MASTER

ELEVATED TEMPERATURE INELASTIC ANALYSIS
OF METALLIC MEDIA UNDER TIME-VARYING LOADS
USING STATE VARIABLE THEORIES

Subrata Mukherjee¹ 201 6000
Virendra Kumar²
Kaung Jain Chang³ 950 3033

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

- 1 Department of Theoretical and Applied Mechanics
Cornell University, Ithaca, N.Y. 14853, U.S.A.
- 2 Department of Materials Science and Engineering
Cornell University, Ithaca, N.Y. 14853, U.S.A.
- 3 Rock Mechanics and Explosives Research Center
University of Missouri, Rolla, Missouri 65401 U.S.A.
Formerly: Department of Theoretical and Applied Mechanics
Cornell University, Ithaca, N.Y. 14853

ERDA REPORT NO. C00-2733-12
✓September 1977

Prepared For
The U.S. Energy Research and Development Administration
Under Contract No. EY-76-S-02-2733

EB
DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

ABSTRACT

A general method for analysing the time-dependent inelastic response of three-dimensional metallic bodies to arbitrary time-varying mechanical loads and temperature is presented in this paper. The method is capable of using any of a number of constitutive relations using state variables that have been recently proposed by other researchers. For the purpose of illustration, the problems of a hollow sphere, cylinder and thin circular disc subjected to various combinations of internal and external pressures, axial force (or constraint) and torque are analysed using the proposed analysis procedure. The constitutive equations due to Hart are used in the numerical calculations. Hart's model has been previously shown to accurately simulate the deformation behavior of uniaxial metallic specimens subjected to various histories of stress and strain. Numerical results are presented here for the mechanical response of a sphere to various prescribed internal pressure or internal displacement histories. Various experimentally observed phenomena in creep and plasticity such as constant stress creep, stress redistribution, history dependence, creep recovery, plastic work hardening, strain rate sensitivity and Bauschinger's effect are shown to be qualitatively predicted by Hart's model for this multiaxial problem.

1. INTRODUCTION

The classical theories of creep described in the books by Rabotnov [1], Penny and Marriott [2] and Odqvist [3], for example, have several drawbacks as discussed by Krempl [4-6]. To overcome shortcomings of classical theories and to have more faithful representations of the time-dependent non-elastic deformation of metals and alloys at elevated temperatures, a broad class of newer constitutive relations have been proposed in recent years in terms of state variables by various researchers such as Hart [7-8], Miller [9-10], Lagenborg-Robinson [11-12], and Bodner-Partom [13]. The present authors recently proposed [14-16] an efficient method for inelastic analysis of three-dimensional bodies subjected to steady loadings and temperature distributions, using these state variable theories. In the present paper, a general time-dependent inelastic analysis procedure for three-dimensional bodies subjected to arbitrary time-varying mechanical loads and temperatures, is presented. The loads are supposed to vary slowly in time so that inertia effects are negligible. A brief review of the salient features of state variable theories is presented in the next section. The reader is referred to [7-13] for a more detailed discussion. This is followed by a presentation of the computational scheme for general three-dimensional problems and then several illustrative problems involving spheres, cylinders and discs. The hollow spherical shell is chosen as an example and numerical results are presented for the response of a 304 stainless steel sphere at 400° C to a variety of internal pressure and internal displacement histories. Hart's complete constitutive relations are used in these calculations. The significance of phenomena such as creep,

stress redistribution, history dependence, creep recovery, plastic work hardening, strain-rate sensitivity and Bauschinger's effect are discussed in the context of Hart's theory.

2. STATE VARIABLE THEORIES

The mathematical structure of the state variable theories of inelastic deformation [7-13] has some salient features which permit a simple and efficient method of time-dependent, three dimensional inelastic analysis. According to these theories, the total strain tensor $\underline{\epsilon}$ at any time consists of the sum of the elastic strain $\underline{\epsilon}^e$, the nonelastic strain $\underline{\epsilon}^n$ and the thermal strain $\underline{\epsilon}^T$. Thus, in terms of rates

$$\dot{\underline{\epsilon}} = \dot{\underline{\epsilon}}^e + \dot{\underline{\epsilon}}^n + \dot{\underline{\epsilon}}^T \quad (2.1)$$

The nonelastic strain $\underline{\epsilon}^n$ is time and history dependent and includes both the time-independent plastic (in the sense of classical plasticity) and time-dependent creep strains. There is no yield surface and $\underline{\epsilon}^n$ is assumed to exist at all stages of the deformation process.

The nonelastic behavior is described in terms of certain well defined state variables $\underline{q}_i$ ($i = 1, \dots, n$) which may be scalars or tensors. These state variables are assumed to completely characterize the present deformation state of the material and differentiate between geometrically identical specimens with different initial deformation states. They vary along the deformation path according to certain laws and their values at the current time depend upon the deformation history upto this time. Moreover, the history dependence of the rate of nonelastic deformation upto the current time is completely taken into account by the values of $\underline{q}_i$ at this time and no further knowledge of the prior deformation history is required. With this in view, the constitutive equations assume the forms

$$\dot{\underline{\epsilon}}^n = \underline{f}(\underline{\sigma}, \underline{q}_i, T) \quad (2.2)$$

$$\dot{q}_i = g_i(\sigma, q_i, T) \quad (2.3)$$

where σ is the stress tensor and T the temperature. Also, nonelastic deformation is assumed deviatoric, i.e.

$$\dot{\epsilon}_{kk}^n = 0 \quad (2.4)$$

Because of this, the stress σ in eqs. (2.2) and (2.3) is often replaced by its deviatoric part defined as

$$s = \sigma - (1/3)\sigma_{kk}I \quad (2.5)$$

In the above, a tensor A has cartesian components A_{ij} and, as usual, a repeated index implies summation over that index. I is the identity tensor. The initial deformation state of the structure must be specified by prescribing initial values of the state variables q_i .

Many of these state variable theories try to include various phenomena such as strain rate sensitivity, work-hardening, Bauschinger's effect, history dependence, creep recovery, material softening etc. that are observed in metals at elevated temperatures. The state variables q_i have different meanings in different theories. The proposed computational scheme depends only upon the mathematical structure of the constitutive relations as described above and not upon their explicit forms. Therefore, these specific forms are not given here. The specific forms of Hart's constitutive equations, used in numerical calculations, are presented later in the paper.

3. COMPUTATIONAL METHOD

A computational method for solving boundary value problems for three-dimensional metallic bodies subjected to time-varying loads and temperature distributions is presented below. The proposed method consists of formulating a given problem in terms of real time rates which leads to an inhomogeneous linear boundary value problem for the stress or displacement rates. The linear boundary value problem thus obtained is solved to find the rates of the variables of interest throughout the body at any time, and then these rates are integrated forward in time to obtain the time-histories of the variables of interest.

The governing differential equations of the problem in terms of rates are as follows:

$$\text{Kinematic: } \dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^n + \dot{\epsilon}_{ij}^T = \frac{1}{2} (\dot{u}_{i,j} + \dot{u}_{j,i}) \quad (3.1)$$

or, equivalently, the total strain rates must satisfy the compatibility equation

$$\nabla \times \dot{\epsilon} \times \nabla = e_{imp} e_{jsk} \dot{\epsilon}_{ps,mk} = 0 \quad (3.2)$$

$$\text{equilibrium: } \dot{\sigma}_{ij,j} = -\dot{F}_i \quad (3.3)$$

$$\text{constitutive: } \dot{\epsilon}_{ij}^e = \frac{1}{2G} (\dot{\sigma}_{ij} - \frac{\nu}{1+\nu} \dot{\sigma}_{kk} \delta_{ij}) \quad (3.4)$$

$$\dot{\epsilon}_{ij}^n = f(\sigma, q_i, T) \quad (3.5)$$

$$\dot{\epsilon}_{ij}^T = \alpha T \delta_{ij} \quad (3.6)$$

$$\dot{q}_i = g_i(\sigma, q_i, T) \quad (3.7)$$

In the above u_i is the displacement vector, F_i the body force vector per unit volume, ν the Poisson's ratio, G the shear modulus, α the coefficient of linear thermal expansion, e_{ijk} the alternating tensor, δ_{ij} the Kronecker delta and ∇ the gradient operator.

3.1 Stress Rate Formulation.

Using the constitutive equations (3.1) and (3.4), the compatibility equation (3.2) becomes

$$\nabla \times \left[\frac{1}{2G} \left(\dot{\sigma} - \frac{\nu}{1+\nu} \dot{\sigma}_{kk} I \right) \right] \times \nabla = -\nabla \times \dot{\epsilon}^n \times \nabla - \nabla \times [\alpha \dot{T} I] \times \nabla \quad (3.8)$$

The traction boundary condition is

$$\dot{\sigma}_{ij} n_j = \dot{\tau}_i \quad (3.9)$$

where $\underline{n}$ is the unit outward normal to the surface S of the body and $\underline{\tau}$ the surface traction vector prescribed on S .

3.2 Displacement Rate Formulation.

The equilibrium equation (3.3), written in terms of displacement rates using eq. (3.1) and eqs. (3.4, 3.6), becomes

$$\dot{u}_{i,jj} + \frac{1}{1-2\nu} \dot{u}_{j,ji} = -\frac{\dot{F}_i}{G} + 2\dot{\epsilon}_{ij,j}^n + \frac{2(1+\nu)}{1-2\nu} \alpha \dot{T}_{,i} \quad (3.10)$$

The boundary conditions can either involve stresses, i.e.

$$\begin{aligned} \dot{\sigma}_{ij} n_j &= G \left(\frac{2\nu}{1-2\nu} \dot{u}_{j,j} n_i + (\dot{u}_{i,j} + \dot{u}_{j,i}) n_j - 2\dot{\epsilon}_{ij,j}^n - \frac{2(1+\nu)}{1-2\nu} \alpha \dot{T} n_i \right) \\ &= \dot{\tau}_i \quad \text{on } S \end{aligned} \quad (3.11)$$

or displacements, i.e.

$$\dot{u}_i = \dot{\Delta}_i \text{ on } S \quad (3.12)$$

where Δ is the prescribed displacement vector on the surface of the body.

The initial conditions are obtained by specifying the initial distribution of state variables $q_i(x,0)$ throughout the body, and by taking the initial nonelastic strains to be zero. Thus the total initial strain only has elastic and thermal components so that the initial stresses are given by the solution of the corresponding thermoelastic problem. Thus

$$\begin{aligned} \sigma_{ij}(x,0) &\equiv \sigma_{ij}^0(x), \quad \epsilon_{ij}(x,0) \equiv \epsilon_{ij}^0(x) = \epsilon_{ij}^e(x,0) + \epsilon_{ij}^T(x,0) \\ \epsilon_{ij}^n(x,0) &= 0, \quad q_i(x,0) = q_i^0(x) \end{aligned} \quad (3.13)$$

where σ_{ij}^0 and ϵ_{ij}^0 correspond to the thermoelastic solution.

Thermomechanical coupling is ignored. The temperature on the surface of the body S is assumed to change slowly in time and thermal steady state conditions are assumed to exist throughout the body at each time. Thus, at any time, the temperature $T(x,t)$ throughout the body is obtained by solving the steady state heat conduction equation subject to prescribed (time varying) thermal boundary conditions on the surface S . The body force distribution is also assumed to be completely prescribed throughout the body at all times.

The resulting boundary value problems discussed in Sections 3.1 and 3.2 are linear in stress rates and displacement rates respectively. The inhomogeneous terms on the right hand sides of eqs. (3.3), (3.8) and (3.10) are known at any time provided that the spatial distributions of stresses,

state variables, temperature, and body forces are known at that time. The structure of these boundary value problems is seen to be very similar to that of classical thermoelasticity.

3.3 Integration in Time.

The initial distributions of stresses and displacements in a body are first obtained by solving the corresponding thermoelasticity problem. The boundary value problems formulated in Sections 3.1 and 3.2 are then solved for stress and displacement rates. These rates must be integrated in time to find time-histories of stresses, displacements, and other variables of interest. This integration procedure is now described in terms of Euler's method of integration.

The stresses, displacements and state variables at a new time Δt are obtained by using, for example, Euler's method ($\sigma|_{\Delta t} = \sigma^0 + \dot{\sigma}|_{t=0} \Delta t$ etc.). These new stresses and state variables are now used to obtain the rates at time Δt and so on, and the process continued upto the desired final time. Thus, stresses and state variables at time t are used to find the rates at time t by solving the appropriate boundary value problem. These rates are again integrated to find stresses and state variables at $t + \Delta t$. It should be pointed out that these boundary value problems may require use of finite difference or finite element methods for problems involving complicated geometrical shapes. In these cases, initial stresses and displacements may also have to be obtained by such numerical methods.

Because the non-elastic constitutive relations are very stiff in their structure, the Euler's method can not be used for integrating relevant equations. A modified Euler predictor-corrector method which adjusts the step size automatically is used to find numerical results for the sphere problem discussed later in the paper.

If the loadings have discontinuities at some points in time, then elastic loading or unloading must also be considered at those points.

4. ILLUSTRATIVE PROBLEMS

Formulations for boundary value problems involving spheres, cylinders and discs subjected to time varying loads and temperature distributions are presented in this section.

4.1 Sphere under Internal and External Pressures and Radial Temperature Gradient.

A thick spherical shell of internal and external radii a and b is subjected to internal and external pressures $p(t)$ and $q(t)$. The internal temperature is $T_a(t)$ and the external temperature $T_b(t)$ (Fig. 1a). Because of spherical symmetry, the only nonzero displacement component is the radial displacement $u(r,t)$ and the nonzero stress and strain components are σ_r , $\sigma_\theta = \sigma_\phi$, ϵ_r and $\epsilon_\theta = \epsilon_\phi$.

The kinematic and equilibrium equations have the usual forms given in Kumar and Mukherjee [15,16]. The constitutive equations are:

$$\epsilon_r = [\sigma_r - 2\nu\sigma_\theta]/E + \epsilon_r^n + \alpha T \quad (4.1)$$

$$\epsilon_\theta = [(1-\nu)\sigma_\theta - \nu\sigma_r]/E + \epsilon_\theta^n + \alpha T \quad (4.2)$$

$$\dot{\epsilon}_r^n = f_r(\sigma, q_i) \quad , \quad \dot{\epsilon}_\theta^n = f_\theta(\sigma, q_i) \quad (4.3)$$

$$\dot{\epsilon}_r^n + 2\dot{\epsilon}_\theta^n = 0 \quad (4.4)$$

In the above, E is the Young's Modulus. To this list eq. (2.3) must be added. The boundary conditions are as given in [16] with p , q , T_a , T_b functions of time. The initial conditions are

$$q_i(r,0) = q_i^0(r) \quad (4.5)$$

together with the zero time thermoelastic solution to the problem given in Timoshenko and Goodier [17] (see also Kumar [18]).

Using the displacement rate formulation outlined in Section 3.2 the displacement rate and then the stress rates are obtained. The stress rates are given by the following equations

$$\begin{aligned} \dot{\sigma}_r = & \left(\frac{E}{1-\nu} \right) \left(\int_a^r \frac{\dot{\epsilon}_r^n}{n} d\eta - \frac{(r^3-a^3)}{(b^3-a^3)} \frac{b^3}{r^3} \int_a^b \frac{\dot{\epsilon}_r^n}{n} d\eta \right) + \frac{1}{r^3(a^3-b^3)} [\dot{p}a^3(b^3-r^3) + \dot{q}b^3(r^3-a^3)] \\ & + \frac{\alpha E(\dot{T}_a - \dot{T}_b)ab}{(1-\nu)(b^3-a^3)} \left[a+b - \frac{(a^2+ab+b^2)}{r} + \frac{a^2b^2}{r^3} \right] \end{aligned} \quad (4.6)$$

$$\begin{aligned} \dot{\sigma}_\theta = & \left(\frac{E}{1-\nu} \right) \left(\int_a^r \frac{\dot{\epsilon}_r^n}{n} d\eta - \frac{(2r^3+a^3)}{2(b^3-a^3)} \frac{b^3}{r^3} \int_a^b \frac{\dot{\epsilon}_r^n}{n} d\eta + \frac{\dot{\epsilon}_r^n}{2} \right) + \frac{1}{2r^3(a^3-b^3)} [-\dot{p}a^3(2r^3+b^3) + \dot{q}b^3(2r^3+a^3)] \\ & + \frac{\alpha E(\dot{T}_a - \dot{T}_b)ab}{(1-\nu)(b^3-a^3)} \left[a+b - \frac{a^2+ab+b^2}{2r} - \frac{a^2b^2}{2r^3} \right] \end{aligned} \quad (4.7)$$

$$\dot{T} = \dot{T}_b + (\dot{T}_a - \dot{T}_b) \frac{a}{(b-a)} \left(\frac{b}{r} - 1 \right) \quad (4.8)$$

The first term in either of eqs. (4.6) or (4.7) is due to the presence of nonelastic strain rates. The second is the time derivative of Lamé's elastic solution of the problem and the third is the time derivative of the thermoelastic component of the stress.

The problem can now be solved using the integration scheme outlined in Section 3.3 and the time histories of the variables of interest obtained.

Another problem of interest is the response of a sphere to prescribed internal radial displacement and external pressure. The boundary conditions for this problem are

$$u(a,t) = u_a(t) \quad (4.9)$$

$$\sigma_r(b,t) = -q(t) \quad (4.10)$$

For simplicity, the temperature is assumed to be uniform throughout the sphere and steady in time. In this case, the stress rate equations (4.6 and 4.7) must be replaced by

$$\dot{\sigma}_r = \left(\frac{E}{1-\nu} \right) \left(\int_a^r \frac{\dot{\epsilon}_r^n}{n} d\eta - \frac{f(r)}{f(b)} \frac{b^3}{r^3} \int_a^b \frac{\dot{\epsilon}_r^n}{n} d\eta \right) + \frac{2E}{f(b)} a^3 \left(1 - \frac{b^3}{r^3} \right) \frac{\dot{u}_a}{a} - \frac{f(r)}{f(b)} \frac{b^3}{r^3} \dot{q} \quad (4.11)$$

$$\dot{\sigma}_\theta = \left(\frac{E}{1-\nu} \right) \left(\int_a^r \frac{\dot{\epsilon}_r^n}{n} d\eta + \frac{\dot{\epsilon}_r^n}{2} + \frac{g(r)}{f(b)} \frac{b^3}{r^3} \int_a^b \frac{\dot{\epsilon}_r^n}{n} d\eta \right) + \frac{Ea^3}{f(b)} \left(2 + \frac{b^3}{r^3} \right) \frac{\dot{u}_a}{a} + \frac{g(r)}{f(b)} \frac{b^3}{r^3} \dot{q} \quad (4.12)$$

where

$$f(r) = 2(1-2\nu)a^3 + (1+\nu)r^3 \quad (4.13)$$

$$g(r) = (1-2\nu)a^3 - (1+\nu)r^3 \quad (4.14)$$

4.2 Thick Cylinder Subjected to Internal and External Pressures, Axial Force and Torque, and Radial Temperature Gradient.

A long thick-walled cylinder of circular cross section with internal and external radii a and b is subjected to internal and external pressures $p(t)$ and $q(t)$, axial force $P(t)$ and torque $C(t)$. The internal temperature is $T_a(t)$ and the external temperature $T_b(t)$ (Fig. 1(a), (b)).

Neglecting end effects, the radial, tangential and axial displacements have the form (see [15])

$$u_r = u_r(r,t) , \quad u_\theta = \psi(t)rz , \quad u_z = f(t)z \quad (4.15)$$

where $\psi(t)$ is the twist of a cross section per unit length of the cylinder. This gives rise to nonvanishing strains ϵ_r , ϵ_θ , ϵ_z and $\epsilon_{z\theta}$ and corresponding stresses σ_r , σ_θ , σ_z and $\sigma_{z\theta}$.

The kinematic and equilibrium equations have the usual forms (see [15]) with P and C functions of time. The constitutive equations become

$$\epsilon_r = [\sigma_r - \nu(\sigma_\theta + \sigma_z)]/E + \epsilon_r^n + \alpha T \quad (4.16)$$

$$\epsilon_\theta = [\sigma_\theta - \nu(\sigma_r + \sigma_z)]/E + \epsilon_\theta^n + \alpha T \quad (4.17)$$

$$\epsilon_z = [\sigma_z - \nu(\sigma_r + \sigma_\theta)]/E + \epsilon_z^n + \alpha T \quad (4.18)$$

$$\epsilon_{z\theta} = (1+\nu)\sigma_{z\theta}/E + \epsilon_{z\theta}^n \quad (4.19)$$

$$\dot{\epsilon}_r^n = f_r(\sigma, q_i) , \quad \dot{\epsilon}_\theta^n = f_\theta(\sigma, q_i) \quad (4.20, 21)$$

$$\dot{\epsilon}_z^n = f_z(\sigma, q_i) , \quad \dot{\epsilon}_r^n + \dot{\epsilon}_\theta^n + \dot{\epsilon}_z^n = 0 \quad (4.22, 23)$$

together with eq. (2.3).

The boundary conditions are as given in [15]. The initial conditions on the state variables must be specified as in eq. (4.5) and the initial stresses and strains are given by the thermoelastic solution to the problem given in [17,18].

Using the displacement rate formulation outlined in Section 3.2, the displacement rate and then the stress rates are obtained. These stress rates become

$$\begin{aligned} \dot{\sigma}_r = & \frac{E}{2(1-\nu^2)} \left(\int_a^r \frac{\dot{\epsilon}_r^n - \dot{\epsilon}_\theta^n}{\eta} d\eta - \frac{(r^2 - a^2)}{(b^2 - a^2)} \frac{b^2}{r^2} \int_a^b \frac{\dot{\epsilon}_r^n - \dot{\epsilon}_\theta^n}{\eta} d\eta \right) + \frac{E(1-2\nu)}{2(1-\nu^2)} \frac{1}{r^2} \left(\int_a^r \eta \dot{\epsilon}_z^n d\eta - \frac{r^2 - a^2}{b^2 - a^2} \int_a^b \eta \dot{\epsilon}_z^n d\eta \right) \\ & + \frac{1}{r^2(a^2 - b^2)} [\dot{p}a^2(b^2 - r^2) + \dot{q}b^2(r^2 - a^2)] + \frac{\alpha E}{2(1-\nu)(b^2 - a^2)} \left[\dot{T}_b b^2 - \dot{T}_a a^2 + \frac{a^2 b^2}{r^2} (\dot{T}_a - \dot{T}_b) - \dot{T}(b^2 - a^2) \right] \end{aligned} \quad (4.24)$$

$$\begin{aligned} \dot{\sigma}_\theta = & \frac{E}{2(1-\nu^2)} \left(\int_a^r \frac{\dot{\epsilon}_r^n - \dot{\epsilon}_\theta^n}{\eta} d\eta - \frac{(r^2 + a^2)}{(b^2 - a^2)} \frac{b^2}{r^2} \int_a^b \frac{\dot{\epsilon}_r^n - \dot{\epsilon}_\theta^n}{\eta} d\eta \right) - \frac{E(1-2\nu)}{2(1-\nu^2)} \frac{1}{r^2} \left(\int_a^r \eta \dot{\epsilon}_z^n d\eta + \frac{r^2 + a^2}{b^2 - a^2} \int_a^b \eta \dot{\epsilon}_z^n d\eta \right) \\ & - \frac{E}{1-\nu^2} (\dot{\epsilon}_\theta^n + \nu \dot{\epsilon}_z^n) + \frac{1}{r^2(a^2 - b^2)} [-\dot{p}a^2(r^2 + b^2) + \dot{q}b^2(r^2 + a^2)] + \frac{\alpha E}{2(1-\nu)(b^2 - a^2)} \times \\ & \left[\dot{T}_b b^2 - \dot{T}_a a^2 - \frac{a^2 b^2}{r^2} (\dot{T}_a - \dot{T}_b) + \frac{(\dot{T}_a - \dot{T}_b)(b^2 - a^2)}{\ln(b/a)} - \dot{T}(b^2 - a^2) \right] \end{aligned} \quad (4.25)$$

$$\begin{aligned} \dot{\sigma}_z = & \frac{E\nu}{1-\nu^2} \left(\int_a^r \frac{\dot{\epsilon}_r^n - \dot{\epsilon}_\theta^n}{\eta} d\eta - \frac{b^2}{b^2 - a^2} \int_a^b \frac{\dot{\epsilon}_r^n - \dot{\epsilon}_\theta^n}{\eta} d\eta \right) + \frac{E(2-\nu)}{(1-\nu^2)} \frac{1}{(b^2 - a^2)} \int_a^b \eta \dot{\epsilon}_z^n d\eta + \frac{E}{1-\nu^2} [\nu \dot{\epsilon}_r^n - (1-\nu) \dot{\epsilon}_z^n] \\ & + \frac{\dot{P}}{\pi(b^2 - a^2)} + \frac{\alpha E}{(1-\nu)(b^2 - a^2)} \left[\dot{T}_b b^2 - \dot{T}_a a^2 + \frac{(\dot{T}_a - \dot{T}_b)(b^2 - a^2)}{2\ln(b/a)} - \dot{T}(b^2 - a^2) \right] \end{aligned} \quad (4.26)$$

$$\dot{\sigma}_{r\theta} = \frac{2\pi E r}{J(1+\nu)} \int_a^b \eta^2 \dot{\epsilon}_{z\theta}^n d\eta - \frac{E}{1+\nu} \dot{\epsilon}_{z\theta}^n + \frac{\dot{C}r}{J} \quad (4.27)$$

$$\dot{T}(r) = \frac{1}{\ln(b/a)} [\dot{T}_b \ln(r/a) + \dot{T}_a \ln(b/r)] \quad (4.28)$$

where $J = \pi(b^4 - a^4)/2$, is the polar moment of inertia of the cross section.

As in the case of the sphere, a stress rate equation, say eqn. (4.24), contains terms involving the nonelastic strain rates together with the time derivative of the thermoelastic solution of the problem.

The time histories of the variables of interest can now be obtained by using the computational scheme outlined in Section 3.3.

4.3 Thin Circular Disc Subjected to Internal and External Pressures and Radial Temperature Gradient.

The final example consists of a thin uniform circular disc of internal and external radii a and b , thickness h , subjected to internal and external pressures $p(t)$ and $q(t)$. As before, the temperatures on the surfaces $r = a$ and $r = b$ are $T_a(t)$ and $T_b(t)$ respectively (Fig. 1(c)). Plane stress conditions are assumed to prevail so that $\sigma_z = 0$. The nonzero displacement components are $u_r(r, t)$ and u_z which give rise to non-vanishing stresses and strains $\sigma_r, \sigma_\theta, \epsilon_r, \epsilon_\theta$ and ϵ_z . The stresses and strains are functions of r and t only. The kinematic and equilibrium equations have the same form as in [15] (with the angular velocity $\omega = 0$). The constitutive equations have the form

$$\epsilon_r = [\sigma_r - \nu \sigma_\theta] / E + \epsilon_r^n + \alpha T \quad (4.29)$$

$$\epsilon_\theta = [\sigma_\theta - \nu \sigma_r] / E + \epsilon_\theta^n + \alpha T \quad (4.30)$$

$$\epsilon_z = -\nu(\sigma_r + \sigma_\theta) / E + \epsilon_z^n + \alpha T \quad (4.31)$$

$$\dot{\epsilon}_r^n = f_r(\sigma, q_i) \quad , \quad \dot{\epsilon}_\theta^n = f_\theta(\sigma, q_i) \quad (4.32, 33)$$

$$\dot{\epsilon}_z^n = f_z(\sigma, q_i) \quad , \quad \dot{\epsilon}_r^n + \dot{\epsilon}_\theta^n + \dot{\epsilon}_z^n = 0 \quad (4.34, 35)$$

together with eq. (2.3).

The boundary conditions have the same form as in the cylinder problem. The initial conditions on the state variables must be specified and the initial stresses and strains obtained, as usual, from the corresponding thermoelastic solution [17-18].

The stress rates and $\dot{\epsilon}_z$ are obtained in the usual manner. The equations for $\dot{\sigma}_r$ and $\dot{\sigma}_\theta$ have exactly the same form as eqns. (4.24) and (4.25) with the Poisson's ratio ν set to zero. The equation for $\dot{T}$ is the same as eqn. (4.28) while $\dot{\epsilon}_z$ is given by the following equation

$$\begin{aligned} \dot{\epsilon}_z = & -\nu \left(r \int_a^r \frac{\dot{\epsilon}_r - \dot{\epsilon}_\theta}{\eta} d\eta - \frac{b^2}{b^2 - a^2} \int_a^b \frac{\dot{\epsilon}_r - \dot{\epsilon}_\theta}{\eta} d\eta \right) + \frac{\nu}{b^2 - a^2} \int_a^b \eta \dot{\epsilon}_z d\eta + \dot{\epsilon}_z + \nu \dot{\epsilon}_\theta + \frac{\nu}{E(a^2 - b^2)} (\dot{p}a^2 - \dot{q}b^2) \\ & - \frac{\alpha \nu}{(b^2 - a^2)} \left[\dot{T}_b b^2 - \dot{T}_a a^2 + \frac{(\dot{T}_a - \dot{T}_b)(b^2 - a^2)}{2 \ln(b/a)} \right] + \alpha \dot{T} (1 + \nu) \end{aligned} \quad (4.36)$$

The change of disc thickness is governed by the equation

$$\dot{h} = h \dot{\epsilon}_z \quad (4.37)$$

and the problem can be solved by using the computational scheme outlined in Section 3.3.

5. HART'S CONSTITUTIVE RELATIONS

The constitutive relations proposed by Hart [7,8,19] are used in numerical calculations for the problems discussed in the previous section. Hart's equations have been used to accurately predict uniaxial deformation behavior of several metals and alloys. Recently, uniaxial creep and tensile tests were carried out on Nickel and 304 stainless steel specimens at various temperatures for various values of stress and strain rate and the results compared with predictions from Hart's constitutive equations. The correlation between theory and experiment was found to be very good. These results are available in [20] and [21]. A brief review of Hart's constitutive relations is presented in this section. The details can be obtained from [7,8,19].

According to Hart, the nonelastic strain rate $\dot{\underline{\epsilon}}^n$ consists of three components

$$\dot{\underline{\epsilon}}^n = \dot{\underline{\epsilon}}^a + \dot{\underline{\epsilon}}^p + \dot{\underline{\epsilon}}^g \quad (5.1)$$

Here $\underline{\epsilon}^a$ is the anelastic strain due to grain matrix deformation, a stored strain that is eventually completely recoverable upon unloading. $\underline{\epsilon}_g^a$ is the anelastic strain due to grain boundary sliding and this also is completely recoverable upon unloading. $\dot{\underline{\epsilon}}^p$ is the permanent strain rate and the corresponding permanent strain $\underline{\epsilon}^p$ is path dependent and completely irrecoverable.

The state variables are the anelastic strain tensor $\underline{\epsilon}^a$ and scalar hardness σ^* . Let $\underline{s}$ represent the deviatoric part of stress tensor $\underline{\sigma}$. The deviatoric stress tensor $\underline{s}$ is composed of two auxiliary tensors $\underline{s}^a$ and $\underline{s}^f$

$$\underline{s} = \underline{s}^a + \underline{s}^f \quad (5.2)$$

The flow relations are

$$\dot{\underline{\epsilon}}^a = \frac{3}{2} \frac{\underline{\epsilon}^a}{\sigma^a} \underline{s}^a \quad (5.3)$$

$$\dot{\epsilon}^p = \frac{3}{2} \frac{\dot{\epsilon}^p}{\sigma^a} s^a \quad (5.4)$$

$$\dot{\epsilon}^n = \frac{3}{2} \frac{\dot{\epsilon}^n}{\sigma^f} s^f \quad (5.5)$$

$$\dot{\epsilon}_g^a = \frac{3}{2} \frac{\dot{\epsilon}_g^a}{\sigma} s \quad (5.6)$$

where $\dot{\epsilon}^n$, $\dot{\epsilon}^p$, ϵ^a , σ , σ^a and σ^f are scalars defined as

$$\begin{aligned} \dot{\epsilon}^n &\equiv \sqrt{\frac{2}{3} \dot{\epsilon}_{ij}^n \dot{\epsilon}_{ij}^n}, \quad \dot{\epsilon}^p \equiv \sqrt{\frac{2}{3} \dot{\epsilon}_{ij}^p \dot{\epsilon}_{ij}^p}, \quad \epsilon^a \equiv \sqrt{\frac{2}{3} \epsilon_{ij}^a \epsilon_{ij}^a} \\ \sigma &\equiv \sqrt{\frac{3}{2} s_{ij} s_{ij}}, \quad \sigma^a \equiv \sqrt{\frac{3}{2} s_{ij}^a s_{ij}^a}, \quad \sigma^f \equiv \sqrt{\frac{3}{2} s_{ij}^f s_{ij}^f} \end{aligned} \quad (5.7)$$

These scalars are related to each other through the uniaxial relations

$$\sigma^a = M \epsilon^a \quad (5.8)$$

$$\dot{\epsilon}^n = \dot{a}^* \left(\frac{\sigma^f}{M} \right) \quad (5.9)$$

$$\dot{\epsilon}^p = \dot{\epsilon}^* \left(\ln \frac{\sigma^*}{\sigma^a} \right)^{-1/\lambda} \quad (5.10)$$

$$\dot{\sigma}^* = \dot{\epsilon}^p \sigma^* \Gamma(\sigma^*, \sigma^a) \quad (5.11)$$

$$\dot{\epsilon}_g^a = \dot{a}_g^* \left(\frac{\sigma}{M_g} - \epsilon_g^a \right) \quad (5.12)$$

where

$$\dot{\epsilon}^* \equiv \left(\frac{\sigma}{G} \right)^{m^*} \exp\left(-\frac{Q}{RT}\right) \quad (5.13)$$

In the above, M , M , m , λ , f , $\dot{a}^*$, $\dot{a}_g^*$ and M_g are flow parameters for a material and some of these are functions of temperature, R is the gas constant, Q the activation energy for atomic self-diffusion and G the shear modulus at temperature T . Γ is the work hardening function. The flow parameters are determined experimentally from a series of load relaxation and load change tests and the work hardening function from a series of strain hardening tests. It can be verified easily that the above constitutive relations have the same structure as described by eqs. (2.2)-(2.4).

The viscoplastic limit.

The viscoplastic limit is defined to be the region where $\sigma^a \approx \sigma^*$ [20]. In the region $\sigma^a \approx \sigma^*$ eq. (5.10) becomes nearly singular, and, for computational purposes, must be replaced by

$$\dot{\epsilon}^p = \frac{s_{ij}^a \dot{\epsilon}_{ij}^n / \sigma^a}{1 + \frac{\sigma^* \Gamma}{M}} \quad (5.14)$$

which is obtained by using the condition $\dot{\sigma}^a = \dot{\sigma}^*$ shown to be true in the viscoplastic region. For loading histories without sharp discontinuities, a simpler criterion can be used to define the viscoplastic limit. This is given by

$$\dot{\epsilon}^p > \dot{\epsilon}_c^p \quad \text{in the viscoplastic limit}$$

where

$$\dot{\epsilon}_c^p = \dot{\epsilon}_{cd}^p (\sigma^* / \sigma_d^*)^m \quad (5.15)$$

and $\dot{\epsilon}_{cd}^p$ and σ_d^* are experimentally obtained reference strain rate and hardness parameters that are material and temperature dependent.

6. NUMERICAL RESULTS AND DISCUSSION

Results of numerical calculations carried out for a 304 stainless steel sphere at a uniform temperature of 400°C are presented in this section. Grain boundary sliding is neglected so that $\epsilon_g^a = 0$ throughout the sphere for all time. The work hardening function in this case has the form

$$\Gamma(\sigma^*, \sigma^a) = (\beta/\sigma^*)^\delta (\sigma^a/\sigma^*)^{\beta/\sigma^*} \quad (6.1)$$

The values of the various flow parameters are [20,21]

$$\begin{aligned} \lambda &= 0.15 & M &= 7.8 & m &= 5 \\ \dot{a}^* &= 0.6937 \times 10^{25} \text{ sec}^{-1} & M &= 0.132 \times 10^8 \text{ psi} \\ \dot{\epsilon}^* &= 0.252 \times 10^{-20} (\sigma^*/45,670)^5 \text{ sec}^{-1} \end{aligned}$$

The work hardening parameters are

$$\beta = 0.179 \times 10^6 \text{ psi} \quad \delta = 1.33$$

and the parameters defining the viscoplastic limit are

$$\dot{\epsilon}_d^{pc} = 0.2123 \times 10^{-8} \text{ sec}^{-1} \quad \sigma_d^* = 10,000 \text{ psi}$$

Finally, the elastic constants are

$$E = 0.244 \times 10^8 \text{ psi} \quad \nu = 0.298$$

In the above, the hardness and stress are in units of psi and time is in seconds.

The appropriate equations are nondimensionalized through the use of the following dimensionless variables

$$\bar{\gamma} = \gamma/\Sigma \quad (6.2)$$

where γ is any variable having the dimensions of stress and $\Sigma = 10,000$ psi.

Also,

$$\xi = r/a, \quad k = b/a, \quad \bar{u} = u/a \quad (6.3)$$

Either the internal pressure or the internal displacement is assumed to be prescribed in time and the external pressure q is taken to be zero for all time in all cases. The initial stresses and strains are obtained from Lamé's elastic solution to the appropriate problem. The initial anelastic and permanent strains are taken to be zero in all cases while the initial hardness distribution is prescribed at an uniform value throughout the sphere.

The radius ratio k from the sphere is 1.5 and the dimensionless shell thickness $k-1$ is divided into 20 equal segments for these numerical calculations. Spatial integrals are evaluated using Simpson's rule. Integration in time is carried out using a modified Euler predictor-corrector scheme of the type

$$\frac{dy}{dt} = f(y,t) \quad (6.4)$$

$$y^P(t+\Delta t) = y(t) + \Delta t f(y,t) \quad (6.5)$$

$$y^C(t+\Delta t) = y(t) + 1/2 \Delta t (f(y,t) + f(y^P, t+\Delta t)) \quad (6.6)$$

$$y(t+\Delta t) = (4/5)y^C(t+\Delta t) + (1/5)y^P(t+\Delta t) \quad (6.7)$$

The maximum percentage difference between predictor and corrector values for any variable being integrated, at any spatial location within

the sphere, at any time step, is limited to .05%. A much tighter error bound does not alter the results significantly. Typical computing times for single calculations are between one to two minutes on an IBM 370/168 Computer.

Numerical results for several prescribed histories of internal pressure and internal displacement are presented in Figs. 2-14. Results for constant internal pressure are shown in Figs. 2-5. It is seen from Fig. 2 that an initially hardened sphere (11% initial cold work) creeps less than an identical annealed sphere at the same pressure (15 ksi). As expected, larger pressures cause more "initial" deformation and creep and the response is highly nonlinear. For a pressure of 20 ksi, for example, the hoop strain at the inner wall of the sphere is about 1.55% within a very short time. This is the sum of the elastic and "instantaneous plastic" strain of classical plasticity. Redistribution of stress invariants and hardness and of radial and tangential stress are shown in Figs. 3-5. The anelastic stress invariant σ^a , which is initially zero, increases till it equals the initial hardness (17 ksi) at the inner wall (Fig. 3). The hardness is then driven up until $\sigma^a \approx \sigma^*$ throughout the sphere. At all times, $\sigma^* \geq \sigma^a$ and $\sigma \geq \sigma^a$. The "instantaneous" deformation of classical plasticity is accumulated very quickly during the saturation of the anelastic stress invariant σ^a . Redistribution of the radial and tangential stresses (Figs. 4-5) is simulated as expected. A discontinuity in the slope of the tangential stress is seen to propagate quickly from the inside to the outside of the sphere even though Hart's is a theory without a yield surface.

The response to an interrupted load history is shown in Fig. 6. There is creep recovery at zero load. The effect of previous history on subsequent

deformation is evident from the displacement at the inner wall of the sphere upon reloading. The material hardens and the creep rate is lower than at the start. The change of hardness and the anelastic stress invariant at the inner wall with time for this case is shown in Fig. 7. The state variables have different values at the point of reloading compared to those at the start and this causes the difference in the response of the sphere.

The response to a load change is shown in Fig. 8. The creep strain rate increases upon increase of load and the displacement for the case of a constant internal pressure of 20 ksi is greater than that for the case where the pressure is 15 ksi for the first 3 hours and is then raised to 20 ksi.

A comparison with classical plasticity for the case of pressure increasing at a constant rate is shown in Fig. 9. The prescribed pressure rate corresponds to an initial tangential strain rate at the inner wall of 10^{-3}sec^{-1} . The yield stress and slope of the stress-strain curve for the rigid-linear work hardening plasticity model is obtained from uniaxial work hardening experiments, performed at Cornell [20]. The correlation for two values of initial hardness is excellent. Redistribution of radial and tangential stress for this case is shown in Figs. 10 and 11. The expected propagation of the discontinuity in the slope of the tangential stress is seen in Fig. 11.

Strain rate sensitivity is shown in Fig. 12. This time the internal displacement history is prescribed. The strain rate sensitivity of 304 SS at 400°C is known to be small from experiments and this is seen in Fig. 12. The higher the strain rate, the larger is the pressure required to get the same accumulated strain.

Finally, the response to cyclic internal displacement is shown in Fig. 13. Bauschinger's effect and cyclic hardening is simulated by the theory. The nature of the stress-strain curve is qualitatively similar to that obtained from uniaxial cyclic straining experiments on metals (see, for example, [22]). The behavior of the hardness and the anelastic stress invariant for this case is shown in Fig. 14. There are periods where the two are equal (analogous to tensile or compressive yielding in classical plasticity) interspersed with periods in which σ^a drops to zero and increases again.

7. CONCLUSIONS

The proposed computational scheme is very efficient for analysing the time-dependent inelastic response of structures obeying Hart's constitutive relations. Hart's model, which has been previously shown to accurately simulate quantitative uniaxial experimental results on metals for a variety of stress and strain histories, is here shown to qualitatively simulate a wide variety of experimentally observed phenomena in creep and plasticity for a structural element subjected to a multiaxial state of stress. The scalar state variable called hardness and the tensor state variable called anelastic strain is found to be adequate for representing the effect of previous deformation history on subsequent deformation behavior.

ACKNOWLEDGEMENTS

This research was supported by contract No. EY-76-S-02-2733.#000 of the Energy Research and Development Administration, Washington, D.C., with Cornell University.

REFERENCES

1. Rabotnov, Y.N., "Creep Problems in Structural Members," North-Holland, Amsterdam, 1969.
2. Penny, R.K., and Marriott, D.L., "Design for Creep," McGraw-Hill, London, 1971.
3. Odqvist, F.K.G., "Mathematical Theory of Creep and Creep Rupture". Oxford Clarendon Press, 1966.
4. Krempl, E., "The Interaction of Rate and History-dependent Effects and its Significance for Slow Cyclic Inelastic Analysis at Elevated Temperatures," Nuclear Engineering and Design, Vol. 29, No. 1, Nov. 1974, pp. 125-134.
5. Krempl, E., "Cyclic Creep - an Interpretive Literature Survey," Welding Research Council Bulletin, No. 195, June 1974, pp. 63-123.
6. Krempl, E., "On the Interaction of Rate and History Dependence in Structural Metals," Acta Mechanica, Vol. 22, No. 1-2, 1975, pp. 53-90.
7. Hart, E.W., Li, C.Y., Yamada, H., and Wire, G.L., "Phenomenological Theory: A Guide to Constitutive Relations and Fundamental Deformation Properties," in "Constitutive Equations in Plasticity," (ed. A.S. Argon), M.I.T. Press, Cambridge, Mass., 1976, pp. 149-197.
8. Hart, E.W., "Constitutive Relations for the Non-Elastic Deformation of Metals", Journal of Engineering Materials and Technology, Trans. ASME, Vol. 98, No. 3, 1976, pp. 193-202.
9. Miller, A., "An Inelastic Constitutive Model for Monotonic, Cyclic and Creep Deformation: Part I - Equations Development and Analytical Procedures," Journal of Engineering Materials and Technology, Trans. ASME, Vol. 98, No. 2, 1976, pp. 97-105.
10. Miller, A., "An Inelastic Constitutive Model for Monotonic, Cyclic and Creep Deformation: Part II - Application to type 304 Stainless Steel," Journal of Engineering Materials and Technology, Trans. ASME, Vol. 98, No. 2, 1976, pp. 106-113.
11. Lagneborg, R., "A Modified Recovery-Creep Model and Its Evaluation," Metal Science Journal, Vol. 6, July 1972, pp. 127-133.
12. Robinson, D.N., "A Candidate Creep-Recovery Model For 2 1/4 Cr-1 Mo Steel and its Experimental Implementation" Oak Ridge National Laboratory, Report No. ORNL-TM-5110, 1975.

13. Bodner, S.R. and Partom, Y., "Constitutive Equations for Elastic Viscoplastic Strain-Hardening Materials," *Journal of Applied Mechanics*, Vol. 42, No. 2, Trans. ASME, Vol. 97, Series E, June 1975, pp. 385-389.
14. Kumar, V. and Mukherjee, S., "Creep Analysis of Structures using a New Equation of State Type Constitutive Relation," *Computers & Structures*, Vol. 6, 1976, pp. 429-437.
15. Kumar, V., and Mukherjee, S., "Time-Dependent Inelastic Analysis of Metallic Media Using Constitutive Relations with State Variables," *Nuclear Engineering and Design*, Vol. 41, March 1977, pp. 27-43.
16. Kumar, V., and Mukherjee, S., "Creep Analysis of Metallic Structures in the Presence of Thermal Gradients Using Newer Constitutive Relations," *Journal of Pressure Vessel Technology*, Trans. ASME, Vol. 99, Series J, May 1977, pp. 272-280.
17. Timoshenko, S.P. and Goodier, J.N., "Theory of Elasticity," 3rd ed., McGraw Hill, 1970.
18. Kumar, V., "Time-dependent Inelastic Analysis of Metallic Media using Constitutive Relations with State Variables," Ph.D. Thesis, Cornell University, Ithaca, New York, May 1977.
19. Hart, E.W., "Constitutive Relations for Nonelastic Deformation," Invited paper No. L 1/1, Trans. 4th SMiRT Conference Jager T.A., and Boley, B.A. ed., Comm. of the European Communities, Brussels, Vol. L, August 1977.
20. Li, C-Y, "Deformation in Type 304 Stainless Steel," Third Quarterly Report to EPRI under Project No. RP697-1, Department of Materials Science and Engineering, Cornell University, Ithaca, New York, March 1977.
21. Li, C-Y, "Deformation in Type 304 Stainless Steel". Fourth Quarterly Report to EPRI under Project No. RP697-1, Department of Materials Science and Engineering, Cornell University, Ithaca, New York. Under preparation.
22. Corum, J.M., "Material Property Data for Elastic-Plastic-Creep Analyses of Stiffened Shear Leg Panel," Appendix C to ORNL-SUB-3754-1 by R.L. Egger, 1974, p. 222.

Figure 1. Structural elements with loads. (a) Sphere (a,b) Cylinder (c) Disc.

Figure 2. Creep-radial displacements at inner wall as functions of time for various values of constant internal pressure and initial hardness.

Figure 3. Creep - redistribution of stress invariants and hardness. $\bar{p} = 2.0$, $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 4. Creep - redistribution of radial stress. $\bar{p} = 2.0$, $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 5. Creep - redistribution of tangential stress. $\bar{p} = 2.0$, $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 6. Interrupted load - radial displacement at the inner wall as a function of time.
 $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 7. Interrupted load - hardness and anelastic stress invariant at the inner wall as functions of time. $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 8. Load change - radial displacement at the inner wall as a function of time. $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 9. Work hardening - internal pressures as functions of radial displacements at the inner wall for two values of initial hardness.

Figure 10. Work hardening - redistribution of radial stress. $\dot{\bar{p}} = 2.2338$, $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 11. Work hardening - redistribution of tangential stress. $\dot{\bar{p}} = 2.2338$, $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 12. Strain rate sensitivity - internal pressures as functions of radial displacements at the inner wall for two values of internal displacement rate. $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 13. Cyclic strain - radial stress as a function of radial displacement at the inner wall. $\bar{\sigma}_0^* = 1.7$ (annealed).

Figure 14. Cyclic strain - hardness and anelastic stress invariant at the inner wall as functions of time. $\bar{\sigma}_0^* = 1.7$ (annealed).

List of Illustrations

- Figure 1. Structural elements with loads. (a) Sphere (a,b) Cylinder (c) Disc.
- Figure 2. Creep - radial displacements at inner wall as function of time for various values of constant internal pressure and initial hardness.
- Figure 3. Creep - redistribution of stress invariants and hardness. $\bar{p} = 2.0$, $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 4. Creep - redistribution of radial stress. $\bar{p} = 2.0$, $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 5. Creep - redistribution of tangential stress. $\bar{p} = 2.0$, $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 6. Interrupted load - radial displacement at the inner wall as a function of time. $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 7. Interrupted load - hardness and anelastic stress invariant at the inner wall as functions of time. $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 8. Load change - radial displacement at the inner wall as a function of time $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 9. Work hardening - internal pressures as functions of radial displacements at the inner wall for two values of initial hardness.
- Figure 10. Work hardening - redistribution of radial stress. $\bar{p} = 2.2338$, $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 11. Work hardening - redistribution of tangential stress. $\bar{p} = 2.2338$, $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 12. Strain rate sensitivity - internal pressures as functions of radial displacements at the inner wall for two values of internal displacement rate. $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 13. Cyclic strain - radial stress as a function of radial displacement at the inner wall. $\bar{\sigma}_0^* = 1.7$ (annealed).
- Figure 14. Cyclic strain - hardness and anelastic stress invariant at the inner wall as functions of time. $\bar{\sigma}_0^* = 1.7$ (annealed).