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**Coherent Synchro-Betatron Oscillations
Excited by the Beam-Beam Interaction***

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1. Introduction

The beam-beam interaction in an electron storage ring can potentially excite coherent motions of the bunches in the two colliding beams. The simplest case of these beam-beam excited coherent motions is that of dipole betatron oscillation in which all bunches execute rigid transverse betatron motions.^{1,2,3} These dipole motions of bunches are coupled through the beam-beam kicks and when the betatron tune is close to an integer, these beam-beam induced dipole motions become unstable and beams will be lost.

A second type of coherent beam-beam motion occurs when all bunches execute transverse quadrupole oscillations.⁴ The center of each bunch does not move and all bunches collide head-on, but the shape of each bunch oscillates in time, leading to loss of luminosity. This beam-beam coherent quadrupole instability occurs when the betatron tunes are close to one of the following resonances: $\nu_x = n/2$, $\nu_y = n/2$ and $\nu_x + \nu_y = n$.

In this note, a third possible coherent instability excited by the beam-beam interaction will be discussed. This time it involves the longitudinal motion of the bunches. What is studied is the possibility of beam-beam excited longitudinal dipole oscillations or coherent synchro-betatron oscillations in which the longitudinal and transverse motions are coupled through the beam-beam collisions. We suggest that such modes of oscillations exist when there is a finite dispersion at the interaction point.

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The beam-beam induced synchro-betatron coupling effects on single particle motion has been studied in Ref. 6. It was pointed out that the coupling mechanism can be a finite dispersion at the interaction point or beams colliding with a crossing angle. The present work concerns the effect on coherent beam motions due to the finite dispersion mechanism. In the calculation below, the beam-beam force is linearized. A matrix techniques is then applied to study the stability problem, as was similarly done in Refs. 1-4. It is found that the strength of such coherent synchro-betatron instabilities is rather weak, provided that the dispersion at the interaction points is not excessively large and that one operates away from the $\nu_x - \nu_s = n$ resonances. In the present work, we have not studied whether the nonlinear effects (and the corresponding multiple stopbands $\nu_x - k \nu_s = n$) will make the instability effects much more enhanced.

It may be useful to mention here that there is a possibility that the coherent beam-beam effects are more relevant than the incoherent ones (the "strong-weak" case) in determining the beam-beam stability threshold.⁷⁻¹⁰ The reason may be argued as follows. For each incoherent beam-beam instability mechanism, there can be found a corresponding mechanism that causes a coherent instability. In coherent motions, the separation between one piece of the beam and the corresponding piece of the on-coming beam is effectively twice the separation when one of the beams does not move, as would be the case in the strong-weak case. The beam-beam kicks are therefore effectively stronger for coherent motions and as a result, the coherent effect sets a tighter stability limit than the corresponding incoherent one. It is hoped that the present work provides an underline to this possibility.

2. The Coherent Synchro-betatron Coupling Mechanism

To see how coherent longitudinal motions can be excited by the beam-beam interaction, consider first two rigid bunches with energy errors δ_1 and δ_2 . The two bunches collide with a transverse separation of $\eta(\delta_1 - \delta_2)$ where η is the dispersion function at the interaction point (either by design or due to errors). This separation then gives rise to angular kicks*

* The beam separation of course contains also a betatron component. This will be treated in the next section when we perform the complete analysis.

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$$\Delta x'_1 = -\Delta x'_2 = \xi \eta (\delta_1 - \delta_2) \quad (1)$$

where ξ is the usual beam-beam strength parameter. These kicks then change the path length of the two bunches in the subsequent revolution by the amounts $\Delta z_1 = \eta \Delta x'_1$ and $\Delta z_2 = \eta \Delta x'_2$, i.e.

$$\Delta z_1 = -\Delta z_2 = \xi \eta^2 (\delta_1 - \delta_2) \quad (2)$$

If we consider the longitudinal motions of the two bunches, the beam-beam kicks provide coupling among the longitudinal quantities δ_1 , δ_2 , z_1 and z_2 of the two beams. When ξ is large enough, this will make coherent longitudinal oscillations unstable.

One can also consider the more general problem of coupled longitudinal and betatron motions. Then the quantities coupled by the beam-beam kicks are x_1 , x'_1 , z_1 , δ_1 , x_2 , x'_2 , z_2 and δ_2 . The object of this work is to study the effects of this beam-beam induced synchro-betatron coupling.

Detailed calculations are presented in the next section. The beam-beam force is linearized and after doing so, the treatment can be done by first setting up the transformation matrices and then looking for the eigenvalues of the total matrix that describe the motion for one revolution. Instability occurs when any one of the eigenvalues has an absolute value larger than unity.

3. Calculations

The beam bunches are represented as rigid charge distributions specified by their centers of charge coordinates (x, x', z, δ) . The storage ring is assumed to be periodic with one interaction point per period. We also assume one rf cavity in each period (although different rf cavity arrangements should not change the conclusions significantly).

Consider the motion of one of the bunches in, say, the electron beam. The transformation matrix from one interaction point (IP) to the next is given by

$$R_- = N T_e M \quad (3)$$

where M is the 4×4 matrix from IP to the rf cavity, T_c is the matrix for the cavity and N is the matrix from cavity to the next IP. More explicitly,

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 & x_m \\ m_{21} & m_{22} & 0 & y_m \\ a_m & b_m & 1 & -c_m - z_m \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

$$N = \begin{bmatrix} n_{11} & n_{12} & 0 & y_n \\ n_{21} & n_{22} & 0 & y_n \\ a_n & b_n & 1 & -c_n - z_n \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

$$T_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & q & 1 \end{bmatrix} \quad (6)$$

where

$$x_m = \eta_c - m_{11} \eta_p - m_{12} \eta'_p, \quad a_m = m_{21} \eta_c - m_{11} \eta'_c + \eta'_p$$

$$y_m = \eta'_c - m_{21} \eta_p - m_{22} \eta'_p, \quad b_m = m_{22} \eta_c - m_{12} \eta'_c - \eta_p$$

$$z_m = a_m \eta_p + b_m \eta'_p, \quad c_m = \int_{IP}^{cav} \frac{\eta}{\rho} ds$$

$$x_n = \eta_p - n_{11} \eta_c - n_{12} \eta'_c, \quad a_n = n_{21} \eta_p - n_{11} \eta'_p + \eta'_c$$

$$y_n = \eta'_p - n_{21} \eta_c - n_{22} \eta'_c, \quad b_n = n_{22} \eta_p - n_{12} \eta'_p - \eta_c$$

$$z_n = a_n \eta_c + b_n \eta'_c, \quad c_n = \int_{cav}^{IP} \frac{\eta}{\rho} ds$$

with η_c the dispersion function at the cavity, η_p the dispersion function at IP and q given by the slope of the rf voltage at the bunch arrival time.

Similarly the transformation matrix R_+ from IP to IP for the positron bunches can be calculated. If we then set up the $8N$ -dimensional vector (N = number of bunches per beam)

$$(x_1, x'_1, z_1, b_1, x_2, x'_2, z_2, b_2, \dots)$$

whose elements are the coordinates of all the beam bunches, the transformation between collisions is simply given by the $8N \times 8N$ matrix which is made up of the diagonal array of 4×4 matrices each describing the motion of one bunch.

Coupling among bunches is provided at the beam-beam interaction points. To calculate this, one first identifies the N colliding pairs. For each pair, the part of the $8N \times 8N$ matrix that describes their motion is given by

$$T_{bb} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & -k & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \quad (7)$$

where $k = 4\pi \xi$.

Once the $8N \times 8N$ matrices are established, one successively multiplies these matrices together for half a revolution (no need to compute it for one revolution) of the storage ring. Eigenvalues of the resulting matrix are then calculated and stability region determined. A computer program has been written to perform these calculations. In the appendix, we also offer the result of an analytic calculation for the case of one bunch per beam. In the next section, the numerical results for the general cases are presented.

4. Results

The model reduces to that of the coherent dipole beam-beam instability studied in Refs. 1-3 when the cavity and the IP are non-dispersive. Indeed, Figs. 1-4 give the expected sawtooth stability diagrams for 1, 2, 3 and 4 bunches per beam. Note that Figs. 1 to 4 are periodic in ν_β with periods 1, 2, 3 and 4, respectively. Shaded region indicates instability.

The next calculation is to take the case of Fig. 1 and let the IP be dispersive with $\bar{\eta}_p \equiv \eta_p / \sqrt{\beta_p \alpha R / \nu_s} = 0.5$.^{*} This rather excessive $\bar{\eta}_p$ is chosen to make the effects pronounced for our studies. The cavities are inserted at the midpoints between IP's but are assumed to be nondispersive. The synchrotron tune is chosen to be $\nu_s = 0.05$. The resulting stability diagram is shown as Fig. 5.

Note that the period in Fig. 5 is $\nu_\beta = 2$ as compared with a period of 1 in Fig. 1. The sawteeth of Fig. 1 still persist in Fig. 5 except that the beam-beam induced synchro-betatron motion has introduced another stability boundary superimposing that given by Fig. 1.

Starting with Fig. 6, we move the rf cavities to the IP's where $\bar{\eta}_p = 0.5$ to show the effects of dispersive cavities. Figure 8 is the result. The main change from Fig. 5 is a stopband near the difference resonance $\nu_\beta - \nu_s = 0$ irrespective of the beam-beam strength. This is the single-particle synchro-betatron instability discussed in Ref. 6. In addition, a narrow beam-beam induced instability is born near the resonance.

Figures 7 and 8 are similar to Fig. 6 except that $\bar{\eta}_p$ has been reduced to 0.2 and 0.05, respectively. As can be seen, the synchro-betatron effects on Fig. 8 is rather weak.

Again assuming the cavities are located at the IP's where $\bar{\eta}_p = 0.2$, we show in Figs. 9, 10 and 11 the sawtooth diagrams for 2, 3 and 4 bunches per beam. These figures are periodic with periods $\nu_\beta = 4, 6$ and 8.

The sawtooth diagrams are stability diagrams in the (ξ, ν_β) plane for a given ν_s . One can also give stability diagrams in the (ν_s, ν_β) plane for given values of ξ . Figure 12 is such a calculation for the one bunch per beam case and $\xi = 0.02$ and 0.06. Again $\bar{\eta}_c = \bar{\eta}_p = 0.2$.

^{*} $\bar{\eta}_p$ is the "normalized" dispersion at the IP. If $\beta_p = 1m$, $\alpha = 0.003$, $R = 350m$, then $\bar{\eta}_p = 2.3m$.

Appendix

In this appendix, we give the result of an analytic calculation of the instability conditions for the case of one bunch per beam. The calculation is straightforward and we will only briefly sketch some of the algebraic procedures and omit the details.

The total 8×8 matrix that describe the motion of the two bunches for half a revolution can be written as

$$T = T_{bb} \begin{bmatrix} R_- & 0 \\ 0 & R_+ \end{bmatrix} \quad (A1)$$

where R_+ and R_- are the 4×4 matrices for electrons and positrons (R_- is given by eq. (3)), T_{bb} is the 8×8 matrix given by eq. (7).

Instead of directly looking for the eigenvalues of the matrix T , we will solve for the eigenvalues of $T + T^{-1}$, where T^{-1} is obtained by $T^{-1} = -S \tilde{T} S$ using the symplecticity property of T . The eigenvalues Λ of $T + T^{-1}$ are determined from

$$\det |T + T^{-1} - \Lambda| = 0 \quad (A2)$$

A straightforward algebraic manipulation shows that (A2) can be written as the form

$$\det \begin{vmatrix} A & N \\ \tilde{N} & B \end{vmatrix} = 0 \quad (A3)$$

where A, N are 4×4 matrices with B proportional to the unit matrix. The lower left corner $\tilde{N} = -S \tilde{N} S$ is the symplectic transform of N . Equation (A3) is equivalent to

$$\det |BA - N \tilde{N}| = 0, \quad (A4)$$

eq. (A4) is much simpler than eq. (A2), which involves calculating the determinant of an 8×8 matrix. The same technique in fact can be carried for one more step since the matrix $BA - N \tilde{N}$ is of the form

$$\begin{bmatrix} a & b \\ \tilde{b} & c \end{bmatrix}$$

where a, b and c are 2×2 with a and c proportional to the unit matrix. Equation (A4) then becomes equivalent to

$$|ac - b \tilde{b}| = 0, \quad (A5)$$

which involves only 2×2 matrices.

When these manipulations are carried out, the final result is that the eigenvalues Λ satisfy

$$\begin{aligned} \Lambda^2 - (2 \cos \mu_s + 2 \cos \mu + q H \sin \mu) \Lambda \\ + 4 \cos \mu \cos \mu_s + 2 q H \sin \mu = 0 \end{aligned} \quad (A6)$$

or

$$\begin{aligned} \Lambda^2 - [2 \cos \mu_s + 2 \cos \mu - 2 k \sin \mu + q H \sin \mu - A] \Lambda \\ + 4 \cos \mu_s (\cos \mu - k \sin \mu) + 2 q H \sin \mu \\ - 4 k q (1 - \cos \mu) \frac{\eta_p^2}{\beta_p} - 4 k q A = 0 \end{aligned} \quad (A7)$$

where μ and μ_s are the betatron and the synchrotron phase advances from one IP to the next, subscripts c and p refer to the cavity and the IP locations, subscripts m and n refer to the M and N arc sections described in the text,

$$H = \left[\eta_c^2 + (\beta_c \eta_c' + \alpha_c \eta_c)^2 \right] / \beta_c \quad (A8)$$

$$\begin{aligned} A = - \frac{\eta_p^2}{\beta_p} - \cos \mu \frac{\eta_c^2}{\beta_c} + \sin \mu_m \sin \mu_n H \\ + \frac{1}{\sqrt{\beta_c \beta_p}} \left[(\cos \mu_m + \cos \mu_n) \eta_p \eta_c + (\sin \mu_n - \sin \mu_m) \eta_p (\beta_c \eta_c' + \alpha_c \eta_c) \right. \\ \left. + \sin(\mu_m - \mu_n) \eta_c (\beta_c \eta_c' + \alpha_c \eta_c) \right] \end{aligned} \quad (A9)$$

The cavity voltage parameter q is related to the synchrotron phase advance by

$$\cos \mu_s = 1 - \frac{q}{2} \oint ds \frac{\eta}{\rho} \quad (A10)$$

The modes whose eigenvalues are given by eq. (A6) are such that the two bunches move transversely always in phase so that the beam-beam interaction does not influence them. Indeed, eq. (A6) is independent of the beam-beam strength parameter $k = 4\pi \xi$. The synchro-betatron coupling described by (A6) is the same as that of Ref. 6.

The two modes described by eq. (A7), on the other hand, are perturbed by the beam-beam interaction. Note that (A7) reduces to (A6) if $k = 0$. Another special case is when $k \neq 0$ but the cavity and the IP are non-dispersive. Then $H=0$, $A=0$ and the 4 eigenvalues are readily solved to give

$$\Lambda_{1,2,3,4} = 2\cos\mu_s, 2\cos\mu, 2\cos\mu_s, 2(\cos\mu - k \sin\mu). \quad (A11)$$

The modes corresponding to Λ_1 and Λ_3 describe the free synchrotron oscillation; the beam-beam interaction does not influence these modes when IP is non-dispersive. The mode corresponding to Λ_2 describes the free "0-mode" betatron motions. The mode Λ_4 describes the " π -mode" betatron oscillations.

In general, the stability of the modes is determined by the condition

$$|\Lambda| = \left| \lambda + \frac{1}{\lambda} \right| \leq 2. \quad (A12)$$

For given μ, μ_s, η_p etc., eq. (A11) thus determines the highest possible value of ξ beyond which at least one of the coherent beam-beam modes is unstable. These analytic results are checked satisfactorily against the numerical calculations described in the text.

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Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12