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*TSAAS: Finite-Element Thermal and
Stress Analysis of Plane and
Axisymmetric Solids with
Orthotropic Temperature-Dependent
Material Properties*

Los Alamos

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TSAAS: Finite-Element Thermal and Stress Analysis of Plane and Axisymmetric Solids with Orthotropic Temperature-Dependent Material Properties

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TSAAS:
FINITE ELEMENT THERMAL AND STRESS ANALYSIS OF
PLANE AND AXISYMMETRIC SOLIDS WITH ORTHOTROPIC
TEMPERATURE-DEPENDENT MATERIAL PROPERTIES

by

R. V. Browning and C. A. Anderson

ABSTRACT

The finite element method is used to determine the temperatures, displacements, stresses, and strains in axisymmetric solids with orthotropic, temperature-dependent material properties under axisymmetric thermal and mechanical loads. The mechanical loads can be surface pressures, surface shears, and nodal point forces as well as an axial or centripetal acceleration. The continuous solid is replaced by a system of ring elements with triangular or quadrilateral cross sections. Accordingly, the method is valid for solids that are composed of many different materials and that have complex geometry. Nonlinear mechanical behavior as typified by plastic, locking, or creeping materials can be approximated. Two dimensional mesh generation, plotting, and editing features allow the computer program to be readily used.

In addition to a stress analysis program that is based on a modified version of the SAAS code, TSAAS can carry out a transient thermal analysis with the finite element mesh used in stress analysis. An implicit time differencing scheme allows the use of arbitrary time steps with consequent fast running times. At specified times, the program will return to SAAS for thermal stress analysis. Nonlinear thermal properties and Arrhenius reaction kinetics are also incorporated into TSAAS.

Several versions of TSAAS are in use at Los Alamos, running on CDC-7600, CRAY-1 and VAX 11/780 computers. This report describes the nominal TSAAS; other versions may have some unique features.

I. INTRODUCTION*

The finite element terminology as applied to the analysis of continua was first used in 1960.¹ Before 1960, two-dimensional elements were used in conventional methods of structural analysis as a means of improving the stiffness idealization of complex aircraft structures.² Since the introduction of the finite element method for the stress analysis of plane stress bodies, the technique has been successfully applied to plates,³ axisymmetric solids,^{4,5,6} axisymmetric shells,^{7,8} three-dimensional solids,^{9,10} and other boundary value problems (e. g., Refs. 11 and 12). Reference 13 is an up-to-date text on the application of the finite element method to many linear and nonlinear field problems in engineering science.

In the finite element approximation of solids, the continuum is replaced by a system of elements that are interconnected at their corners (nodes). In the case of an axisymmetric solid, the nodes are actually circles and are called nodal circles. A finite element idealization of a simple axisymmetric solid using triangular elements is shown in Fig. 1. For axisymmetric loading, two equilibrium equations, which are expressed in terms of unknown nodal circle displacements, are derived for each node of the finite element system. A solution of the resulting set of linear algebraic equations constitutes an equilibrium solution to the finite element approximation of the axisymmetric solid.

The advantages of the finite element method over other methods of analysis are numerous. The finite element method affords nearly complete generality in the specification of geometrical and material properties, i. e., geometrically complex bodies of many different materials are easily represented. Displacement or stress boundary

*Portions of Sections I and II and Appendix A have been lifted from the interim report, SAAS IIa, Finite Element Stress Analysis of Axisymmetric Solids with Orthotropic, Temperature-Dependent Material Properties, Sandia Corporation.

Fig. 1. The finite element idealization.

conditions can be specified at any node of the finite element system. Thermal and mechanical loads can be specified at nodes and, if the number of nodes is sufficient, nearly arbitrary distributions of the loads can be represented. In addition, the body as a whole can be subjected to an axial acceleration and/or an angular velocity.

The present report is concerned with transient thermal stress analysis using the finite element method. In Section II, the finite element method is applied to the determination of stresses, strains, and displacements in arbitrary axisymmetric bodies with orthotropic, temperature-dependent material properties subjected to arbitrary axisymmetric mechanical and thermal loading. The mechanical loads can be surface pressures, surface shears, and nodal point forces as well as an axial acceleration or angular velocity. Using triangular and quadrilateral finite elements, the equilibrium equations for a

finite element system are derived from the minimum potential energy theorem for elastic solids.

In Section III, the finite element method is applied to the determination of transient temperature fields in axisymmetric solids with time-dependent boundary conditions starting from specified initial conditions. Again temperature-dependent thermal properties as well as nonlinear temperature-dependent heat generation (such as provided by reaction kinetics) are incorporated into the analysis.^{14,15} The finite element discretized equations are again based on triangles and quadrilaterals allowing the use of a single finite element mesh to be used in both problems. Finally, an implicit time differencing scheme allows the use of an arbitrary time increment without loss of numerical stability in the discretized equations; the net result is a great saving in computer time compared with finite element methods that are restricted by a stability requirement.

TSAAS, Thermal Stress Analysis of Axisymmetric Solids, incorporates the theory of Sections II and III into a practical code. The stress analysis portion of TSAAS is based on the SAAS code¹⁶ together with two-dimensional automatic mesh generation. This is an important aid to the user since it drastically reduces the number of input data cards. The thermal analysis portion of TSAAS has been written by the authors and uses several initial condition and boundary condition options that will be an important aid to the user. The overlay program SAASPL provides a plotting and editing capability for the enormous amount of data (displacements, temperatures, stresses, and strains) generated in TSAAS. In addition, TSAAS has a multiple case capability whereby a problem can be run using a previously generated mesh.

Section IV discusses various special features of TSAAS such as mesh generation, skew boundaries, and orthotropic elastic properties. In Section V are discussed the various material

nonlinearities that have been adapted to TSAAS. Basically, TSAAS carries out a linear elastic analysis, but by changing elastic material properties for each element, forming a new stiffness matrix, and then re-solving, nonlinear material behavior can be approximated. In this way plastic, locking, and "no tension"¹⁷ material behavior have been approximated in TSAAS. In addition, the initial strain method has been incorporated into a version of TSAAS which allows for the calculation of both thermal and creep behavior of axisymmetric solids.

Input instructions for TSAAS are described in Appendix A while Appendix B describes the TSAAS computer output. Finally, Appendix C illustrates several examples of stress and thermal problems solved by using TSAAS.

II. METHOD OF STRESS ANALYSIS

The finite element method is described in Ref. 13 and the general equations that govern the equilibrium of an elastic system are given there. However, for completeness and in order to define the various terms involved, the equations are briefly rederived here. A standard matrix notation will be used with $\{v\}$ representing a vector, $[A]$ a matrix, etc.

A. Equilibrium Equations

The potential energy for an elastic solid is given by

$$V = U - \int_{\text{Vol}} \{v\}^T \{F\} dV - \int_{\text{Area}} \{v\}^T \{P\} ds, \quad (1)$$

where U is the total strain energy for the solid, and is given by the integral

$$U = \int_{\text{Vol}} \left[\int_0^{\epsilon_i} \{\sigma\}^T \{d\epsilon\} \right] dV. \quad (2)$$

In Eqs. (1) and (2), $\{v\}$ is the displacement vector, $\{F\}$ is the body force vector, and $\{P\}$ is the surface traction. For a finite element system that is composed of M elements, the potential energy can be written as a summation over all the elements

$$V = \sum_{m=1}^M \left[U_m^m - \int_{Vol} \{F\}_m^T \{v\}_m dV - \int_{Area} \{P\}_m^T \{v\}_m ds \right]. \quad (3)$$

The surface integral exists only if the m th element is on the boundary of the body and subjected to surface tractions $\{P\}$.

In the finite element displacement approach, it is necessary to assume a solution for the displacement field in each element, that is, in general, taken so as to satisfy compatibility between elements of the system. For an axially symmetric system, for instance, the radial displacement in the m th element $u_m(r, z)$ can be written down

$$u_m(r, z) = [A(r, z)]_m \{u\}_m, \quad (4)$$

where $[A]_m$ is the so-called shape function matrix and $\{u\}_m$ is the vector of the radial nodal displacements. In the displacement approach, an expression for the element strains $\{\epsilon\}_m$ is obtained from the assumed displacement field

$$\{\epsilon\}_m = [B]_m \{\delta\}_m, \quad (5)$$

where $[B]_m$ is the strain-displacement matrix and $\{\delta\}_m$ is the vector of all of the element nodal point displacements

$$\{\delta\}_m = \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ \vdots \end{Bmatrix}. \quad (6)$$

In Eq. (6), u_i , w_i represent the radial and axial components of displacement of node i of element m , etc. In transposed form, Eq. (5) becomes

$$\{\epsilon\}_m^T = \{\delta\}_m^T [B]_m^T. \quad (7)$$

For an elastic material, the element stresses are expressed in terms of the corresponding strains and thermal effects by the linear thermo-elastic stress-strain relationship

$$\{\sigma\}_m = [C]_m \{\epsilon\}_m - \{\tau\}_m, \quad (8)$$

where $\{\tau\}$ are thermal stresses.

Using Eq. (8) and some manipulation with transposes, the strain energy, Eq. (2), can be written in matrix form as

$$U^m = \frac{1}{2} \int_{Vol} \{\epsilon\}_m^T [C]_m \{\epsilon\}_m dV - \int_{Vol} \{\epsilon\}_m^T \{\tau\}_m dV. \quad (9)$$

The total potential energy, Eq. (3), can then be written in matrix form as

$$V = \frac{1}{2} \{\delta\}^T [K] \{\delta\} - \{\delta\}^T \{Q\}, \quad (10)$$

$[K]$ is the stiffness matrix for the assembled finite element system:

$$[K] = \sum_{m=1}^M [k]_m. \quad (11)$$

The individual element stiffness matrix is

$$[k]_m = \int_{Vol} [B]_m^T [C]_m [B]_m dV, \quad (12)$$

and the load vector $\{Q\}$ is defined as

$$\{Q\} = \sum_{m=1}^M \{L\}_m + \sum_{m=1}^M \{R\}_m. \quad (13)$$

The element body force is

$$\{L\}_m = \int_{Vol} \left[[A]_m^T \{F\}_m + [B]_m^T \{\tau\}_m \right] dV, \quad (14)$$

and the element load vector due to surface forces is

$$\{R\}_m = \int_{Area} [A]_m^T \{P\}_m ds. \quad (15)$$

The potential energy is made stationary by requiring that

$$\frac{\partial V}{\partial \delta_i} = 0 \quad i = 1 \dots N \quad (16)$$

where N is the total number of unknown nodal point displacements. After consideration of the rules for matrix differentiation with respect to a vector, the application of Eq. (10) yields the following linear system equations for the equilibrium of a finite element system:

$$[K] \{\delta\} = \{Q\} \quad (17)$$

B. Boundary Conditions

Before the displacement boundary conditions are imposed, the stiffness matrix $[K]$ is singular and, thus, a unique solution does not exist for the displacements. The minimum number of boundary conditions that must be used are those sufficient to restrain the rigid body motion of the structure.

The method used herein is that which has been used by Wilson, and that is the diagonal term of the stiffness matrix of the corresponding restraint is replaced by unity, while the remaining off-diagonal elements in that row and column are set to zero. The corresponding force term is the prescribed displacement while the other force terms are modified by subtracting from them the product of the corresponding off-diagonal term of the stiffness matrix and the prescribed displacement. This is obviously done before the off-diagonal terms are set to zero. The subroutine MODIFY, which is used to perform these steps, consists of only 17 FORTRAN statements.

Incidentally, this technique does not destroy the band width and, in fact, the boundary conditions are imposed on the matrix that is stored in a rectangular array. Using this method, the number of equations that must be "solved" is equal to the total number of degrees of freedom; however, the number of restraints is usually relatively small and the band solution proceeds at a fast rate.

C. Triangular Element

The cross section of a typical triangular ring element is shown in Fig. 2. The displacements (u, w) in the r - z plane in the element are assumed to be of the following form:

$$u(r, z) = b_1 + b_2 r + b_3 z \quad (17a)$$

$$w(r, z) = b_4 + b_5 r + b_6 z \quad (17b)$$

This linear displacement field assures continuity between elements since lines which are initially straight remain straight in their displaced position.

If Eqs. (17a) and (17b) are evaluated at the three vertices (r_i, z_i) , (r_j, z_j) , (r_k, z_k) of the triangular ring element for the nodal point displacements (u^i, w^i) etc., then, the following matrix equation is obtained

$$\begin{bmatrix} u^i & w^i \\ u^j & w^j \\ u^k & w^k \end{bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_k & z_k \end{bmatrix} \begin{bmatrix} b_1 & b_4 \\ b_2 & b_5 \\ b_3 & b_6 \end{bmatrix} \quad (18)$$

A solution of Eq. (18) yields the following expression for the constants b_i ($i = 1, 2, \dots, 6$) in terms of the displacements at the vertices:

$$\begin{bmatrix} b_1 & b_4 \\ b_2 & b_5 \\ b_3 & b_6 \end{bmatrix} = [D] \begin{bmatrix} u^i & w^i \\ u^j & w^j \\ u^k & w^k \end{bmatrix} \quad (19a)$$

where

$$[D] = \frac{1}{\lambda} \begin{bmatrix} r_j z_k - r_k z_j & r_k z_i - r_i z_k & r_i z_j - r_j z_i \\ z_j - z_k & z_k - z_i & z_i - z_j \\ r_k - r_j & r_i - r_k & r_j - r_i \end{bmatrix} \quad (19b)$$

in which $\lambda = r_j(z_k - z_i) + r_i(z_j - z_k) + r_k(z_i - z_j)$. Equation (19a) can be written in an expanded form where the constants and displacements are column vectors,

$$\{b\} = [h] \{\delta\} \quad (20a)$$

Fig. 2. Triangular Element

and the shape function matrix $[A]$ given in Eq. (4) is seen to be

$$[A(r, z)]^T = [D]\{e\} \quad (20b)$$

where

$$\{e\}^T = \{1, r, z\}.$$

The element strains are obtained from Eqs. (17a) and (17b). Letting θ be the hoop direction,

$$\epsilon_{rr} = \frac{\partial u}{\partial r} = b_2$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = b_6$$

$$\epsilon_{\theta\theta} = \frac{u}{r} = \frac{1}{r} b_1 + b_2 + \frac{z}{r} b_3$$

$$\epsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = b_3 + b_5$$

which can be written in matrix form as

$$\begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ \epsilon_{\theta\theta} \\ \epsilon_{rz} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{r} & 1 & \frac{z}{r} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{Bmatrix} \quad (21)$$

or, symbolically,

$$\{\epsilon\} = [g]\{b\} \quad (22)$$

The substitution of Eq. (20a) into Eq. (22) yields

$$\{\epsilon\} = [g][h]\{\delta\} \quad (23)$$

Thus, the strain-displacement transformation

matrix $[B]$, as defined in Eq. (5), is

$$[B] = [g][h] \quad (24)$$

With this definition of $[B]$, the element stiffness

matrix, Eq. (12), is rewritten as

$$[k] = \int_{Vol} [h]^T [g]^T [C] [g] [h] dV \quad (25)$$

Since $[h]$ is not a function of r and z , Eq. (25) for the element stiffness matrix becomes

$$[k] = [h]^T \left[\int_{Vol} [g]^T [C] [g] dV \right] [h] \quad (26)$$

and the body force vector, Eq. (14), reduces to

$$\{L\} = [h]^T \int_{Vol} [g]^T \{\tau\} + \{e\}^T \{F\} dV \quad (27)$$

where again $\{e\}^T = \{1, r, z\}$.

For axisymmetric bodies with orthotropic material properties whose principal axes are not aligned with the r - z coordinates, the stress-strain relations in the principal N-S coordinates are

$$\begin{Bmatrix} \sigma_{NN} \\ \sigma_{SS} \\ \sigma_{\theta\theta} \\ \sigma_{NS} \end{Bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & 0 \\ C'_{12} & C'_{22} & C'_{23} & 0 \\ C'_{13} & C'_{23} & C'_{33} & 0 \\ 0 & 0 & 0 & C'_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_{NN} \\ \epsilon_{SS} \\ \epsilon_{\theta\theta} \\ \epsilon_{NS} \end{Bmatrix} \quad (28)$$

where

$$\begin{aligned} \tau_{NN} &= T (C'_{11} \alpha_N + C'_{12} \alpha_S + C'_{13} \alpha_\theta) \\ \tau_{SS} &= T (C'_{12} \alpha_N + C'_{22} \alpha_S + C'_{23} \alpha_\theta) \\ \tau_{\theta\theta} &= T (C'_{13} \alpha_N + C'_{23} \alpha_S + C'_{33} \alpha_\theta) \end{aligned} \quad (29)$$

and T is the element temperature. The N-S coordinates are defined in terms of the r - z coordinates in Fig. 3.

Fig. 3. N-S and r-z coordinates.

Equation (28) can be abbreviated as

$$\{\sigma\}_{NS} = [C]_{NS} \{\epsilon\}_{NS} - \{\tau\}_{NS} \quad (30)$$

where the subscripts refer to the coordinate system in which the quantities are expressed. The stresses in the local N-S coordinates are transformed into the body r-z coordinates by use of the transformation

$$\{\sigma\}_{rz} = [t]^T \{\sigma\}_{NS} \quad (31)$$

where $[t]^T$ is the transpose of

$$[t] = \begin{bmatrix} \cos^2 \beta & \sin^2 \beta & 0 & \sin \beta \cos \beta \\ \sin^2 \beta & \cos^2 \beta & 0 & -\sin \beta \cos \beta \\ 0 & 0 & 1 & 0 \\ -2 \sin \beta \cos \beta & 2 \sin \beta \cos \beta & 0 & \cos^2 \beta - \sin^2 \beta \end{bmatrix} \quad (32)$$

The strains in the local N-S coordinates are transformed into the body r-z coordinates by use of the transformation

$$\{\epsilon\}_{rz} = [t]^{-1} \{\epsilon\}_{NS} \quad (33)$$

The inverse of Eq. (38) is

$$\{\epsilon\}_{NS} = [t] \{\epsilon\}_{rz} \quad (34)$$

Upon substitution of Eqs. (31) and (34) in Eq. (30), it is seen that

$$\{\sigma\}_{rz} = [C]_{rz} \{\epsilon\}_{rz} - \{\tau\}_{rz} \quad (35)$$

where

$$[C]_{rz} = [t]^T [C]_{NS} [t] \quad (36)$$

$$\{\tau\}_{rz} = [t]^T \{\tau\}_{NS} \quad (37)$$

Equation (35) can be expanded to read

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ \epsilon_{\theta\theta} \\ \epsilon_{rz} \end{Bmatrix} - \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{Bmatrix} \quad (38)$$

where

$$\begin{aligned} \tau_1 &= T(C_{11} \alpha_r + C_{12} \alpha_z + C_{13} \alpha_\theta) \\ \tau_2 &= T(C_{12} \alpha_r + C_{22} \alpha_z + C_{23} \alpha_\theta) \\ \tau_3 &= T(C_{13} \alpha_r + C_{23} \alpha_z + C_{33} \alpha_\theta) \\ \tau_4 &= T(C_{14} \alpha_r + C_{24} \alpha_z + C_{34} \alpha_\theta) \end{aligned} \quad (39)$$

The coefficients of thermal expansions, α_r , α_z , and α_θ , are in the r, z, and θ directions, respectively, and T is the temperature change within the element. For this type of material, the terms under the integral in Eq. (26) are

$$[g]^T[C][g] = \begin{bmatrix} \frac{1}{r^2}C_{33} & \frac{1}{r}(C_{13} + C_{33}) & \frac{z}{r^2}C_{33} + \frac{1}{r}C_{34} & 0 & \frac{1}{r}C_{34} & \frac{1}{r}C_{23} \\ & C_{11} + 2C_{13} + C_{33} & \frac{z}{r}(C_{13} + C_{33}) + C_{14} + C_{34} & 0 & C_{14} + C_{34} & C_{12} + C_{23} \\ & & \frac{z^2}{r^2}C_{33} + \frac{2z}{r}C_{34} + C_{44} & 0 & \frac{z}{r}C_{34} + C_{44} & \frac{z}{r}C_{23} + C_{24} \\ & \text{Symmetrical} & & 0 & 0 & 0 \\ & & & & C_{44} & C_{24} \\ & & & & & C_{22} \end{bmatrix} \quad (40)$$

and

$$[g]^T\{\tau\} + [e]^T\{F\} = \begin{bmatrix} F_r + \frac{1}{r}\tau_3 \\ rF_r + \tau_1 + \tau_3 \\ zF_r + \frac{z}{r}\tau_3 + \tau_4 \\ F_z \\ rF_z + \tau_4 \\ zF_z + \tau_2 \end{bmatrix} \quad (41)$$

In the case of an angular frequency, ω , the body force in the r -direction is $F_r = m\omega^2 r$ and, for an axial acceleration, a_z , the body force in the z -direction is given by $F_z = -ma_z$ where m is the mass density of the material.

Equations (40) and (41) can be programmed directly, since the integrals over the volume of the element for the various functions are most conveniently evaluated numerically in the computer program. The element stiffness matrix, Eq. (26), and the body force vector, Eq. (27), are then formed by standard matrix operations.

D. Quadrilateral Element

A typical quadrilateral element is composed of four triangular elements as illustrated in Fig. 4. The ten equilibrium equations for the quadrilateral are developed by the application of Eq. (10) and can be written in the following matrix form

$$\begin{bmatrix} k_{aa} & k_{ab} \\ k_{ba} & k_{bb} \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = \begin{bmatrix} q_a \\ q_b \end{bmatrix} \quad (42)$$

where u_a and q_a are associated with points 1 to 4 and u_b and q_b are associated with point 5.

Equation (42) can be written as two matrix equations,

$$[k_{aa}]\{u_a\} + [k_{ab}]\{u_b\} = \{q_a\} \quad (43)$$

$$[k_{ba}]\{u_a\} + [k_{bb}]\{u_b\} = \{q_b\} \quad (44)$$

Equation (44) can be solved for the displacements

u_b :

$$\{u_b\} = -[k_{bb}]^{-1}[k_{ba}]\{u_a\} + [k_{bb}]^{-1}\{q_b\} \quad (45)$$

If Eq. (45) is substituted in Eq. (43), an expression is found which relates the forces at points 1 to 4 to the unknown displacements at points 1 to 4 and the known thermal loads.

Fig. 4. Quadrilateral Element

$$\{\bar{q}_a\} = [\bar{k}_{aa}] \{u_a\} \quad (46)$$

where the quadrilateral stiffness matrix is

$$[\bar{k}_{aa}] = [k_{aa}] - [k_{ab}] [k_{bb}]^{-1} [k_{ba}] \quad (47)$$

and the modified load matrix is

$$\{\bar{q}_a\} = \{q_a\} - [k_{ab}] [k_{bb}]^{-1} \{q_b\} \quad (48)$$

The use of the quadrilateral as a separate element is desirable since the resulting set of equilibrium equations has fewer unknowns for a given number of triangular elements.

III. METHOD OF THERMAL ANALYSIS

A. Basic Equations

Development of the matrix equations governing finite element transient heat conduction may proceed in several ways as typified in Refs. 12, 14, or 18. Since the potential energy concept was employed for elastic stress analysis in Section II, our approach here will use "temperature potentials." Adopting this approach we begin, as before, by approximating the temperature field $\phi(r, z, t)$ over the finite element shown in Fig. 2 with functions of the form

$$\phi_m(r, z, t) = [A(r, z)]_m \{\phi(t)\}_m, \quad (49)$$

where $[A]_m$ is the shape function matrix (for the triangle, $[A]$ is given by 19b and 20b) and $\{\phi(t)\}_m$ are the nodal point temperature values that are to be determined. The conductivity potential V_k and heat capacity potential V_c can be written¹⁸ in matrix notation as

$$V_k = \frac{1}{2} \{\phi\}^T [K] \{\phi\} - \{\phi\}^T \{Q\} \quad (50)$$

$$V_c = \frac{1}{2} \{\dot{\phi}\}^T [C] \{\dot{\phi}\}, \quad (51)$$

where a $\{\dot{\phi}\}$ represents the time derivative of $\{\phi\}$. In Eqs. (50) and (51), $[K]$ and $[C]$ are the conductivity and heat capacity matrices for the assembled finite element system and $\{Q\}$ is the thermal flux (load) vector for the assembled system:

$$[K] = \sum_{m=1}^M [k]_m, \quad [C] = \sum_{m=1}^M [c]_m, \quad \{Q\} = \sum_{m=1}^M \{q\}_m. \quad (52)$$

The individual element conductivity and heat capacity matrices are

$$[k]_m = \frac{1}{2} \int_{VOL} [a]_m^T [\eta]_m [a]_m dV, \quad (53)$$

$$[c]_m = \frac{1}{2} \int_{VOL} \rho_m c_m [A]_m^T [A]_m dV, \quad (54)$$

where $[\eta]_m$ is the element thermal conductivity matrix

$$[\eta]_m = \begin{bmatrix} \eta_{rr} & \eta_{rz} \\ \eta_{rz} & \eta_{zz} \end{bmatrix}, \quad (55)$$

$\rho_m c_m$ is the product of the element density and the specific heat capacity, and $[a]_m$ is the matrix of gradients of the shape functions,

$$[a]_m = \begin{bmatrix} \frac{\partial [A]_m}{\partial r} \\ \frac{\partial [A]_m}{\partial z} \end{bmatrix}. \quad (56)$$

Finally, the element thermal flux vector $\{q\}_m$ is given by

$$\{q\}_m = - \int_{VOL} \rho_m [A]_m^T Q^* dV - \int_{AREA} [A]_m^T q^* ds, \quad (57)$$

where Q^* is the internal heat generation per unit of mass and q^* is the surface flux intensity (note that the second integral occurs only when the element possesses a side on the boundary).

The stationary conditions of the variational approach to finite element transient heat conduction then require that

$$\frac{\partial V_k}{\partial \phi_i} + \frac{\partial V_c}{\partial \dot{\phi}_i} = 0, \quad i = 1, 2, \dots, N, \quad (58)$$

for the N nodal point temperatures. Invoking these conditions on the potentials defined in Eqs. (50) and (51) then leads to the matrix differential equation for transient heat condition

$$[C] \{\dot{\phi}\} + [K] \{\phi\} = \{Q\} . \quad (59)$$

B. Triangular Element and Quadrilateral Element

The temperature distribution within the element is given, in terms of the nodal point values, by Eq. (49) where the shape function matrix $[A(r, z)]$ for triangular finite elements has been defined previously by Eqs. (19b) and (20b). From Eq. (19b) it is easily seen that the gradient matrix $[a]_m$ is constant in each element and is of the form

$$[a]_m = \frac{1}{\lambda} \begin{bmatrix} (z_j - z_k) & (z_k - z_i) & (z_i - z_j) \\ (r_k - r_j) & (r_i - r_k) & (r_j - r_i) \end{bmatrix} \quad (60)$$

where (r_i, z_i) , (r_j, z_j) , (r_k, z_k) are the nodal coordinates of the triangle and $\lambda = r_j(z_k - z_i) + r_i(z_j - z_k) + r_k(z_i - z_j)$ as before.

Observing that the gradient matrix $[a]_m$ does not depend on r or z and using the representation (20b) for the shape function matrix, we can write down expressions for the element conductivity and heat capacity matrices as well as the thermal flux vector:

$$[k]_m = [a]_m^T [\eta]_m [a]_m \cdot \text{Vol} \quad (61)$$

$$[c]_m = \rho_m c_m [D]^T \left[\int_{\text{VOL}} \{e\} \{e\}^T dV \right] [D] \quad (62)$$

$$\begin{aligned} \{q\}_m &= -\rho_m [D]^T \int_{\text{VOL}} \{e\} Q^* dV \\ &\quad - [D]^T \int_{\text{AREA}} \{e\} q^* dS . \end{aligned} \quad (63)$$

For the quadrilateral element composed of four triangular elements shown in Fig. 4, nodal point reduction is carried out as discussed in Section II. D and an equivalent 4 by 4 element conductivity matrix $[k]_m$ and thermal load vector $\{q\}_m$ are derived. For the heat capacity matrix, $[c]_m$ the capacitance of the fifth interior node is equally distributed among the quadrilateral corner nodes.

The element thermal conductivity matrix $[\eta]$ for orthotropic materials is expressed along principal N-S coordinates. In terms of r, z coordinates, the element conductivity matrix is

$$[\eta]_{rz} = [\hat{t}]^T [\eta]_{NS} [\hat{t}] , \quad (64)$$

where the transformation matrix $[\hat{t}]$ is

$$[\hat{t}] = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \quad (65)$$

where β is again the inclination of the N-S coordinate system to the $r-z$ coordinate system.

C. Boundary and Initial Conditions

A well-posed problem of transient heat conduction requires the specification of initial values of the temperature field ϕ_i , $i = 1, 2, \dots, N$. This is done by setting the values at $t = 0$.

In addition to initial conditions, boundary conditions must be specified. For temperature boundary conditions subroutine MODIFY is used as discussed in Section II. B. For flux boundary conditions, the appropriate thermal "load" vector is given by second integral in Eq. (63). Finally, for many practical situations the boundary condition

$$\eta_r \frac{\partial \phi}{\partial r} + \eta_z \frac{\partial \phi}{\partial z} + \gamma (\phi - \phi_a) = 0, \quad (66)$$

typifying Newton cooling into ambient air at temperature ϕ_a with surface heat transfer coefficient γ , is useful. For this case to the element conductivity matrix $[k]_m$ and the flux vector $\{q\}_m$ must be added the terms

$$\gamma \int_{\text{AREA}} [A]_m^T [A]_m dS$$

and

$$-\gamma \varphi_a \int_{\text{AREA}} [A]_m^T dS, \quad (67)$$

respectively. Of course, these contributions are taken only if the element possesses a side along the boundary where Eq. (66) applies.

D. Transient Solution by the Crank-Nicolson Method

In order to solve the transient equation

$$[K]\{\dot{\varphi}\} + [C]\{\dot{\varphi}\} - [Q] = \{0\}, \quad (68)$$

with specified initial condition $\{\varphi\} = \{\varphi\}_0$ at $t = 0$, we proceed by discretizing time pointwise, i.e., we consider a sequence of points in time $t_0 = 0, t_1, t_2, \dots, t_n, \dots$. At a time t_n , say, we can write a midinterval approximation for the vector $\{\dot{\varphi}\}$,

$$\{\dot{\varphi}\}_{n+\frac{1}{2}} = \frac{\{\varphi\}_{n+1} - \{\varphi\}_n}{\Delta t}$$

where $\Delta t = t_{n+1} - t_n$. If we evaluate the remaining terms in Eq. (68) at the midinterval (for instance, $\{\varphi\}_{n+\frac{1}{2}} = \frac{1}{2}(\{\varphi\}_n + \{\varphi\}_{n+1})$, etc.) we obtain the equation

$$\begin{aligned} \frac{1}{2}[K](\{\varphi\}_{n+1} + \{\varphi\}_n) + \frac{[C]}{\Delta t}(\{\varphi\}_{n+1} - \{\varphi\}_n) \\ - \frac{1}{2}([Q]_n + [Q]_{n+1}) = \{0\}. \end{aligned} \quad (69)$$

We now make the further assumption that the conductivity and heat capacity matrices $[K]$ and $[C]$ and the flux vector $[Q]$ do not vary greatly with φ so that they can be regarded as constant with respect to the change in φ occurring in the time interval Δt . Equation (69) can then be rearranged, putting the components of $\{\varphi\}_{n+1}$ on one side, to give

$$\begin{aligned} ([K] + \frac{2}{\Delta t}[C])\{\varphi\}_{n+1} = -([K] - \frac{2}{\Delta t}[C])\{\varphi\}_n \\ + 2[Q]_n \end{aligned} \quad (70)$$

where the right-hand side is known since $\{\varphi\}_n$ is known at t_n .

Equation (70) provides the algorithm for advancing in time from the initial $\{\varphi\}_0$ in solving the transient Eq. (68). In contrast to explicit methods of solution, this algorithm is unconditionally stable for linear problems and corresponds to the Crank-Nicolson method of Ref. 19 for finite difference equations. This stability feature is very important, since it allows the use of arbitrary size time steps Δt .

In addition, the truncation error associated with the Crank-Nicolson procedure is of order $(\Delta t)^2$ contrasted with (Δt) for the usual explicit time differencing method.

IV. SPECIAL COMPUTER PROGRAM FEATURES

Certain special computer program features which are not implied by the report title are discussed in this section. The special features include finite element mesh generation, skew boundaries, material properties, and a transient thermal loading capability. In addition, it can be noted from Appendix A that a constant temperature can be specified for the body (a feature which can be utilized in cool-down problems), or a constant material temperature can be specified and free thermal strains can be input as an alternative to coefficients of thermal expansion.

A. Finite Element Mesh Generation

The mesh generator incorporated in TSAAS has been designed to simplify the meshing process for fairly complex bodies and to allow for quickly modifying an existing mesh. The boundaries of the region may be described in terms of straight or circular line segments, as well as general spline curves. All internal nodal point locations can be determined automatically by the mesher, and nodal points can be allowed to slide along the boundary curves, providing a more uniform mesh than obtainable with uniform nodal point spacing. All boundary conditions may be specified over rows of

nodal points. The mesh generation instructions and most boundary conditions are saved, and on subsequent cases, only changes to the previous information is input to define a different mesh.

This mesher is an elaboration of one written by Perry Stern of Lockheed Palo Alto Research Laboratory and originally adapted to SAAS at Sandia Laboratories.

The first step in the mesh generation process is to define and number all of the external and internal material boundaries of the two-dimensional region. These line segments may be defined as straight lines, circular, circular curves, or spline curves. Additional spline curves may be defined as being a constant distance from a previously defined reference spline. (Offsets from other curve types are handled in the nodal point definitions, see below.)

The second step in the generation process is to establish a mapping from a logical or I-J index space to the real r-z space. In this mapping, straight lines between I-J nodes are associated with the previously defined line segments. The nodes can be uniformly spaced along the curve, or they can be constrained to lie on the curve with their actual location selected by the mesher. In addition, a row of nodes may be offset by a constant distance from a given nonspline type reference curve, greatly simplifying the meshing of shell geometries.

The material type for each element is initialized to 1. Changes to other material numbers can be made for rectangular blocks of elements (in the I-J space) by giving the indices of diagonally opposed corner nodes on a material block card.

Triangular elements may be specified either in the interior or on the boundary. In the I-J space, a square or diagonal row of squares is split into one or two triangular elements, depending on whether it is an internal or a boundary element. With the currently used nodal point

numbering scheme, there is little advantage to triangular elements and their use is not recommended.

Given the above information, the mesher can generate the geometrical information needed by TSAAS. The final location of all unspecified interior nodes and constrained nodes are determined by an iterative process which approximates the construction of a conformal mapping from the I-J space to the r-z space, subject to the given constraints.

Figures 5-8 show an example mesh. Normally a drawing or sketch of the material boundaries of the region to be meshed, as shown in Fig. 6, is the starting point. An estimate of the required number of elements in the mesh is made, and a sketch of the I-J logical mesh is made as shown in Fig. 5. All the lines should be numbered except lines offset from linear, circular or circular curves. Thus, the external curve is to be a spline curve and a reference for spline curve 5; however, only one of the inner circular curves is numbered and the other is specified as offset from it on the nodal point definition card. Figures 7 and 8 show the card deck and resulting mesh. Other examples are given in the test problems in Appendix C.

To generate one dimensional nodal point and element numbers in accordance with the requirements of TSAAS, nodal points are assigned sequential numbers starting with $J = 1$ and going in the increasing I-direction, then numbering the $J = 2$ row, and so forth. This means that the bandwidth of the equations in TSAAS is determined by the width of the mesh in the I-direction. The range of J values is limited to 200 by the mesher, however, this can be readily increased by changing some array dimensions in the program. The more realistic limitation is 1600 total nodal points, the limit for TSAAS.

Because the mesher automatically fills in the interior of any specified boundary, it is not possible to create a multiply connected mesh. Interior voids must be included in the mesh and then assigned a

Fig. 5. I-J representation of the example, a thick-walled composite vessel.

Fig. 6. Material boundaries in r - z coordinates. Circles on outer boundary indicate spline knots.

material with properties approximating those of a void. Also, the filling-in is done by assuming the existence of nodes from the smallest value of I to the largest I, for each J that appears on a nodal point definition card (but not material block or other cards). This implies that indentations in the logical mesh in the J-direction must be handled like interior voids as mentioned above.

Mechanical and thermal boundary conditions can be assigned over rows or blocks of nodal points in the I-J index space. For proper operation, pressure shear, flux and convective conditions should be applied to rows of nodal points. Nodal point constraints and temperature conditions can be applied to blocks of nodal points as required. The mechanical conditions and initial thermal conditions are saved and can be reused in multiple case problems. The conditions following the TSAAS control card, that give time varying boundary conditions, are not saved and must be input for each case.

Concave indentations in a region can result in meshes with distinct nodal points occupying the same location or even negative area elements. Although this problem is aggravated by the use of the sliding boundary node option, it is basically caused by the iterative procedure used to locate unconstrained interior points. A nodal point is located at the average location of its four neighbors in the I and J-direction. In a concave region, if the node spacing is too great, this average location may fall outside the desired boundary. To fix this problem, either more nodes must be used in the region or else additional constraints must be applied to the interior nodes.

B. Skew Boundaries

Mixed constraints can be specified in a sloping direction, for example, as required by a sliding sloping boundary. To use this option the value of CODE, usually set to 0, 1, 2, or 3, should be set to a negative value, giving the slope in degrees. When this is done the constraint values are

interpreted as follows:

XR is the specified load in the 1-direction.

XZ is the specified displacement in the 2 direction.

Furthermore, the displacements printed by the program are given with respect to the rotated frame, as follows:

U_r is the displacement in the 1-direction.

U_z is the displacement in the 2-direction

The stresses and strains are calculated in the r and z-directions, as usual. (See Example 1.)

C. Orthotropic Material Properties

Material properties versus temperature are input in tabular form for each material. The properties for each element in the system are then evaluated by interpolation according to the temperature of the element. The mass density of a material is required. The coefficients of thermal expansion are required only for thermal stress analysis. Finally, the yield stresses are required only if nonlinear material properties are specified.

The orthotropic mechanical properties are specified in the principal N-S directions of the material in terms of the definitions displayed in the following stress-strain relation:

$$\begin{bmatrix} \epsilon_{NN} \\ \epsilon_{SS} \\ \epsilon_{\theta\theta} \\ \epsilon_{NS} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_N} & -\frac{\nu_{SN}}{E_S} & -\frac{\nu_{\theta N}}{E_\theta} & 0 \\ -\frac{\nu_{NS}}{E_N} & \frac{1}{E_S} & -\frac{\nu_{\theta S}}{E_\theta} & 0 \\ -\frac{\nu_{N\theta}}{E_N} & -\frac{\nu_{S\theta}}{E_S} & \frac{1}{E_\theta} & 0 \\ 0 & 0 & 0 & \frac{1}{G_{NS}} \end{bmatrix} \begin{bmatrix} \sigma_{NN} \\ \sigma_{SS} \\ \sigma_{\theta\theta} \\ \sigma_{NS} \end{bmatrix} + T \begin{bmatrix} \alpha_N \\ \alpha_S \\ \alpha_\theta \\ 0 \end{bmatrix} \quad (71)$$

The above matrix is symmetrical; therefore,

$$E_i \nu_{ji} = E_j \nu_{ji} \text{ see Ref. 20.}$$

The orthotropic thermal conductivity matrix is specified in the principal N-S directions of the material as

$$\begin{bmatrix} k_N & 0 \\ 0 & k_S \end{bmatrix} \quad (72)$$

The thermal conductivity matrix and the material heat capacity are also input in tabular form as a function of temperature.

D. Transient Thermal Loading Capability

A practical problem of thermal stress analysis is the situation where a body is subjected to a time-dependent temperature or Newton cooling (see III. C) boundary condition. Practically, this situation is simulated in so-called environmental tests. One can model this situation using TSAAS by two different methods. In the first method, new boundary condition cards (flux, temperature, or Newton cooling) can be read in before each entry into the transient thermal calculation subroutine; these boundary conditions are then taken to be constant over the time interval until exit from the subroutine. In the second method, a set of boundary condition cards is read in, and $N (\leq 100)$ sets of multiplicative thermal factors and times are read in. During any time interval linear interpolation is used to determine the ambient air temperature (Newton cooling) or the surface temperature. This method provides for a continuous variation in temperature boundary conditions. The method is shown schematically in Fig. 9.

E. Plane Strain and Plane Stress Options

TSAAS has been modified to carry out the thermal stress analysis of plane two dimensional solids. The changes to TSAAS include modification of the incremental volume dV and incremental area dS occurring in such equations as (1)-(3), (9), (12) etc., in Section II as well as (53), (54), and (57) of Section III and changes in the elastic constitutive matrix Eq. (71) discussed in Section IV. C. For the plane stress case the matrix corresponding to Eq. (71) is

$$\begin{bmatrix} \epsilon_{NN} \\ \epsilon_{SS} \\ \epsilon_{NS} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_N} & -\frac{\nu_{SN}}{E_S} & 0 \\ -\frac{\nu_{NS}}{E_N} & \frac{1}{E_S} & 0 \\ 0 & 0 & \frac{1}{G_{NS}} \end{bmatrix} \begin{bmatrix} \sigma_{NN} \\ \sigma_{SS} \\ \sigma_{NS} \end{bmatrix} \quad (73)$$

while for the plane strain case TSAAS uses the constitutive matrix described by Eq. (71) and sets the hoop strain to zero. In both cases the conductivity matrix Eq. (72) is unchanged. The remaining modifications to TSAAS consist of changes to heading formats. All of these changes are accomplished by an update deck.

Fig. 9. Specification of continuously varying ambient air temperature time - thermal load factors (t_i, f_i) , $i = 1, 2, \dots, N$ and initial temperature T_0 .

V. NONLINEAR MATERIAL PROPERTIES

In this section, an attempt is made to define the behavior of orthotropic and isotropic materials which have a bilinear effective stress-effective strain curve. This work should not be construed as being anything more than a rough approximation since practical orthotropic materials may or may not be representable by the effective stress function used here. Accordingly, this theory should be applied with considerable reservation when orthotropic materials are treated. However, for isotropic materials, the effective stress function is well founded (it is related to the von Mises yield criterion), and can be used with confidence. This section concludes with a discussion of creep analysis using the initial strain method.

A method of successive approximation is used to solve for displacements, stresses, and strains in bodies with nonlinear material properties. Basically, this involves the repeated solution of the following equation for the displacements of the system:

$$\{Q\}_{i-1} = [K]_{i-1} \{u\}_i$$

where

- $[K]_{i-1}$ = an estimate of the effective stiffness of the system based on the previous solution ($i - 1$)
- $\{Q\}_{i-1}$ = an estimate of the loads acting on the system (since the thermal loads are a function of the stiffness)
- $\{u\}_i$ = the displacements of the system for the i th approximation.

The load and stiffness used in the first approximation ($i = 1$) are based on the initial linear material properties. Since deformations are assumed to be small, the development of the effective stiffness depends only on the estimation of an effective stress-strain relationship for each element in the system.

It is apparent that this approach has certain disadvantages. First, the procedure is not guaranteed to converge. However, experience has indicated that, for systems where the nonlinear effects

are small compared to the initial linear analysis, the procedure does converge. Second, to obtain a unique solution, the method is restricted to elastic materials, in other words, materials with single-valued stress-strain relationships. Incremental plasticity is not approximated by this method and the plasticity capability of TSAAS corresponds to the so-called deformation theory of plasticity. Thus, proportional loading problems are done correctly, however, unloading is not handled at all; see Ref. 21.

A. Plastic Materials

The von Mises yield condition for isotropic material is given by

$$\sigma_y = \frac{1}{\sqrt{2}} \sqrt{(\bar{\sigma}_1 - \bar{\sigma}_2)^2 + (\bar{\sigma}_1 - \bar{\sigma}_3)^2 + (\bar{\sigma}_2 - \bar{\sigma}_3)^2} \quad (74)$$

where $\bar{\sigma}_1$, $\bar{\sigma}_2$, and $\bar{\sigma}_3$ are the principal stresses, and σ_y is the uniaxial yield stress. Equation (74) can be rewritten in normalized form:

$$1 = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{\bar{\sigma}_1}{\sigma_y} - \frac{\bar{\sigma}_2}{\sigma_y}\right)^2 + \left(\frac{\bar{\sigma}_1}{\sigma_y} - \frac{\bar{\sigma}_3}{\sigma_y}\right)^2 + \left(\frac{\bar{\sigma}_2}{\sigma_y} - \frac{\bar{\sigma}_3}{\sigma_y}\right)^2} \quad (75)$$

For orthotropic materials, Eq. (75) is modified without justification to

$$1 = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{\bar{\sigma}_1}{\sigma_{1y}} - \frac{\bar{\sigma}_2}{\sigma_{2y}}\right)^2 + \left(\frac{\bar{\sigma}_1}{\sigma_{1y}} - \frac{\bar{\sigma}_3}{\sigma_{3y}}\right)^2 + \left(\frac{\bar{\sigma}_2}{\sigma_{2y}} - \frac{\bar{\sigma}_3}{\sigma_{3y}}\right)^2} \quad (76)$$

where σ_{1y} , σ_{2y} , and σ_{3y} are the yield stresses in the principal directions.

Now, for all values of stress, the "normalized effective stress" for orthotropic materials is defined as

$$\sigma_e = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{\sigma_1}{\sigma_{1y}} - \frac{\sigma_2}{\sigma_{2y}}\right)^2 + \left(\frac{\sigma_1}{\sigma_{1y}} - \frac{\sigma_3}{\sigma_{3y}}\right)^2 + \left(\frac{\sigma_2}{\sigma_{2y}} - \frac{\sigma_3}{\sigma_{3y}}\right)^2} \quad (77)$$

The normalized effective stress-strain relationship is shown in Fig. 10. An approximate consideration of strain hardening can be included in

the formulation as shown in the figure. The solution procedure for the nonlinear plasticity is as follows:

1. After each approximate solution, the normalized effective stress is calculated for each element (linear properties are used in the first approximation).

2. The ratio of nonlinear properties to linear properties for each element for a given approximation i is defined by

$$R_i = \frac{\sigma_i}{e_i} \quad (78)$$

Therefore, the corresponding normalized effective strain is

$$e_i = \frac{\sigma_i}{R_i} \quad (79)$$

(R_i is set equal to 1 for the first approximation.)

3. If the strains are assumed not to change, an estimation of the plasticity ratio is

$$R_{i+1} = \frac{1 + n(e_i - 1)}{e_i} \quad (80)$$

4. In the next approximation, the linear properties for E_N , E_θ , E_S , and G_{NS} are multiplied by the ratio in Eq. (80) before they are used. The Poisson's ratios are changed according to the following relation:

$$\nu_{\alpha\beta_{i+1}} = 1/2 - (1/2 - \nu_{\alpha\beta_i}) * R_{i+1} \quad (81)$$

Fig. 10. Effective stress-strain relationship.

The procedure must be repeated until convergence is obtained. The displacements, stresses, and strains are not printed by the program after each approximation. The number of approximations required will depend on the specific problem and must be specified as computer input. Hence, a certain amount of experience is required in using the nonlinear option. An arbitrary number of approximations can be requested, but the program will stop when R_{i+1} for all elements changes by 0.5 percent or less.

B. Hardening or Locking Materials

In TSAAS, nonlinear hardening or locking material behavior can be simulated by using a plasticity ratio n , greater than one. The material properties E_N , E_S , E_θ , and G_{NS} are multiplied at each iteration by the new plasticity ratios whereas the Poisson's ratios are unchanged [i.e., Eq. (81) is suppressed].

C. Creep Behavior

TSAAS provides for an initial strain distribution, which can be caused, for example, by differential thermal expansion. Thus, TSAAS is easily adapted to the determination of stresses and deformation in materials exhibiting creep behavior where the creep strain-rates are determined from the stress, temperature, and time. The creep version of TSAAS, called SANGRE uses a nonlinear creep law of the form

$$\dot{\epsilon}_e = A e^{-Q/R\phi} N_{G_e} \quad (82)$$

where σ_e is the effective stress [given by Eq. (77), for example], ϕ is the temperature, and A , Q/R and N are material constants. The components of the element creep strain-rates for axisymmetric creep are then calculated by means of

$$\dot{\epsilon}_r^c = \frac{\dot{\epsilon}_e}{\sigma_e} \left[\sigma_r - \frac{1}{2} (\sigma_\theta + \sigma_z) \right]$$

$$\dot{\epsilon}_\theta^c = \frac{\dot{\epsilon}_e}{\sigma_e} \left[\sigma_\theta - \frac{1}{2} (\sigma_r + \sigma_z) \right]$$

$$\dot{\epsilon}_z^c = \frac{\dot{\sigma}_e}{\sigma_e} [\sigma_z - \nu_c (\sigma_r + \sigma_\theta)] \quad (83)$$

$$\dot{\epsilon}_{rz}^c = \frac{3\dot{\epsilon}_e}{\sigma_e} \sigma_{rz}$$

where ν_c is the effective Poisson's ratio in creep.

The creep version of TSAAS (called SANGRE) combines the stress and thermal analysis programs of TSAAS with the element creep Eqs. (82) and (83) into one program. SANGRE uses a nearly incompressible formulation of the element stiffness and an implicit time stepping method to advance the creep solution. Also, large rotations can be accounted for in the nonlinear creep analysis. Finally, SANGRE adjusts the time intervals so that both temperatures and stresses vary only within prescribed limits over the given time interval, otherwise a halving or doubling of the time interval takes place. Long increments in time indicate the steady creep is occurring. Details are given in LA-7888-MS, "A Finite Element Method for Studying the Transient Nonlinear Thermal Creep of Geological Structures."

D. No-Tension Materials

This feature is intended to model an unbonded interface between two materials. A thin layer of material on one side of the interface is set with a no-tension flag. If the normal stress on the interface becomes positive (tensile), then the Young's moduli for that element are reduced by a factor, simulating the release of the interface.

The feature is enabled by setting the plastic ratio PEMR to a large negative number. When the tension condition is detected, the moduli are divided by the absolute value of PEMR and the Poisson's ratios are unchanged. The tensile stress is checked in the β direction (that is, on σ_{NN}). Note that for those elements for which this model applies, the angle β must be carefully set by the normal to the interface.

APPENDIX A COMPUTER PROGRAM INPUT INSTRUCTIONS

The first step in the analysis of an axisymmetric solid is to select a finite element representation of the two-dimensional cross section of the solid. The definition of positive directions of the r-z coordinate system is shown in Fig. 1 in the main text. The I-J coordinate system used by the mesher must have the same orientation as the r-z system, in order to generate elements with positive areas. The cards described below define the problem.

If stress calculation alone is desired (i.e., a 1 in Column 3 of the problem control card) then the setup deck is described in Appendix A I. For transient thermal analysis (with or without accompanying stress analysis) the problem setup deck must also include the cards described in Appendix A II. After each data case, a card with END OF CASE beginning in Column 1 must appear. After the last data case, a card with END OF DATA must appear (after END OF CASE).

Plotting control cards are described in Appendix A III.

I. PROBLEM DECK - SAAS

NUMBER OF CASES CARD

Format (I5)

Cols. 1-5 The number of cases to be run.

TITLE CARD

Format (8A10)

Cols. 1-80 Case Title

CONTROL CARD

Format (20I1, 6I5, 3F10.0)

Cols. 1-15 Output option switches. A 1 in the corresponding column enables the option, a 0 suppresses the option.

Card Column	Parameter	
1	ISTART	Start after mesh description using previous mesh.
2	IMESH	Generate mesh.
3	ISAAS	Stress calculation only.
4	NUMTC	Material temperature initialization. Set to 0 to set all temperatures to TREF, set to 1 to initialize by material, see p. 26.
5-6	5-6	Not used.
7	IPT2	Write calculation results on Tape 11 — for material boundary plots, contour plots, or edited stress and strain listings produced by STRAPP or SAASPL.
8	IPT3	Suppress printing of mesh generation information.
9	IPT4	Suppress printing of nodal point listing.
10	IPT5	Suppress printing of element listing.
11	IPT6	Suppress printing of displacement.
12	IPT7	Suppress printing of stresses.
13	IPT8	Suppress printing of strains.
14	IPT9	Print (nonsuppressed items) after every iteration, and dump results on Tape 11 every iteration if enabled.

15	IPT10	Suppress printing of temperatures in thermal option.	71-80	Angular velocity in rad/s.
16-20		Not used.	<p>*It is not necessary to specify this value if the mesh is generated. But it must be specified if a generated mesh is being reused in the multiple case option.</p>	
21-25	NUMNP	Number of nodal points. (1600 maximum)	<p><u>PLANE PROBLEM CONTROL CARD</u></p>	
26-30	NUMEL	Number of elements. (1600 maximum)	<p>Format (15)</p>	
31-35	NUMMAT	Number of different materials. (20 maximum). When doing multiple cases this is the number of materials to be added or changed, use zero if the same material definitions are to be used.	Cols. 1-5	NPP
			0	Axisymmetric
			-1	Plane Strain
			+1	Plane Stress
			<p><u>MESH GENERATION CONTROL CARD</u></p>	
			<p>If the mesh is not generated (1MESH = 0) skip this card and go to the Element Cards. If 1START = 1 then go to the material property cards.</p>	
			<p>Format (1615)</p>	
36-40	NUMPC	Number of boundary pressure cards. (200 maximum). If negative, then the first IABS (NUMPC) number of previous pressure conditions is used.	Cols. 1-5	NSFR Should be set to 1 for compatibility with other codes, however, TSAAS does not use this parameter.
			6-10	MESHOP Restart Option.
			=0	Start generating new mesh.
			=1	*Add following cards to previous mesh.
41-45	NUMSC	Number of boundary shear cards. (200 maximum). If negative, then the first IABS (NUMSC) number of previous shear conditions is used.	11-15	NLS Number of line segments or curves (≤ 200).
			16-20	NNPD Number of nodal point definition cards.
46-50	NNLA	Number of nonlinear approximations ≥ 1 .	21-25	NMB Number of material block cards.
51-60	TREF	Reference temperature.	26-30	NTD Number of triangle definition cards.
61-70	ACELZ	Axial acceleration. If the density is in kilograms per cubic metre, pressure in pascals, and nodal coordinates are in metres, then ACELZ should be in g's times 9.80 665.	31-35	NNPC Number of nodal point constraint cards.
			36-40	NPB Number of pressure boundary conditions.

41-45	NSB	Number of shear boundary conditions.
46-50	NQB	Number of flux condition card (initial values).
51-55	NCB	Number of convective condition cards (initial values).
56-60	NTB	Number of temperature boundary condition cards (initial values).
76-80	BETA	Mesh generator algorithm parameter.

*Note: On multiple case problems when MESHOP = 1, the numbers in Cols. 11-60 refer to the number of additional line segments or description cards to be added to the previous mesh definition. In all cases, contradictory definitions result in the last definition being used. The main limitation is that the logical I-J mesh size cannot be reduced and triangles can not be deleted from a mesh.

LINE SEGMENT CARDS

These cards define the various basic curves required to define the boundaries and fixed nodal point locations required in a mesh. If an object has constant thickness shells in it, then only one shell surface need be defined in detail; the other may be specified as a constant thickness away from the basic surface.

Format (2I5, 6E10.0)

Cols. 1-5	IN	Curve index number.
6-10	IT	Type of space curve (see definition below). Make negative to indicate values are R- θ (in degrees) polar coordinates, but do not use with special curves.
11-20	R1	R-coordinate of first point (if both R1 and Z1 are zero, then end point of previous line segment is used).

21-30	Z1	Z- or θ -coordinate (If both R1 and Z1 are zero, then end point of previous line segment is used.)
31-40	R2	R-coordinate of second point.
41-50	Z2	Z- or θ -coordinate.
51-60	R3	R-coordinate of third point.
61-70	Z3	Z- or θ -coordinate.

The curve type number in Cols. 6-10 can take the following values:

- IT = 1 straight line, input first and second points only.
- IT = 2 circle specified by first and second points at the ends of the arc and the third point at some intermediate point (e. g., the mid-point) of the arc.
- IT = 3 circle specified by first and second points at the ends of the arc, and the third point at the center of the circle. The arc runs counter clockwise from 1 to 2.
- IT = 4 as in 3, except arc runs clockwise from 1 to 2.
- IT = 5 circle specified by radius (R1), angle in degrees from R-axis about center to first end (Z1), angle to second end (R2), and center coordinate Z2 and R3. Arc is counter clockwise from 1 to 2. Do not use negative type number.
- IT = 6 as in 5, except that arc runs clockwise from 1 to 2.
- IT = 7 Not used.
- IT = 8
- IT = 9 a spline curve parallel to a previously defined spline curve. For this particular case the line segment card format is:

Cols. 1-5	IN	Curve index number.
6-10	IT	9
11-20	R1	Index of reference spline curve.

21-30 Z1 Normal offset from reference spline curve. Positive if to right of reference, negative if to left, as seen looking from first point to last point.

IT = 10 piecewise circilinear curve. That is, a curve composed of a sequence of straight lines and circular arcs. On this card R1 (Cols. 11-20) designates the number of lines and arcs in the curve. The lines and arcs are defined on R1 number of following continuation cards. The continuation cards follow the same format as used for the line segment types 1-6 with two exceptions. First, the curve index number (IN) is ignored, and secondly the coordinates of the first point (R1 and Z1) are not used except on the first continuation card. Thus, the first point of every straight line or circular arc (except for the first one) is automatically defined as the second point of the previous line or arc.

IT = 11 spline curve — in this case R1 contains the number of points (NUMSG) defining the curve. The points are given on succeeding continuation cards. A continuation card for spline definition contains either one or two points as follows:

Format (I5, 5X, 6E10.0)

Cols. 1-5	Blank or zero.
6-10	Not used.
11-20 R1	R-coordinate of first point.
21-30 Z1	Z-coordinate of first point.
31-40 R2	Slope of curve (in degrees) at this point. A value is required at the first and last point. For intermediate points, a zero value (or blank) can be given and a cubic spline curve will be generated.

41-50 Z2	Same as column 11-40
51-60 R3	except define next point.
61-70 Z3	If <u>all</u> values are zero, then these fields are ignored.

Note that NUMSG specifies the number of points to be read not the number of cards on which points are defined.

NODAL POINT DEFINITION CARDS

These associate lines of nodal points, in the logical I-J space, with actual R-Z locations as specified by the preceding space curve cards. The line of nodal points may be specified as being uniformly spaced along the curve (this is only approximate for the spline curves) or can simply be constrained to have all nodal points lie along the curve with the first and last nodal points located at the ends of the curve. At least enough cards must be used to define the outer boundary of the region. Interior nodal points are normally constrained only along material boundaries.

Format (4E5.0, 2I5, 2E10.0)

Cols. 1-5	I1	I-coordinate of first nodal point.	} If both are zero, then I, J, coordinate of last point (Col. 11-20) from previous card are used. Initialized to (1, 1).
6-10	J1	J-coordinate.	
11-15	I2	I-coordinate of last nodal point.	
16-20	J2	J-coordinate.	
21-25	IM	Space curve index number. If not positive, then fixes one node (I1, J1).	
26-30	IC	Constraint type number (see below).	

- 31-40 T Normal offset from reference space curve. Positive if to right of reference curve as seen looking from first point to last point. If $IM = 0$ use as R-coordinate of $I1$, $J1$.
- 41-50 If $IM = 0$ use as Z-coordinate of $I1$, J .

Constraint type number—the number in Cols. 26-30 can be:

- IC = 1 then the nodal points are uniformly spaced along the curve.
- IC = 2 the nodal points are constrained to lie along the curve.

MATERIAL BLOCK ASSIGNMENT CARDS

Each card assigns a material definition number to a block of elements defined by the I-J coordinates. The default assignment is to material 1 with $\beta = 0$. The angle β , defined by Fig. 3, if specified as -1, is set to the average I-direction through the element. (See Fig. A-1.)

Format (5I5, 5X, F100)

- Cols. 1-5 Material definition number.
- 6-10 Minimum I.
- 11-15 Maximum I.
- 16-20 Minimum J.
- 21-25 Maximum J.
- 26-30 Reserved.
- 31-40 Material property angle β , in degrees $0 \leq \beta \leq 360$. (If $\beta = -1$ then mesher sets β to the average I-direction in element.)

Fig. A-1. Automatic definition of β .

TRIANGLE DEFINITION CARDS

A diagonal line in the I-J space is specified by giving a nodal point and a count of elements to be split. This creates one triangular element for each partly defined boundary quadrilateral or two triangular elements for a fully defined quadrilateral. (See examples.)

Format (3I5)

- Cols. 1-5 I-coordinate of nodal point.
- 6-10 J-coordinate of nodal point.
- 11-15 Count of number of logical elements to be split — positive if line goes upward and rightward, negative if upward and leftward.

Example A: Circles show previously defined nodal points.

Card image: _ _ _ 1 _ _ _ 1 _ _ _ 1

Example B:

Card image: _ _ _ 4 _ _ _ 1 _ _ _ -3

NODAL POINT CONSTRAINT CARDS

Format (4I5, 3F10.0)

Cols.	1-5	I1	I coordinate of first point.
	6-10	I2	I coordinate of last point (if zero set equal to I1).
	11-15	J1	J coordinate of first point.
	16-20	J2	J coordinate of last point (if zero set equal to J1).
	21-30	CODE	Boundary condition code (see below).
	31-40	XR	Radial boundary condition.
	41-50	XZ	Axial boundary condition.

The boundary condition codes are

CODE = 0	{	XR is the specified R-load and XZ is the specified Z-load.
CODE = 1	{	XR is the specified R-displacement and XZ is the specified Z-load.
CODE = 2	{	XR is the specified R-load and XZ is the specified Z-displacement.
CODE = 3	{	XR is the specified R displacement and XZ is the specified Z displacement.

All loads are total forces acting on a one radian segment. If CODE is other than 0, 1, 2, or 3, it is taken as the angle of a sloping boundary restraint, as discussed in Sec. IV.B. A negative number gives the angle in degrees of the slope measured clockwise from the horizontal. XR specifies tangential force and XZ specifies normal displacement.

PRESSURE, SHEAR, FLUX, CONVECTIVE, AND TEMPERATURE BOUNDARY CONDITIONS

Format (4E5.0, E10.0)

Cols.	1-5	I1	First I.
	6-10	I2	Last I (if zero, assumed equal to I1).
	11-15	J1	First J.
	16-20	J2	Last J (if zero, assumed equal to J1).
	21-30		Pressure, shear, flux or temperature value. Not used for convective condition.

Note: Pressure, shear, and flux conditions are assumed to apply to the element(s) to the left of the nodal points, when going from the first nodal point to the last nodal point.

ELEMENT CARDS

These cards are not included if a mesh is generated.

One card is required for each element in numerical sequence.

Format (6I5, F10.0)

Cols.	1-5	Element number.	1. Order nodal points counter clockwise (see Fig. 1 of main text) around element.
	6-10	Nodal point I.	2. Maximum difference between nodal point numbers must be less than 40.
	11-15	Nodal point J.	
	16-20	Nodal point K.	
	21-25	Nodal point L.	
	26-30	Material identification.	
	31-40	Material angle β - in degrees.	

If element cards are omitted, the program generates the omitted elements by incrementing by 1 the preceding I, J, K, and L. The material identification code for the generated elements is the value specified on the first nongenerated card. The last element card must be supplied.

Triangular elements are specified by repeating the last nodal point number, i.e., I, J, K, K.

NODAL POINT CARDS

One card is required for each nodal point in numerical sequence. These cards are not included if a mesh is generated.

Format (I5, F5.0, 5F10.0)

Cols.	1-5	Nodal point number.
	6-10	Boundary condition code.
	11-20	R-coordinate.
	21-30	Z-coordinate.
	31-40	XR.
	41-50	XZ.
	51-60	Temperature (if not given on temperature cards or on tape).

If the number in Cols. 6-10 is

0. $\left\{ \begin{array}{l} \text{XR is the specified R-load and} \\ \text{XZ is the specified Z-load.} \end{array} \right.$
1. $\left\{ \begin{array}{l} \text{XR is the specified R-displacement and} \\ \text{XZ is the specified Z-load.} \end{array} \right.$
2. $\left\{ \begin{array}{l} \text{XR is the specified R-load and} \\ \text{XZ is the specified Z-displacement.} \end{array} \right.$
3. $\left\{ \begin{array}{l} \text{XR is the specified R-displacement and} \\ \text{XZ is the specified Z-displacement.} \end{array} \right.$

All loads are total forces acting on a one-radian segment.

If nodal point cards are omitted, nodal points will be generated at equal intervals along a straight line between the defined nodal points. The boundary condition code is then 0, along with XR and XZ. The temperatures are linearly interpolated from the defined temperatures.

TEMPERATURE INITIALIZATION BY MATERIAL

If material temperature initialization is specified (i. e., a 1 in Col. 4 of the problem control card) then a constant value of temperature is given on a single card and all nodes are initialized to this temperature. In addition, NMAT material blocks can be initialized to different temperatures.

Format (F10.0, I10)

First card

Cols. 1-10 Initial temperature T_0

11-20 Number of material blocks, NMAT, initialized differently from T_0 .

Next NMAT cards

Cols. 1-10 Temperature.

11-20 Material number.

Material Property Information

The following group of cards must be specified for each of a maximum of twenty materials.

MATERIAL IDENTIFICATION CARD

Format (2I5, 2F10.0)

Cols. 1-5 Material identification number.

6-10 Number of temperatures for which properties are given (six maximum).

11-20 Mass density of material (if required).

21-30 Thermal expansion parameter (if 1, free thermal expansions are on the material property cards. Otherwise, coefficients of thermal expansion are on the material property cards.)

MECHANICAL PROPERTY CARDS*

Two cards are required for each temperature.

Cards are to be in order of ascending temperature.

First card

Format (8F10.0)

Cols. 1-10 Temperature

11-20 Modulus of elasticity, E_N .

21-30 Modulus of elasticity, E_S^{**} .

31-40 Modulus of elasticity, E_θ^{**} .

41-50 Poisson's ratio ν_{SN} .

51-60 Poisson's ratio, $\nu_{\theta N}^{**}$.

61-70 Poisson's ratio, $\nu_{\theta S}^{**}$.

71-80 Shear modulus, G_{NS}^{**} .

(For an isotropic elastic material $E_N = E_S = E_\theta = E$,

$$\nu_{SN} = \nu_{\theta N} = \nu_{\theta S} = \nu \text{ and } G$$

$$= \frac{E}{2(1 + \nu)} .$$

Second card

Format (7F10.0)

Cols. 1-10 Free thermal strain, $\alpha_N T$, or

Coefficient of thermal expansion,

α_N

11-20 Free thermal strain, $\alpha_S T$, or

Coefficient of thermal expansion,

α_S^{**} .

- 21-30 Free thermal strain, a_T , or coefficient of thermal expansion, a_T^{**} .
- 31-40 Yield stress, σ_N , or yield strain, ϵ_N , if PEMR is greater than one.
- 41-50 Yield stress, σ_S , or yield strain, ϵ_S , if PEMR is greater than one**.
- 51-60 Yield stress, σ_T , or yield strain, ϵ_T , if PEMR is greater than one.
- 61-70 Ratio of effective plastic to elastic modulus, PEMR. This number is less than one for plastic materials and greater than one for locking materials.
Negative for non-tension materials

*In general, for an orthotropic material there are nine independent material constants, but due to symmetry conditions (axisymmetric body and loading) there are only seven independent constants — the input for first card described above. Note that the Poisson's ratios are defined by the following symmetry expression,

$$E_i \nu_{ji} = E_j \nu_{ij}.$$

If the material is transversely isotropic with the N- θ plane being the isotropic plane then

$$E_N = E_\theta, \nu_{N\theta} = \nu_{\theta N}, \nu_{S\theta} = \nu_{\theta S} = \nu_{NS} \frac{E_S}{E_\theta}$$

i.e., there are only five independent constants. Furthermore, [20],

$$\left| \nu_{SN} \right| < \left(\frac{E_S}{E_N} \right)$$

$$-1 < \nu_{N\theta} < 1 - 2 \left(\frac{E_N}{E_S} \right) \nu_{SN}^2.$$

**If left blank then the value is automatically set to the isotropic value.

THERMAL PROPERTY CARDS

One card required for each temperature.

Each card follows the two mechanical property cards. If a one appears in Col. 3 of the problem control card (i.e., stress calculation only) then the thermal property cards are omitted.

Format (6F10.0)

Cols. 1-10 Thermal conductivity in N-direction.

11-20 Thermal conductivity in S-direction**.

21-30 Heat capacity.

31-40 Internal heat generation rate per unit mass. If a chemically reactive material, this is QZ.

41-50 For chemically reactive materials, the activation energy divided by gas constant $\frac{E}{R}$.
Use zero for nonreacting materials.

51-60 Maximum temperature (TMAX) change before recomputing conductivity and heat capacity matrices.

BOUNDARY PRESSURE CARDS

Used only if conditions are not input via automatic mesher cards. One card is required for each boundary element which is subjected to a normal pressure.

Format (2I5, F10.0)

Cols. 1-5 Nodal point I.

6-10 Nodal point J.

11-20 Normal pressure.

As shown in the sketch below, the boundary element must be on the left when progressing from I to J to obtain the correct sense of the pressure. Surface normal tensile force is input as a negative pressure.

BOUNDARY SHEAR CARDS

Used only if conditions are not input via automatic mesher cards. One card is required for each boundary element which is subjected to surface shear.

Format (2I5, F10.0)

Cols. 1-5 Nodal point I.
6-10 Nodal point J.
11-20 Surface shear.

As shown in the sketch below, the boundary element must be on the left when progressing from I to J. The positive sense of the shear is from I to J.

II. PROBLEM DECK--TSAAS

TSAAS requires the control cards, mesh description cards, and material property cards described above, together with the additional cards described below. The entry subroutine is called TEM1. It is important for the user to recognize that thermal boundary conditions for TSAAS can be specified in two distinct ways. In the one method boundary condition cards specifying boundary temperatures, fluxes or convection coefficients are read in and applied before each entry into TEM1. In the other method a set of boundary condition cards consisting of times, multiplication temperature factors, and convection coefficients are read in on the TSAAS control card. These options are enabled by IPT13.Eq. 1 or IPT13.NE.1, respectively.

TSAAS CONTROL CARD

Format (5I1, 5X, 6F10.0)

Cols.			
1	IPT11	If 1 uses variable time step.	
2	IPT12	If 1 uses the thermal properties varying with temperature.	
3	IPT13	If 1 boundary conditions are specified by THERMAL LOAD CONTROL CARDS; if not equal to 1 then boundary conditions are specified by THERMAL HISTORY CARDS.	
4	IPT14	If 1 determines ambient air temperature by subroutine.	
5-10		Not used.	
11-20	DT	Starting value of time increment.	
21-30	DTMIN	Minimum value of time increment.	
31-40	TDMIN	Minimum nodal point temperatures change per calculational cycle.	
41-50	TDMAX	Maximum nodal point temperature change per calculational cycle.	
51-60	TDEXP	Ignition temperature change per calculational cycle.	

61-70 TZERO Zero Offset for
absolute temperature
for reactive kinetics.

INCREMENT CONTROL

Format (15) NINC This card controls the
number of times the
transient thermal
subroutine TEM1 is
entered. NINC can be
at most 100 and NINC-1
entries into subroutine
TEM1 take place.

OUTPUT CONTROL

Format (16I5) NOUT(I)

This card (or cards
sets up the NOUT (I), I
= 1, NINC array. Upon
exit from TEM1 at
increment I, a stress
analysis is carried out
using the temperature
field calculated in
TEM1 if NOUT (I + 1) =
1. Otherwise reentry
into TEM1 takes place.
NOUT(1) controls
whether a stress
analysis is carried out
for the initial
temperature field.

THERMAL HISTORY CARDS

There are NINC thermal history cards only if
IPT13 doesn't equal 1. If IPT13 = 1, go to the
Thermal Load Control Cards.

Format (I10, 4F10.0)

Cols.

1-10	I	Thermal load number
11-20	FAC(I,1)	Time
21-30	FAC(I,2)	Thermal load factor*
31-40	FAC(I,3)	Film coefficient
41-50	FAC(I,4)	Ambient temperature

*If equal to zero then the film coefficient and
ambient temperature govern the boundary condition.

THERMAL LOAD CONTROL CARDS

Format (4I5, 4F10.0) NINC cards occur here
only if IPT13 = 1 on the TSAAS control card.

Cols. 1-5	NUMQC	Number of flux boundary cards, negative if old flux boundary con- ditions are used.
6-10	NUMSCC	Number of convective boundary cards, nega- tive if old convective conditions are used.
11-15	NUMTB	Number of temperature boundary cards, negative if old temperature boundary conditions are used.
16-20	NYCLE	Number of calculation- cycles per print cycle.
20-30	FC	Coefficient of surface heat transfer.
31-40	ATEM	Ambient air temperature.
41-50	TIMMAX	Time at which exit from TEM1 takes place. If equal to 0.0 only the equilibrium state is calculated.

The following three formats are the boundary
condition card formats and use the TSAAS nodal
point numbering reference system (that is, they
are not defined with respect to the logical mesh).
As with the pressure and shear boundary condition
cards of the SAAS code, no more than 200 of each
of these boundary condition cards can be used.
Keep in mind that flux and convection are pre-
scribed across the surface of an element, tempcr-
ature is fixed on a nodal point. Also, positive
flux and convective conditions are assumed to apply
to the elements to the left of the nodal points
when going from the first nodal point to the last
nodal point.

FLUX BOUNDARY CARDS*

NUMQC in number where NUMQC $\leq$ 200.

Format (2I5, F10.1)

Cols.

1-5 Nodal point I.

6-10 Nodal point J.

11-20 Flux.

CONVECTIVE BOUNDARY CARDS*

NUMSCC in number, where NUMSCC $\leq$ 200. If the film coefficient given below is zero, then default values of FC and ATEM given on the THERMAL LOAD CONTROL card are used.

Format (2I5, 2F10.0)

Cols.

1-5 Nodal point I.

6-10 Nodal point J.

11-20 Film coefficient.

21-30 Ambient fluid temperature.

TEMPERATURE BOUNDARY CARDS*

NUMTB in number where NUMTB $\leq$ 200.

Format (I5, F10.0)

Cols.

1-5 Nodal point.

6-15 Nodal point temperature ($^{\circ}$ C).

*Flux and convection are prescribed across the surface of an element, temperature is fixed on a nodal point. Positive flux and convective conditions are assumed to apply to the elements to the left of the nodal points when going from the first nodal point to the last nodal point.

APPENDIX B

COMPUTER PROGRAM OUTPUT

In a computer program as comprehensive as the present program, it is easy to generate enough information to overwhelm the interpretive capabilities of the user. Consequently, much thought was given to the problem of making the assimilation task as easy as possible. There are two types of output: printed and plotted. The plotted output serves as a quick guide to the more extensive information which is printed by the program.

I. PRINTED OUTPUT

The following information can be printed by the program:

- a. Input data. (An image of each data card is printed prior to execution of any data case. Most input data are also printed out and described for each data case as it is executed.)
- b. Mesh generation information.
- c. Nodal point locations and interpolated temperatures.
- d. Element makeup (with respect to nodal points) and interpolated temperatures.
- e. Nodal point displacements.
- f. Stresses and mechanical strains (total strain minus free thermal strain) at the center of each element.
- g. The approximate value of the weight or mass and strain energy in each material as well as the total values.

II. PLOTTED OUTPUT: SAASPL

An associated plotting and editing program called SAASPL has been prepared. This program can produce a plot of the mesh, a plot of the material boundaries, and contour plots of the temperature field, displacements, stresses (σ_r , σ_z , σ_{θ} , σ_{rz} , $\sigma_{\max}$, $\sigma_{\min}$, σ_{avg} , $\sigma_{\text{max shear}}$), and strains as well as edited listings of the stresses and strains.

Two listings are prepared for each material type showing stress and strain components for each selected element. The elements selected

are those which have the largest positive value, the largest negative value, and the five largest absolute values of a stress component. A similar criterion is used to choose elements for the strain listing. On each of these listings there is a column marked SEL IND. There are eight octal numbers corresponding to the eight listed stresses or strains. If the number is in the range of 7 to 3, it indicates that the corresponding value is the largest down to the fifth largest in absolute value. A 2 or 1 indicates the largest positive and largest negative value, respectively, while a zero indicates that this value played no role in the selection of this element.

The primary inputs to SAASPL are the files TAPE 10 and TAPE 11 generated by SAAS (assuming the second option switch is set to 1). An optional card deck can be used to control the range values of the contour plots. This is done by making a card with the desired ranges for each plot that is to be controlled. The plots are considered to be indexed 1, 2, 3, ..., 19 in the order plotted, that is temperature, r-displacement, z-displacement, the stresses (σ_r , σ_z , σ_{θ} , σ_{rz} , $\sigma_{\max}$, $\sigma_{\min}$, σ_{avg} , $\sigma_{\text{max shear}}$), and the strains (same order as stresses). The control cards are read with a (3X, I2, 2E10.0, I5) format to obtain the index number, the minimum limit, and the maximum limit, and a control number. If the control number is 0, then the contours will be done at 20 equally spaced values between the specified limits. If the control number is negative, the frame is skipped. If the control number is positive, then the contour plotting is done only in the material with that number, with maximum and minimum limits automatically generated for that material. Plots for which there are no control cards will be automatically scaled as before.

APPENDIX C

EXAMPLE PROBLEMS

Two numerical examples are presented to illustrate the use of the various capabilities of the TSAAS program.

I. THICK SPHERICAL SHELL WITH STEADY HEAT FLOW

TSAAS results are compared with an exact solution for a thick spherical shell under a condition of steady heat flow. The shell has an internal radius of 20, the material of the shell is elastic with an elastic modulus of 70 000, a Poisson's ratio of 0.33, and a coefficient of linear expansion of 0.001. The thermal conductivity of the materials is 0.00092. A constant flux of 0.02 is applied to the inner surface and the temperature of the external surface is fixed at 0.

The finite element solution is obtained for a wedge-shaped ring of the spherical shell so that full advantage is taken of the spherical symmetry

of the problem. A schematic diagram of the wedge-shaped ring is shown in Fig. C-1. The boundary conditions on the sloping lines of Fig. C-1 are zero shear stress and normal displacement.

The input cards for a 50-element wedge-shaped ring are shown in Fig. C-2. Note that the wedge-shaped ring must be treated by use of the skew boundaries capability described in Sec.

IV.B. Figure C-3 illustrates the results of SAASPL editing program described in Appendix B2.

Table C-1 shows the close agreement between the finite element and analytical results for the radial stress σ_r , the hoop stress σ_θ , and temperature T.

TABLE C-1
COMPARISON OF TSAAS AND EXACT VALUES

Radius	σ_r (Exact)	σ_r (TSAAS)	σ_θ (Exact)	σ_θ (TSAAS)	T(Exact)	T(TSAAS)
12.193	-173	-173	-316	-315	69.54	69.47
14.192	-173	-172	- 54	- 53	44.45	44.38
16.191	-124	-123	119	120	25.55	25.49
18.190	- 60	- 59	241	242	10.81	10.75

II. INTERNAL HEAT GENERATION BY A SPHERE IMBEDDED IN A CYLINDER WITH TEMPERATURE AND CONVECTIVE BOUNDARY CONDITIONS

This problem illustrates a time-dependent thermal stress calculation for a radioactive sphere of radius 8 embedded in an inert cylinder of radius 12 and height 48. The material (material 2) in the sphere has a heat generation rate of 0.005. The ends of the cylinder are fixed

at 0° and the side of the cylinder convects to air at 0° with a surface coefficient of heat transfer at 0.0001.

The input cards for this problem are shown in Fig. C-4. Table C-2 shows the elastic and thermal properties that were used in the analysis. Figure C-5 illustrates the finite element mesh and the thermal boundary conditions. Figure C-6 is a plot of isotherms at 5000 s, obtained from SAASPL, while Fig. C-7 illustrates the edited stresses at this time.

TABLE C-2

MATERIAL PROPERTY DATA FOR COMPOSITE PROBLEM

Material	Elastic Modulus	Poisson's Ratio	Coefficient of Thermal Expansion	Heat Capacity	Thermal Conductivity
1	13 000	0.3	50×10^{-6}	0.3	0.01
2	6 000 000	0.3	8×10^{-6}	0.03	0.1

Fig. C-1. Finite element mesh.

```

*****      TSAAS-II      JUN81      LASL-7600      VER 3      *****
      10      20      30      40      50      60      70      80
123456789012345678901234567890123456789012345678901234567890
1
THERMAL STRESS ANALYSIS OF A SPHERE
0101 110000000      1      1      0.
1 0 3 3 0 0 2 0 0 1 0 1
1 1 9.9778 0.6667 19.9555 1.3333
2 1 10.0000 0.0000 20.0000 0.0000
3 1 9.9778 -0.6667 19.9555 -1.3333
1 1 1 26 1 1
2 1 2 26 2 1
3 1 3 26 3 1
1 1 1 26 -176.18
3 3 1 26 -3.82
1 3 1 1 .02
1 3 26 26 0.
0.
1 1 2.
0. 70.E3 .33
.0001 700. 1.
.00092 .248 5.
001
1
1
2
END OF CASE
END OF DATA

```

Fig. C-2. Data input to TSAAS.

THERMAL STRESS ANALYSIS OF A SPHERE

		MATERIAL	1	PRINT CYCLE				1	TIME				O 000000
EL	R LOC	Z LOC	SIG R	SIG Z	SIG TH	SIG RZ	SIG MX	SIG MN	SIG AV	SIG MS	SIG IND		
1	10 188677	340016	-30	751	751	24	29	752	-511	361	06760666		
2	10 188677	340016	30	751	751	24	29	752	-511	361	07670777		
3	10 588231	353348	-79	-643	643	19	79	643	-455	282	04440444		
4	10 588231	353348	-79	643	643	-19	-79	-643	-455	282	05550555		
5	10 987785	366680	-116	-546	-546	14	-115	-547	-403	216	03300030		
6	10 987785	- 366680	-116	546	-546	-14	-115	-547	-403	216	00030303		
14	12 586001	- 420008	-180	252	-252	2	-180	-252	-228	36	30000000		
15	12 985555	433340	-183	-196	-196	0	-183	-196	-191	7	70001000		
16	12 985555	- 433340	-183	-196	-196	-0	-183	-196	-191	7	60000000		
17	13 385109	446672	-182	144	-144	-1	144	-182	-156	19	40000000		
18	13 385109	- 446672	-182	-144	144	1	-144	-182	-156	19	50000000		
45	18 978865	633320	-32	281	281	10	281	-33	176	157	00003000		
47	19 378419	646652	-19	299	299	-11	299	-19	193	159	00005000		
48	19 378419	646652	-19	299	299	11	299	19	193	159	00004000		
49	19 777973	659984	-6	316	316	-11	316	-6	209	161	00006000		
50	19 777973	659984	-6	316	316	11	316	6	209	161	02207020		

THERMAL STRESS ANALYSIS OF A SPHERE

MEAN OF SIGMAV BY MATERIAL

MAT	MEAN
1	-76

Fig. C.3. SAASPL edit.

THERMAL STRESS ANALYSIS OF A SPHERE

		MATERIAL		1		PRINT CYCLE				1		TIME		O 000000								
EL		R LOC		Z LOC		FPS R		EPS Z		FPS TH		FPS RZ		EPS MX		EPS MN		EPS AV		EPS MS		SEL IND
1	10	188677		340016		6654	-	7047		7050		0915		6669		7062		2481		6866		76767666
2	10	188677	-	340016		6654	-	7047	-	7050	-	0915		6669	-	7062	-	2481		6866		67676777
3	10	588231		353348		4926	-	5776		5778		0714		4938	-	5788	-	2209		5363		54544444
4	10	588231	-	353348		4926	-	5776	-	5778	-	0714		4938	-	5788	-	2209		5363		45455555
5	10	987785		366680		3495	-	4683	-	4684		0546		3504	-	4692		1957		4098		00000030
6	10	987785	-	366680		3495	-	4683	-	4684	-	0546		3504	-	4692	-	1957		4098		33333303
15	12	985555		433340	-	0764	-	1013	-	1010		0017		0764	-	1013	-	0929		0125		00001000
43	18	579311		619988	-	3118		2716		2724	-	0390		2723	-	3125		0774		2924		10000000
50	19	777973	-	659984		3065		3049		3057		0408		3056		3072		1014		3064		02200020


```

***** TSAAS II JUN81 LASL-7600 VER 3 *****
      10      20      30      40      50      60      70      80
1234567890123456789012345678901234567890123456789012345678901234567890
1
INTERNAL HEAT GENERATION WITH TEMP AND CONV BOUNDARY CONDITIONS
1 1 1 2 20
1 0 8 8 1 0 3 0 0 0 1 1
1 1 1 E-8 1 E-8 8 1 E-8
2 5 8 0 45 1 E-8 1 E-8
3 5 8 45 90 1 E-8 1 E-8
4 1 1 E-8 8 1 E-8 1 E-8
5 1 8 1 E-8 12 1 E-8
6 1 12 24
7 1 1 E-8 24
8 1 1 E-8 8
1 1 9 1 1 2
9 1 13 1 5 2
13 13 25 6 2
1 25 7 2
1 9 8 2
2 1 9 1 9
1 9 1 1 2
1 1 1 13 1
1 1 1 1 3
13 13 1 25
1 13 25 25 0
1 1 2
20 1 3E5 3
50 E-6 1
10 E-3 30 10 1
2 1 16
20 6 E6 3
8 E-6 1
0 1 03 5 0E 3 0 0 5 1
1 1 1 0 10
3 1 1
0 1 1
1 0 0001 0 0
2 500 0 0001 0 0
3 5000 0 0001 0 0
END OF CASE
END OF DATA

```

Fig. C-4. Data input to TSAAS.

Fig. C-5. Finite element mesh.

Fig. C-6. Temperature contours.

INTERNAL HEAT GENERATION WITH TEMP AND CONV BOUNDARY CONDITIONS

MATERIAL 1			PRINT CYCLE 1				TIME				500 000021		
EL	R LOC	Z LOC	SIG R	SIG Z	SIG TH	SIG RZ	SIG MX	SIG MN	SIG AV	SIG MS		SEL IND	
9	8 448299	403797	123	2	14	58	147	-21	46	84		30057027	
12	11 474033	395332	2	4	85	0	4	2	29	44		00200000	
21	8 377924	1 205142	1	19	-34	67	59	-77	-18	68		00070006	
22	9 368028	1 191041	7	29	44	56	76	39	27	57		00040000	
33	8 240386	1 988733	-49	25	53	62	27	-100	-42	64		00060005	
34	9 274935	1 974409	30	33	29	49	60	56	11	58		00000003	
45	8 041648	2 745739	-78	-27	-63	53	7	-112	56	59		00030004	
72	11 361500	4 375933	-8	59	72	2	59	8	41	40		02000000	
97	447436	8 368020	-110	-104	-110	-4	-102	-112	108	5		07706060	
98	1 333285	8 279561	-113	-103	-110	-10	-97	-119	-109	11		06605070	
99	2 192941	8 108678	-117	-98	-109	-15	-90	-125	-108	18		05504050	
100	3 011970	7 866135	-123	90	-107	-18	-82	-131	-107	24		04403340	
101	3 780486	7 567346	-128	-82	-104	-19	-75	-135	-104	30		40300430	
102	4 494984	7 232587	-131	-73	-100	-19	-67	-137	-101	35		50000500	
103	5 161589	6 890362	-133	-63	-93	-19	58	-138	-96	40		70000600	
104	5 805165	6 591549	-132	-50	-82	-24	-44	-138	88	47		60000700	
109	444648	9 192454	-46	90	-46	-6	-46	-90	-61	22		03000000	
116	6 358034	7 983308	-39	10	12	-47	24	-74	-13	49		00010000	
221	4 414805	18 133473	6	5	7	0	6	5	6	1		00000200	

INTERNAL HEAT GENERATION WITH TEMP AND CONV BOUNDARY CONDITIONS

MATERIAL 2			PRINT CYCLE 1				TIME 5000 000069				
EL	R LOC	Z LOC	SIG R	SIG Z	SIG TH	SIG RZ	SIG MX	SIG MN	SIG AV	SIG MS	SEL IND
8	7 412809	408670	-14	-64	148	117	81	-158	24	153	11020100
44	6 990665	2 777191	495	431	593	-132	598	328	506	135	04000400
56	6 728422	3 507711	513	445	622	-144	627	331	527	148	06354540
61	442797	5 216031	356	4	356	-10	357	4	239	176	00000003
68	6 418493	4 202123	521	436	633	-156	640	316	530	162	45466350
73	441604	6 271398	402	46	403	-14	403	45	284	179	00000007
74	1 318970	6 203945	405	59	411	-32	408	56	292	177	00000004
77	3 816739	5 658985	464	169	498	93	491	143	377	178	00000005
78	4 586903	5 401049	514	234	554	-95	544	205	434	175	30000000
79	5 330432	5 130893	585	429	660	82	620	394	558	133	63603660
80	6 066584	4 865714	510	422	638	-167	639	294	523	173	00575030
90	4 462162	6 240365	486	268	553	137	552	202	436	176	00030000
91	5 097584	5 874790	545	332	614	-142	616	261	497	178	50040006
92	5 667436	5 518434	736	617	789	-120	811	543	714	134	77707770

ACKNOWLEDGMENTS

This report is a revision of an original report, LA-5599-MS, "TSAAS: Finite Element Thermal and Stress Analysis of Axisymmetric Solids with Orthotropic Temperature-Dependent Material Properties".

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