accept NIM or $\overline{\text{NIM}}$ pulses as input. The leading edge of a NIM pulse opens a gate in the unit. If the other n-1 NIM pulses arrive within this gate, the output is a NIM pulse, otherwise no output results. If one of the inputs is a $\overline{\text{NIM}}$ pulse the unit interprets this as a veto and regardless of the state of the other inputs there is no output. Usually there is also a separate VETO input that vetos the output upon being hit with a NIM pulse.

Gate generators accept NIM pulses as input. The output is a -.75V pulse whose length can be varied from nanoseconds to seconds. Another output is a "delayed" NIM pulse fired sometime after the input signal. The time interval between the input and the "delayed out" is adjustable. In addition gate generators have a NIM and $\overline{\text{NIM}}$ outputs which fire immediately after being hit with an input pulse.

Analog to Digital Converters (ADC) have two kinds of inputs. There is a GATE input and several numbered input channels. As long as there is an input signal going into the GATE the ADC will digitize each of the inputs into the numbered channels. That is the ADC assigns a number to each channel that is proportional to the amount of charge that flowed in through the input.

Time to Digital Converters (TDC) also have two kinds of inputs. There is a START input and several numbered input channels. The TDC converts the time difference between the arrival of the START input and the arrival of an input into a numbered channel to a number proportional to that time difference. Although the START signal is common to all channels, the stopping signal applies only to the particular channel that it goes into.

II. SUMMER STUDIES

A. First Test Module

The first model of the polarimeter was designed to show how feasible is the reconstruction of the direction of a particle's motion from its Čerenkov light. The model was a watertight aluminium box of dimensions 4"x12"x20". Two of the sides, bottom (12"x20") and front (4"x12") were constructed with transparent lucite walls and waveshifter slabs so as to be sensitive to light (fig 2.1).

The box was mounted so that it could be tilted with respect to the floor. A cosmic ray trigger was then constructed. One 2" diameter scintillation counter (T1) was placed above the box and an identical counter (T2) was placed immediately underneath it. Under T2 degrader was put in place and under the degrader was another scintillation counter (TL) with a diameter of 3.8" (fig 2.2).

The degrader consisted of 4.2" of Heavimet, this corresponds to a muon momentum of 350 MeV/c (at this altitude most high energy cosmic rays are muons). Since $\beta = v/c$ of a particle is given by

$$(2.1) \quad \beta = \frac{p}{(p^2 + m^2)^{1/2}}$$

where p is the momentum and m is the particle mass, then for a muon ($m_\mu = 105.659$ MeV) β corresponding to the minimum momentum of 350 MeV is $\beta = .96$. The box is filled with water ($n_w = 1.33$) so that using eq. A1.1 the threshold velocity for Čerenkov radiation is

$$(2.2) \quad \beta_{th} = \frac{1}{n_w} = .75$$

Fig. 2.1

Fig. 2.2

θ is the tilt angle

Hence any muon traversing the trigger (T1•T2•TL) will have $\beta > \beta_{th}$ and will $\hat{C}$ erenkov radiate.

For a particle with $\beta \approx 1$ the Cerenkov angle in water, using eq. A1.1, is 41° . The angle of total internal reflection for a water-air interface (as was remarked earlier the intervening lucite does not change this angle) is

$$(2.3) \quad \theta_r = \sin^{-1}(n_{air} / n_w) = 49^\circ .$$

Then unless the muon moves almost exactly perpendicularly to the bottom of the box, much of the $\hat{C}$ erenkov cone will intersect the interface at an angle greater than 49° and will be reflected. It is this phenomenon that lies at the heart of the device. Tilting the box, and thus varying the direction of motion of the muon with respect to the box changes the relative orientation of the Cerenkov cone. Since the amount of light that is reflected or transmitted at any interface depends on the angle of incidence, different tilt angles will cause differing amounts of light to be transmitted across the water-air interface on each side. Only the transmitted light is absorbed in the waveshifter so the waveshifter output will be proportional to this transmitted fraction. The box was sensitive to light hitting its bottom or front sides. From the ratio of the amounts of light seen at the two sides, R, it was hoped to see the dependence of R on the tilt angle.

B. Preparatory Work

Before the module was tested the counters and waveshifters had to be timed in properly and calibrated.

1) Timing

All counters and waveshifters have intrinsic response times that are often quite different from each other. This is due to variations in the construction of phototubes and their bases.

A typical circuit used for timing in counters looks as follows:

The two counters are placed such that their scintillators overlap. Thus when a cosmic ray strikes one of them, it should also hit the other one. The number of coincidences of counter #1 and #2 are counted for a constant time interval as a function of the relative delay of the two counters as measured by the settings on the delay boxes. Timing curves for T1 and T2 are shown in fig. 2.3, those for T1 and TL in fig. 2.4.

2) Optimal Voltage Selection

Once the counters are timed in, the optimal operating voltage must be selected for each one. Too high a voltage will make the noise level of the tube unacceptable, while too small a voltage will make the tube unresponsive.

The quantity of interest is the efficiency of a counter.

The efficiency of a counter C is defined as $\frac{\# \text{ of counts of } A \cdot B \cdot C}{\# \text{ of counts of } A \cdot B}$.

Counters A and B are placed on either side of C so that whenever a particle moves through both of them it must have gone through C as well. A typical circuit used for this purpose is shown below:

Efficiency curves for TL and T1 are shown in figs. 2.5 and 2.6 respectively.

C. The Experiment

The circuit used during the running of the experiment is shown in fig. 2.7. The trigger, signaling that a particle went through the box, was a coincidence $T1 \cdot T2 \cdot TL$ (output of coincidence unit #2). There are three outputs from the trigger. One opens a 150 nsec gate of the ADC that looks at the output from the waveshifters. The second fires gate generators #2 and #3, which after a delay of 100 nsec turn on the teletype and after a delay of 200 msec turn on gate generator #4 which effectively clears both the scalers and the ADC after the teletype has recorded the event. The third output turns on gate generator #1 which then in turn disables the trigger for 8 sec, long enough to allow the teletype to finish typing out the event. For each event the following information is collected and typed out on the teletype: number of

Fig. 2.3

TIMING CURVE OF T1 AND T2

Fig. 2.4

TIMING CURVE OF T1 AND TL

Fig. 2.5

EFFICIENCY CURVE OF TL

Fig. 2.6

EFFICIENCY CURVE OF T1

Fig. 2.7

coincidences of $T1 \cdot T2$, number of counts from each of the three counters $T1$, $T2$ and TL and the digitized pulse heights from the two waveshifters WB (bottom) and WS (side). The trigger counter information is used to make sure that the counters as well as the circuit components are functioning correctly. The waveshifter information is the real data of the experiment.

Since the counters were small and separated by a fairly large distance compared with their diameters, the trigger subtended only a small solid angle and the event rate was low. The runs were typically about 8 hours long and yielded roughly 50 events, an average of one every ten minutes. For each run there were several events where one or both ADC channels would overflow, this further cut down the number of good events.

After gathering data for tilts of 0° , 30° , 45° and 60° a change was made in the construction of the module. In order to cut down on any stray reflections from the aluminium sides of the box, the inside of the box was coated with black painted aluminium foil. Hence, unless the light hit WB or WS , it would be absorbed. The runs for all the tilt angles were repeated with the new arrangement.

D. Monte Carlo, Results and Analysis

Concurrently with the experiment, a Monte Carlo computer simulation of the test was developed. In the program, a high energy particle was generated at a fixed point on the top of the module. To simulate the finite

solid angle acceptance of the trigger, the direction of motion is smeared out for a sample of particles by rotating their velocity vector randomly in the x-z plane by an angle between 0° and 8° and then rotating it about the z axis by an arbitrary amount (fig 2.3 - note: the z axis is perpendicular to the floor).

Fig. 2.3

The program then generates 16 rays of light at the Čerenkov angle about the direction of motion. The rays are followed inside the box, using Fresnel's equations, until they are either absorbed by WS or WB or are lost due to absorption in other sides of the box or attenuation.

In fig. 2.9 and 2.10 the results of experimental work and Monte Carlo predictions for the pulse heights of WB and WS are displayed. Except for one anomalous data point (0° , WB, aluminium walls), the experimental data reproduces the behavior predicted by the Monte Carlo.

The key result from the experiment is to see whether we can experimentally reproduce the angular dependence of $R = WS / WB$ predicted by the Monte Carlo. Fig. 2.11 R plotted against the tilt angle for the experimental and predicted data. Although the experimental data reproduces the predicted shape of the curve well,

R_{exp} is consistently lower than R predicted by the Monte Carlo. This systematic error was found to be caused by the unequal response of WB and WS. Calibration of the waveshifters at their operating voltage, 1800 V, showed the ADC channel of the peak of the pulse height distribution of WB was 1.44 times as large as WS. After correcting for this bias, the experimental and predicted values of R are found to be in close agreement (fig 2.12).

The ADC pulse height distribution also furnishes information about the number of photoelectrons at the phototube cathode. This number divided by the phototube efficiency (10% in this case) yields the approximate number of photons that get to the tube. Using the relation

$$(2.4) \quad \frac{\sigma}{p} = (N_{\text{pe}})^{-1/2}$$

where σ is the standard deviation of the distribution, p the ADC channel of the peak and N_{pe} the number of photoelectrons, N_{pe} can easily be evaluated. The tables below summarize the results.

N_{pe} with Al walls			
Tilt	WB	WS	Sum
0°	4	>1	4
30°	5	1	6
45°	5	2	7
60°	6	2	8

N_{pe} with black walls			
Tilt	WB	WS	Sum
0°	5	>1	5
30°	8	2	10
45°	9	6	15
60°	8	2	10

Fig. 1.9

Fig. 1.10

Fig. 2.11

RATIO OF WS TO WB

Fig. 2.12

RATIO $R=WS/WB$

As expected all runs with black walls have significantly fewer photoelectrons than corresponding runs with aluminium walls.

From eq. A1.4 it is possible to compute a rough estimate of how many photons of Čerenkov light will be produced by a particle going through the box. The optical properties of water, lucite and waveshifter give $\lambda_1 = 3800\text{\AA}$ and $\lambda_2 = 4400\text{\AA}$. L is 4-6" depending on the tilt angle and for a particle with $\beta \approx 1$, $\sin^2 \theta_c = .43$. Substituting into eq. A1.4 we get $N_{ph} = 808$. Of that number only about 50% will be transmitted through the water and lucite and absorbed in the waveshifter, 33% will be propagated through the waveshifter to the phototube and the phototube efficiency is 10%. Taking all this into account $N_{pe} = 13$, which is in good agreement with the observed values.

Despite the small number of events the observed results correlate well with those predicted by the Monte Carlo or predicted theoretically. Thus we are led to conclude that our experimental technique is capable of differentiating between particles moving in separate directions from their Čerenkov light patterns.

III. SOME THEORETICAL CONSIDERATIONS

A. Source of Muons

The second polarimeter experiment was carried out at the A2 Test Beam at the Brookhaven AGS. The primary beam is a beam of protons that are accelerated in the AGS ring. Every three seconds, for a period of about one second, part of the proton beam is extracted from the ring and aimed at a production target. Dipole bending magnets behind the target select a given momentum of the recoil particles and direct them to the experimental areas. The vast majority of particles produced by the proton-target collision are nucleons, pions and electrons. The pions are the source of muons used in this experiment.

The reaction of interest is $\pi^+ \rightarrow \mu^+ + \nu$, which is by far the most predominant mode of pion decay. In the rest frame of the pion the percentage of pions that decay N_d is given by

$$(3.1) \quad N_d = (N_0 - N_0 e^{-t/\tau}) / N_0 = 1 - e^{-t/\tau}$$

where N_0 is the number of pions at $t=0$ and τ is the pion lifetime. In the laboratory frame the pion is in motion so that eq. 3.1 becomes

$$(3.2) \quad N_d = 1 - e^{-t/\gamma\tau} ; \quad \gamma = (1 - \beta^2)^{-1/2}$$

where $\beta = v/c$ and v is the pion velocity in the lab frame. If the pions are created at $t=0$ at the production target and move with a velocity v , then since the polarimeter is a distance $d = 3.8 \times 10^3$ cm away from the target we can

replace t by dc/β and write

$$(3.3) \quad N_d = 1 - \exp\{-d/\gamma\beta c\tau\}.$$

The product $c\tau$ is a well known number, 780.4 cm. During most of the testing a range of momenta between 0.8 GeV/c and 1.2 GeV/c was used. Using eq. 2.1 to find the corresponding values of $\beta\gamma$ and substituting into eq. 3.3 we get

$N_d(0.8 \text{ GeV/c}) = .57$ and $N_d(1.2 \text{ GeV/c}) = .43$. Thus roughly half the pions created at the target decay before the polarimeter.

The reaction $\pi \rightarrow \mu + \nu$ is a two body decay and its kinematic parameters can be solved for exactly.

Fig. 3.1

Using 4-momentum conservation to solve for the muon momentum and energy in the muon rest frame (same as the center-of-mass frame) we get

$$(3.4) \quad p_{\mu}^{\text{cm}} = \frac{m_{\pi}^2 - m_{\mu}^2}{2m_{\pi}} \quad ; \quad E_{\mu}^{\text{cm}} = \frac{m_{\pi}^2 + m_{\mu}^2}{2m_{\pi}}.$$

Since $m_{\pi} = 139.569 \text{ MeV}$ and $m_{\mu} = 105.659 \text{ MeV}$, then substituting these values in eq. 3.4 $p_{\mu}^{\text{cm}} = 29.796 \text{ MeV/c}$ and $E_{\mu}^{\text{cm}} = 109.779 \text{ MeV/c}$.

An additional feature of this decay is the simplicity of the spin analysis in the CM (center-of-mass) frame. The pion is spinless while both the neutrino and the muon are spin 1/2 particles. The neutrino is massless and therefore always fully polarized, its spin either parallel or anti-parallel to the momentum vector. Since angular momentum must be conserved, the muon must also be fully polarized in the CM frame. However, while the neutrino is fully polarized in all reference frames, the massive muon is not.

The component of muon spin along the beam direction given by $\frac{1}{2} \cos\theta$ in the CM frame, varies as a function of the parent pion momentum in the lab frame (the pion momenta in the experiment were sufficiently large so that regardless of the decay angle θ , the muon moved essentially along the beam line). The spin component referred to above is obviously not one obtained in a single measurement, since the spin projection of a muon along any direction measured is always $\pm 1/2$. This is a spin measurement averaged over a large sample of muons, while some muons will have spin $+1/2$ others will have $-1/2$ their numbers in general need not be equal. Then we can conveniently define the average spin component as $(N_{+1/2} - N_{-1/2}) / (N_{+1/2} + N_{-1/2})$ and now can talk meaningfully about spin components being something other than $\pm 1/2$ along a given direction.

B. Muon Polarization Calculation

In this section a calculation of the average muon polarization along the beam direction as a function of the pion beam momentum and degrader thickness is carried out. Part of the calculation is based on a paper by S. Hayakawa³.

Let the muon polarization be represented by the unit vector $\hat{n}$, the spin 4-vector by $s_\mu = (\vec{s}, s_0)$. When the particle is at rest $s_0 = 0$ and the space part of s_μ , $\vec{s}$, is identical with $\hat{n}$. When the particle is in motion, $\vec{s}$ picks up a component along the direction of momentum. Since $\vec{s}$ is a pseudovector its component along the momentum

$\vec{p}$ must be a pseudovector. The only pseudovector that can be formed out of quantities of interest is $(\hat{n} \cdot \vec{p}) \cdot \vec{p}$, hence the component of s along p must be proportional to it.

$$(3.5) \quad \vec{s} = \hat{n} + k(\hat{n} \cdot \vec{p}) \cdot \vec{p}$$

To find k we need three relations.

$$(3.6) \quad s_\mu p_\mu = \vec{s} \cdot \vec{p} - s_0 E = 0$$

this is clearly true since in the rest frame $\vec{p} = 0 = s_0$ and $s \cdot p$ is a Lorentz scalar and thus invariant in all frames. In special relativity a 4-vector $V = (\vec{V}, V_0)$ that is given by $V = V_R = (\vec{V}_R, 0)$ in some frame R , transforms to any other frame according to

$$(3.7) \quad V = \left(\vec{V}_R + \beta \left(\frac{V_0}{\gamma + 1} \right) \vec{\beta}, \gamma \vec{\beta} \cdot \vec{V}_R \right) \\ = \left(\vec{V}_R + \beta \left(\frac{V_0}{\gamma + 1} \right) \vec{\beta}, \frac{\vec{\beta} \cdot \vec{V}_R}{\gamma} \right)$$

then, using eq. 3.6 and 3.7 and the fact that $s_x = (\hat{n}, 0)$ we can write

$$(3.8) \quad s_0 = \hat{n} \cdot \vec{p} / m$$

Finally,

$$(3.9) \quad s_\mu s_\mu = s_0^2 - \vec{s}^2 = -1$$

since in the rest frame $s_0 = 0$, $\vec{s}$ is a unit vector and $s_\mu s_\mu$ is a Lorentz scalar. Using eqs. 3.6, 3.8 and 3.9 we get $k = 1/m(E+m)$ and hence

$$(3.10) \quad s = \hat{n} + (\hat{n} \cdot \vec{p}) \vec{p} / m(E+m)$$

In the rest frame of the parent pion the muon is fully polarized (for definiteness assume the spin is parallel to the momentum), then $p = |\vec{p}| \hat{n}$ and eq. 3.10 simplifies to

$$(3.11) \quad \vec{s}^{\text{cm}} = \hat{n}^{\text{cm}} E^{\text{cm}} / m_{\mu}$$

Transforming the spin 4-vector to the lab frame using eq. 3.7 and simplifying we get

$$(3.12) \quad \vec{s}^{\text{L}} = (E_{\mu}^{\text{cm}} / m_{\mu}) \hat{n}^{\text{cm}} + (\gamma_{\pi} - 1) (E_{\mu}^{\text{cm}} / m_{\mu} \beta_{\pi}) (\hat{n}^{\text{cm}} \cdot \vec{\beta}_{\pi}) \vec{\beta}_{\pi} + \gamma_{\pi} (p_{\mu}^{\text{cm}} / m_{\mu}) \vec{\beta}_{\pi} \\ s_0 = \gamma_{\pi} (p_{\mu}^{\text{cm}} / m_{\mu}) + (E_{\mu}^{\text{cm}} / m_{\mu}) (\vec{\beta}_{\pi} \cdot \hat{n}^{\text{cm}})$$

where $\beta_{\pi} = v_{\pi}/c$, the velocity of the pion in the lab and $\gamma_{\pi} = (1 - \beta_{\pi}^2)^{-1/2}$. The quantity of interest is the component of polarization along the momentum, $\hat{n} \cdot \hat{n}^{\text{cm}}$ ($\hat{n} = \vec{p}/|\vec{p}|$). Using eq. 3.8 and 3.12 we obtain

$$(3.13) \quad (\hat{n} \cdot \hat{n}^{\text{cm}})^{\text{L}} = s_0^{\text{L}} m_{\mu} / p_{\mu}^{\text{L}} = \frac{\gamma_{\pi}}{p_{\mu}^{\text{L}}} \left[p_{\mu}^{\text{cm}} + \beta_{\pi} E^{\text{cm}} (\vec{\beta}_{\pi} \cdot \hat{n}^{\text{cm}}) \right]$$

but $\hat{n}^{\text{cm}}$ is parallel to $\vec{p}^{\text{cm}}$ so that $\vec{\beta}_{\pi} \cdot \hat{n}^{\text{cm}}$ is the cosine of the angle between the decay angle of the muon in the pion rest frame and the pion beam direction, this is simply the angle θ^{cm} in fig. 3.1. Thus we can write

$$(3.14) \quad (\hat{n} \cdot \hat{n}^{\text{cm}})^{\text{L}} = p_{\mu} = \frac{\gamma_{\pi}}{p_{\mu}^{\text{L}}} \left[p_{\mu}^{\text{cm}} + \beta_{\pi} E_{\mu}^{\text{cm}} \cos \theta^{\text{cm}} \right]$$

The last term to be evaluated is p_{μ}^{L} , by the Lorentz transformation it is given by

$$(3.15) \quad p_{\mu}^{\text{L}} \cos \theta^{\text{L}} = \gamma_{\pi} \left[p_{\mu}^{\text{cm}} \cos \theta^{\text{cm}} + \beta_{\pi} E_{\mu}^{\text{cm}} \right]$$

with

$$(3.16) \quad \tan \theta^{\text{L}} = \tan \theta^{\text{cm}} / \gamma_{\pi} \left[1 + (\beta_{\pi} / \beta_{\mu}^{\text{cm}}) \sec \theta^{\text{cm}} \right]$$

$\beta_{\mu}^{\text{cm}} = .272$ and with the high momenta 0.8 GeV/c to 1.4 GeV/c used in the experiment $\cos \theta^{\text{L}} = .99$, this reflects the fact that the momentum of the muon in the pion rest frame p_{μ}^{cm} is so low in comparison to the momentum of the pion in the lab frame, that regardless of the decay angle in CM, the muon

moves along the beam line in the lab frame. Making the approximation $\cos\theta$ eq. 3.14 becomes

$$(3.17) \quad P_{\mu} = \left[p_{\mu}^{\text{cm}} + \beta_{\pi} E^{\text{cm}} \cos\theta^{\text{cm}} \right] / \left[p_{\mu}^{\text{cm}} \cos\theta^{\text{cm}} + \beta_{\pi} E_{\mu}^{\text{cm}} \right].$$

From eq. 3.15 it is clear that each muon momentum in the lab corresponds to a unique decay angle in the pion rest frame. Thus, if the polarimeter stops a certain range of momenta, say between p_{max} and p_{min} , this corresponds to a unique range in the decay angle between $\cos\theta_{\text{max}}$ and $\cos\theta_{\text{min}}$. The average spin polarization P of such a sample of muons is given by

$$(3.18) \quad \bar{P}_{\mu} = \frac{\int P_{\mu}(\cos\theta) f(\cos\theta) d(\cos\theta)}{\int f(\cos\theta) d(\cos\theta)}$$

where the limits of integration are $\cos\theta_{\text{max}}^{\text{cm}}$ and $\cos\theta_{\text{min}}^{\text{cm}}$ and $f(\cos\theta)$ is the probability distribution function of $\cos\theta^{\text{cm}}$. Since the decay of pions is isotropic in their rest frame, all $\cos\theta^{\text{cm}}$ are equiprobable and $f(\cos\theta) = 1$. Substituting the result from eq. 3.17 into 3.18 and integrating we obtain the solution for the average polarization

$$(3.19) \quad \bar{P}_{\mu} = \frac{A}{a} ; \quad A = \frac{a \cdot E_{\mu}^{\text{cm}}}{p_{\mu}^{\text{cm}}} + \left[1 - \left(\frac{\beta_{\pi} E^{\text{cm}}}{p_{\mu}^{\text{cm}}} \right)^2 \right] \ln \left(\frac{\beta_{\pi} E^{\text{cm}} + \cos\theta_{\text{max}}^{\text{cm}} p_{\mu}^{\text{cm}}}{\beta_{\pi} E^{\text{cm}} + \cos\theta_{\text{min}}^{\text{cm}} p_{\mu}^{\text{cm}}} \right)$$

where $a = \cos\theta_{\text{max}}^{\text{cm}} - \cos\theta_{\text{min}}^{\text{cm}}$. In addition since the full range of $\cos\theta = 2$, the percentage of all the beam muons that stop in the polarimeter, P_{stop} is given by

$$(3.20) \quad P_{\text{stop}} = 100(a/2).$$

The computation of $\cos\theta_{\text{max}}^{\text{cm}}$ and $\cos\theta_{\text{min}}^{\text{cm}}$ for a given pion momentum requires the knowledge of the experimental set up that was used. Degradar, in form of lead bricks,

was placed between the production target and the polarimeter. This served the dual purpose of slowing down muons and degrading out most of the other beam particles. The thickness of the degrader was variable but the most frequently used thickness was 16", corresponding to a momentum loss of 680 MeV/c for muons. The 22" of trichloroethane in the polarimeter represents a range corresponding to 270 MeV/c (the range was computed on the assumption that trichloroethane, CH_3CCl_3 , has the same stopping power as a carbon sample with the same density as CH_3CCl_3 , 1.35 gm/cm^3). Only muons that stop in the polarimeter are those with a momentum in the range $0 \leq p \leq 270 \text{ MeV}$ after the degrader. Using momentum-range tables⁴ we can obtain the momenta of muons before the degrader that correspond to the desired range after it. For each pion momentum, the corresponding decay angles are computed using eq. 3.15. These are the desired values of $\cos\theta_{\text{max}}^{\text{cm}}$ and $\cos\theta_{\text{min}}^{\text{cm}}$. Expected values of muon polarization, computed with this method are displayed in fig. 3.2, the dashed line (plotted on a scale 0 to 1) is the fraction of the beam muons that stop in the box according to eq. 3.20.

C. Čerenkov Light Output

Since the reaction $\mu \rightarrow e + \nu + \bar{\nu}$ is a three body decay, the electron can have any momentum between 0 and $E_{\text{max}}^{\text{el}}$, $E_{\text{max}}^{\text{el}} = 53 \text{ MeV}$ experimentally. If the energy of the electron is measured, by measuring the total amount of light it emits, than the parameter α in eq. 1.1 need not be

PION MOMENTUM = 0.8 GeV/c

PION MOMENTUM = 1.2 GeV/c

averaged over, which results in a savings by a factor of 4 of the number of events needed for the determination of P_{μ} to a given accuracy⁵. In order to calibrate and check the light measurements, a light source whose output can be controlled and is known to the experimenters is necessary. Unfortunately, the LED's installed for this purpose inside the polarimeter failed to function properly and could not be used. Another possible method of calibration involves the Čerenkov light of the parent muon.

From eq. A1.1, the threshold velocity of a particle in trichloroethane is $\beta_{th} = 1/1.44 = .7$. For a muon this corresponds to a momentum of 100 MeV/c, hence all muons with a momentum p , after the degrader, such that $0 \leq p \leq 270$ MeV/c will Čerenkov radiate. Using eq. A1.4 and given a momentum, it is possible to calculate the photon yield in the wavelength range of interest, 3600 Å to 4400 Å (this is the sensitive range of the detector as will be discussed later). The results are graphically shown in fig. 3.3, the number of photons produced is plotted against the muon momentum before the degrader. Equation A1.4 cannot be used in a straightforward way to compute the total light output, since not only does the polarimeter accept a broad range of momenta but the relative numbers of muons in a particular momentum range vary with the pion momentum.

An approximation scheme was devised to approximate the photon yield N_{ph} for a given pion momentum and degrader thickness. The polarimeter was subdivided into six segments (fig 3.4), the first five are 10 cm and the last is 6 cm long, total length of the module is 22" = 56 cm.

LIGHT OUTPUT AS A FUNCTION OF MOMENTUM

Fig. 3.3

Fig. 3.4

Polarimeter Segments

The average amount of light expected from each segment L_k is computed and so is the percentage P_k of all stopping muons that stop in each segment. The sum

$$(3.21) \quad \sum_{k=1}^6 L_k P_k = N_{ph}$$

gives an approximation of the average number of photons produced in a given experimental setup.

First, each segment is associated with a range of momenta of muons that will stop in it Δp_k , for example segment 3 corresponds to $\Delta p_3 = 160 - 192 \text{ MeV/c}$. Depending on the thickness of degrader Δp_k corresponds to a unique range of momenta Δp_k^b before the degrader, for 16" of lead $\Delta p_3^b = 730 - 760 \text{ MeV/c}$. This range now corresponds, depending on pion momentum, to some range in decay angle $\Delta \cos \theta_k$, for $p_\pi = 1.2 \text{ GeV/c}$, $\Delta \cos \theta_3 = -.5712 \text{ to } -.4445 = -.1267$. Then using eq. 3.20, the percentage of muons stopping in a given segment is given by

$$(3.22) \quad P_k = (\Delta \cos \theta_k / 2a) \cdot 100\% .$$

To compute the average amount of light from a given segment it is necessary to find the appropriate values of L and $\sin^2 \theta_c$. L is most naturally defined as the range in cm equivalent to the average value of momentum of all particles that stop in that segment. Since all momenta are equiprobable, the averaging procedure is straightforward. For segment 3, $\bar{p}_3^b = (730 + 760)/2 \text{ MeV/c}$, after traversing the degrader this becomes $\bar{p}_3 = 176 \text{ MeV/c}$. L as a function of momentum after the degrader is shown in fig. 3.5. The parameter $\sin^2 \theta_c$ is averaged for each segment by

$$(3.23) \quad \overline{\sin^2 \theta_c^k} = \frac{\int_{\beta_0}^{\beta_k} \sin^2 \theta_c^k d\beta}{\int_{\beta_0}^{\beta_k} d\beta} = \frac{\beta_k - \beta_0 + (1/n^2) \left[(1/\beta_k) - (1/\beta_0) \right]}{\beta_k - \beta_0}$$

where $\sin^2 \theta_c = 1 - (1/n\beta)^2$. β_k is the velocity that corresponds to $\bar{p}_k$ and $\beta_0 = .7$, the threshold velocity for Čerenkov radiation. The rationale behind this step is that while the muon enters the module with the velocity β_k it is degraded to a stop $\beta = 0$ within the polarimeter, and it stops radiating if $\beta < \beta_0$. Thus $\sin^2 \theta_c$ varies continuously along the path. The above averaging procedure was carried out on the assumption that the change in β in any given segment is linear with distance. Although this is not in general true, it is a good assumption if the distance over which β is to change is small.

D. Photoelectron Yield

It is not the photons that are directly measured, however, but the photoelectrons that the photons kick out at the cathode of the phototube. The factors affecting the result are the optical properties of glass, trichloroethane, and the waveshifter and the phototube quantum efficiency. The absorption of light in glass and trichloroethane and transmission of light through the waveshifter is the first factor, it is plotted as a function of wavelength in fig. 3.6. If the cutoff wavelength is defined such that at least 1/3 of the light is absorbed in the waveshifter, then the limits are $\lambda_1 = 3600\text{\AA}$ and $\lambda_2 = 4400\text{\AA}$. The average value of this absorption parameter

EFFECTIVE RADIATING RANGE

Fig. 3.6

COEFFICIENT OF ABSORPTION IN BBQ

on the interval $3600-4400\text{\AA}$ is .76, but this is a misleading result. Since the light output of Cerenkov light in any wavelength range goes as $1/\lambda^2$, most of the light is produced in the poor response region near $3600\text{\AA}$. A more reasonable value of this parameter is .50.

Of all the light absorbed in the waveshifter only about $1/3$ is reemitted and transmitted to the phototube. Finally, the phototube quantum efficiency in the characteristic reëmission region of the waveshifter is .10. Hence, out of N_{ph} photons we should see on the order of $.5 \times .33 \times .1 \times N_{ph}$ or $.017N_{ph}$ photoelectrons. The table below shows the results for various momenta with 16" of lead degrader.

p (GeV/c)	P_{stop}	N_{ph}	N_{pe}
0.8	33	965	16
0.9	36	1320	22
1.0	33	1320	22
1.1	30	1320	22
1.2	27	1320	22
1.3	16	1948	32
1.4	5	2695	45

IV. BROOKHAVEN TESTS

A. Experimental Setup

The polarimeter, degrader and all the trigger counters (see fig 4.1) were mounted on a movable chasis so that the position of the beam with respect to the polarimeter could be varied by moving the entire assembly. The waveshifters were mounted on plywood sheets for physical support and the plywood was screwed to the metal polarimeter frame, pressing the waveshifter against the glass. In order to light-tight the module all the edges and corners were taped over with black tape. This effort, however, was not entirely successful as some of the waveshifters had a noise signal rate on the order of a 100K counts/sec. A tent of black mylar was constructed around the polarimeter, and with it in place the noise on all waveshifters was at an acceptable rate of 3 to 4K counts/sec. As a precaution against a possibility of a catastrophic spill of the 65 gallons of trichloroethane, the module was put in an aluminium spill pan. All counters, T1, T2, T4, and T5 are scintillation counters. T1 and T2 are circular with a 2" diameter, T4 and T5 are large rectangular counters covering almost the entire 22"x22" side.

The circuit used in the experiment is shown in fig. 4.2. The trigger for an event, a stopping particle, was a coincidence $T1 \cdot T2 \cdot \overline{T4}$. The trigger has three outputs. One starts the TDC. A second one opens a 150 nsec "prompt" gate in the ADC. The ADC then looks at the summed signal from all six waveshifters. The light seen within this

Fig. 4.1

GG#1 - .5 sec delayed out
GG#2 - 10 sec gate

gate is the $\hat{\text{Cerenkov}}$ light of the stopping particle. The third output is delayed by $0.5 \mu\text{sec}$ ($\mu\text{sec} = 10^{-6} \text{sec}$) and then fires a $10 \mu\text{sec}$ gate into a coincidence circuit of $(T1 \cdot T2 \cdot \overline{T4}) \text{ dly } WS \cdot \overline{T5}$. If triggered, the coincidence circuit sends a 150 nsec gate to the ADC. The reason for the delay is two-fold. One is to allow the original particle to come to rest and all its Cerenkov light to be absorbed. More importantly, if the stopping particle had been a pion than, according to eq. 3.1, after $0.5 \mu\text{sec}$ there would only be a $4.4 \times 10^{-7} \%$ chance that the pion had not yet decayed (pion lifetime is 26 nsec), whereas if the particle had been a muon there would be an 30% chance. The delay gives a high degree of confidence that the decay seen is a muon decay rather than a pion decay.

$\overline{T5}$ is in the circuit to protect from stray beam particles entering the box while the ADC is looking for the decay electron's light. If any particle fires $T5$ during the $10 \mu\text{sec}$ gate, it will veto the entire event. When the delayed gate is opened, each waveshifter feeds into its individual ADC channel. In addition, each waveshifter has its own STOP channel at the TDC, so that when it sees any light it will issue a STOP command.

The tests were conducted during several weeks so that many changes both in the electronics and the physical setup were made. An example of the latter is the $1/4"$ thick lead sheet that was placed between the box and $T5$ halfway through the testing. Its purpose was to stop

any decay electrons that came out through the front of the module and prevent them from vetoing good events by firing T5. Thus the above description of the circuit and setup is meant to show the general features of the experiment which remained constant throughout, rather than concentrate on details which may have undergone changes in the course of the work.

All the data collecting devices, ADC's, TDC's and scalars, were interfaced with a computer and data was written on magnetic tape. In summary, the sets of data that were collected and their uses are as follows. ADC "prompt" gate looked at the light, from the sum of the waveshifters, from $t=0$ to $t=150$ nsec. This light came from the beam particle that that stopped in the module. The total amount of light is a measure of the incident particle's energy and thus can be used to help determine its stopping position within the box as well as help in calibrating the light measurements (as outlined in Chapter #3). ADC "delayed" gate looked at the light from each individual waveshifter for a period of 150 nsec sometime between $0.5 \mu\text{sec}$ and $10.5 \mu\text{sec}$. This is the decay particle's light, from it we hope to reconstruct the direction of motion of the particle. The 6 TDC channels measured the time intervals between the arrival of the triggering particle in the box until a waveshifter fired, that is saw the decay electron's light. From this data we should be able to reproduce the characteristic exponential decay curve, from which the muon lifetime can be computed. This

provides a direct check that we are in fact looking at muon decays.

During the running time of the experiment we made 109 beam runs at varying beam (pion) momenta and degrader thicknesses, except for three runs the beam was centered on the front face of the module. The momenta varied from 0.7 GeV/c to 1.4 GeV/c, the degrader from 7.5" to 16" of lead. Most of our data was collected with the degrader setting of 16" and momenta at 0.8 GeV/c and 1.2 GeV/c. We also performed several runs at very high momentum, up to 4 GeV/c, triggering on T1·T2·T4, that is particles with $\beta = 1$ going all the way through the box. The purpose of these runs was to collect data useful for calibration of light measurements.

B. Waveshifter Calibration

Before the tests with trichloroethane begun, the waveshifters were calibrated, so that by adjusting their high voltages each would give the same response to an identical light signal. The box was filled with water and a waveshifter was mounted on the side downstream from the beam (back panel), all the other sides were light-tighted with black mylar and tape. The trigger was T1 T2 T4, and there was no degrader. The Čerenkov angle in water is $\cos^{-1}(1/1.33) = 41^\circ$, it is less than the angle of total internal reflection $\sin^{-1}(1/1.33) = 49^\circ$ so that all the Čerenkov light from the straight-through particle hits the back waveshifter.

The high voltage settings were varied and the position (in ADC channels) of the peak of the distribution of the

pulse heights is recorded. In fig. 4.3 the calibration curves for the waveshifters are shown.

In runs 100 through 150, the high voltage settings were such as to give the peak in ADC channel 460, mode A. In the later runs the settings were changed to shift the peak channel to 700, mode B. The exception is the back waveshifters, its signal was attenuated in the circuitry more than the others and to compensate for this the high voltage was set so as to give the peak in an ADC channel 1.25 times as big as the other waveshifters.

WS	Position*	High Voltage	
		Mode A	Mode B
1	Bottom	2125	2225
2	Top	2060	2155
3	Right	1975	2055
4	Front	2010	2090
5	Left	1950	2055
6	Back	2045	2130

* Sides of the box as viewed while standing upstream from the box and looking downstream.

Fig. 4.3

WAVESHIFTER No.1

WAVESHIFTER No.2

WAVESHIFTER No.3

48.

WAVESHIFTER No.4

WAVESHIFTER No.5

WAVESHIFTER No.6

APPENDIX I.

A charged particle moving through a medium, with a refractive index $n(\lambda)$, with a velocity $\beta = v/c$ greater than that of light in that medium will emit radiation along a surface of a cone about the direction of its motion. The opening angle of the cone is given by

$$(A1.1) \quad \cos\theta = 1 / n(\lambda)\beta$$

where θ is the opening angle.

The number of Čerenkov photons emitted per unit length of path is given by

$$(A1.2) \quad \frac{dN}{dL} = 2\pi\alpha Z^2 \int_{\lambda_1}^{\lambda_2} \left[1 - (1/n^2\beta^2) \right] \frac{d\lambda}{\lambda^2}$$

where α is the fine structure constant and Z the charge of the particle. Photon yield is inversely proportional to the wavelength and since the lower limit of λ_1 appears to be 0, it looks as if the integral diverges. This is not so, however, since the refractive index $n(\lambda)$ becomes effectively less than one for wavelengths shorter than ultraviolet. Since $\beta < 1$, eq. A1.1 shows that for $n < 1$ there is no radiation, otherwise $\cos\theta > 1$ which is clearly impossible. Thus $\lambda_1 > 0$ and the integral is strictly convergent. Note that from eq. A1.2 it follows that most of the radiation will be emitted in the region near λ_1 , in the ultraviolet.

To obtain the number of photons expected from a particle moving through a distance L integrate eq. A1.2

$$(A1.3) \quad N = 2\pi a Z^2 \int_0^L \int_{\lambda_1}^{\lambda_2} \left[1 - (1/n^2 \rho^2) \right] \frac{d\lambda}{\lambda^2} dL.$$

Often, $n(\lambda)$ is a constant in the interval of interest, then, eq. A1.3 can be written as

$$(A1.4) \quad N = 2\pi a Z^2 \sin^2 \theta_c L \int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\lambda^2}$$

using the fact that $1 - (1/n^2 \rho^2) = 1 - \cos^2 \theta_c = \sin^2 \theta_c$.

APPENDIX II.

Consider a situation where a light ray travels from a substance of refractive index n_1 , through a substance with a refractive index n_2 , to a substance with an index of n_3 and $n_2 > n_1 > n_3$.

This is precisely the situation with a light ray moving from water through lucite to an air layer.

If substance 2 is not there, the angle of total internal reflection is given by

$$(A2.1) \quad \theta_{\text{crit}} = \sin^{-1}(n_3 / n_1) .$$

The critical angle for a ray going from 2 to 3 is

$$(A2.2) \quad \psi_{\text{crit}} = \sin^{-1}(n_3 / n_2) .$$

But since that ray originated in 1, the angle in 1 that corresponds to the angle ψ_{crit} is

$$(A2.3) \quad \theta'_{\text{crit}} = \sin^{-1}\left(\frac{n_2}{n_1} \sin \psi_{\text{crit}}\right) .$$

Using eq. A2.2 we can write

$$(A2.4) \quad \theta'_{\text{crit}} = \sin^{-1}\left(\frac{n_2}{n_1} \frac{n_3}{n_2}\right) = \sin^{-1}(n_3 / n_1) .$$

Hence $\theta'_{\text{crit}} = \theta_{\text{crit}}$ as required.

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