LA-UR -86-2917

CONF-8606202--2

MASTER

TITLE: EXTENSION OF THE METHOD OF THE SMALL ANGLE APPROXIMATION: OPEN DETECTOR

LA-UR--86-2917

DE86 015308

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SUBMITTED TO: FOR PROCEEDINGS OF THE 1986 CRDEC CONFERENCE ON OBSCURATION AND AEROSOL RESEARCH HELD IN ABERDEEN, MD ON JUNE 23-27, 1986.

AUGUST 18, 1986

Los Alamos Los Alamos National Laboratory
Los Alamos, New Mexico 87545

FORM NO 836 R4
ST NO 2629 5/81

Extension of the method of the small angle approximation : Open Detector.

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A) "Diffusion of a Laser Beam : Open detector", to be submitted to JOSA

ABSTRACT.

We use the radiative transfer equation to study the multiple scattering undergone by a laser beam propagating through a turbid medium. During the propagation, we view the beam as first scattering into a narrow forward cone, and then into a diffuse pattern. To describe this process, we propose a systematic and practical method to combine the small angle approximation with the diffusion approximation. The method works when the scattering cross-section describing scattering from aerosols can be written as the sum of a gaussian σ_s to describe scattering into small angles, and a term σ_d , that can be represented by the first two terms of a Legendre expansion to describe scattering into large/diffuse angles. We use a Green's function formalism to perform partial resummations and set up a hierarchy of approximations in the form of coupled radiative transfer equations to describe the scattering of radiation from small angles into large angles.

The adjoint operator formalism then provides a simple way to obtain the net flux received by an open detector at any given point. Our approximations may be described rigorously as a power series expansion in $\sigma^0{}_d/\sigma^0{}_s$, the ratio of the diffusion scattering cross-section to the forward scattering cross-section. Thus our technique works well when small angle scattering dominates.

1. INTRODUCTION.

The small angle approximation has found a good measure of success in radiative transfer theory[1,2,3] in describing the scattering of radiation from relatively large particles (compared to the wavelength) into small angles in the forward direction. In fact, a recent comparison of the theory with experimental results[4] shows excellent agreement between the two for angles of the order of one radian. For larger scattering angles, however, the theory is not meant to work, for it assumes that the phase function is described by a sharply peaked Gaussian, fitted to a realistic phase function[3,4] in the vicinity of the forward direction. In general, the scattering cross-section (or equivalently, the phase function) may be described by a combination of two terms[5]:

$$\sigma(\Omega,\Omega') = \sigma_s(\Omega,\Omega') + \sigma_d(\Omega,\Omega') \qquad (1.1)$$

where σ is connected to the phase function $p(\Omega,\Omega')$ by:

$$\sigma(\Omega,\Omega') = (\sigma^0/4\pi)\ p(\Omega,\Omega') \qquad (1.2)$$

and

$$\sigma_s(\Omega,\Omega') = (\sigma^0{}_s/4\pi)\, p_s(\Omega-\Omega') \qquad (1.3)$$

with

$$p_s(\Omega-\Omega') = (\alpha^2/\pi)\exp(-\alpha^2(\Omega-\Omega')^2) \qquad (1.4)$$

while

$$\sigma_d(\Omega,\Omega') = (\sigma^0{}_d/4\pi)\, p_d(\Omega\cdot\Omega') \qquad (1.5)$$

with $\sigma^0=\sigma^0{}_s+\sigma^0{}_d$ and

$$p_d(\Omega\cdot\Omega') = 1 + 3a_1 P_1(\Omega\cdot\Omega') \qquad (1.6)$$

where Ω , Ω' are the final and initial scattering angles respectively. σ is the scattering **cross-section**, $\sigma^0{}_s$ is the **forward scattering cross-section** (the area under the gaussian part of the scattering cross-section), σ^0_d is the **diffusion scattering cross-section** (the area under the "smooth" part of the scattering cross-section), a_1 is the first moment in the Legendre expansion of $\sigma_d(\Omega,\Omega')$, and $P_1(\theta)=\cos(\theta)$ is the Legendre polynomial of the first order.

Equation (1.4) describes scattering into small angles, and Zardecki[3] has shown that the forward lobe of the Mie scattering cross-section of aerosols may be well-fitted by $\alpha^{-1} \sim 10$ degrees for wavelengths around 0.5μm. As mentioned earlier, neglecting the second term in Eq. (1.1) in solving the radiative transfer equation is adequate for describing small-angle (forward) scattering. But for scattering into large angles it is not appropriate to eliminate σ_d from Eq. (1.1).

If σ_s in Eq. (1.1) were to be neglected, then the phase function is seen to be fairly *flat* , indicating that the radiation will be scattered in a fairly diffuse manner. The diffusion aproximation would be appropriate in that case to solve the radiative transfer equation,[6] provided the source was fairly isotropic. In the diffusion approximation, only the first two Legendre moments of the specific intensity $I(r,\Omega)$ are retained viz., the energy density $\int I(r,\Omega)d^2\Omega$ and the net flux $\int I(r,\Omega)\Omega d^2\Omega$.

From a physical point of view, we may imagine that the radiation is first scattered by the aerosols into a *forward* cone, and this radiation acts as a secondary source, which is subsequently scattered into a diffuse pattern. Our intuitive ideas are similar to those of Tam and Zardecki[7] and those of Weinman.[8] But in our formulation we have an entire hierarchy of approximations which encompasses the range of the ratio σ^0_d/σ^0_s, and the error involved in each step of the hierarchy may be estimated.

2. FORMALISM.

We shall use a Green's function formalism to combine the small angle approximation and the diffusion approximation. This method not only yields a useful physical insight into the scattering process, but provides us in the end with a set of radiative transfer equations to solve. These equations may then be solved using standard techniques. Thus a practical scheme for the combination of the small angle and diffusion approximations is obtained.

Let $I(\mathbf{r},\mathbf{\Omega})$ be the specific intensity of the radiation being propagated through a scattering medium whose total cross-section (absorption + scattering) is σ_t. If there were no scattering processes, then the loss in $I(\mathbf{r},\mathbf{\Omega})$ at r, in the direction $\mathbf{\Omega}$ is given by[9]:

$$\mathbf{\Omega}\cdot\nabla\, I(\mathbf{r},\mathbf{\Omega}) = -\,\sigma_t\, I(\mathbf{r},\mathbf{\Omega}) + Q(\mathbf{r},\mathbf{\Omega}) \tag{2.1}$$

where $Q(\mathbf{r},\mathbf{\Omega})$ is taken to be a gaussian, mono-directional source of radiation, located at $z=z_s$.

$$Q(\mathbf{r},\mathbf{\Omega}) = q_0\, \delta_\gamma(\boldsymbol{\rho}-\boldsymbol{\rho}_s)\,\delta(z-z_s)\,\delta(\mu-\mu_s)\,\delta(\varphi-\varphi_s) \tag{2.1a}$$

$$\delta_\gamma(\boldsymbol{\rho}-\boldsymbol{\rho}_s) = (\gamma^2/\pi)\exp(-\gamma^2(\boldsymbol{\rho}-\boldsymbol{\rho}_s)^2) \tag{2.1b}$$

where φ is the azimuthal angle, and γ is the initial width of the beam. If the radiation were to reappear due to some scattering process then the net change in $I(\mathbf{r},\boldsymbol{\Omega})$ is[9]:

$$\boldsymbol{\Omega}\cdot\nabla\, I(\mathbf{r},\boldsymbol{\Omega}) = -\,\sigma_t\, I(\mathbf{r},\boldsymbol{\Omega}) + \int d^2\Omega'\, \sigma(\boldsymbol{\Omega},\boldsymbol{\Omega}')\, I(\mathbf{r},\boldsymbol{\Omega}') + Q(\mathbf{r},\boldsymbol{\Omega}) \qquad (2.2)$$

where $\sigma(\boldsymbol{\Omega},\boldsymbol{\Omega}')$ is given by Eq. (1.1).

The reason we have chosen a volumetric source rather than a boundary source is that we would like to have the capability of locating our laser source of arbitrary strength at arbitrary positions in the atmosphere.

Equation (2.2) may be witten in operator form as:

$$(\,\mathbf{L_0} - \boldsymbol{\sigma}\,)\,I = Q \qquad (2.3)$$

where

$$\mathbf{L_0} = \boldsymbol{\Omega}\cdot\nabla + \sigma_t\,, \qquad (2.4)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma_s} + \boldsymbol{\sigma_d} \qquad (2.5)$$

and the matrix elements of $\boldsymbol{\sigma}$, $\boldsymbol{\sigma_s}$, $\boldsymbol{\sigma_d}$ are:

$$\boldsymbol{\sigma}(\boldsymbol{\Omega},\boldsymbol{\Omega}') = \sigma(\boldsymbol{\Omega},\boldsymbol{\Omega}') \qquad (2.6)$$

$$\boldsymbol{\sigma_s}(\boldsymbol{\Omega},\boldsymbol{\Omega}') = \sigma_s(\boldsymbol{\Omega},\boldsymbol{\Omega}') \qquad (2.7)$$

$$\boldsymbol{\sigma_d}(\boldsymbol{\Omega},\boldsymbol{\Omega}') = \sigma_d(\boldsymbol{\Omega},\boldsymbol{\Omega}') \qquad (2.8)$$

where $\sigma(\boldsymbol{\Omega},\boldsymbol{\Omega}')$, $\sigma_s(\boldsymbol{\Omega},\boldsymbol{\Omega}')$, $\sigma_d(\boldsymbol{\Omega},\boldsymbol{\Omega}')$ are given by Eqns. (1.1)-(1.6). I and Q in Eq. (2.3) are column vectors, whose components are given by $I(\mathbf{r},\boldsymbol{\Omega})$ and $Q(\mathbf{r},\boldsymbol{\Omega})$ respectively.

The formal solution to Eq. (2.3) is

$$I = G\,Q \qquad (2.9)$$

where G is the resolvent

$$G = (\,\mathbf{L_0} - \boldsymbol{\sigma_s} - \boldsymbol{\sigma_d}\,)^{-1} \qquad (2.10)$$

Equation (2.9) may be rewritten as

$$I = (G_d^{-1} - \sigma_s)^{-1} Q \tag{2.11}$$

$$= (1 - G_d \sigma_s)^{-1} G_d Q \tag{2.12}$$

where

$$G_d \equiv (L_0 - \sigma_d)^{-1} \tag{2.13}$$

By adding and subtracting $G_d Q$ to the right-hand-side of Eq. (2.12) we obtain after a few simple manipulations, without using series expansions:

$$I = G_d Q + G_d \sigma_s \ (1- G_d \sigma_s)^{-1} G_d Q \tag{2.14}$$

$$= G_d Q' \tag{2.15}$$

where Q' is a modified source:

$$Q' = [1 + \sigma_s \ (G_d^{-1} - \sigma_s)^{-1}] Q \tag{2.16}$$

which may be recast (using $(G_d^{-1} - \sigma_s)=(G_s^{-1} - \sigma_d)$) as

$$Q' = [1 + \sigma_s \ (1- G_s \sigma_d)^{-1} G_s] Q \tag{2.17}$$

where

$$G_s \equiv (L_0 - \sigma_s)^{-1} \tag{2.18}$$

If we expand the second term on the right hand side of Eq. (2.17) in powers of $G_s \sigma_d$ (this is possible when $\sigma_d/\sigma_s << 1$; see reference A for details), and retain only the first two terms for the sake of exposition, we get

$$I \approx G_d Q + (G_d \sigma_s)(G_s Q) + (G_d \sigma_s)(G_s \sigma_d)(G_s Q) \qquad (2.19)$$

Denoting $G_s Q$ by I_s and $G_d Q$ by I_d, we get

$$I \approx I_d + G_d(\sigma_s I_s) + G_d(\sigma_s (G_s \sigma_d) I_s) \qquad (2.20)$$

Of course, given the definitions of G_s and G_d, I_d and I_s are solutions to :

$$(L_0 - \sigma_d) I_d = Q \qquad (2.21)$$

and

$$(L_0 - \sigma_s) I_s = Q \qquad (2.22)$$

respectively.

In the second term on the right hand side of Eq. (2.20) the small angle-solution acts as a secondary source. The third term gives us cross-scattering between the forward angles and the diffuse angles. The higher order terms we neglected would give us greater amounts of cross-scattering. These higher order terms scatter the radiation in an increasingly diffuse manner.

In the lowest approximation then,

$$I \approx I_d + G_d \sigma_s I_s \qquad (2.23)$$

The terms we neglected are of the order of σ^0_d/σ^0_s (see reference A). This ratio clearly depends on kinds of aerosols (scatterers) we choose to look at, and also the wavelength at which we operate. Thus our approximation can be accurate, depending upon the nature of the aerosol and the wavelength of the incident light.

Since I_d is given by $G_d\, Q = (\mathbf{L_0}-\boldsymbol{\sigma_d})^{-1}\, Q$, and knowing that $|\boldsymbol{\sigma_d}| << \sigma_t$ (see reference A), we approximate I_d by:

$$I_d \approx \mathbf{L_0}^{-1}\, Q = \delta(\boldsymbol{\Omega}-\boldsymbol{\Omega_s})\, \delta_\gamma(\boldsymbol{\rho}-\boldsymbol{\rho_s})\, \exp(-\sigma_t(z-z_s)) \qquad (3.1)$$

with $\delta_\gamma(\boldsymbol{\rho}-\boldsymbol{\rho_s})$ being a gaussian of width γ, as defined in Eqn(2.1b).

In other words, we get the unscattered, attenuated part of the beam explicitly. The scattered part of the beam is given by the second term in Eq.(2.23).

Now the second term on the right hand side of Eq.(2.23) is the solution to

$$(\, \mathbf{L_0} - \boldsymbol{\sigma_d}\,)\, I \approx \boldsymbol{\sigma_s}\, I_s = Q' \qquad (2.24)$$

This may be written in its full form as

$$\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\, I(\mathbf{r},\boldsymbol{\Omega}) + \sigma_t\, I(\mathbf{r},\boldsymbol{\Omega}) - \int d^2\Omega'\, \sigma_d(\boldsymbol{\Omega},\boldsymbol{\Omega}')\, I(\mathbf{r},\boldsymbol{\Omega}')$$

$$= \int d^2\Omega'\, \sigma_s(\boldsymbol{\Omega}-\boldsymbol{\Omega}')\, I_s(\mathbf{r},\boldsymbol{\Omega}') \qquad (2.25)$$

where $I_s(\mathbf{r},\boldsymbol{\Omega})$ is given by

$$\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\, I_s(\mathbf{r},\boldsymbol{\Omega}) + \sigma_t\, I_s(\mathbf{r},\boldsymbol{\Omega}) = Q(\mathbf{r},\boldsymbol{\Omega}) + \int d^2\Omega'\, \sigma_s(\boldsymbol{\Omega}-\boldsymbol{\Omega}')\, I_s(\mathbf{r},\boldsymbol{\Omega}') \qquad (2.26)$$

If $\sigma_s(\boldsymbol{\Omega}-\boldsymbol{\Omega}')$, the small angle phase function is sharply peaked (in the forward direction), one may employ the small angle approximation to solve Eq. (2.26), and utilize that answer in Eq. (2.25).

We take μ_s=1 in Eq.(2.1a), so that in the small angle approximation, $\boldsymbol{\Omega}\cdot\boldsymbol{\nabla} \approx \boldsymbol{\phi}\cdot\boldsymbol{\nabla}_\rho + \partial/\partial z$ where $\boldsymbol{\phi} = (\phi_x, \phi_y, 0)$ and $\boldsymbol{\nabla}_\rho = (\, \nabla_x\, ,\, \nabla_y\, ,\, 0\,)$. Eq. (2.26) then takes the form

$$\partial I_S(\mathbf{r},\boldsymbol{\phi})/\partial z + \boldsymbol{\phi}\cdot\nabla_{\rho} I_S(\mathbf{r},\boldsymbol{\phi}) + \sigma_t\, I_S(\mathbf{r},\boldsymbol{\phi}) = Q(\mathbf{r},\boldsymbol{\Omega}) + {}_{-\infty}\int^{+\infty} d^2\boldsymbol{\phi}'\; \sigma_S(\boldsymbol{\phi}-\boldsymbol{\phi}')\, I_S(\mathbf{r},\boldsymbol{\phi}) \quad (2.27)$$

If we now try to use the diffusion approximation directly on Eq.(2.25), we find that we can accomodate at most the first moment[6] of the $\sigma_S \bullet I_S$ term (on the right hand side of Eq. (2.25)). But this may be a bad approximation, especially near the source, when the beam has not yet had a chance to spread out sufficiently. An alternative is to implement (numerically) a P_N approximation.[6] But this is a nontrivial job in three dimensions, and it is not clear how many moments of $\sigma_S \bullet I_S$ we may need to perform an adequate job.

On the other hand, if we need the net flux at the target position i.e.,we have an open detector, we need the following angular average:

$$P(\mathbf{r_0}) = \int d^2\Omega\; I(\mathbf{r},\boldsymbol{\Omega})\; \boldsymbol{\Omega}\cdot\mathbf{n} \quad (2.28)$$

where $\mathbf{n}$ is a unit vector in the direction toward which the open detector is pointing.

We can now use the adjoint operator formalism[10] (reference A) to show that:

$$P(\mathbf{r_0}) = (\mathbf{n}\cdot\nabla\,|_{\mathbf{r0}}) \int d^3r\; I_0^{+}(\mathbf{r},\mathbf{r_0}) \int d^2\Omega\; Q'(\mathbf{r},\boldsymbol{\Omega}) \quad (2.29)$$

where (see reference A),

$$I^{+}_0(\mathbf{r},\mathbf{r_0}) = -(3/8\pi)\, \exp(-\surd(\sigma^{(0)}/D))\, |\mathbf{r}-\mathbf{r_0}|) \,/\, |\mathbf{r}-\mathbf{r_0}| \quad (2.30)$$

and the angular average of the modified source $Q'(\mathbf{r},\boldsymbol{\Omega})$ is

$$\int d^2\Omega\, Q'(\mathbf{r},\mathbf{\Omega}) = {}_{-\infty}\int^{+\infty} d^2\Omega \; {}_{-\infty}\int^{+\infty} d^2\Omega' \sigma_S(\mathbf{\Omega}-\mathbf{\Omega}') I_S(\mathbf{r},\mathbf{\Omega}') \qquad (2.31)$$

$$= {}_{-\infty}\int^{+\infty} d^2\Omega' \sigma_S(\mathbf{\Omega}-\mathbf{\Omega}') \; {}_{-\infty}\int^{+\infty} d^2\Omega I_S(\mathbf{r},\mathbf{\Omega}')$$

$$= \sigma_S{}^0 \; {}_{-\infty}\int^{+\infty} d^2\Omega I_S(\mathbf{r},\mathbf{\Omega}') \qquad (2.31a)$$

From Eqn(2.29), we see that $P(\mathbf{r}_0)$ is simply the flux obtained by solving the diffusion equation with $\int d^2\Omega\, Q'(\mathbf{r},\mathbf{\Omega})$ as the source.

We must emphasize that this trick applies only when we need the response of an open detector, when in a manner of speaking, angular effects are averaged out.

We therefore used the code **DIF3D**[11] to solve the diffusion equation in a cylindrical geometry, assuming azimuthal symmetry, with Eq(2.31a) as a volumetric source.

4. NUMERICAL RESULTS.

When a laser beam traverses a turbid medium for a few optical depths or less, one notices the following characteristics:

(a) The intensity is concentrated in the vicinity of the beam axis. This is the unscattered part of the beam.

(b) Multiple scattering effects enhance the intensity in the "wings" of the beam.

A true test of a multiple-scattering theory is not simply to match experimental or simulation results close to the beam axis, but also in the wings of the beam where scattering effects dominate, especially at large optical depths.

Our numerical results indicate that our method is successful in this regard, in that our results compare favorably with Monte Carlo[5] results for large optical depths.

We shall illustrate our technique of combining the small-angle approximation with the diffusion approximation by studying the propagation of a collimated laser beam through a polydisperse distribution of heavy dust for optical depths of τ = 3.75 and 7.5. The relevant phase functions were constructed by using the code **AGAUS**.[5]

We have compared our results with those from Monte Carlo solutions to the radiative transfer equation (provided by the code **MSCAT**[5]).

To put the comparison into perspective, we note that:

(1) We chose a polydisperse distribution of heavy dust since this gives us $\sigma_s^0/\sigma_d^0 = 7.156 \times 10^3$ (>>1) for λ=0.55µm. Thus one of the criteria for implementing our combination scheme is satisfied. The single scattering albedo is 0.999.

(2) The Monte Carlo code has a flat beam rather than a gaussian beam.

(3) The true phase function which is used by the code is not simply the sum of a sharp gaussian in the forward direction and a "flat" part.

We found that at τ = 7.5 we got good agreement with the Monte Carlo results (Fig.1). On the other hand for τ = 3.75, where one might expect good agreement from the small angle approximation alone, we find that our use of the diffusion aprorximation gives an "average" result (Fig. 2). At τ = 9.0 The Monte Carlo code does not converge even for photons of the order of 10^5 and we have not shown the results for this case. In Fig. 3 we have shown the results from our theory for three optical depths on the same graph, which clearly shows the beam as it succumbs to absorption and multiple scattering.

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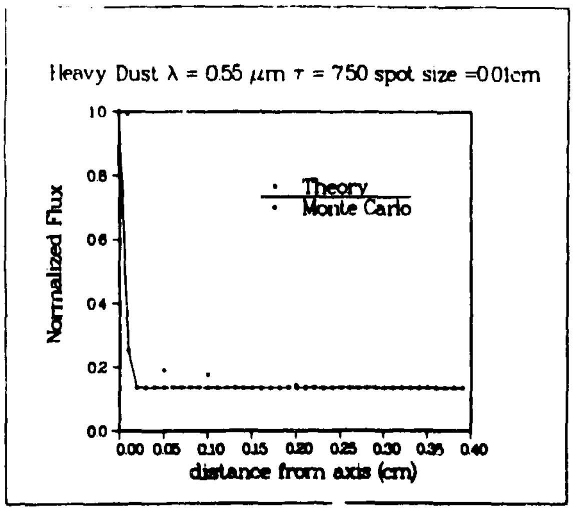

Fig. 1: For τ = 7. we get good agreement with Monte Carlo calculations.

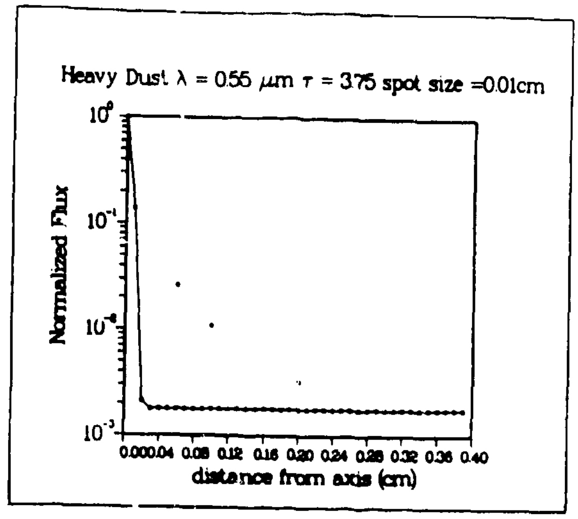

Fig. 2: For τ = 3.75 we get an "average" agreement with Monte Carlo calculations.

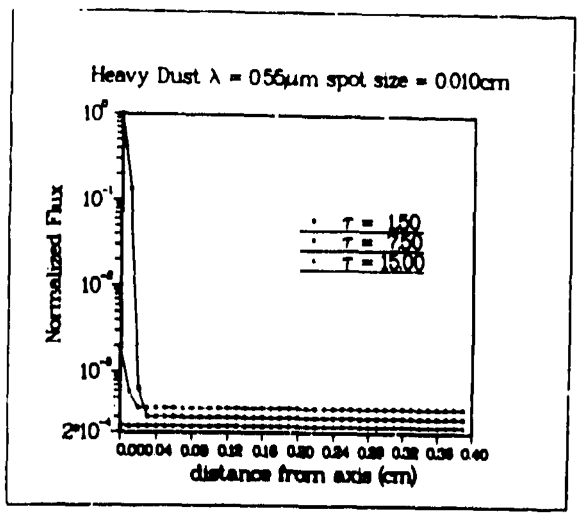

Fig. 3: The results from our theory for three optical depths.